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# Profinite groups with restricted centralizers of commutators

**Eloisa Detomi**

Dipartimento di Matematica, Università di Padova  
Via Trieste 63, 35121 Padova, Italy  
(detomi@math.unipd.it)

**Marta Morigi**

Dipartimento di Matematica, Università di Bologna  
Piazza di Porta San Donato 5, 40126 Bologna, Italy  
(marta.morigi@unibo.it)

**Pavel Shumyatsky**

Department of Mathematics, University of Brasilia  
Brasilia-DF, 70910-900 Brazil (pavel@unb.br)

**ABSTRACT.** A group  $G$  has restricted centralizers if for each  $g$  in  $G$  the centralizer  $C_G(g)$  either is finite or has finite index in  $G$ . A theorem of Shalev states that a profinite group with restricted centralizers is abelian-by-finite. In the present article we handle profinite groups with restricted centralizers of word-values. We show that if  $w$  is a multilinear commutator word and  $G$  a profinite group with restricted centralizers of  $w$ -values, then the verbal subgroup  $w(G)$  is abelian-by-finite.

*Keywords:* group words, profinite groups, centralizers, FC-groups

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*Dedicated to Aner Shalev on the occasion of his 60th birthday.*

## 1. Introduction

A group  $G$  is said to have restricted centralizers if for each  $g$  in  $G$  the centralizer  $C_G(g)$  either is finite or has finite index in  $G$ . This notion was introduced by Shalev in [25] where he showed that a profinite group with restricted centralizers is finite-by-abelian-by-finite. Note that a finite-by-abelian profinite group is necessarily abelian-by-finite

so Shalev's theorem essentially states that a profinite group with restricted centralizers is abelian-by-finite.

In the present article we handle profinite groups with restricted centralizers of word-values. Given a word  $w$  and a group  $G$ , we denote by  $G_w$  the set of all values of  $w$  in  $G$  and by  $w(G)$  the subgroup generated by  $G_w$ . In the case where  $G$  is a profinite group  $w(G)$  denotes the subgroup topologically generated by  $G_w$ .

Recall that multilinear commutator words are words which are obtained by nesting commutators, but using always different variables. Such words are also known under the name of outer commutator words and are precisely the words that can be written in the form of multilinear Lie monomials.

The main purpose of this paper is to prove the following theorem.

**THEOREM 1.1.** *Let  $w$  be a multilinear commutator word and  $G$  a profinite group in which all centralizers of  $w$ -values are either finite or open. Then  $w(G)$  is abelian-by-finite.*

From the above theorem we can deduce the following results.

**COROLLARY 1.2.** *Under the hypothesis of Theorem 1.1, the group  $G$  has an open subgroup  $T$  such that  $w(T)$  is abelian. In particular  $G$  is soluble-by-finite.*

**COROLLARY 1.3.** *Let  $w$  be a multilinear commutator word and  $G$  a profinite group in which every nontrivial  $w$ -value has finite centralizer. Then either  $w(G) = 1$  or  $G$  is finite.*

The proof of Theorem 1.1 is fairly complicated. We will now briefly describe some of the tools employed in the proof.

Recall that a group  $G$  is an FC-group if the centralizer  $C_G(g)$  has finite index in  $G$  for each  $g \in G$ . Equivalently,  $G$  is an FC-group if each conjugacy class  $g^G$  is finite. A group  $G$  is a BFC-group if all conjugacy classes in  $G$  are finite and have bounded size. A famous theorem of B. H. Neumann says that the commutator subgroup of a BFC-group is finite [21]. Shalev used this to show that a profinite FC-group has finite commutator subgroup [25].

In Section 4 we generalize Shalev's result by showing that if  $w$  is a multilinear commutator word and  $G$  is a profinite group in which all  $w$ -values are FC-elements, then  $w(G)$  has finite commutator subgroup. In fact, we establish a much stronger result involving the marginal subgroup introduced by P. Hall (see Section 4 for details). The results of Section 4 enable us to reduce Theorem 1.1 to the case where all  $w$ -values have finite order.

A famous result by Zelmanov says that periodic profinite groups are locally finite [34]. Recall that a group is said to locally have some property if all its finitely generated subgroups have that property. There is a conjecture stating that for any word  $w$  and any profinite group  $G$  in which all  $w$ -values have finite order, the verbal subgroup  $w(G)$  is locally finite. The conjecture is known to be correct in a number of particular cases (see [28, 17, 3]). In Section 5 we obtain another result in this direction. Namely, let  $p$  be a prime,  $w$  a multilinear commutator word and  $G$  a profinite group in which all  $w$ -values have finite  $p$ -power order. We prove that the abstract subgroup generated by all  $w$ -values is locally finite.

The proof of the above result relies on the techniques created by Zelmanov in his solution of the Restricted Burnside Problem [35]. While the result falls short of proving that  $w(G)$  is locally finite, it will be shown to be sufficient for the purposes of the present paper. Indeed, in Section 6 we prove that if a profinite group  $G$  satisfies the hypotheses of Theorem 1.1 and has all  $w$ -values of finite order, then  $w(G)$  is locally finite. This is achieved by combining results of previous sections with the ones obtained in [17] and [3].

In Section 7 we finalize the proof of Theorem 1.1. At this stage without loss of generality we can assume that  $w(G)$  is locally finite and at least one  $w$ -value has finite centralizer. With these assumptions, a whole range of tools (in particular, those using the classification of finite simple groups) become available. We appeal to Wilson's theorem on the structure of compact torsion groups which implies that in our situation  $w(G)$  has a finite series of closed characteristic subgroups in which each factor either is a pro- $p$  group for some prime  $p$  or is isomorphic (as a topological group) to a Cartesian product of finite simple groups.

Recall that the famous Ore's conjecture, stating that every element of a nonabelian finite simple group is a commutator, was proved in [20]. It follows that for each multilinear commutator word  $w$  every element of a nonabelian finite simple group is a  $w$ -value. If a group  $K$  is isomorphic to a Cartesian product of nonabelian finite simple groups and has restricted centralizers of  $w$ -values, then actually all centralizers of elements in  $K$  are either finite or of finite index and so, by Shalev's theorem [25],  $K$  is finite. We use this observation to conclude that under our assumptions the verbal subgroup is (locally soluble)-by-finite. Finally, an application of the results on FC-groups obtained in Section 4 completes the proof of Theorem 1.1.

The next section contains a collection of mostly well-known auxiliary lemmas which are used throughout the paper. In Section 3 we

describe combinatorial techniques developed in [8, 3, 4] for handling multilinear commutator words. We also prove some new lemmas which are necessary for the purposes of the present article. Throughout the paper, unless explicitly stated otherwise, subgroups of profinite groups are assumed closed.

## 2. Auxiliary lemmas

Multilinear commutator words are words which are obtained by nesting commutators, but using always different variables. More formally, the word  $w(x) = x$  in one variable is a multilinear commutator; if  $u$  and  $v$  are multilinear commutators involving different variables then the word  $w = [u, v]$  is a multilinear commutator, and all multilinear commutators are obtained in this way.

An important family of multilinear commutator words is formed by so-called derived words  $\delta_k$ , on  $2^k$  variables, defined recursively by

$$\delta_0 = x_1, \quad \delta_k = [\delta_{k-1}(x_1, \dots, x_{2^{k-1}}), \delta_{k-1}(x_{2^{k-1}+1}, \dots, x_{2^k})].$$

Of course  $\delta_k(G) = G^{(k)}$  is the  $k$ -th term of the derived series of  $G$ .

We recall the following well-known result (see for example [27, Lemma 4.1]).

**LEMMA 2.1.** *Let  $G$  be a group and let  $w$  be a multilinear commutator word on  $n$  variables. Then each  $\delta_n$ -value is a  $w$ -value.*

The following is Lemma 4.2 in [27]

**LEMMA 2.2.** *Let  $w$  be a multilinear commutator word and  $G$  a soluble group in which all  $w$ -values have finite order. Then the verbal subgroup  $w(G)$  is locally finite.*

If  $x$  is an element of a group  $G$ , we write  $x^G$  for the conjugacy class of  $x$  in  $G$ . More generally, if  $S$  is a subset of  $G$ , we write  $S^G$  for the set of conjugates of elements of  $S$ . On the other hand, if  $K$  is a subgroup of  $G$ , then  $K^G$  denotes the normal closure of  $K$  in  $G$ , that is, the subgroup generated by all conjugates of  $K$  in  $G$ , with the usual convention that if  $G$  is a topological group then  $K^G$  is a closed subgroup.

Recall that if  $G$  is a group,  $a \in G$  and  $H$  is a subgroup of  $G$ , then  $[H, a]$  denotes the subgroup of  $G$  generated by all commutators of the form  $[h, a]$ , where  $h \in H$ . It is well-known that  $[H, a]$  is normalized by  $a$  and  $H$ .

We will denote by  $\Delta(G)$  the set of FC-elements of  $G$ , i.e.

$$\Delta(G) = \{x \in G \mid |x^G| < \infty\}.$$

Obviously  $\Delta(G)$  is a normal subgroup of  $G$ . Note that if  $G$  is a profinite group,  $\Delta(G)$  needs not be closed.

LEMMA 2.3. *Let  $G$  be a group. For every  $x \in \Delta(G)$  the subgroup  $[\Delta(G), x]^G$  is finite.*

PROOF. Let  $\Delta = \Delta(G)$ . Note that  $\Delta'$  is locally finite (see [24, Section 14.5]). The subgroup  $[\Delta, x]$  is generated by finitely many commutators  $[y, x]$  where  $y \in \Delta$ . Hence  $[\Delta, x]$  is finite. Further, each commutator  $[y, x]$  is an FC-element and so  $C_G([\Delta, x])$  has finite index in  $G$ . Consequently,  $[\Delta, x]^G$  is a product of finitely many conjugates of  $[\Delta, x]$ . The conjugates of  $[\Delta, x]$  normalize each other so  $[\Delta, x]^G$  is finite.  $\square$

LEMMA 2.4. *Let  $G$  be a locally nilpotent group containing an element with finite centralizer. Suppose that  $G$  is residually finite. Then  $G$  is finite.*

PROOF. Choose  $x \in G$  such that  $C_G(x)$  is finite. Let  $N$  be a normal subgroup of finite index such that  $N \cap C_G(x) = 1$ . Assume that  $N \neq 1$  and let  $1 \neq y \in N$ . The subgroup  $\langle x, y \rangle$  is nilpotent and so the center of  $\langle x, y \rangle$  has nontrivial intersection with  $N$ . This is a contradiction since  $N \cap C_G(x) = 1$ .  $\square$

Lemma 1.6.1 in [16] states that if  $G$  is a finite group,  $N$  is a normal subgroup of  $G$  and  $x$  an element of  $G$ , then  $|C_{G/N}(xN)| \leq |C_G(x)|$ . We will need a version of this lemma for locally finite groups.

LEMMA 2.5. *Let  $G$  be a locally finite group and  $x$  an element of  $G$  such that  $C_G(x)$  is finite of order  $m$ . If  $N$  is a normal subgroup of  $G$ , then  $|C_{G/N}(xN)| \leq m$ .*

PROOF. Arguing by contradiction, assume that  $C_{G/N}(xN)$  contains  $m + 1$  pairwise distinct elements  $b_1N, \dots, b_{m+1}N$ . Let  $K = \langle x, b_1, \dots, b_{m+1} \rangle$  and  $N_0 = N \cap K$ . Note that  $K$  is a finite group and  $C_{K/N_0}(xN_0)$  contains the  $m + 1$  distinct elements  $b_1N_0, \dots, b_{m+1}N_0$ . This contradicts Lemma 1.6.1 in [16].  $\square$

LEMMA 2.6. *Let  $d, r, s$  be positive integers. Let  $G$  be a soluble group of derived length  $d$  generated by a set  $X$  such that every element in  $X$  has finite order dividing  $r$  and has at most  $s$  conjugates in  $G$ . Then  $G$  has finite exponent bounded by a function of  $d, r, s$ .*

PROOF. The proof is by induction on the derived length of  $G$ . If  $G$  is abelian then  $G$  has exponent dividing  $r$ . Note that  $G'$  is generated by all conjugates of the set  $\{[y, z] | y, z \in X\}$ . As  $y, z \in X$  have at most  $s$  conjugates in  $G$  it follows that  $[y, z]$  has at most  $s^2$  conjugates. Note

that the center of  $\langle y, z \rangle$  coincides with  $C_{\langle y, z \rangle}(y) \cap C_{\langle y, z \rangle}(z)$  so it has index at most  $s^2$ , thus the order of the derived subgroup of  $\langle y, z \rangle$  is bounded by a function of  $s$  by Schur's theorem [24, 10.1.4]. By induction, the exponent of  $G'$  is finite and bounded by a function of  $d, r, s$ . As  $G/G'$  has exponent at most  $r$ , the result follows.  $\square$

Throughout the paper, we will use without explicit references the following result.

**LEMMA 2.7.** *Let  $G$  be a finite-by-abelian profinite group. Then  $G$  is central-by-finite.*

**PROOF.** Let  $T$  be a finite normal subgroup of  $G$  such that  $G/T$  is abelian and let  $N$  be an open normal subgroup of  $G$  such that  $N \cap T = 1$ . Then  $N \cap G' = 1$  and so  $N$  is central in  $G$ .  $\square$

### 3. Combinatorics of commutators

We will need some machinery concerning combinatorics of commutators, so we now recall some notation from the paper [4].

Throughout this section,  $w = w(x_1, \dots, x_n)$  will be a fixed multilinear commutator word. If  $A_1, \dots, A_n$  are subsets of a group  $G$ , we write

$$\mathcal{X}_w(A_1, \dots, A_n)$$

to denote the set of all  $w$ -values  $w(a_1, \dots, a_n)$  with  $a_i \in A_i$ . Moreover, we write  $w(A_1, \dots, A_n)$  for the subgroup  $\langle \mathcal{X}_w(A_1, \dots, A_n) \rangle$ . Note that if every  $A_i$  is a normal subgroup of  $G$ , then  $w(A_1, \dots, A_n)$  is normal in  $G$ .

Let  $I$  be a subset of  $\{1, \dots, n\}$ . Suppose that we have a family  $A_{i_1}, \dots, A_{i_s}$  of subsets of  $G$  with indices running over  $I$  and another family  $B_{l_1}, \dots, B_{l_t}$  of subsets with indices running over  $\{1, \dots, n\} \setminus I$ . We write

$$w_I(A_i; B_l)$$

for  $w(X_1, \dots, X_n)$ , where  $X_k = A_k$  if  $k \in I$ , and  $X_k = B_k$  otherwise. On the other hand, whenever  $a_i \in A_i$  for  $i \in I$  and  $b_l \in B_l$  for  $l \in \{1, \dots, n\} \setminus I$ , the symbol  $w_I(a_i; b_l)$  stands for the element  $w(x_1, \dots, x_n)$ , where  $x_k = a_k$  if  $k \in I$ , and  $x_k = b_k$  otherwise.

The following lemmas are Lemma 2.4, Lemma 2.5 and Lemma 4.1 in [4].

**LEMMA 3.1.** *Let  $w = w(x_1, \dots, x_n)$  be a multilinear commutator word. Assume that  $H$  is a normal subgroup of a group  $G$ . Let*



$g_1, \dots, g_n \in G$ ,  $h \in H$  and fix  $s \in \{1, \dots, n\}$ . Then there exist  $y_j \in g_j^H$ , for  $j = 1, \dots, n$ , such that

$$w_{\{s\}}(g_s h; g_l) = w(y_1, \dots, y_n) w_{\{s\}}(h; g_l).$$

LEMMA 3.2. Let  $G$  be a group and let  $w = w(x_1, \dots, x_n)$  be a multilinear commutator word. Assume that  $M, A_1, \dots, A_n$  are normal subgroups of  $G$  such that for some elements  $a_i \in A_i$ , the equality

$$w(a_1(A_1 \cap M), \dots, a_n(A_n \cap M)) = 1$$

holds. Then for any subset  $I$  of  $\{1, \dots, n\}$  we have

$$w_I(A_i \cap M; a_i(A_i \cap M)) = 1.$$

LEMMA 3.3. Let  $G$  be a group and let  $w = w(x_1, \dots, x_n)$  be a multilinear commutator word. Let  $A_1, \dots, A_n$  and  $M$  be normal subgroups of  $G$ . Let  $I$  be a subset of  $\{1, \dots, n\}$ . Assume that

$$w_J(A_i; A_l \cap M) = 1$$

for every proper subset  $J$  of  $I$ . Suppose we are given elements  $g_i \in A_i$  for  $i \in I$  and elements  $h_k \in A_k \cap M$  for  $k \in \{1, \dots, n\}$ . Then we have

$$w_I(g_i h_i; h_l) = w_I(g_i; h_l).$$

LEMMA 3.4. Let  $w = w(x_1, \dots, x_n)$  be a multilinear commutator word. Assume that  $T$  is a normal subgroup of a group  $G$  and  $a_1, \dots, a_n$  are elements of  $G$  such that every element in  $\mathcal{X}_w(a_1 T, \dots, a_n T)$  has at most  $m$  conjugates in  $G$ . Then every element in  $T_w$  has at most  $m^{2^n}$  conjugates in  $G$ .

PROOF. We will first prove the following statement:

(\*) Assume that for some  $g_1, \dots, g_n \in G$  every element in the set  $\mathcal{X}_w(g_1 T, \dots, g_n T)$  has at most  $t$  conjugates in  $G$ , and let  $s \in \{1, \dots, n\}$ . Then every element of the form  $w_{\{s\}}(h_s; g_l h_l)$ , where  $h_1, \dots, h_n \in T$ , has at most  $t^2$  conjugates.

Choose an element  $z = w_{\{s\}}(h_s; g_l h_l)$  as above. By Lemma 3.1

$$w_{\{s\}}(g_s h_s; g_l h_l) = w(y_1, \dots, y_n) w_{\{s\}}(h_s; g_l h_l),$$

where  $y_j \in (g_j h_j)^T \subseteq g_j T$ , for  $j = 1, \dots, n$ .

As both  $w_{\{s\}}(g_s h_s; g_l h_l)$  and  $w(y_1, \dots, y_n)$  lie in  $\mathcal{X}_w(g_1 T, \dots, g_n T)$ , they have at most  $t$  conjugates in  $G$ . Thus

$$z = w(y_1, \dots, y_n)^{-1} w_{\{s\}}(g_s h_s; g_l h_l)$$

has at most  $t^2$  conjugates in  $G$ . This proves (\*).

We will now prove that every element in

$$\mathcal{X}_w(T, \dots, T, a_i T, \dots, a_n T)$$

has at most  $m^{2^i}$  conjugates, by induction on  $i$ . The lemma will follow by taking  $i = n$ .

If  $i = 1$  the statement is true by the hypotheses. So assume that  $i \geq 2$  and every element in  $\mathcal{X}_w(T, \dots, T, a_{i-1}T, \dots, a_nT)$  has at most  $m^{2^{i-1}}$  conjugates. By applying  $(*)$  with  $g_1 = \dots = g_{i-1} = 1$ ,  $t = 2^{i-1}$  and  $s = i$  we get the result.  $\square$

**LEMMA 3.5.** *Let  $w = w(x_1, \dots, x_n)$  be a multilinear commutator word. Assume that  $H$  is a normal subgroup of a group  $G$ . Then there exist a positive integer  $t_n$  depending only on  $n$  such that for every  $g_1, \dots, g_n \in G$ ,  $h_1, \dots, h_n \in G$  the  $w$ -value  $w(g_1h_1, \dots, g_nh_n)$  can be written in the form:  $w(g_1h_1, \dots, g_nh_n) = ah$ , where  $a$  is a product of at most  $t_n$  conjugates of elements in  $\{g_1^{\pm 1}, \dots, g_n^{\pm 1}\}$  and  $h \in H_w$ .*

**PROOF.** The proof is by induction on the number  $n$  of variables appearing in  $w$ . If  $n = 1$  then  $w = x$  and the result is true.

If  $n > 1$ , then  $w$  is of the form  $w = [u, v]$ , where  $u = u(x_1, \dots, x_r)$ ,  $v = v(x_{r+1}, \dots, x_n)$  are multilinear commutator words. By induction,  $u(g_1h_1, \dots, g_rh_r) = a_1h_1$ ,  $v(g_{r+1}h_{r+1}, \dots, g_nh_n) = a_2h_2$ , where  $a_1$  (resp.  $a_2$ ) is a product of at most  $t_r$  (resp.  $t_{n-r}$ ) conjugates of elements in  $S = \{g_1^{\pm 1}, \dots, g_n^{\pm 1}\}$ ,  $h_1 \in H_u$  and  $h_2 \in H_v$ . By the standard commutator formulas we have that:

$$\begin{aligned} w(g_1h_1, \dots, g_nh_n) &= [a_1h_1, a_2h_2] = [a_1, a_2h_2]^{h_1} [h_1, a_2h_2] = \\ &= [([a_1, h_2][a_1, a_2]^{h_2})^{h_1} [h_1, h_2][h_1, a_2]^{h_2} = \\ &= [a_1, h_2]^{h_1} [a_1, a_2]^{h_2h_1} [h_1, a_2]^{h_2} [h_1, h_2]^{[h_1, a_2]^{h_2}}, \end{aligned}$$

where  $[a_1, h_2] = a_1^{-1}a_1^{h_2}$ ,  $[a_1, a_2] = a_1^{-1}a_1^{a_2}$  are products of at most  $2t_r$  conjugates of elements in  $S$ ,  $[h_1, a_2] = (a_2^{-1})^{h_1}a_2$  is a product of at most  $2t_{n-r}$  conjugates of elements in  $S$  and  $[h_1, h_2]^{[h_1, a_2]^{h_2}} \in H_w$ . So the result follows taking  $t_n$  to be the maximum of the set  $\{4t_r + 2t_{n-r} | r = 1, \dots, n-1\}$ .  $\square$

#### 4. Profinite groups in which $w$ -values are FC-elements.

The famous theorem of B. H. Neumann says that the commutator subgroup of a BFC-group is finite [21]. This was recently extended in [5] as follows. Let  $w$  be a multilinear commutator word and  $G$  a group in which  $|x^G| \leq m$  for every  $w$ -value  $x$ . Then the derived subgroup of  $w(G)$  is finite of order bounded by a function of  $m$  and  $w$ . The case where  $w = [x, y]$  was handled in [6].

In the present article we require a profinite (non-quantitative) version of the above result. We show that if  $G$  is a profinite group in which

all  $w$ -values are FC-elements, then the derived subgroup of  $w(G)$  is finite. In fact we establish a stronger result, which uses the concept of marginal subgroup.

Let  $G$  be a group and  $w = w(x_1, \dots, x_n)$  a word. The marginal subgroup  $w^*(G)$  of  $G$  corresponding to the word  $w$  is defined as the set of all  $x \in G$  such that

$$w(g_1, \dots, xg_i, \dots, g_n) = w(g_1, \dots, g_ix, \dots, g_n) = w(g_1, \dots, g_i, \dots, g_n)$$

for all  $g_1, \dots, g_n \in G$  and  $1 \leq i \leq n$ . It is well known that  $w^*(G)$  is a characteristic subgroup of  $G$  and that  $[w^*(G), w(G)] = 1$ .

Note that marginal subgroups in profinite groups are closed.

Let  $S$  be a subset of a group  $G$ . Define the  $w^*$ -residual of  $S$  of  $G$  to be the intersection of all normal subgroups  $N$  such that  $SN/N$  is contained in the marginal subgroup  $w^*(G/N)$ .

For multilinear commutator words the  $w^*$ -residual of a normal subgroup has the following characterization.

**LEMMA 4.1.** *Let  $w$  be a multilinear commutator word,  $G$  a group and  $N$  a normal subgroup of  $G$ . Then the  $w^*$ -residual of  $N$  in  $G$  is the subgroup generated by the elements  $w(g_1, \dots, g_n)$  where at least one of  $g_1, \dots, g_n$  belongs to  $N$ .*

This follows from [29, Theorem 2.3]. For the reader's convenience, we will give here a proof in the spirit of Section 3.

**PROOF.** Let  $N_i = \langle w(g_1, \dots, g_n) \mid g_1, \dots, g_n \in G \text{ and } g_i \in N \rangle$  and let  $R = N_1 N_2 \dots N_n$ . Clearly, if  $M$  is a normal subgroup of  $G$  such that  $N/M$  is contained in  $w^*(G/M)$  then  $N_i \leq M$  for every  $i = 1, \dots, n$ . Therefore  $R$  is contained in the  $w^*$ -residual of  $N$ .

On the other hand, it follows from Lemma 3.3 that if  $N_i = 1$ , then

$$w(g_1, \dots, g_i h, \dots, g_n) = w(g_1, \dots, g_i, \dots, g_n)$$

for every  $g_1, \dots, g_n \in G$  and every  $h$  in  $N$ . Thus we have

$$w(g_1, \dots, g_i h, \dots, g_n)R = w(g_1, \dots, g_i, \dots, g_n)R$$

for every  $i = 1, \dots, n$ , for every  $g_1, \dots, g_n \in G$  and every  $h \in N$ . So  $N/R$  is contained in  $w^*(G/R)$ . This implies the result.  $\square$

It follows from Lemma 4.1 that if  $w$  is a multilinear commutator word and  $N$  is a normal subgroup of a group  $G$  which does not contain nontrivial  $w$ -values, then  $N$  is contained in  $w^*(G)$  and, in particular, it centralizes  $w(G)$ . Indeed in this case, by Lemma 4.1, the  $w^*$ -residual of  $N$  in  $G$  is trivial.

A word  $w$  is concise if whenever  $G$  is a group such that the set  $G_w$  is finite, it follows that also  $w(G)$  is finite. Conciseness of multilinear commutators was proved by J. C. R. Wilson in [31] (see also [8]).

LEMMA 4.2. *Let  $w$  be multilinear commutator word,  $G$  a profinite group and  $N$  an open normal subgroup of  $G$ . Then the  $w^*$ -residual of  $N$  is open in  $w(G)$ .*

PROOF. Let  $K$  be the  $w^*$ -residual of  $N$ . As  $N/K$  is contained in  $w^*(G/K)$  and it has finite index in  $G/K$ , we deduce that the set of  $w$ -values of  $G/K$  is finite. It follows from the above result of Wilson that  $w(G/K)$  is finite, as desired.  $\square$

As above,  $\Delta(G)$  denotes the set of FC-elements of  $G$ . In what follows we will denote by  $H$  the topological closure of  $\Delta(G)$  in a profinite group  $G$ .

The goal of this section is to prove the following theorem.

THEOREM 4.3. *Let  $w$  be a multilinear commutator word,  $G$  a profinite group and  $T$  a normal subgroup of  $G$  such that every  $w$ -value of  $G$  contained in  $T$  is an FC-element. Then the  $w^*$ -residual of  $T$  has finite commutator subgroup.*

It is straightforward that the  $w^*$ -residual of  $G$  is precisely  $w(G)$ . Thus Theorem 4.3 has the following consequence.

COROLLARY 4.4. *Let  $w$  be a multilinear commutator word and  $G$  a profinite group in which every  $w$ -value is an FC-element. Then  $w(G)$  has finite commutator subgroup.*

The key result of the remaining part of this section is the next proposition, from which Theorem 4.3 will be deduced.

PROPOSITION 4.5. *Let  $w = w(x_1, \dots, x_n)$  be a multilinear commutator word,  $G$  a profinite group and  $H$  the topological closure of  $\Delta(G)$  in  $G$ . Assume that  $A_1, \dots, A_n$  are normal subgroups of  $G$  with the property that*

$$\mathcal{X}_w(A_1, \dots, A_n) \subseteq \Delta(G).$$

*Then  $[H, w(A_1, \dots, A_n)]$  is finite.*

The following lemma can be seen as a development related to Lemma 2.4 in [6] and Lemma 4.5 in [30].

LEMMA 4.6. *Assume the hypotheses of Proposition 4.5, with  $A_1, \dots, A_n$  being normal subgroups of  $G$  with the property that  $\mathcal{X}_w(A_1, \dots, A_n) \subseteq \Delta(G)$ . Let  $M$  be an open normal subgroup of  $G$  and  $a_i \in A_i$  for*

$i = 1, \dots, n$ . Then there exist elements  $\tilde{a}_i \in a_i(A_i \cap M)$  and an open normal subgroup  $\tilde{M}$  of  $M$ , such that the order of

$$[H, w(\tilde{a}_1(A_1 \cap \tilde{M}), \dots, \tilde{a}_n(A_n \cap \tilde{M}))]^G$$

is finite.

PROOF. Throughout the proof, whenever  $K$  is a subgroup of  $G$  we write  $K_i$  for  $A_i \cap K$ .

For each natural number  $j$  consider the set  $\Delta_j$  of elements  $g \in G$  such that  $|G : C_G(g)| \leq j$ . Note that the sets  $\Delta_j$  are closed (see for instance [19, Lemma 5]). Consider the sets

$$C_j = \{(y_1, \dots, y_n) \mid y_i \in a_i M_i \text{ and } w(y_1, \dots, y_n) \in \Delta_j\}.$$

Each set  $C_j$  is closed, being the inverse image in  $a_1 M_1 \times \dots \times a_n M_n$  of the closed set  $\Delta_j$  under the continuous map  $(g_1, \dots, g_n) \mapsto w(g_1, \dots, g_n)$ . Moreover the union of the sets  $C_j$  is the whole  $a_1 M_1 \times \dots \times a_n M_n$ . By the Baire category theorem (cf. [15, p. 200]) at least one of the sets  $C_j$  has nonempty interior. Hence, there exist a natural number  $m$ , some elements  $z_i \in a_i M_i$  and a normal open subgroup  $Z$  of  $G$  such that

$$w(z_1 Z_1, \dots, z_n Z_n) \subseteq \Delta_m.$$

By replacing  $Z$  with  $Z \cap M$ , if necessary, we can assume that  $Z \leq M$ .

Choose in  $\mathcal{X}_w(z_1 Z_1, \dots, z_n Z_n)$  an element  $a = w(\tilde{a}_1, \dots, \tilde{a}_n)$  such that the number of conjugates of  $a$  in  $H$  is maximal among the elements of  $\mathcal{X}_w(z_1 Z_1, \dots, z_n Z_n)$ , that is,  $|a^H| \geq |g^H|$  for any  $g \in \mathcal{X}_w(z_1 Z_1, \dots, z_n Z_n)$ .

Since  $\Delta(G)$  is dense in  $H$ , we can choose a right transversal  $b_1, \dots, b_r$  of  $C_H(a)$  in  $H$  consisting of FC-elements. Thus  $a^H = \{a^{b_i} \mid i = 1, \dots, r\}$ , where  $a^{b_i} \neq a^{b_j}$  if  $i \neq j$ . Let  $\tilde{M}$  be the intersection of  $Z$  and all  $G$ -conjugates of  $C_G(b_1, \dots, b_r)$ :

$$\tilde{M} = \left( \bigcap_{g \in G} C_G(b_1, \dots, b_r)^g \right) \cap Z$$

and note that  $\tilde{M}$  is open in  $G$ .

Consider the element  $w(\tilde{a}_1 v_1, \dots, \tilde{a}_n v_n)$  where  $v_i \in \tilde{M}_i$  for  $i = 1, \dots, n$ . As  $w(\tilde{a}_1 v_1, \dots, \tilde{a}_n v_n) \tilde{M}_i = a \tilde{M}_i$  in the quotient group  $G/\tilde{M}_i$ , we have

$$w(\tilde{a}_1 v_1, \dots, \tilde{a}_n v_n) = va,$$

for some  $v \in \tilde{M} \leq C_G(b_1, \dots, b_r)$ . It follows that  $(va)^{b_i} = va^{b_i}$  for each  $i = 1, \dots, r$ . Therefore the elements  $va^{b_i}$  form the conjugacy class  $(va)^H$  because they are all different and their number is the allowed maximum. So, for an arbitrary element  $h \in H$  there exists

$b \in \{b_1, \dots, b_r\}$  such that  $(va)^h = va^b$  and hence  $v^h a^h = va^b$ . Therefore  $[h, v] = v^{-h}v = a^h a^{-b}$  and so  $[h, v]^a = a^{-1}a^h a^{-b}a = [a, h][b, a] \in [H, a]$ . Thus  $[H, v]^a \leq [H, a]$  and

$$[H, va] = [H, a][H, v]^a \leq [H, a].$$

Therefore  $[H, w(\tilde{a}_1 \tilde{M}, \dots, \tilde{a}_n \tilde{M})] \leq [H, a]$ . Lemma 2.3 states that the abstract group  $[\Delta(G), a]^G$  has finite order and thus the same holds for  $[H, a]^G$ . The result follows.  $\square$

For the reader's convenience, the most technical part of the proof of Proposition 4.5 is isolated in the following proposition.

**PROPOSITION 4.7.** *Assume the hypotheses of Proposition 4.5, with  $A_1, \dots, A_n$  being normal subgroups of  $G$  such that  $\mathcal{X}_w(A_1, \dots, A_n) \subseteq \Delta(G)$ . Let  $I$  be a nonempty subset of  $\{1, \dots, n\}$  and assume that there exist a normal subgroup  $U$  of  $G$  of finite order and an open normal subgroup  $M$  of  $G$  such that*

$$[H, w_J(A_i; A_i \cap M)] \leq U \quad \text{for every } J \subsetneq I.$$

*Then there exist a finite normal subgroup  $U_I$  of  $G$  containing  $U$  and an open normal subgroup  $M_I$  of  $G$  contained in  $M$  such that*

$$[H, w_I(A_i; A_i \cap M_I)] \leq U_I.$$

**PROOF.** For each  $i = 1, \dots, n$  consider a right transversal  $C_i$  of  $A_i \cap M$  in  $A_i$ , and let  $\Omega$  be the set of  $n$ -tuples  $\underline{c} = (c_1, \dots, c_n)$  where  $c_r \in C_r$  if  $r \in I$  and  $c_r = 1$  otherwise. Note that the set  $\Omega$  is finite, since  $C_r$  is finite for every  $r$ . For any  $n$ -tuple  $\underline{c} = (c_1, \dots, c_n) \in \Omega$ , by Lemma 4.6, there exist elements  $d_i \in c_i(A_i \cap M)$  and an open normal subgroup  $M_{\underline{c}}$  of  $G$  such that the order of

$$[H, w(d_1(A_1 \cap M_{\underline{c}}), \dots, d_n(A_n \cap M_{\underline{c}}))]^G$$

is finite. Let

$$\begin{aligned} M_I &= M \cap \left( \bigcap_{\underline{c} \in \Omega} M_{\underline{c}} \right), \\ U_I &= U \prod_{\underline{c} \in \Omega} [H, w(d_1(A_1 \cap M_{\underline{c}}), \dots, d_n(A_n \cap M_{\underline{c}}))]^G. \end{aligned}$$

As  $\Omega$  is finite, it follows that  $M_I$  is open in  $G$  and  $U_I$  has finite order.

Let  $Z/U_I$  be the center of  $HU_I/U_I$  in the quotient group  $G/U_I$  and let  $\bar{G} = G/Z$ . We will use the bar notation to denote images of elements or subgroups in the quotient group  $\bar{G}$ .

Let us consider an arbitrary generator  $w_I(k_i, h_l)$  of  $w_I(A_i; A_l \cap M_I)$ , where  $k_i \in A_i$  and  $h_l \in A_l \cap M_I$ . Let  $\underline{c} = (c_1, \dots, c_n) \in \Omega$  be the  $n$ -tuple such that

$$k_i \in c_i(A_i \cap M)$$

if  $i \in I$  and  $c_i = 1$  otherwise. Let  $d_1, \dots, d_n$  be the elements as above, corresponding to the  $n$ -tuple  $\underline{c}$ . Then, by definition of  $U_I$ ,

$$[H, w(d_1(A_1 \cap M_I), \dots, d_n(A_n \cap M_I))] \leq U_I,$$

that is

$$\overline{w(d_1(A_1 \cap M_I), \dots, d_n(A_n \cap M_I))} = 1,$$

in the quotient group  $\bar{G} = G/Z$ . We deduce from Lemma 3.2 that

$$(1) \quad \overline{w_I(d_i(A_i \cap M_I); (A_l \cap M_I))} = 1.$$

Moreover, as  $c_i(A_i \cap M) = d_i(A_i \cap M)$ , we have that  $k_i = d_i v_i$  for some  $v_i \in A_i \cap M$ . It also follows from our assumptions that

$$\overline{w_J(A_i; A_l \cap M)} = 1$$

for every proper subset  $J$  of  $I$ . Thus we can apply Lemma 3.3 and obtain that

$$w_I(\bar{k}_i; \bar{h}_l) = w_I(\bar{d}_i \bar{v}_i; \bar{h}_l) = w_I(\bar{d}_i; \bar{h}_l) = 1,$$

where in the last equality we have used (1). Since  $w_I(k_i, h_l)$  was an arbitrary generator of  $w_I(A_i; A_l \cap M_I)$ , it follows that

$$\overline{w_I(A_i; A_l \cap M_I)} = 1,$$

that is

$$[H, w_I(A_i; A_l \cap M_I)] \leq U_I,$$

as desired.  $\square$

**PROOF OF PROPOSITION 4.5.** Recall that  $w = w(x_1, \dots, x_n)$  is a multilinear commutator word,  $G$  is a profinite group,  $H$  is the closure of  $\Delta(G)$  and  $A_1, \dots, A_n$  are normal subgroup of  $G$  with the property that

$$\mathcal{X}_w(A_1, \dots, A_n) \subseteq \Delta(G).$$

We want to prove that  $[H, w(A_1, \dots, A_n)]$  is finite.

We will prove that for every  $s = 0, \dots, n$  there exist a finite normal subgroup  $U_s$  of  $G$  and an open normal subgroup  $M_s$  of  $G$  such that whenever  $I$  is a subset of  $\{1, \dots, n\}$  of size at most  $s$  we have

$$[H, w_I(A_i; A_l \cap M_s)] \leq U_s.$$

Once this is done, the proposition will follow taking  $s = n$ .

Assume that  $s = 0$ . We apply Lemma 4.6 with  $M = G$  and  $a_i = 1$  for every  $i = 1, \dots, n$ . Thus there exist  $\tilde{a}_1, \dots, \tilde{a}_n \in G$  and an open normal subgroup  $M_0$  of  $G$ , such that the order of

$$U_0 = [H, w(\tilde{a}_1(A_1 \cap M_0), \dots, \tilde{a}_n(A_n \cap M_0))]^G$$

is finite.

Let  $Z/U_0$  be the center of  $HU_0/U_0$  in the quotient group  $G/U_0$  and let  $\bar{G} = G/Z$ . We have that

$$\overline{w(\tilde{a}_1(A_1 \cap M_0), \dots, \tilde{a}_n(A_n \cap M_0))} = 1,$$

so it follows from Lemma 3.2 that

$$\overline{w(A_1 \cap M_0, \dots, A_n \cap M_0)} = 1,$$

that is,  $[H, w(A_1 \cap M_0, \dots, A_n \cap M_0)] \leq U_0$ . This proves the proposition in the case where  $s = 0$ .

Now assume that  $s \geq 1$ . Choose  $I \subseteq \{1, \dots, n\}$  with  $|I| = s$ . By induction, the hypotheses of Proposition 4.7 are satisfied with  $U = U_{s-1}$  and  $M = M_{s-1}$ , so there exist a finite normal subgroup  $U_I$  of  $G$  containing  $U_{s-1}$  and an open normal subgroup  $M_I$  of  $G$  contained in  $M_{s-1}$  such that

$$[H, w_I(A_i; A_i \cap M_I)] \leq U_I.$$

Let

$$M_s = \bigcap_{|I|=s} M_I, \quad U_s = \prod_{|I|=s} U_I,$$

where the intersection (resp. the product) ranges over all subsets  $I$  of  $\{1, \dots, n\}$  of size  $s$ .

As there is a finite number of choices for  $I$ , it follows that  $U_s$  (resp.  $M_s$ ) has finite order (resp. finite index in  $G$ ). Note that  $M_s \leq M_{s-1}$  and  $U_{s-1} \leq U_s$ . Therefore

$$[H, w_I(A_i; A_i \cap M_s)] \leq U_s$$

for every  $I \subseteq \{1, \dots, n\}$  with  $|I| \leq s$ . This completes the induction and the proof of the proposition.  $\square$

**PROOF OF THEOREM 4.3.** Let  $w = w(x_1, \dots, x_n)$  be a multilinear commutator word,  $G$  a profinite group and  $T$  a normal subgroup of  $G$ . For  $i = 1, \dots, n$ , let  $X_i$  be the set of  $w$ -values  $w(g_1, \dots, g_n)$  such that  $g_i$  belongs to  $T$ . Obviously  $X_i \subseteq T$  and therefore  $X_i \subseteq \Delta(G)$  for every  $i$ . It follows from Proposition 4.5 that  $[H, \langle X_i \rangle]$  is finite for every  $i$ . By Lemma 4.1, the  $w^*$ -residual of  $T$  is the subgroup  $N$  generated by the set  $X = X_1 \cup \dots \cup X_n$ . Thus  $[H, N] = \prod_{i=1}^n [H, \langle X_i \rangle]$  is finite. Finally, note that  $N \leq H$  and so  $N' \leq [H, N]$  is also finite.  $\square$



**COROLLARY 4.8.** *Let  $w$  be a multilinear commutator word and let  $G$  be a profinite group with restricted centralizers of  $w$ -values. If  $G$  has a  $w$ -value of infinite order, then  $w(G)$  is abelian-by-finite.*

**PROOF.** Let  $x$  be a  $w$ -value of  $G$  of infinite order. As  $C_G(x)$  is open, it contains an open normal subgroup  $C$  of  $G$ . Let  $K$  be the  $w^*$ -residual of  $C$  in  $G$ . Since all  $w$ -values contained in  $C$  have infinite centralizers, we apply Theorem 4.3 and conclude that  $K$  is finite. Being finite-by-abelian,  $K$  is also abelian-by-finite. It follows from Lemma 4.2 that  $K$  has finite index in  $w(G)$  and so  $w(G)$  is abelian-by-finite.  $\square$

## 5. Pronilpotent groups with restricted centralizers of $w$ -values

In the present section we use the techniques created by Zelmanov to deduce a theorem about pronilpotent groups with restricted centralizers of  $w$ -values (see Theorem 5.7). A combination of this result with Corollary 4.8 yields a proof of Theorem 1.1 for pronilpotent groups.

For the reader's convenience we collect some definitions and facts on Lie algebras associated with groups (see [26] or [35] for further information). Let  $L$  be a Lie algebra over a field. We use the left normed notation; thus if  $l_1, \dots, l_n$  are elements of  $L$  then

$$[l_1, \dots, l_n] = [\dots [[l_1, l_2], l_3], \dots, l_n].$$

An element  $y \in L$  is called ad-nilpotent if  $ady$  is nilpotent, i.e. there exists a positive integer  $n$  such that  $[x, {}_n y] = 0$  for all  $x \in L$ . If  $n$  is the least integer with the above property then we say that  $y$  is ad-nilpotent of index  $n$ . Let  $X$  be any subset of  $L$ . By a commutator in elements of  $X$  we mean any element of  $L$  that could be obtained from elements of  $X$  by repeated operation of commutation with an arbitrary system of brackets, including the elements of  $X$ . Here the elements of  $X$  are viewed as commutators of weight 1. Denote by  $F$  the free Lie algebra over the same field as  $L$  on countably many free generators  $x_1, x_2, \dots$ . Let  $f = f(x_1, \dots, x_n)$  be a nonzero element of  $F$ . The algebra  $L$  is said to satisfy the identity  $f \equiv 0$  if  $f(a_1, \dots, a_n) = 0$  for any  $a_1, \dots, a_n \in L$ . In this case we say that  $L$  is *PI*. We are now in position to quote a theorem of Zelmanov [35, 36] which has numerous important applications to group theory. A detailed proof of this result recently appeared in [37].

**THEOREM 5.1.** *Let  $L$  be a Lie algebra generated by finitely many elements  $a_1, \dots, a_m$  such that all commutators in  $a_1, \dots, a_m$  are ad-nilpotent. If  $L$  is *PI*, then it is nilpotent.*

Let  $G$  be a group. Recall that the lower central word  $[x_1, \dots, x_k]$  is usually denoted by  $\gamma_k$ . The corresponding verbal subgroup  $\gamma_k(G)$  is the familiar  $k$ th term of the lower central series of the group  $G$ . Given a prime  $p$ , a Lie algebra can be associated with the group  $G$  as follows. We denote by

$$D_i = D_i(G) = \prod_{jp^k \geq i} (\gamma_j(G))^{p^k}$$

the  $i$ th dimension subgroup of  $G$  in characteristic  $p$  (see for example [12, Chap. 8]). These subgroups form a central series of  $G$  known as the Zassenhaus-Jennings-Lazard series. Set  $L(G) = \bigoplus D_i/D_{i+1}$ . Then  $L(G)$  can naturally be viewed as a Lie algebra over the field  $\mathbb{F}_p$  with  $p$  elements. For an element  $x \in D_i \setminus D_{i+1}$  we denote by  $\tilde{x}$  the element  $xD_{i+1} \in L(G)$ .

LEMMA 5.2 (Lazard, [18]). *For any  $x \in G$  we have  $(ad \tilde{x})^p = ad(\tilde{x}^p)$ .*

The next proposition follows from the proof of the main theorem in the paper of Wilson and Zelmanov [33].

PROPOSITION 5.3. *Let  $G$  be a group satisfying a coset identity. Then  $L(G)$  is PI.*

Let  $L_p(G)$  be the subalgebra of  $L(G)$  generated by  $D_1/D_2$ . Often, important information about the group  $G$  can be deduced from nilpotency of the Lie algebra  $L_p(G)$ .

PROPOSITION 5.4. [26, Corollary 2.14] *Let  $G$  be a group generated by elements  $a_1, a_2, \dots, a_m$  such that every  $\gamma_k$ -value in  $a_1, a_2, \dots, a_m$  has finite order, for every  $k$ . Assume that  $L_p(G)$  is nilpotent. Then the series  $\{D_i\}$  becomes stationary after finitely many steps.*

Let  $P$  be a Sylow subgroup of a finite group  $G$ . An immediate corollary of the Focal Subgroup Theorem [9, Theorem 7.3.4] is that  $G' \cap P$  is generated by commutators. A weaker version of this fact for multilinear commutator words was proved in [1, Theorem A].

PROPOSITION 5.5. *Let  $G$  be a finite group and  $P$  a Sylow subgroup of  $G$ . If  $w$  is a multilinear commutator word, then  $w(G) \cap P$  is generated by powers of  $w$ -values.*

PROPOSITION 5.6. *Let  $p$  be a prime,  $w$  a multilinear commutator word and  $G$  a profinite group in which all  $w$ -values have finite  $p$ -power order. Let  $K$  be the abstract subgroup of  $G$  generated by all  $w$ -values. Then  $K$  is a locally finite  $p$ -group.*

PROOF. It follows from Proposition 5.5 that  $w(G)$  is a pro- $p$  group. Indeed if  $Q$  is a Sylow  $q$ -subgroup of  $w(G)$ , then the image of  $Q$  in any finite continuous image of  $G$  is generated by powers of  $w$ -values, which are  $p$ -elements, hence  $Q = 1$  unless  $q = p$ .

By Lemma 2.1 there exists an integer  $k$  such that each  $\delta_k$ -value is a  $w$ -value. It is sufficient to prove that the abstract subgroup  $R$  generated by all  $\delta_k$ -values is locally finite. Indeed, the abstract group  $G/R$  is a soluble group such that all  $w$ -values have finite order. Hence  $w(G/R)$  is locally finite by Lemma 2.2.

Let  $X$  be the set of  $\delta_k$ -values of  $G$ . Every finitely generated subgroup of  $R$  is contained in a subgroup generated by a finite subset of  $X$ . So we choose finitely many elements  $a_1, \dots, a_s$  in  $X$  and consider the subgroup  $H$  topologically generated by  $a_1, \dots, a_s$ . It is sufficient to prove that  $H$  is finite.

Note that  $H$  is a pro- $p$  group, since it is a subgroup of  $w(G)$ . For every positive integer  $t$ , consider the set

$$S_t = \{(h_1, \dots, h_{2^k}) \mid h_i \in H \text{ and } \delta_k(h_1, \dots, h_{2^k})^{p^t} = 1\}.$$

These sets are closed and their union is the whole Cartesian product of  $2^k$  copies of  $H$ . By the Baire category theorem at least one of the sets  $S_t$  has nonempty interior. Hence, there exist a natural number  $m$ , some elements  $y_i \in H$  and a normal open subgroup  $Z$  of  $H$  such that

$$\delta_k(y_1 Z, \dots, y_{2^k} Z)^{p^m} = 1.$$

In particular  $H$  satisfies a coset identity.

Let  $L = L_p(H)$  be the Lie algebra associated with the Zassenhaus-Jennings-Lazard series  $\{D_i\}$ . Then  $L$  is generated by  $\tilde{a}_i = a_i D_2$  for  $i = 1, \dots, s$ . Let  $b$  any Lie-commutator in  $\tilde{a}_1, \dots, \tilde{a}_s$  and let  $c$  be the group-commutator in  $a_1, \dots, a_s$  having the same system of brackets as  $b$ . Since  $X$  is commutator closed,  $c$  is a  $\delta_k$ -value and so it has finite order. By Lemma 5.2 this implies that  $b$  is ad-nilpotent. As  $H$  satisfies a coset identity, it follows from Proposition 5.3 that  $L$  satisfies some nontrivial polynomial identity. By Theorem 5.1 we conclude that  $L$  is nilpotent. As every  $\gamma_k$ -value in  $a_1, \dots, a_s$  has finite order, Proposition 5.4 shows that the series  $\{D_i\}$  has only finitely many nontrivial terms. Since  $H$  is a pro- $p$  group, it follows that the intersection of all  $D_i$ 's is trivial. Taking into account that each  $D_i$  has finite index in  $H$ , we deduce that  $H$  is finite. This proves that  $R$  is locally finite and the proposition follows.  $\square$

**THEOREM 5.7.** *Let  $w$  be a multilinear commutator word and let  $G$  be a pronilpotent group with restricted centralizers of  $w$ -values in which*

every  $w$ -value has finite order. Then the derived subgroup of  $w(G)$  is finite.

PROOF. First assume that  $G$  is a pro- $p$  group. Let  $K$  be the abstract subgroup of  $G$  generated by all  $w$ -values. By Proposition 5.6  $K$  is a locally finite  $p$ -group. If a  $w$ -value of  $G$  has finite centralizer, then  $K$  is finite by Lemma 2.4. Since  $K$  is dense in  $w(G)$ , we conclude that  $w(G)$  is finite. Therefore we can assume that every  $w$ -value in  $G$  is an FC-element and so the result follows from Corollary 4.4.

When  $G$  is pronilpotent, it is the Cartesian product of its Sylow subgroups. Let  $\mathcal{P}$  be the set of primes  $p$  such that  $w(P) \neq 1$  where  $P$  is the Sylow  $p$ -subgroup of  $G$ . If  $\mathcal{P}$  is infinite, then  $G$  has a  $w$ -value of infinite order, against our assumption. Thus  $\mathcal{P}$  is finite. If  $P$  is a Sylow  $p$ -subgroup of  $G$ , then the derived subgroup of  $w(P)$  is finite by what we proved above. Therefore the derived subgroup of  $w(G) = \prod_{p \in \mathcal{P}} w(P)$  is finite, as desired.  $\square$

## 6. Local finiteness of $w(G)$

The goal of the present section is to show that if the hypotheses of Theorem 1.1 hold and all  $w$ -values have finite order, then  $w(G)$  is locally finite. There is a long-standing conjecture stating that each torsion profinite group has finite exponent (cf. Hewitt and Ross [11]). The conjecture can be easily proved for soluble groups (cf. [23, Lemma 4.3.7]). In [3] this was extended as follows.

PROPOSITION 6.1. [3, Theorem 3] *Let  $w$  be a multilinear commutator word and  $G$  a soluble-by-finite profinite group in which all  $w$ -values have finite order. Then  $w(G)$  is locally finite and has finite exponent.*

We remark that the above result does not follow from Lemma 2.2 and its proof is significantly more complicated.

Given a word  $w$  and a subgroup  $P$  of a profinite group  $G$ , we denote by  $W(P)$  the closed subgroup generated by all elements of  $P$  that are conjugate in  $G$  to elements of  $P_w$ :

$$W(P) = \langle P_w^G \cap P \rangle.$$

Let  $\mathcal{Y}_w$  be the class of all profinite groups  $G$  in which all  $w$ -values have finite order and the subgroup  $W(P)$  is periodic for any Sylow subgroup  $P$  of  $G$ .

The following theorem was implicitly established in [17]. We will now reproduce the proof.

THEOREM 6.2. *Let  $w$  be a multilinear commutator word and let  $G$  be a profinite group in the class  $\mathcal{Y}_w$ . Then  $w(G)$  is locally finite.*

PROOF. Recall that finite groups of odd order are soluble by the Feit-Thompson theorem [7]. Combining this with [17, Theorem 1.5] (applied with  $p = 2$ ), we deduce that  $G$  has a finite series of closed characteristic subgroups

$$(2) \quad G = G_0 \geq G_1 \geq \cdots \geq G_s = 1$$

in which each factor either is prosoluble or is isomorphic to a Cartesian product of nonabelian finite simple groups. There cannot be infinitely many nonisomorphic nonabelian finite simple groups in a factor of the second kind, since this would give a  $w$ -value of infinite order. Indeed, by a result of Jones [13], any infinite family of finite simple groups generates the variety of all groups; therefore, the orders of  $w$ -values cannot be bounded on such an infinite family. Thus, we can assume in addition that each nonprosoluble factor in (2) is isomorphic to a Cartesian product of isomorphic nonabelian finite simple groups. We use induction on  $s$ . If  $s = 0$ , then  $G = 1$  and the result follows. Let  $s \geq 1$ . By induction,  $w(G_1)$  is locally finite. Passing to the quotient  $G/w(G_1)$ , we can assume that  $G_1$  is soluble. If  $G/G_1$  is isomorphic to a Cartesian product of isomorphic nonabelian finite simple groups, then  $G/G_1$  is locally finite and the result follows from [17, Lemma 5.6]. If  $G/G_1$  is prosoluble, then so is  $G$ , and then by [17, Proposition 5.12]  $G$  has a series of finite length with pronilpotent quotients. In this case,  $w(G)$  is locally finite by [17, Lemma 5.7], as required.  $\square$

PROPOSITION 6.3. *Let  $w$  be a multilinear commutator word and let  $G$  be a profinite group with restricted centralizers of  $w$ -values. Assume that every  $w$ -value has finite order. Then  $w(G)$  is locally finite.*

PROOF. By Lemma 2.1 there exists an integer  $k$  such that each  $\delta_k$ -value is a  $w$ -value. Set  $u = \delta_{2k}$ . Let us show that  $G \in \mathcal{Y}_u$ , that is,

$$U(P) = \langle P_u^G \cap P \rangle$$

is periodic for every Sylow subgroup of  $P$  of  $G$ .

Let  $P$  be a Sylow subgroup of  $G$ . It follows from Theorem 5.7 that  $w(P)'$  is a finite  $p$ -group, so  $w(P)$  is soluble. In view of Lemma 2.1 we have  $P^{(k)} \leq w(P)$  and so  $P$  is soluble. By Proposition 6.1,  $P^{(k)}$  is locally finite and has finite exponent. In particular  $P^{(k)}$  is locally nilpotent.

If  $P$  is finite then also  $U(P)$  is finite so we can assume that  $P$  is infinite.

If some element  $x \in P_{\delta_k}$  has finite centralizer we get a contradiction, because on the one hand  $x^P$  is infinite, on the other hand  $x^P$  is contained in  $P^{(k)}$ , which is finite by Lemma 2.4. Thus we can assume that

the centralizer of each element in  $P_{\delta_k}$  is infinite. As  $G$  has restricted centralizers of  $w$ -values and every  $\delta_k$ -value is also a  $w$ -value, it follows that each element in  $P_{\delta_k}$  has centralizer of finite index in  $G$ . Consider the sets

$$C_j = \{(y_1, \dots, y_{2^k}) \mid y_i \in P \text{ and } |\delta_k(y_1, \dots, y_{2^k})^G| \leq j\}.$$

Note that each set  $C_j$  is closed. Moreover their union is the whole Cartesian product of  $2^k$  copies of  $P$ . By the Baire category theorem at least one of the sets  $C_j$  has nonempty interior. Hence, there exist a natural number  $m$ , some elements  $a_i \in P$  and an open normal subgroup  $T$  of  $P$  such that

$$\mathcal{X}_{\delta_k}(a_1 T, \dots, a_{2^k} T) \subseteq C_m.$$

We deduce from Lemma 3.4 that there exists a positive integer  $m_1$  such that each element in  $T_{\delta_k}$  has at most  $m_1$  conjugates. Let  $T_0 = T \cap P^{(k)}$ . As  $P^{(k)}$  is topologically generated by  $P_{\delta_k}$ , we can choose a right transversal  $b_1, \dots, b_r$  of  $T_0$  in  $P^{(k)}$  consisting of finite products of elements in  $P_{\delta_k}$ . Of course  $b_1, \dots, b_r$  are FC-elements and thus there exists a positive integer  $m_2$  such that each  $b_i$  has at most  $m_2$  conjugates. Let  $x \in P_u$ . We have

$$x = \delta_k(c_1, \dots, c_{2^k}),$$

where  $c_i \in P_{\delta_k}$  for  $i = 1, \dots, 2^k$ . Now each  $c_i$  is of the form  $c_i = g_i h_i$  where  $g_i \in \{b_1, \dots, b_r\}$  and  $h_i \in T_0$ .

It follows from Lemma 3.5 that  $x = ah$  where  $a$  is the product of at most  $t_{2^k}$  conjugates of elements in  $\{b_1^{\pm 1}, \dots, b_r^{\pm 1}\}$  and  $h \in T_{\delta_k}$ .

As each  $b_i$  has at most  $m_2$  conjugates and  $h$  has at most  $m_1$  conjugates it follows that  $x$  has at most  $m_3$  conjugates for some positive integer  $m_3$  which does not depend on  $x$ . So each  $x \in P_u$  has order dividing  $e$ , where  $e$  is the exponent of  $P^{(k)}$ , and has at most  $m_3$  conjugates.

Recall that  $U(P) = \langle P_u^G \cap P \rangle$ . It follows from Lemma 2.6 that  $U(P)$  has finite exponent. This proves that  $G \in \mathcal{Y}_u$ .

We deduce from Theorem 6.2 that  $G^{(2^k)}$  is locally finite. Thus we can pass to the quotient group  $G/G^{(2^k)}$  and assume that  $G^{(2^k)} = 1$ . Now the result follows from Proposition 6.1.  $\square$

## 7. Proof of Theorem 1.1

We recall that the Hirsch-Plotkin radical of an (abstract) group is defined as the maximal normal locally nilpotent subgroup. In a profinite group the Hirsch-Plotkin radical need not be closed. However, in the particular case where the profinite group is locally finite, the

Hirsch-Plotkin radical is closed. Indeed the closure of an abstract locally nilpotent subgroup is pronilpotent in any profinite group, and so it is locally nilpotent if the group is locally finite.

An important result about profinite torsion groups is the following theorem due to J. S. Wilson.

**THEOREM 7.1.** [32, Theorem 1] *Let  $G$  be a compact Hausdorff torsion group. Then  $G$  has a finite series*

$$1 = G_0 \leq G_1 \leq \cdots \leq G_s \leq G_{s+1} = G$$

*of closed characteristic subgroups, in which each factor  $G_{i+1}/G_i$  either is a pro- $p$  group for some prime  $p$  or is isomorphic (as a topological group) to a Cartesian product of finite simple groups.*

In particular, a profinite locally soluble torsion group has a finite series of characteristic subgroups in which each factor is a pro- $p$  group for some prime  $p$ .

**PROOF OF THEOREM 1.1.** Recall that  $w$  is a multilinear commutator word and  $G$  a profinite group with restricted centralizers of  $w$ -values. We want to prove that  $w(G)$  is abelian-by-finite.

If  $G$  has a  $w$ -value of infinite order, then by Corollary 4.8 the subgroup  $w(G)$  is abelian-by-finite. So we can assume that every  $w$ -value has finite order. It follows from Proposition 6.3 that  $w(G)$  is locally finite.

By Theorem 7.1,  $w(G)$  has a finite series of characteristic subgroups

$$1 = A_0 \leq A_1 \leq \cdots \leq A_s \leq A_{s+1} = w(G)$$

in which each factor either is a pro- $p$  group for some prime  $p$  or is isomorphic to a Cartesian product of finite simple groups. Let  $A/B$  be a factor in the series which is isomorphic to a Cartesian product of finite simple groups. Recall that the famous Ore's conjecture, stating that every element of a nonabelian finite simple group is a commutator, was proved in [20]. It follows that every element of a nonabelian finite simple group is a  $w$ -value, therefore every element in  $A/B$  is a  $w$ -value. We deduce from Lemma 2.5 that  $A/B$  is a profinite group with restricted centralizers. By Shalev's result [25],  $A/B$  is abelian-by-finite and therefore finite.

Since all non-pronilpotent factors in the above series are finite, we derive that  $w(G)$  is prosoluble-by-finite. Moreover  $w(G)$  has an open characteristic subgroup  $K$ , which in turn has a finite characteristic series

$$1 = F_0 \leq F_1 \leq F_2 \leq \cdots \leq F_r \leq F_{r+1} = K$$

where  $F_{i+1}/F_i$  is the Hirsch-Plotkin radical of  $K/F_i$ , for every  $i$ .

Alternatively, the existence of such a subgroup  $K$  could be shown using theorems of Hartley [10] and Dade [2].

Let  $j$  be the maximal index such that all  $w$ -values contained in  $F_j$  are FC-elements. If  $j = r + 1$ , then by Corollary 4.4 we conclude that  $w(G)$  is finite-by-abelian, hence abelian-by-finite.

So assume now that  $j \leq r$ . Then there exists a  $w$ -value whose centralizer in  $G$  is finite. As  $w(G)$  is locally finite, Lemma 2.5 guarantees that  $F_{j+1}/F_j$  has an element with finite centralizer. Thus  $F_{j+1}/F_j$  satisfies the hypothesis of Lemma 2.4, hence it is finite. Since  $F_{j+1}/F_j$  is the Hirsch-Plotkin radical of  $K/F_j$ , it contains its centralizer in  $K/F_j$ . Taking into account that  $F_{j+1}/F_j$  is finite, we conclude that its centralizer in  $K$  has finite index. Therefore  $F_{j+1}$  has finite index in  $K$ . We deduce that  $F_j$  has finite index in  $w(G)$ .

Let  $T$  be the  $w^*$ -residual of  $F_j$ . Since every  $w$ -value in  $F_j$  is an FC-element, we can apply Theorem 4.3 and we obtain that  $T$  is finite. Hence,  $T$  is abelian-by-finite. Note that  $F_j/T$  is contained in  $w^*(G/T)$ , hence it centralizes  $w(G/T)$ . By Lemma 2.5 the verbal subgroup  $w(G/T)$  has an element with finite centralizer, so we deduce that  $F_j/T$  is finite. Thus  $T$  is open in  $w(G)$  and we conclude that  $w(G)$  is abelian-by-finite, as desired.  $\square$

In the sequel, we will use the fact that an abelian-by-finite group contains a characteristic abelian subgroup of finite index (see [22, Ch. 12, Lemma 1.2] or [14, Lemma 21.1.4]).

**PROOF OF COROLLARY 1.2.** Recall that  $w$  is a multilinear commutator word and  $G$  a profinite group in which centralizers of  $w$ -values are either finite or open. It follows from Theorem 1.1 that  $w(G)$  is abelian-by-finite. In particular  $w(G)$  has an open characteristic abelian subgroup  $N$ . As  $w(G)/N$  is finite, there exists an open normal subgroup  $T$  of  $G$  containing  $N$ , such that  $T/N$  intersects  $w(G)/N$  trivially. Since  $w(T) \leq T \cap w(G) \leq N$ , we conclude that  $w(T)$  is abelian, as desired. The solubility of  $T$  is immediate from Lemma 2.1.  $\square$

**PROOF OF COROLLARY 1.3.** Recall that  $w$  is a multilinear commutator word and  $G$  a profinite group in which every  $w$ -value has finite centralizer. Assume that  $w(G) \neq 1$ . It follows from Theorem 1.1 that  $w(G)$  is abelian-by-finite. In particular,  $w(G)$  has an open characteristic abelian subgroup  $N$ . If  $N$  contains a nontrivial  $w$ -value, then  $N$  is finite, by assumption. Therefore we can assume that  $N \cap G_w = 1$ . It follows from the remark following Lemma 4.1 that  $N$  is contained in  $w^*(G)$ . Since the marginal subgroup centralizes  $w(G)$ , we deduce that  $N$  is finite. This proves that  $w(G)$  is finite. Hence,  $C_G(w(G))$  has finite



index in  $G$ . We see that  $C_G(w(G))$  is both finite and of finite index, which proves that  $G$  is finite.  $\square$

As a final remark, we point out that in [25] Shalev actually proved that if  $G$  is a profinite group with restricted centralizers then  $\Delta(G)$  has finite index in  $G$  and finite commutator subgroup. Our proof of Theorem 1.1 implies that if  $w$  is a multilinear commutator word and  $G$  a profinite group with restricted centralizers of  $w$ -values, then the closed subgroup generated by  $G_w \cap \Delta(G)$  has finite index in  $w(G)$  and finite commutator subgroup.

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