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On a general model system related to affine stochastic differential equations

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Abstract

We link a general method for modeling random phenomena using systems of stochastic differential equations to the class of affine stochastic differential equations. This general construction emphasises the central role of the Duffie-Kan system [5] as a model for first order approximations of a wide class of nonlinear systems perturbed by noise. We also specialise to a two dimensional framework and propose a direct proof of the Duffie-Kan theorem which does not passes through the comparison with an auxiliary process. Our proof produces a scheme to obtain an explicit representation of the solution once the one dimensional square root process is assigned.

Key words and phrases: stochastic differential equations, square root process, Feller condition

AMS 2000 classification: 60H10, 60H30

1 Introduction

Stochastic differential equations (SDEs, for short) with Hölder-continuous coefficients appear in the modeling of several evolutionary systems perturbed by noise. The most important instance is probably the so-called *square root process* defined to be the unique strong solution of the following one dimensional SDE

$$dX_t = (aX_t + b)dt + \sigma\sqrt{X_t}dW_t, \quad X_0 = x \tag{1.1}$$

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where $a, b \in \mathbb{R}$, $\sigma, x \in]0, +\infty[$ and $\{W_t\}_{t\geq 0}$ denotes a standard one dimensional Brownian motion. This equation is very popular in interest rate modeling due to the properties of its solution. We refer the reader to the book Cairns [4] for a detailed analysis of this topic (see also Mao [12]). SDEs with Hölder-continuous coefficients appear in the description of certain epidemic models as well: in this case the solution process represents the number of susceptible individuals in a given population. We mention the papers Greenhalgh et al. [7] and Bernardi et al. [2] which consider models described by SDEs with random and Hölder-continuous coefficients.

From a mathematical point of view the analysis of existence and uniqueness for strong solutions of SDEs with Hölder-continuous coefficients is quite challenging. In the one dimensional case, resorting to the famous Yamada-Watanabe principle (i.e. weak existence plus pathwise uniqueness implies strong existence) one can prove the existence of a unique strong solution for SDEs where the drift coefficient is locally Lipshiptz-continuous while the diffusion coefficient is of the type $\sigma(x) = |x|^\alpha$ for $\alpha \in [1/2, 1]$. The hard part of this proof is the pathwise uniqueness which heavily relies on an *ad hoc* technique introduced by Yamada and Watanabe [15] (see also the books Ikeda and Watanabe [8] and Karatzas and Shreve [9] for comparison theorems obtained with a similar approach). When we move to systems of SDEs with Hölder-continuous coefficients, then only few particular cases can be found in the literature; in fact, the lack of a multidimensional version of the Yamada-Watanabe technique to prove pathwise uniqueness forced the authors of those papers to consider equations that can be investigated with a slight modification of the one dimensional approach. The most important paper in this stream of results is certainly Duffie and Kan [5] where the authors, motivated by financial applications, consider a multidimensional version of the square root process (1.1). They prove existence, uniqueness and positivity for the strong solution of an SDE where the components of the drift vector are affine functions of the solution and the diffusion matrix is a constant matrix times a diagonal matrix with entries being square roots of affine functions of the solution. Their proof is based on a suitable application of the comparison theorem mentioned above, which we recall is based on the one dimensional Yamada-Watanabe technique. We now mention a series of results where the Yamada-Watanabe approach has been utilised in some multidimensional problems: Graczyk and J. Malecki [6] and Kumar [10] consider SDEs where for $i \in \{1, ..., m\}$ the i-th row of the diffusion matrix depends only on the i-th component of the solution; Luo [11] investigates a nested system of SDEs where the i-th row of the diffusion matrix depends only on the first i components of the solution; Wand and Zhang [14] introduce an integrability condition involving the determinant of the diffusion matrix and an auxiliary function fulfilling certain requirements.

The aim of the present paper is to link the general method presented in the book Allen [1] for modeling random phenomena using SDEs to the multidimensional system studied in Duffie and Kan [5]. More precisely, in Allen [1] pages 138-139 it is shown how, assigning probabilities to the possible changes of a general two dimensional system, one

can deduce a Fokker-Planck partial differential equation for the candidate density of the system and from that a suitable SDE describing the random motion of the system. Following this procedure we consider an m-dimensional system with some prescribed admissible (i.e. with positive probability) changes and we deduce after some simplifying assumptions an m-dimensional SDE with Hölder continuous coefficients. Then, Taylor-expanding up to the first order the coefficients of the SDE around the initial condition, we end up with the multidimensional SDE investigated in Duffie-Kan [5] for which the existence of a unique strong solution is guaranteed under proper restrictions (we also present a detailed proof of this result, elaborating some technical aspects missing in the original proof). Therefore, this general construction emphasises the central role of the Duffie-Kan SDE as a model for first order approximations of a wide class of nonlinear systems perturbed by noise. We also remark that the positivity property guaranteed by the Duffie-Kan theorem entails the consistency of our procedure: in fact, such property will ensure the positivity of the probabilities originally assigned to the m-dimensional system according to the Allen's method. We then specialise to the two dimensional case and we suggest a direct proof of the Duffie-Kan theorem which does not passes through the comparison with an auxiliary process. Our proof is based on the sole properties of the one dimensional square root process (1.1) and produces a scheme to obtain an explicit solution of the two dimensional system once the process in (1.1) is assigned.

The paper is organised as follows: in Section 2 we adapt the Allen's procedure to an m-dimensional system assigning probabilities of admissible changes and making some simplifying assumptions; Section 3 contains the description of the first order approximation, link to the Duffie-Kan SDE, statement and detailed proof of the Duffie-Kan theorem; lastly, in Section 4 we specialise to the two dimensional framework and propose a constructive alternative proof of the Duffie-Kan theorem.

2 A general m-dimensional system

Let us consider a model system with $m \in \mathbb{N}$ different states evolving in time according to some probabilistic rules specified below. We write

$$S_t = (S_t^1, S_t^2, ..., S_t^m)^T, \quad t \geq 0$$

to represent the values of the m states of the system at time t.
It is assumed that in a small time interval $[t, t + \Delta t]$ every state can change by $-\Delta x$, 0 or $+\Delta x$ where Δx is a small positive constant. This produces a total of 3^m possible different changes (the number of vectors of length m with components taking values in the set $\{-\Delta x, 0, \Delta x\}$). We let $\Delta S_t := S_{t+\Delta t} - S_t$ be the global change of the system in the time interval $[t, t + \Delta t]$; for instance, $\Delta S_t = (-\Delta x, 0, \Delta x, 0, ..., 0)^T$ means that in the time interval $[t, t + \Delta t]$ state S^1 has decreased of Δx, state S^3 has increased of Δx while all the other states remained unchanged. As illustrated in Figure 1, we denote

$$r_j(t, x) \;:=\; \mathbb{P}(\Delta S_t = -\Delta x e_j + \Delta x e_{j+1} | S_t = x)/\Delta t, \quad j \in \{1, ..., m-1\} \qquad (2.1)$$

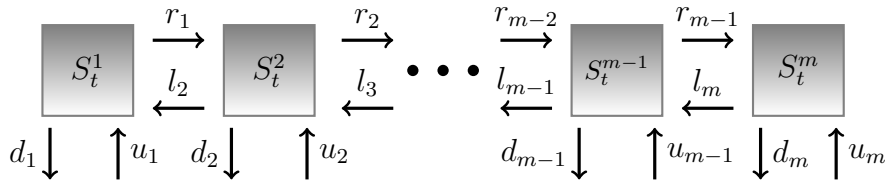

Figure 1: An m-state dynamical process

$$
\begin{aligned}
l_j(t,x) &:= \mathbb{P}(\Delta S_t = -\Delta x e_j + \Delta x e_{j-1} | S_t = x)/\Delta t, \quad j \in \{2,...,m\} && (2.2)\\
d_j(t,x) &:= \mathbb{P}(\Delta S_t = -\Delta x e_j | S_t = x)/\Delta t, \quad j \in \{1,...,m\} && (2.3)\\
u_j(t,x) &:= \mathbb{P}(\Delta S_t = \Delta x e_j | S_t = x)/\Delta t, \quad j \in \{1,...,m\} && (2.4)\\
p_0(t,x) &:= 1 - \Delta t \cdot \sum_{j=1}^{m} (r_j(t,x) + l_j(t,x) + d_j(t,x) + u_j(t,x)) && (2.5)
\end{aligned}
$$

where $\{e_1,...,e_m\}$ denotes the canonical base of $\mathbb{R}^m$ and $r_m(t,x) = l_1(t,x) \equiv 0$. We remark that the probabilities associated to those changes not specified by (2.1)-(2.5) are identically zero. We also observe that $p_0(t,x)$ represents the probability of no changes during the interval $[t, t+\Delta t]$ given that $S_t = x$. According to Figure 1, the evolution of the states of the system is determined by interactions between neighboring states (r_j's and l_j's) and exchanges with the outside world (u_j's and d_j's).
Given the probabilities (2.1)-(2.5) one can introduce, following Allen [1] pages 137-139, a Fokker-Planck equation solved by the density $p(t,x) := \mathbb{P}(S_t = x)$ of the system which in turn is related to the stochastic differential equation

$$
\begin{cases} dS_t = \mu(t,S_t)dt + B(t,S_t)dW_t \\ S_0 = s, \end{cases} \tag{2.6}
$$

where $\{W_t\}_{t\geq 0}$ is an m-dimensional standard Brownian motion,

$$
\mu(t,x) := \mathbb{E}\left[\Delta S_t | S_t = x\right]/\Delta t
$$

is the mean vector and $B(t,x)$ denotes the symmetric square root of the covariance matrix

$$
V(t,x) := \mathbb{E}\left[(\Delta S_t)(\Delta S_t)^T | S_t = x\right]/\Delta t.
$$

According to equations (2.1)-(2.4) we can write

$$
\begin{aligned}
\mu(t,x) &= (-r_1(t,x) + l_2(t,x) + u_1(t,x) - d_1(t,x))\, e_1 \\
&\quad + \sum_{j=2}^{m-1} (r_{j-1}(t,x) - r_j(t,x) + l_{j+1}(t,x) - l_j(t,x) + u_j(t,x) - d_j(t,x))\, e_j \\
&\quad + (r_{m-1}(t,x) - l_m(t,x) + u_m(t,x) - d_m(t,x))\, e_m \qquad (2.7)
\end{aligned}
$$

and

$$V(t,x) = \sum_{j=1}^{m}(u_j(t,x) + d_j(t,x))e_j \otimes e_j + \sum_{j=1}^{m-1}(r_j(t,x) + l_{j+1}(t,x))M_j, \tag{2.8}$$

where for $j \in \{1, ..., m-1\}$ we set $M_j := (e_j - e_{j+1}) \otimes (e_j - e_{j+1})$. We remark that the previous general system has been proposed in Bernardi et al. [3] as a model to study risks aggregation in a Bonus-Malus migration system. To proceed in the analysis of the SDE (2.6) we need to find the symmetric square root of the matrix $V(t,x)$. To this aim we assume the following.

Assumption 2.1 *For any $i, j \in \{1, ..., m\}$ we have*

$$u_i(t,x) + d_i(t,x) = u_j(t,x) + d_j(t,x) =: \gamma(t,x)$$

and for any $i, j \in \{1, ..., m-1\}$ we have

$$r_i(t,x) + l_{i+1}(t,x) = r_j(t,x) + l_{j+1}(t,x) =: \theta(t,x).$$

Assumption 2.1 introduces some symmetries in the evolution of our system. More precisely, the first condition implies that each state has the same probability of an exchange with the outside, while the second condition means that the probability of exchanges between neighboring states does not depend on the specific states considered. As a result we can now rewrite equation (2.8) in the simplified form

$$V(t,x) = \gamma(t,x)I + \theta(t,x)M, \tag{2.9}$$

where I is the $m \times m$ identity matrix while M is the $m \times m$ matrix defined as

$$M = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 2 & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix}$$

According to Theorem 4 page 73 in Yueh [16] (with $a = c = -1$, $\alpha = \beta = \sqrt{ac} = 1$ and $b = 2$) the matrix M has m distinct eigenvalues of the form

$$\lambda_k = 2 + 2\cos(k\pi/m), \quad k = 1, ..., m \tag{2.10}$$

and hence there exists an orthogonal matrix Σ such that

$$M = \Sigma \mathcal{M} \Sigma^T \quad \text{with} \quad \mathcal{M} = \operatorname{diag}[\lambda_1, \ldots, \lambda_m].$$

Therefore, setting $y(t,x) := \theta(t,x)/\gamma(t,x)$ from equation (2.9) we deduce that

$$\begin{aligned} V(t,x) &= \gamma(t,x)\cdot(I + y(t,x)M) \\ &= \gamma(t,x)\cdot(I + y(t,x)\Sigma\mathcal{M}\Sigma^T) \\ &= \gamma(t,x)\cdot\Sigma(I + y(t,x)\mathcal{M})\Sigma^T. \end{aligned}$$

Since

$$(I + y(t,x)\mathcal{M})^{1/2} = \text{diag}\left[\sqrt{1+y(t,x)\lambda_1}, ..., \sqrt{1+y(t,x)\lambda_m}\right]$$

we conclude that

$$\begin{aligned} B(t,x) &= \sqrt{V(t,x)} \\ &= \sqrt{\gamma(t,x)}\cdot\Sigma\ \text{diag}\left[\sqrt{1+y(t,x)\lambda_1}, ..., \sqrt{1+y(t,x)\lambda_m}\right]\Sigma^T \\ &= \Sigma\ \text{diag}\left[\sqrt{\gamma(t,s)+\theta(t,x)\lambda_1}, ..., \sqrt{\gamma(t,x)+\theta(t,x)\lambda_m}\right]\Sigma^T. \end{aligned} \tag{2.11}$$

To sum up, given the probabilities (2.1)-(2.5) together with Assumption 2.1 our model system evolves according to the stochastic differential equation

$$\begin{cases} dS_t = & \mu(t,S_t)dt \\ & +\Sigma\ \text{diag}\left[\sqrt{\gamma(t,S_t)+\theta(t,S_t)\lambda_1}, ..., \sqrt{\gamma(t,S_t)+\theta(t,S_t)\lambda_m}\right]\Sigma^T dW_t \\ S_0 = & s, \end{cases}$$

or equivalently

$$\begin{cases} dS_t = & \mu(t,S_t)dt \\ & +\Sigma\ \text{diag}\left[\sqrt{\gamma(t,S_t)+\theta(t,S_t)\lambda_1}, ..., \sqrt{\gamma(t,S_t)+\theta(t,S_t)\lambda_m}\right]d\tilde{W}_t \\ S_0 = & s, \end{cases} \tag{2.12}$$

where $\tilde{W}_t := \Sigma^T W_t$ is a new m-dimensional standard Brownian motion (recall that by construction Σ^T is orthogonal) while $\mu(t,S_t)$ and the λ_j's are defined in (2.7) and (2.10), respectively.

3 First order approximation and the Duffie-Kan's theorem

The aim of the present section is to prove the existence of a unique strong solution for an SDE of the type (2.12) under suitable regularity assumptions on the coefficients of the equation. First of all we observe that according to equation (2.7) and Assumption 2.1 the components of the drift coefficient μ and the scalar functions γ and β are linear combinations of the functions r_j's, l_j's, u_j's and d_j's defined in (2.1)-(2.4).
If we assume for simplicity that the functions r_j's, l_j's, u_j's and d_j's are time independent and we expand each of them into its first order Taylor polynomial around the point s (which is the initial condition of the SDE (2.12)), then we obtain a corresponding family

of affine functions on $\mathbb{R}^m$. Linear combinations of these affine functions will result in new affine functions representing the components of the drift coefficient μ and the scalar functions γ and θ. More precisely, introducing the notation $f^\star$ to denote the first order Taylor polynomial around s of the smooth function $f : \mathbb{R}^m \to \mathbb{R}$, i.e

$$\begin{aligned} f^\star : \mathbb{R}^m &\to \mathbb{R} \\ x &\mapsto f^\star(x) := f(s) + \langle \nabla f(s), x - s \rangle, \end{aligned}$$

we approximate the functions r_j's, l_j's, u_j's and d_j's with $r_j^\star$'s, $l_j^\star$'s, $u_j^\star$'s and $d_j^\star$'s, respectively. This results in the first order approximation of μ, γ and θ transforming equation (2.12) into

$$\begin{cases} dS_t = & \mu^\star(S_t)dt \\ & +\Sigma \text{ diag} \left[\sqrt{\gamma^\star(S_t) + \theta^\star(S_t)\lambda_1}, ..., \sqrt{\gamma^\star(S_t) + \theta^\star(S_t)\lambda_m} \right] d\tilde{W}_t \\ S_0 = & s. \end{cases} \tag{3.1}$$

The SDE (3.1) now falls into the class of *affine* stochastic differential equations which is a class of equations having a relevant role in the theory of interest rate models (see for instance Cairns [4]). Existence, uniqueness and positivity for affine SDEs have been investigated in the remarkable paper Duffie and Kan [5]. Here we recall their main theorem together with a detailed proof.

Theorem 3.1 (Duffie and Kan [5]) *Consider the m-dimensional stochastic differential equation*

$$dS_t = (aS_t + b)dt + \Sigma \text{ diag} \left(\sqrt{v_1(S_t)}, \sqrt{v_2(S_t)}, ..., \sqrt{v_m(S_t)} \right) dW_t, \tag{3.2}$$

where $a, \Sigma \in M_{m\times m}$, $b \in \mathbb{R}^m$ *and* $v_i(x) := \alpha_i + \langle \beta_i, x \rangle$ *with* $\alpha_1, ..., \alpha_m \in \mathbb{R}$ *and* $\beta_1, ..., \beta_m \in \mathbb{R}^m$. *Assume that*

1. *If* $x \in \mathbb{R}^m$ *is such that* $v_i(x) = 0$, *then*

$$\langle \beta_i, ax + b \rangle > |\Sigma^T \beta_i|^2/2.$$

2. *For all* $j \in \{1, ..., m\}$ *if* $(\Sigma^T \beta_i)_j \neq 0$, *then* $v_i(x) = v_j(x)$ *for all* $x \in \mathbb{R}^m$.

Then, for any initial condition $S_0 = s \in \mathbb{R}^m$ *belonging to*

$$D := \{x \in \mathbb{R}^m : v_i(x) > 0 \text{ for all } i \in \{1, ...,m\}\}$$

the SDE (3.2) admits a unique global strong solution. Moreover, such solution satisfies for all $i \in \{1, ..., m\}$ *and* $t \geq 0$

$$v_i(S_t) > 0 \text{ almost surely.}$$

PROOF. We first consider the case in which

$$v_i(x) = v(x) = \alpha + \langle \beta, x \rangle \quad \text{for all } i \in \{1, ..., m\}$$

making the second assumption trivially satisfied. In this case equation (3.2) reduces to

$$dS_t = (aS_t + b)dt + \sqrt{v(S_t)}\Sigma dW_t. \tag{3.3}$$

Let $\{\varepsilon_n\}_{n\geq 1}$ be a positive strictly decreasing sequence of numbers converging to zero. For each $n \geq 1$, let $\{S_t^{(n)}\}_{t\geq 0}$ be the unique strong solution of the stochastic differential equation defined by (3.3) for $t \leq \tau_n := \inf\{r \geq 0 : v(S_r^{(n)}) = \varepsilon_n\}$ and by $S_t^{(n)} = S_{\tau_n}^{(n)}$ for $t \geq \tau_n$. This is the process satisfying (3.3) that is absorbed at the boundary $\{x \in \mathbb{R}^m : v(x) = \varepsilon_n\}$. Since the coefficient functions defining (3.3) are uniformly Lipschitz on the domain $\{x \in \mathbb{R}^m : v(x) \geq \varepsilon_n\}$, the process $\{S_t^{(n)}\}_{t\geq 0}$ is well defined and is a strong Markov process by standard SDE results.
With $\tau_0 = 0$ we can now define a unique process $\{S_t\}_{t\geq 0}$ on the closed time interval $[0, +\infty]$ by $S_t = S_t^{(n)}$ for $\tau_{n-1} \leq t \leq \tau_n$ and by $S_t = s$ for $t \geq \tau := \lim_{n\to+\infty} \tau_n$. If $\tau = +\infty$ almost surely, then $\{S_t\}_{t\geq 0}$ uniquely solves (3.3) on $[0, +\infty[$, as desired, and is strong Markov. To prove that $\tau = +\infty$ almost surely we will construct an auxiliary positive process that lower bounds $v(S_t)$. We begin by considering the scalar process

$$V_t := v(S_t) = \alpha + \langle \beta, S_t \rangle, \quad t \geq 0,$$

which clearly satisfies

$$dV_t = \langle \beta, aS_t + b\rangle dt + \sqrt{V_t} \cdot \langle \beta, \Sigma dW_t \rangle. \tag{3.4}$$

If we set

$$\hat{W}_t := \langle \Sigma^T \beta, W_t \rangle / |\Sigma^T \beta|, \quad t \geq 0$$

we see that $\{\hat{W}_t\}_{t\geq 0}$ is a one dimensional Brownian motion and equation (3.4) can be rewritten as

$$dV_t = \langle \beta, aS_t + b\rangle dt + |\Sigma^T \beta| \sqrt{V_t} d\hat{W}_t. \tag{3.5}$$

According to the first assumption the inequality

$$\langle \beta_i, ax + b\rangle - |\Sigma^T \beta|^2/2 > 0$$

holds on the hyper-plane $v(x) = 0$. Therefore, by continuity there exists $\varepsilon > 0$ such that the previous inequality is valid on the strip $\{x \in \mathbb{R}^m : 0 \leq v(x) \leq \varepsilon\}$. We can assume without loss of generality that such ε coincides with ε_1. In particular, we can find a $\delta > 0$ such that

$$\langle \beta_i, ax + b\rangle - |\Sigma^T \beta|^2/2 > \delta \tag{3.6}$$

holds for all x belonging to the aforementioned strip. Denoting by $\bar{\eta} := |\Sigma^T \beta|^2/2 + \delta$ we have that

$$\langle \beta_i, ax + b \rangle > \bar{\eta} > |\Sigma^T \beta|^2/2 \tag{3.7}$$

on the set $\{x \in \mathbb{R}^m : 0 \leq v(x) \leq \varepsilon_1\}$. We can also assume that $V_0 > \varepsilon_1$.
We now introduce the *excursions* of the process V from ε_2 to ε_1. We set $T^\star(0) = 0$ and for $k \geq 1$ we define

$$T(k) := \inf\{t \geq T^\star(k-1) : V_t = \varepsilon_2\} \quad \text{and} \quad T^\star(k) := \inf\{t \geq T(k) : V_t = \varepsilon_1\}.$$

These stopping times realize a partition of $[0, +\infty[$:

$$0 = T^\star(0) < T(1) < T^\star(1) < T(2) < T^\star(2) < \cdots.$$

In addition, we consider the auxiliary process $\{\hat{V}_t\}_{t\geq 0}$ defined as follows:

$$\begin{aligned} \hat{V}_t &= \varepsilon_2 + \int_{T(k)}^{T} \bar{\eta} ds + \int_{T(k)}^{T} |\Sigma^T \beta| \sqrt{\hat{V}_s} d\hat{W}_s, \quad \text{if } t \in [T(k), T^\star(k)] \\ \hat{V}_t &= V_t, \quad \text{if } t \in]T^\star(k), T(k+1)[. \end{aligned}$$

The process $\{\hat{V}_t\}_{t\geq 0}$ satisfies

$$0 < \hat{V}_t \leq V_t \quad \text{for all } t \in [0, +\infty[. \tag{3.8}$$

In fact, when $t \in]T^\star(k), T(k)[$ then $\hat{V}_t = V_t$ and by the construction of the stopping times $V_t \geq \varepsilon_2 > 0$ on that time interval. On the other hand, when $t \in [T(k), T^\star(k)]$ then $\hat{V}_t$ is a one dimensional square root process satisfying the Feller condition $\bar{\eta} > |\Sigma^T \beta|^2/2$ (compare with the second inequality in (3.7)). This gives the positivity of $\hat{V}_t$. Moreover, recalling the dynamic of the process $\{V_t\}_{t\geq 0}$ in (3.5), the first inequality in (3.7) together with Theorem 1.1 page 437 in Ikeda and Watanabe [8] implies $\hat{V}_t \leq V_t$.
We now consider the general case: let $\{\varepsilon_n\}_{n\geq 1}$ be a positive strictly decreasing sequence of numbers converging to zero and define as before for each $n \geq 1$ the process $\{S_t^{(n)}\}_{t\geq 0}$ to be the solution of the stochastic differential equation defined by (3.2) for $t \leq \tau_n := \inf\left\{r \geq 0 : \min_{i\in\{1,\ldots,d\}} v_i(S_r^{(n)}) = \varepsilon_n\right\}$ and by $S_t^{(n)} = S_{\tau_n}^{(n)}$ for $t \geq \tau_n$. This is the process satisfying (3.2) that is absorbed at the boundary $\left\{x \in \mathbb{R}^m : \min_{i\in\{1,\ldots,d\}} v_i(x) = \varepsilon_n\right\}$. Since the coefficient functions defining (3.2) are uniformly Lipschitz on the domain $\left\{x \in \mathbb{R}^m : \min_{i\in\{1,\ldots,d\}} v_i(x) \geq \varepsilon_n\right\}$, the process $\{S_t^{(n)}\}_{t\geq 0}$ is uniquely well defined and is a strong Markov process by standard SDE results.
With $\tau_0 = 0$ we can now define a unique process $\{S_t\}_{t\geq 0}$ on the closed time interval $[0, +\infty]$ by $S_t = S_t^{(n)}$ for $\tau_{n-1} \leq t \leq \tau_n$ and by $S_t = s$ for $t \geq \tau := \lim_{n\to+\infty} \tau_n$. If $\tau = +\infty$ almost surely, then $\{S_t\}_{t\geq 0}$ uniquely solves (3.2) on $[0, +\infty[$. For $i \in \{1, ..., m\}$ let

$$V_t^i := v_i(S_t) = \alpha_i + \langle \beta_i, S_t \rangle, \quad t \geq 0,$$

which clearly satisfies

$$\begin{aligned} dV_t^i &= \langle \beta_i, aS_t + b\rangle dt + \left\langle \beta_i, \Sigma \text{ diag}\left(\sqrt{V_t^1}, \sqrt{V_t^2}, ..., \sqrt{V_t^d}\right) dW_t \right\rangle \\ &= \langle \beta_i, aS_t + b\rangle dt + \left\langle \Sigma^T \beta_i, \text{ diag}\left(\sqrt{V_t^1}, \sqrt{V_t^2}, ..., \sqrt{V_t^d}\right) dW_t \right\rangle \\ &= \langle \beta_i, aS_t + b\rangle dt + \sum_{j=1}^{m} (\Sigma^T \beta_i)_j \cdot \sqrt{V_t^j} dW_t^j \\ &= \langle \beta_i, aS_t + b\rangle dt + \sum_{j\in\mathcal{C}_i} (\Sigma^T \beta_i)_j \cdot \sqrt{V_t^j} dW_t^j, \end{aligned}$$

where

$$\mathcal{C}_i := \left\{ j \in \{1, ..., m\} : (\Sigma^T \beta_i)_j \neq 0 \right\}.$$

According to the second assumption of the theorem, we have that $V_t^j = V_t^i$ for all $j \in \mathcal{C}_i$ and $t \geq 0$. Therefore,

$$\begin{aligned} dV_t^i &= \langle \beta_i, aS_t + b\rangle dt + \sum_{j\in\mathcal{C}_i} (\Sigma^T \beta_i)_j \cdot \sqrt{V_t^j} dW_t^j \\ &= \langle \beta_i, aS_t + b\rangle dt + \sqrt{V_t^i} \sum_{j\in\mathcal{C}_i} (\Sigma^T \beta_i)_j dW_t^j \\ &= \langle \beta_i, aS_t + b\rangle dt + \sqrt{V_t^i} d\hat{W}_t^i, \end{aligned}$$

with

$$\hat{W}_t^i := \sum_{j\in\mathcal{C}_i} (\Sigma^T \beta_i)_j W_t^j / |\Sigma^T \beta_i|$$

being a one dimensional Brownian motion (observe that $|\Sigma^T \beta_i|^2 = \sum_{j\in\mathcal{C}_i} (\Sigma^T \beta_i)_j^2$ by the definition of $\mathcal{C}_i$). One can now proceed as before introducing m auxiliary process $\hat{V}^i$ which satisfy $0 < \hat{V}_t^i \leq V_t^i$ for all $i \in \{1, ..., m\}$ and $t \geq 0$. This completes the proof. ∎

By means of the previous theorem we can now set concrete assumptions on the probabilities (2.1)-(2.4) for the existence of a unique strong solution for the SDE (3.1). These assumptions will also guarantee the non negativity of the probabilities in our original model system making the whole construction consistent. Before stating the result we recall that by Assumption 2.1 we have

$$\gamma(x) = u_j(x) + d_j(x) \quad \text{ for all } j \in 1, ..., m.$$

Corollary 3.2 *If $\theta^\star \equiv 0$, $\gamma(s) > 0$ and the inequality*

$$\langle \nabla\gamma(s), \mu^\star(x)\rangle > |\nabla\gamma(s)|^2/2 \quad \textit{holds true on the set } \{x \in \mathbb{R}^m : \gamma(x) = 0\}, \tag{3.9}$$

then equation (3.1) admits a unique strong solution $\{S_t\}_{t\geq 0}$ such that $\gamma^\star(S_t) > 0$ almost surely for all $t \geq 0$.

PROOF. We have simply to verify that our assumptions imply those of Theorem 3.1. First of all, $\theta^\star \equiv 0$ is by Assumptions 2.1 equivalent to $r_j^\star + l_j^\star \equiv 0$ for all $j \in \{1, ..., m\}$ and hence $r_j^\star = l_j^\star \equiv 0$. With $\theta^\star \equiv 0$ the system (3.1) reduces to

$$\begin{cases} dS_t = & \mu^\star(S_t)dt + \sqrt{\gamma^\star(S_t)}dW_t \\ S_0 = & s. \end{cases} \tag{3.10}$$

Observe that $\Sigma\tilde{W}_t = W_t$ by definition of $\tilde{W}_t$ and orthogonality of Σ. Equation (3.10) trivially satisfies the second assumption of Theorem 3.1 since, in the notation of that theorem, $v_1(x) = \cdots = v_m(x)$. We are left with the verification of the first assumption in Theorem 3.1. We note that

$$\gamma^\star(x) = \gamma(s) + \langle \nabla\gamma(s), x - s\rangle = \alpha + \langle \beta, x\rangle$$

if $\beta := \nabla\gamma(s)$ and $\alpha := \gamma(s) - \langle \nabla\gamma(s), s\rangle$. Since $\mu^\star(x)$ corresponds to $ax + b$ using the orthogonality of Σ we get that (3.9) is equivalent to the first assumption of Theorem 3.1. ∎

We observe that from the previous corollary we get the positivity of

$$\gamma^\star(S_t) = u_j^\star(S_t) + d_j^\star(S_t), \quad j \in \{1, ..., m\},$$

which is the aggregated probability of an increase and a decrease for each single state.

4 Two dimensional system

We now focus our attention on the two dimensional version of the general model system presented above. For the sake of clarity we schematise in Figure 2 below the dynamic investigated in the present section

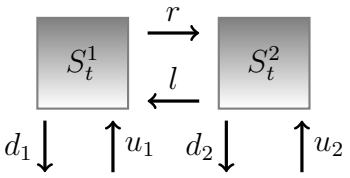

Figure 2: Two dimensional system

and we set

$$r(t,x) := \mathbb{P}(\Delta S_t = (-\Delta x, \Delta x)|S_t = x)/\Delta t \tag{4.1}$$
$$l(t,x) := \mathbb{P}(\Delta S_t = (\Delta x, -\Delta x)|S_t = x)/\Delta t \tag{4.2}$$
$$d_1(t,x) := \mathbb{P}(\Delta S_t = (-\Delta x, 0)|S_t = x)/\Delta t \tag{4.3}$$
$$u_1(t,x) := \mathbb{P}(\Delta S_t = (\Delta x, 0)|S_t = x)/\Delta t \tag{4.4}$$

$$
\begin{aligned}
d_2(t,x) &:= \mathbb{P}(\Delta S_t = (0,-\Delta x)|S_t = x)/\Delta t && (4.5)\\
u_2(t,x) &:= \mathbb{P}(\Delta S_t = (0,\Delta x)|S_t = x)/\Delta t. && (4.6)
\end{aligned}
$$

In addition, we denote

$$
\begin{aligned}
p_0(t,x) &:= \mathbb{P}(\Delta S_t = (0,0)|S_t = x)\\
&= 1 - \Delta t \cdot (r(t,x) + l(t,x) + d_1(t,x) + u_1(t,x) + d_2(t,x) + u_2(t,x))
\end{aligned}
$$

implying that

$$
\mathbb{P}(\Delta S_t = (-\Delta x, -\Delta x)|S_t = x) = \mathbb{P}(\Delta S_t = (\Delta x, \Delta x)|S_t = x) = 0.
$$

According to the scheme presented in the previous sections, if we employ the first order Taylor approximation of the functions defined above (which are assumed to be time independent), then the stochastic differential equation under investigation takes now the form

$$
\begin{aligned}
dS_t &= \mu^\star(S_t)dt + B^\star(S_t)dW_t\\
&= (aS_t + b)dt + \Sigma \begin{bmatrix} \sqrt{v_1(S_t)} & 0 \\ 0 & \sqrt{v_2(S_t)} \end{bmatrix} \Sigma^T dW_t\\
&= (aS_t + b)dt + \Sigma \begin{bmatrix} \sqrt{\alpha_1 + \langle \beta_1, S_t\rangle} & 0 \\ 0 & \sqrt{\alpha_2 + \langle \beta_2, S_t\rangle} \end{bmatrix} d\tilde{W}_t, && (4.7)
\end{aligned}
$$

where for suitable choices of $a \in M_{2\times 2}$, $b, \beta_1, \beta_2 \in \mathbb{R}^2$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ we find that

$$
\begin{aligned}
(aS_t + b)_1 &= u_1^\star(S_t) - d_1^\star(S_t) - r^\star(S_t) + l^\star(S_t)\\
(aS_t + b)_2 &= u_2^\star(S_t) - d_2^\star(S_t) + r^\star(S_t) - l^\star(S_t)
\end{aligned}
$$

(this follows from equation (2.7)) and

$$
\begin{aligned}
\alpha_1 + \langle \beta_1, S_t\rangle &= d_1^\star(S_t) + u_1^\star(S_t) + 2(r^\star(S_t) + l^\star(S_t)) && (4.8)\\
\alpha_2 + \langle \beta_2, S_t\rangle &= d_1^\star(S_t) + u_1^\star(S_t) && (4.9)
\end{aligned}
$$

(which follows from equation (2.11)). We remark that in the present case

$$
\lambda_1 = 2, \quad \lambda_2 = 0, \quad \gamma^\star(x) = d_1^\star(x) + u_1^\star(x) \quad \text{and} \quad \theta^\star(x) = r^\star(x) + l^\star(x)
$$

and Assumption 2.1 reduces to

$$
d_1^\star(x) + u_1^\star(x) = d_2^\star(x) + u_2^\star(x).
$$

Moreover, we have

$$
\Sigma = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.
$$

If we look through the proof of Theorem 3.1, we see that the second assumption in the statement of the theorem, namely

$$\textit{for all } j \in \{1, ..., m\} \textit{ if } (\Sigma^T \beta_i)_j \neq 0, \textit{ then } v_i(x) = v_j(x) \textit{ for all } x \in \mathbb{R}^m \tag{4.10}$$

serves to reduce the diffusion matrix

$$\text{diag}\left(\sqrt{v_1(S_t)}, \sqrt{v_2(S_t)}, ..., \sqrt{v_m(S_t)}\right)$$

to one of the form $\sqrt{v_i(S_t)}I$ where I stands for $m \times m$ identity matrix. Therefore, there is no loss of generality in considering only the case

$$v_1(x) = v_2(x) = \cdots = v_m(x).$$

The next result is the two dimensional version of Theorem 3.1 for the case

$$v_1(x) = v_2(x) = \alpha + \langle \beta, x \rangle. \tag{4.11}$$

The proof is, however, different: it is based on a direct approach rather than the Yamada-Watanabe comparison method utilised in the proof of Theorem 3.1. This direct approach has the advantage of providing an explicit representation of the solution. Let us also point out that condition (4.11) together with (4.8) and (4.9) implies

$$r(x) = l(x) = 0.$$

With reference to Figure 2 this means that the interactions between the two states of the system take place in the probabilities u_1, u_2, d_1 and d_2 rather than from direct exchanges.

Theorem 4.1 *Consider the two dimensional stochastic differential equation*

$$dS_t = (aS_t + b)dt + \sqrt{\alpha + \langle \beta, S_t \rangle} dW_t, \quad S_0 = s \in \mathbb{R}^2, \tag{4.12}$$

where $a \in M_{2\times 2}$, $b, \beta \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$. If the inequality

$$\langle \beta, ax + b \rangle \geq |\beta|^2/2 \quad \textit{holds true on the set } \{x \in \mathbb{R}^2 : \alpha + \langle \beta, x \rangle = 0\}, \tag{4.13}$$

then for any initial condition s satisfying $\alpha + \langle \beta, s \rangle > 0$ the SDE (4.12) admits a unique strong solution $\{S_t\}_{t \geq 0}$ with the property that $\alpha + \langle \beta, S_t \rangle > 0$ almost surely for all $t \geq 0$.

Proof. The idea of the proof is to reduce via an orthogonal transformation the system (4.12) to a system where the equation describing the first component is independent of the second. The first component will turn out to be a one dimensional square root process while the equation for the second component will be explicitly solvable once the first is known.
We may assume without loss of generality that $\beta \neq 0$ (if $\beta = 0$ then equation (4.12)

admits a unique strong solution for any $\alpha \geq 0$). Let $K \in M_{2\times 2}$ be the unique orthogonal matrix such that $K\beta = |\beta|e_1$ and define the stochastic process $Y_t := KS_t$, $t \geq 0$. Then, by the linearity of the Itô differential we can write

$$\begin{aligned} dY_t &= (KaS_t + Kb)dt + \sqrt{\alpha + \langle \beta, S_t\rangle} dKW_t \\ &= (KaK^{-1}KS_t + Kb)dt + \sqrt{\alpha + \langle \beta, K^{-1}KS_t\rangle} d\tilde{W}_t \\ &= (\tilde{a}Y_t + \tilde{b})dt + \sqrt{\alpha + \langle K\beta, Y_t\rangle} d\tilde{W}_t \\ &= (\tilde{a}Y_t + \tilde{b})dt + \sqrt{\alpha + |\beta|Y_t^1} d\tilde{W}_t, \end{aligned} \tag{4.14}$$

where $\tilde{a} := KaK^{-1}$, $\tilde{b} := Kb$ and $\tilde{W}_t := KW_t$ being a new two-dimensional standard Brownian motion. The initial condition is $Y_0 = KS_0 = Ks =: \tilde{s}$. We observe that condition (4.13) corresponds to

$$\tilde{a}_{11}y_1 + \tilde{a}_{12}y_2 + \tilde{b}_1 > |\beta|/2 \quad \text{holds true on the set } \{y \in \mathbb{R}^2 : \alpha + |\beta|y_1 = 0\}. \tag{4.15}$$

Indeed,

$$\begin{aligned} \alpha + \langle \beta, x\rangle &= \alpha + \langle \beta, K^{-1}Kx\rangle \\ &= \alpha + \langle K\beta, Kx\rangle \\ &= \alpha + |\beta|y_1 \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} \langle \beta, ax + b\rangle &= \langle K^TK\beta, ax + b\rangle \\ &= |\beta|\langle e_1, Kax + Kb\rangle \\ &= |\beta|\langle e_1, KaK^{-1}y + \tilde{b}\rangle \\ &= |\beta|\langle e_1, \tilde{a}y + \tilde{b}\rangle \\ &= |\beta|\left(\tilde{a}_{11}y_1 + \tilde{a}_{12}y_2 + \tilde{b}_1\right). \end{aligned}$$

Since the set $\{y \in \mathbb{R}^2 : \alpha + |\beta|y_1 = 0\}$ in (4.15) coincides with $\{y \in \mathbb{R}^2 : y_1 = -\alpha/|\beta|\}$, a substitution of the last condition in the inequality of (4.15) gives

$$\tilde{a}_{12}y_2 + \tilde{b}_1 > |\beta|/2 + (\alpha\tilde{a}_{11})/|\beta|.$$

The last inequality has to be true for all $y_2 \in \mathbb{R}$; hence, we get that $\tilde{a}_{12} = 0$ and

$$\tilde{b}_1 > |\beta|/2 + (\alpha\tilde{a}_{11})/|\beta|. \tag{4.17}$$

Therefore, we can write equation (4.14) as

$$\begin{cases} dY_t^1 = (\tilde{a}_{11}Y_t^1 + \tilde{b}_1)dt + \sqrt{\alpha + |\beta|Y_t^1} d\tilde{W}_t^1, \quad Y_0^1 = \tilde{s}_1 \\ dY_t^2 = (\tilde{a}_{21}Y_t^1 + \tilde{a}_{22}Y_t^2 + \tilde{b}_2)dt + \sqrt{\alpha + |\beta|Y_t^1} d\tilde{W}_t^2 \quad Y_0^2 = \tilde{s}_2. \end{cases} \tag{4.18}$$

Let us study the first equation in (4.18). Setting $\mathcal{Y}_t := |\beta| Y_t^1 + \alpha$ and applying the Itô formula we get

$$d\mathcal{Y}_t = \left(\tilde{a}_{11}\mathcal{Y}_t + \tilde{b}_1|\beta| - \alpha\tilde{a}_{11}\right) dt + |\beta|\sqrt{\mathcal{Y}_t} d\tilde{W}_t^1, \quad \mathcal{Y}_0 = |\beta|\tilde{s}_1 + \alpha.$$

The previous SDE has a unique positive solution (see e.g. Cairns [4]) if

$$\tilde{b}_1|\beta| - \alpha\tilde{a}_{11} \geq |\beta|^2/2,$$

which corresponds to (4.17). The positivity of $\mathcal{Y}_t$ is equivalent to the positivity of $|\beta| Y_t^{(1)} + \alpha$ which in turn is equivalent by (4.16) to the positivity of $\alpha + \langle S_t, \beta\rangle$. We can now solve the equation for Y_t^2 in (4.18), namely

$$\begin{aligned} dY_t^2 &= \left(\tilde{a}_{21}Y_t^1 + \tilde{a}_{22}Y_t^2 + \tilde{b}_2\right) dt + \sqrt{\alpha + |\beta| Y_t^1} d\tilde{W}_t^2 \\ &= \tilde{a}_{22}Y_t^2 dt + \left[\left(\tilde{a}_{21}Y_t^1 + \tilde{b}_2\right) dt + \sqrt{\alpha + |\beta| Y_t^1} d\tilde{W}_t^2\right]. \end{aligned}$$

Its solution is given by the formula

$$Y_t^2 = e^{\tilde{a}_{22}t}\tilde{s}_2 + \int_0^T e^{\tilde{a}_{22}(t-s)} \left[\left(\tilde{a}_{21}Y_s^1 + \tilde{b}_2\right) ds + \sqrt{\alpha + |\beta| Y_s^1} d\tilde{W}_s^2\right].$$

Setting $S_t = K^{-1}Y_t$ we obtain the solution of the original system completing the proof. ∎

We now summarize the construction of the solution of the system (4.12) suggested in the previous proof:

- define the orthogonal matrix K imposing that $K\beta = |\beta| e_1$ and set $\tilde{a} := KaK^{-1}$, $\tilde{b} := Kb$, $\tilde{s} := Ks$ and $\tilde{W}_t := KW_t$;
- let $\{\mathcal{Y}_t\}_{t\geq 0}$ to be unique positive strong solution of the (one dimensional) square root SDE

  $$d\mathcal{Y}_t = \left(\tilde{a}_{11}\mathcal{Y}_t + \tilde{b}_1|\beta| - \alpha\tilde{a}_{11}\right) dt + |\beta|\sqrt{\mathcal{Y}_t} d\tilde{W}_t^1, \quad \mathcal{Y}_0 = |\beta|\tilde{s}_1 + \alpha$$

  (note that the driving noise is $\tilde{W}_t^1$);
- set $Y_t^1 := (\mathcal{Y}_t - \alpha)/|\beta|$ and

  $$Y_t^2 := e^{\tilde{a}_{22}t}\tilde{s}_2 + \int_0^t e^{\tilde{a}_{22}(t-s)} \left[\left(\tilde{a}_{21}Y_s^1 + \tilde{b}_2\right) ds + \sqrt{\alpha + |\beta| Y_s^1} d\tilde{W}_s^2\right]$$

  (note that the driving noise is $\tilde{W}_t^2$);
- the process $S_t := K^{-1}Y_t$ solves (4.12).

In the following example we show that Theorem 3.1 without its second assumption no longer holds in general.

Example 4.2 *We consider the system*

$$\begin{cases} dX_t^1 = 2\sqrt{X_t^2 - 1}dW_t^1, \quad X_0^1 = x_1 \\ dX_t^2 = 3dt + 2\sqrt{X_t^2}dW_t^2, \quad X_0^2 = x_2. \end{cases} \tag{4.19}$$

In the notation of Theorem 3.1 it corresponds to

$$m = 2 \quad a = 0 \quad b = (0,3)^T \quad \Sigma = 2I \quad \alpha = (-1,0)^T \quad \beta_1 = (0,1)^T \quad \beta_2 = (0,1)^T.$$

Recalling that $v_i(x) = \alpha_i + \langle \beta_i, x \rangle$ *for* $i = 1, 2$ *we get*

$$v_1(x) = -1 + x_2 \quad \text{and} \quad v_2(x) = x_2.$$

Since the second component of β_1 *is not zero and* $v_1 \neq v_2$*, the second condition of Theorem 3.1 does not hold. However, since* $a = 0$ *the first condition reduces to*

$$\langle \beta_i, b \rangle > |\beta_i|^2/2, \quad i = 1, 2,$$

which is clearly true. The positivity region D *is now given by* $D = \{x \in \mathbb{R}^2 : x_2 > 1\}$*. If the result of Theorem 3.1 were true we should be able to get a unique strong solution of (4.19) lying in* D *for all* $t \geq 0$ *almost surely.*
We observe that the process X^2 *in* (4.19) *falls in the class of the* squared Bessel processes*, i.e. processes that are strong solutions of SDEs of the form*

$$Z_t = z + 2\int_0^t \sqrt{Z_s}dB_s + \delta t,$$

where $z, \delta \geq 0$ *(see Revuz and Yor [13] for a deep analysis of this family of processes). The parameters* δ *and* $\nu := \frac{\delta}{2} - 1$ *are called* dimension *and* index *of* Z*, respectively. It is well known that the transition density of* Z *is given by the formula*

$$f_t^\delta(z, y) = \frac{1}{2t}\left(\frac{y}{z}\right)^{\frac{\nu}{2}} e^{-\frac{z+y}{2t}} I_\nu\left(\frac{\sqrt{zy}}{t}\right) \mathbb{1}_{\{y>0\}},$$

where $I_\nu(z)$ *stands for the modified Bessel function of the first kind of order* ν*, i.e.*

$$I_\nu(z) = \sum_{n=0}^{\infty} \frac{(\frac{z}{2})^{\nu+2n}}{n!\Gamma(n+\nu+1)}, \quad \nu, z \in \mathbb{C}.$$

From this we see that $\mathbb{P}(0 < X_t^2 < 1) > 0$*, even starting with* $x_2 > 1$*. For instance, taking* $x_2 = 2$ *and* $t = 1$ *we have*

$$\mathbb{P}(0 < X_1^2 < 1) = \int_0^1 f_1^3(2, y)dy \approx 0.08.$$

This violates the positivity condition defined by $D = \{x \in \mathbb{R}^2 : x_2 > 1\}$ *which ensures* $\sqrt{X_t^2 - 1}$ *to be well defined.*

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