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Equilibrium existence in the Hotelling model with convex production costs*

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Abstract

We revisit the spatial duopoly model à la Hotelling (1929), to show that, provided the parameter scaling marginal cost is sufficiently high, quadratic production costs guarantee equilibrium existence in presence of linear transportation costs and a uniform distribution, with minimum product differentiation and no undercutting. We also discuss the conditions under which partial coverage arises. Finally, we extend the duopoly game to generalise the condition for the existence of the pure-strategy equilibrium to a class of convex production cost functions.

JEL Codes: D43, L13

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1 Introduction

As is well known, the original version of the Hotelling model (1929) describing horizontal differentiation along a linear city, which can be interpreted either in a literally geographical sense or as a representation of consumers' preference space, is affected by a lack of concavity (and continuity) due to Hotelling's assumption of a linear disutility of transportation. This fact jeopardises the existence of a subgame perfect equilibrium in pure strategies. After the remedy proposed by d'Aspremont *et al.* (1979), the ensuing literature has extended the discussion to a few other aspects, including the minimum amount of convexity of transportation costs needed to restore equilibrium existence (Economides, 1986), the simultaneous presence of linear and quadratic components in the transportation costs function (Anderson, 1988) and the role of several consumer density functions differing from the uniform one characterising the early literature (Neven, 1986; Caplin and Nalebuff, 1991; Anderson *et al.*, 1997). In particular, Anderson *et al.* (1997) have shown the existence of symmetric equilibria with linear transportation costs, provided the distribution of consumers is not 'too concave'.

The problem with the non-existence of the pure-strategy equilibrium has called for the solution in mixed strategies, characterised by Dasgupta and Maskin (1986) and Osborne and Pitchik (1987). Then, a more recent strand of literature, including Baye and Morgan (1999, 2002) and Kaplan and Wettstein (2000) and culminating in Xefteris (2013) has shown the existence of a subgame perfect price equilibrium in the original formulation of the model, with linear consumer preferences and linear production costs.

We connect ourselves to this debate exploring another route, namely, the possibility that (i) firms operate at decreasing returns to scale with a cost function containing a convex component and, for this reason, (ii) they may set undercutting prices to appropriate either a portion of the rival's demand or all of it. After showing that partial undercutting is never profitable, we prove that the same holds for undercutting to monopoly if marginal production cost is sufficiently steep. This result can be reformulated to illustrate that undercutting is unprofitable if the mark-up is low enough. If so, then the subgame perfect pure-strategy equilibrium exists, although the demand effect still dominates the strategic effect and therefore firms supply the product variety preferred by the median consumer.

Then, as in Economides (1984), Hinloopen and van Marrewijk (1999) and Chirco *et al.* (2003), we deal with the possibility for firms to become independent monopolists for low levels of consumers' gross surplus, and then we characterise the regime prevailing for intermediate levels of the same magnitude, in which full coverage obtains with firms located at one quarter and three quarters of the linear city and playing prices driving to zero the net surplus of consumers at the city boundaries and at the city centre.

Having carried out the bulk of the analysis under quadratic production costs, we come to a close with an extension of the model in the vein of Economides (1986), by generalising the game to admit a class of convex costs, of which parabolic ones are a special case. This last case is investigated under full coverage only, to illustrate a more general condition for the existence of equilibrium.

The remainder of the note is structured as follows. Section 2 contains the basic layout based on a linear-quadratic cost function and the analysis of the two-stage game. Undercutting strategies are investigated in section 3. The arising of partial market coverage and separated monopolies, as well as the 'touching' or 'kink' regime with full coverage existing between separate monopolies and proper duopolistic competition are investigated in section 4. The generalization of the duopoly game to a class of convex production cost functions is in section 5. Brief concluding remarks are in section 6.

2 The model

We examine the following version of the Hotelling (1929) model. Two single-product firms, labelled as 1 and 2, operate along the linear city of length L, being located at $x_i \in [0, L]$, $i = 1, 2$, with $x_2 \geq x_1$. Consumers are uniformly distributed along the city, with a constant density d, in such a way that their total mass is $M = dL$. Firm i chooses location x_i and price p_i to maximise its profits, in a two-stage game (the first for locations, the second for prices) taking place under complete and symmetric information. Moreover, information is imperfect at each stage but perfect between stages, so that locations are public domain before price competition takes place.

The net utility of a generic consumer located at $x \in [0, L]$ is the same as in Hotelling, $U = s - p_i - t\,|x - x_i|$, where $s > 0$ is the gross surplus from consumption and $t > 0$ is the unit transportation rate. For the moment, we assume that the market is fully covered, with each consumer buying one unit of the differentiated good. The indifferent consumer is at

$$\widetilde{x} = \frac{p_2 - p_1 + t(x_1 + x_2)}{2t} \tag{1}$$

which defines the demand for product 1, $y_1 = d\widetilde{x}$, and therefore $y_2 = d(L - \widetilde{x})$.

We depart from the original setup by assuming, for the moment, that each firm bears quadratic production costs $C_i = cy_i^2$. The more general case in which the convex component is not necessarily quadratic will be treated later. With this specification of the cost function, the individual profit function is

$$\pi_i = (p_i - cy_i)\, y_i \tag{2}$$

Keeping in mind that, if products become homogeneous at $x_1 = x_2 = L/2$, consumers may distribute themselves randomly across the two firms, we stipulate that if $x_1 = x_2 = L/2$, then $y_1 = y_2 = dL/2$ and therefore firm i's profits are

$$\pi_i = \left(p_i - c\frac{dL}{2}\right)\frac{dL}{2} \text{ if } x_1 = x_2 = L/2 \tag{3}$$

as in Baye and Morgan (1999). The solution of the game, as usual, is carried out by backward induction, starting from the market stage.

2.1 The price stage

In correspondence of a generic location pair, the system of first order conditions (FOCs) at the market stage

$$\frac{\partial \pi_1}{\partial p_1} = \frac{d}{2t}\left[\frac{p_2(cd+t) - p_1(cd+2t) + t(cd+t)(x_1+x_2)}{t}\right] = 0 \tag{4}$$

$$\frac{\partial \pi_2}{\partial p_2} = \frac{d}{2t}\left[\frac{p_1(cd+t) - p_2(cd+2t) + t(cd+t)(2L - x_1 - x_2)}{t}\right] = 0 \tag{5}$$

is solved by the single pair

$$\begin{aligned} p_1^* &= \frac{(cd+t)\left[2cdL + t(2L + x_1 + x_2)\right]}{2cd+3t} \\ p_2^* &= \frac{(cd+t)\left[2cdL + t(4L - x_1 - x_2)\right]}{2cd+3t} \end{aligned} \tag{6}$$

For future reference, we may now characterise the mark-ups of the two firms, $mu_i \equiv (p_i - mc_i)/mc_i$, where $C_i' = 2cy_i$ is the marginal cost of firm i :

$$
\begin{aligned}
C_1' &= \frac{cd\left[p_2 - p_1 + t\left(x_1 + x_2\right)\right]}{t} \\
C_2' &= \frac{cd\left[p_1 - p_2 + t\left(2L - x_1 - x_2\right)\right]}{t}
\end{aligned}
\tag{7}
$$

At the candidate equilibrium prices (6), $mu_i^* = t/(cd)$ because

$$
C_1'^* = \frac{cd\left[2cdL + t\left(2L + x_1 + x_2\right)\right]}{2cd + 3t};\ C_2'^* = \frac{cd\left[2cdL + t\left(4L - x_1 - x_2\right)\right]}{2cd + 3t}
\tag{8}
$$

in such a way that the mark-up is increasing in the unit transportation rate and decreasing in consumer density and the parameter scaling marginal production cost. More importantly, the mark-up is indeed independent of the degree of product differentiation because p_i^* and $C_i'^*$ are proportional, with $p_i^* = (cd + t)\, C_i'^*/(cd)$. This, combined with the fact that marginal cost is increasing in output, will play a relevant role in the remainder.

Substituting (6) back into profit functions, the latter simplify as follows:

$$
\begin{aligned}
\pi_1^* &= \frac{d\left(cd + 2t\right)\left[2cdL + t\left(2L + x_1 + x_2\right)\right]^2}{4\left(2cd + 3t\right)^2} \\
\pi_2^* &= \frac{d\left(cd + 2t\right)\left[2cdL + t\left(4L - x_1 - x_2\right)\right]^2}{4\left(2cd + 3t\right)^2}
\end{aligned}
\tag{9}
$$

which are the relevant objective functions at the first stage.

2.2 The location stage

The inspection of the partial derivatives of profits (9) at the first stage,

$$
\frac{\partial \pi_1}{\partial x_1} = \frac{dt\left(cd + 2t\right)\left[2cdL + t\left(2L + x_1 + x_2\right)\right]}{2\left(2cd + 3t\right)^2}
\tag{10}
$$

$$
\frac{\partial \pi_2}{\partial x_2} = -\frac{dt\left(cd + 2t\right)\left[2cdL + t\left(4L - x_1 - x_2\right)\right]}{2\left(2cd + 3t\right)^2}
\tag{11}
$$

reveals that $\partial \pi_1/\partial x_1 > 0$ and $\partial \pi_2/\partial x_2 < 0$ for all $x_i \in [0, L]$. Therefore, as in Hotelling (1929), the presence of linear transportation costs entails that the demand incentive prevails on the strategic incentive and firms run to the midpoint of the linear city attracted by the median consumer. Once $x_1 = x_2 = L/2$, prices (6) become $p^* = L(cd + t)$, while quantities and profits are $y^* = dL/2$ and $\pi^* = dL^2 (cd + 2t)/4$. However, marginal cost is $C' = 2cy^* = cdL$, and $p^* > C'$ or, equivalently, the mark-up is positive. This fact calls for the analysis of the undercutting incentive (which, in the original model - see d'Aspremont *et al.* (1979) - is appealing well before firms sit in front of the median consumer), and this is where the convexity of production costs kicks in.

3 Undercutting prices

Since the cost function has a quadratic component, it is not *a priori* obvious that a firm's undercutting price should be designed so as to capture the whole market share of the rival, because doing so would bring about a sharp increase in marginal cost. Hence, we shall also investigate the scenario in which undercutting is aimed at stealing just a portion of the rival's demand.

3.1 Partial undercutting

We take the standpoint of firm 1, for a generic location pair (x_1, x_2) along the linear city. To begin with, take the standpoint of firm 1 and suppose there is a consumer located at $x_1^u \in (L/2, x_2)$ which firm 1 targets when choosing its unilateral undercutting price $p_1^u = p_2^* - t(x_1^u - x_1) - \varepsilon$, where ε is positive and arbitrarily small and therefore negligible. If firm 1 adopts this price, her demand is $y_1^u = d\widehat{x}_1 \in (x_1^u, x_2)$, leaving some positive demand to the rival. That is, any discount on transportation costs apparently tailored onto a consumer u to the left of x_2 makes product 1 appealing also to other individuals located between that consumer and firm 2. In particular, neglecting ε, $\widehat{x}_1$ solves

$$s - p_1^u - t(\widehat{x}_1 - x_1) = s - p_2^* - t(x_2 - \widehat{x}_1) \tag{12}$$

where $\widehat{x}_1 = (x_2 + x_1^u)/2$, i.e., $\widehat{x}_1$ is the midpoint between x_1^u and x_2.

An analogous procedure applies for firm 2, targeting a consumer at $x_2^u \in (x_1, L/2)$ through its undercutting price $p_2^u = p_1^* - t(x_2 - x_2^u) - \varepsilon$. Doing so, firm 2 will attract any consumers from $\widehat{x}_2 = (x_1 + x_2^u)/2$, attaining a demand equal to $y_2^u = d(L - \widehat{x}_2)$. That is, $\widehat{x}_1 = (x_1 + x_2^u)/2$ solves

$$s - p_2^u - t(x_2 - \widehat{x}_2) = s - p_1^* - t(\widehat{x}_2 - x_1) \tag{13}$$

The resulting unilateral undercutting prices simplify as follows:

$$p_1^u = \frac{2c^2d^2L + t^2\left(4L + 2x_1 - x_2 - 3x_1^u\right) + cdt\left(6L + x_1 - x_2 - 2x_1^u\right)}{2cd + 3t} \tag{14}$$

$$p_2^u = \frac{2c^2d^2L + t^2\left(4L + x_1 - x_2 + 2x_2^u\right) + cdt\left(2L + x_1 - 2x_2 + 3x_2^u\right)}{2cd + 3t} \tag{15}$$

The resulting individual partial undercutting profits are $\pi_i^u = \left(p_i^u - cy_i^u\right) y_i^u$, $i = 1, 2$, and partial undercutting is not profitable iff $\pi_i^u \leq \pi_i^*$ for both firms, where the expressions of π_i^* are the same as in (9). Now note that π_i^u is a function of x_i, x_j and x_i^u, and therefore to ascertain whether $\pi_i^u \lesseqgtr \pi_i^*$, it suffices to check that the maximum of π_i^u w.r.t. x_i^u is indeed at most equal to π_i^* for any admissible x_i and x_j. The system

$$\begin{aligned} \frac{\partial \pi_1^u}{\partial x_1^u} &= \frac{d\left(cd + 2t\right)\left[2cd\left(L - x_2 - x_1^u\right) + t\left(2L + x_1 - 2x_2 - 3x_1^u\right)\right]}{2\left(2cd + 3t\right)} = 0 \\ \frac{\partial \pi_2^u}{\partial x_2^u} &= \frac{d\left(cd + 2t\right)\left[2cd\left(L - x_1 - x_2^u\right) + t\left(2L - 2x_1 + x_2 - 3x_2^u\right)\right]}{2\left(2cd + 3t\right)} = 0 \end{aligned} \tag{16}$$

delivers the unique pair of solutions

$$\overline{x}_i^u = \frac{2cd\left(L - x_j\right) + t\left(2L + x_i - 2x_j\right)}{2cd + 3t}\,,\; i, j = 1, 2; i \neq j \tag{17}$$

with second order conditions being satisfied since $\partial^2 \pi_i^u / \partial \left(x_1^u\right)^2 = -d\left(cd + 2t\right)/2 < 0$ always. It is then easily checked that $\pi_i^u\left(\overline{x}_i^u, \overline{x}_j^u\right) = \pi_i^*$ for all admissible x_i and x_j. This amounts to saying that neither firm will unilaterally deviate from its candidate equilibrium price p_i^* to the partial undercutting price p_i^u as the latter delivers at most (but in general less than) the same profit as p_i^*.

This exercise implies the following

Lemma 1 *In presence of linear transportation costs and quadratic production costs, partial undercutting is unprofitable.*

Having proved this, we are left with the case of undercutting to monopoly.

3.2 Undercutting to monopoly

As in d'Aspremont *et al.* (1979), the undercutting price $p_i^u = p_j^* - t\left|x_j - x_i\right| - \varepsilon$, makes firm i a monopolist, with profits

$$\pi_i^u = \left(p_i^u - cdL\right) dL \tag{18}$$

Once again, the analysis is carried out for a generic location pair. The unilateral incentive for either firm to undercut the rival exists if the following expressions are positive:

$$\pi_1^u - \pi_1^* = \alpha \left[4Lt\left(2cd+3t\right)\left(cd\left(x_1+3\left(L-x_2\right)\right)+2t\left(x_1+2\left(L-x_2\right)\right)\right) - \right.$$
$$\left. \left(cd+2t\right)\left(2cdL+t\left(2L+x_1+x_2\right)\right)^2 \right] \tag{19}$$
$$\pi_2^u - \pi_2^* = \alpha \left[4Lt\left(2cd+3t\right)\left(cd\left(L+3x_1-x_2\right)+2t\left(L+2x1-x_2\right)\right) - \right.$$
$$\left. \left(cd+2t\right)\left(2cdL+t\left(4L-x_1-x_2\right)\right)^2 \right] \tag{20}$$

where $\alpha \equiv d/\left[4\left(2cd+3t\right)^2\right] > 0$. Expression (19) is positive for all

$$x_1 > \frac{2L\left(c^2d^2+4t^2\right)+cdt\left(8L-x_2\right)-2\left[\left(2cd+3t\right)\sqrt{2Lt\left(cd+2t\right)\left(L-x_2\right)}+t^2x_2\right]}{t\left(cd+2t\right)} \equiv x_1^u\left(x_2\right) \tag{21}$$

while (20) is positive for all

$$x_2 < \frac{-2L\left(c^2d^2+4t^2\right)-cdt\left(8L-x_2\right)+2\left[\left(2cd+3t\right)\sqrt{2Lt\left(cd+2t\right)x_1}-t^2x_1\right]}{t\left(cd+2t\right)} \equiv x_2^u\left(x_1\right) \tag{22}$$

Solving the system $x_i - x_i^u\left(x_j\right) = 0$ w.r.t. locations $\left(x_1, x_2\right)$, one finds that expressions (19-20) are simultaneously positive for all

$$x_1, x_2 \in \left(\frac{L\left(cd+2t\right)}{8t}, L - \frac{L\left(cd+2t\right)}{8t}\right) \tag{23}$$

This range obviously collapses to $(1/4, 3/4)$ if $L = 1$ and $c = 0$, in which case the model coincides with the original Hotelling (1929) setup. Looking at the undercutting range in (23), one notes two relevant features. The first is that it shrinks as c and d increase, while it expands as t increases. The second is that it has measure zero iff

$$\frac{L\left(cd+2t\right)}{8t} = \frac{L}{2} \tag{24}$$

and this happens for all $c \geq 2t/d \equiv \underline{c}$. That is, if production costs become steeper, or consumer density increases, the range of product differentiation degrees at which undercutting is profitable becomes smaller because either (i) serving the whole demand in a single plant operating at decreasing returns implies an excessively high marginal cost or (ii) the number of consumers at each point along the city is sufficiently high to reduce the gravitational attraction exerted by the median consumer. If the parameter determining the curvature of production costs is sufficiently high, the incentive to undercut disappears altogether. Indeed, the critical threshold of c ensuring this result tends to zero as d tends to infinity (and conversely). In principle, the effect of d on $\underline{c}$ might seem ambiguous, because higher density levels could increase undercutting incentives on the basis of a demand effect, while any output expansion implies a higher marginal cost, which makes undercutting less appealing. Indeed, the balance between the demand and cost effects, whereby $\partial\underline{c}/\partial d < 0$ and $\lim_{d\to\infty}\underline{c} = 0$, reveals that cost considerations prevail in shaping firms' incentives as the market becomes progressively denser.

The above condition on the steepness of the cost function can be reinterpreted in terms of the mark-up $mu_i^* = t/(cd)$, noting that $c \geq 2t/d$ is equivalent to $t/(cd) \leq 1/2$, which says that the undercutting incentive vanishes if the mark-up is at most equal to 50%.

Summing up, we may formulate the following:

Proposition 2 *In presence of linear transportation costs and quadratic production costs, if the latter are steep enough to keep the mark-up below 1/2, then there exists a unique subgame perfect pure-strategy equilibrium with firms locating at the midpoint of the linear city and pricing above marginal cost.*

That is, the minimum differentiation principle dating back to Hotelling (1929) may be restored under sufficiently decreasing returns to scale. Recalling the fact that mark-up is constant, as emerged in section 1.2, one may interpret the above Proposition as the consequence of the interplay between a mark-up independent of product differentiation and a marginal cost increasing in the volume of production: as differentiation decreases due to firms' profit incentives at the location stage, either firm is tempted to undercut the rival's price but refrains from doing so because of decreasing returns to scale.

Having shown that the model admits a subgame perfect equilibrium in pure strategies with minimum differentiation, there remain to stress that this is subject to fulfill the initial assumption of full market coverage. This holds true if and only if consumers at the endpoints of the linear city are able to pay the mill price $p^* = L(cd + t)$ and bear the transportation cost $tL/2$, which requires $s \geq L(2cd + 3t)/2 \equiv \widehat{s}$. Hence, there arises the need of investigating what happens for all $s \in (0, \widehat{s})$, i.e., in the range wherein the gross surplus is not high enough to allow firms to enter the duopolistic regime illustrated thus far.

4 Partial market coverage

As anticipated in the introduction, the emergence of partial market coverage in horizontal differentiation models has often received attention in the established literature. Economides (1984) and Hinloopen and van Marrewijk (1999) investigate the matter in the original Hotelling (1929) setup with linear disutility of transportation, Chirco *et al.* (2003) in the d'Aspremont *et al.* (1979) version with quadratic disutility, and Economides (1989) does so in a circular model as in Salop (1979). Since we are dealing with a variation on the theme of the linear city model, we may confine ourselves to summarising the basic results of Economides (1984) and Hinloopen and van Marrewijk (1999) as follows:

- for sufficiently low levels of the reservation price (i.e., gross surplus s), firms are independent monopolies. Their locations are undetermined, and equilibrium prices are linearly increasing in s;
- for intermediate levels of s, firms compete while covering the whole market, and the extent of differentiation is at least a quarter and at most half the market size. Again, equilibrium prices increase linearly in s;
- if s is sufficiently high to allow for price competition to take place over the whole market, the problem of the absence of equilibrium appears.

In both papers, however, the possibility of 'touching' and competitive equilibria arising under partial coverage, with firms pricing out consumers towards both city boundaries while serving an interval of consumers symmetrically defined around the median one is explicitly considered (see, e.g., Economides, 1986, pp. 353-54, and Hinloopen and van Marrewijk, 1999, pp. 739-42). A partially analogous problem solved in presence of quadratic transportation costs (Chirco *et al.*, 2003), admitting the possibility of partial coverage only under independent monopolies while looking at optimal pricing under full coverage as soon as the demand basins of formerly independent monopolies collapse into a 'touching' equilibrium with full coverage. This analysis yields analogous results for low and high levels of s (except, of course that the pure-strategy equilibrium exists), and delivers a 'kink' equilibrium in prices at the market stage,[1] mimicking collusion, with firms locating at the first and third quartile of the linear city and prices linearly increasing in s.

In the present setting, we set out by characterising the regime where the two firms do not interact and play the optimal price, behaving as local monopolists. We approach the problem by taking the same angle as in Chirco *et al.* (2003), i.e., considering partial coverage as the initial situation associated with separate monopolies. The first step, which indeed replicates what appears in Chirco *et al.* (2003, pp. 562-63) consists in proving that, if firms are independent monopolists, then each of them locates itself in such a way that demand will be symmetric to the left and right of its location.

To this purpose, we may look at firm 1 and suppose that x_1 is close enough to the left boundary of the linear city to ensure that p_1 drives to zero the surplus of a single consumer at $x_0 \in (x_1, L/2)$, i.e., $p_1 = s - t(x_1 - x_0)$. This regime, and the definition of the associated demand function, is admissible as long as $x_1 \in [0, x_0/2]$. Accordingly, demand is $y_1 = dx_0$ and the firm must choose x_0 to maximise

$$\pi_1 = d[s - t(x_1 - x_0)]x_0 - cd^2x_0^2 \tag{25}$$

This happens at

$$x_0^{AM} = \frac{s + tx_1}{2(cd + t)} \tag{26}$$

where superscript AM stands for *asymmetric monopoly*. The corresponding levels of price, demand and profits are

$$p^{AM} = \frac{(2cd + t)(s + tx_1)}{2(cd + t)}\,;\; y^{AM} = dx_0^{AM}\,;\; \pi^{AM} = \frac{d(s + tx_1)^2}{4(cd + t)} \tag{27}$$

Clearly, p^{AM}, y^{AM} and π^{AM} are monotonically increasing in x_1, with profits increasing faster than price and output as x_1 moves to the right in the direction of $L/4$. Consequently, the firm will move rightwards, increasing its profits while doing so. Moreover, this will also increase the portion of demand located along the left *hinterland* of firm 1,

[1] The arising of 'kink' equilibria dates back at least to Beckmann (1972) and is also discussed in detail in Salop (1979).

measured by dx_1. At some point, x_1 has to become large enough to have a symmetric demand on both sides, with marginal consumers at 0 and x^{AM} enjoying zero surplus. Labelling the net surplus of the consumer at 0 as U_0, this happens when

$$U_0 = s - p^{AM} - tx_1 = 0 \tag{28}$$

i.e., at

$$\underline{x}_1 = \frac{s}{4cd + 3t} \leq \frac{L}{4} \qquad \forall\, s \leq \frac{L\,(4cd + 3t)}{4} \equiv \overline{s} \tag{29}$$

and at $s = \overline{s}$ we also have, intuitively, that $x_0^{AM} = 2\underline{x}_1$.

Now we may look at the alternative regime in which firm 1 is sufficiently far from the left city boundary to enjoy a symmetric demand to the left and right of x_1, while possibly s isn't yet large enough to allow consumers at 0 and $L/2$ to buy from firm 1. In this case, demand is $y_1 = 2d\,(x_0 - x_1)$ with $x_0 = (s - p_1 - tx_1)\,/2$ driving to zero the net surplus of the two marginal consumers. Profit maximisation requires

$$\frac{\partial \pi_1}{\partial p_1} = \frac{2d\,[4cd\,(s - p_1) - t\,(2p_1 - s)]}{t^2} = 0 \Rightarrow p^{SM} = \frac{s\,(4cd + t)}{2\,(2cd + t)} \tag{30}$$

with superscript SM standing for *symmetric monopoly*. In correspondence of p^{SM}, demand and profits amount to

$$y^{SM} = \frac{ds}{2cd + t}\,;\, \pi^{SM} = \frac{ds^2}{4cd + 2t} \tag{31}$$

with both equilibrium magnitudes being independent of firm 1's location, as it clears out when one simplifies the relevant expressions. However, the position of the marginal consumer does depend on x_1 :

$$x^{SM} = \frac{s + 2x_1\,(2cd + t)}{2\,(2cd + t)} \tag{32}$$

This is increasing in x_1 as well as in s, and obviously coincides with x_1 iff $s = 0$. The SM regime is admissible, from $s = 0$ to $s = L\,(2cd + t)\,/2 \equiv \underline{s}$, for all $x_1 \in (0, L/4]$ and any $x_2 \in [3L/4, 1)$. At $s = \underline{s}$, the demand basins of the two firms located at $L/4$ and $3L/4$ are exactly large enough to cover the whole market at the margin, with $y^{SM} = dL/2$ and $x^{SM} = L/2$, showing that in this specific condition firms are entering in reciprocal contact and fully appropriate the surplus of consumers located at 0, $L/2$ and L.

Then, regime SM can be compared to regime AM to ascertain that

$$\pi^{SM} > \pi^{AM} \qquad \forall\, x_1 \in (0, x_\pi], \tag{33}$$

$$x_\pi \equiv \frac{s}{t} \cdot \frac{\sqrt{2}\,(cd+t) - \sqrt{(cd+t)\,(2cd+t)}}{\sqrt{(cd+t)\,(2cd+t)}} \tag{34}$$

and conversely outside this range. The last step consists in carrying out some elementary algebra to show that $x_\pi > \underline{x}_1$over the whole admissible parameter range, and $\overline{s} - \underline{s} = Lt/4 > 0$ for all $L, t > 0$. This implies that $\pi^{SM} > \pi^{AM}$, $\forall\, x_1 \in (0, \underline{x}_1]$.

We may consequently draw the following conclusion:[2]

Lemma 3 *In the parameter range wherein it is admissible, the monopoly regime with asymmetric demand basins is strictly dominated by the alternative monopoly regime with symmetric demand basins.*

The above Lemma immediately implies

Proposition 4 *For all $s \in (0, \underline{s})$, firms may choose arbitrary locations (not necessarily symmetric around $L/2$) and behave as separated monopolists. The only requirement they have to fulfil is to pick their respective locations $x_1 \in (0, L/4)$ and $x_2 \in (3L/4, L)$ ensuring symmetric demands. In correspondence of $s = \underline{s}$, the separated monopoly regime collapses to full market coverage and is observationally equivalent to a cartel supplying two varieties at $x_1 = L/4$ and $x_2 = 3L/4$.*

There remains to investigate the optimal behaviour of firms for all $s \in (\underline{s}, \widehat{s})$. In this range, the duopolistic competition investigated in sections 2-3 cannot be performed as the level of the gross surplus is too low. In principle, firms may adopt three different pricing strategies, each paired with a specific location pattern. They may, alternatively,

(i) remain at $x_1 = L/4$ and $x_2 = 3L/4$ and adopt prices increasing linearly in s to keep mimicking the behaviour of a cartel, as in the above Proposition, until s becomes large enough to allow for proper duopolistic competition to take place;

[2] Of course, this holds as well if transportation costs are linear and $c = 0$ (see Hinloopen and van Marrewijk, 1999, p. 739). The analogous result in the Hotelling duopoly with quadratic transportation costs and constant average production costs is in Lemma 1 in Chirco *et al.* (2003, p. 562).

(ii) extract the full surplus of consumers at the city boundaries and relocate inside the central quartiles. This regime is defined for all $x_1 \in (L/4, L/2)$ and $x_2 \in (L/2, 3L/4)$;

(iii) extract the full surplus of the median consumer and relocate outside the central quartiles. This regime is defined for all $x_1 \in [0, L/4)$ and $x_2 \in (3L/4, L]$.

We may set out by looking at case (i), in which the symmetric price is $p^k = s - tL/4$, where superscript k mnemonics for the 'kink equilibrium, and the resulting profits $\pi^k = dL\left[4s - L\left(2cd + t\right)\right]/8$ are positive for all $s > \underline{s}/2$. Then, note that $p^k = p^{SM}$ at $s = \underline{s}$ and $p^k = p^*$ at $s = L\left(4cd + 5t\right)/4 \equiv \widetilde{s} \in (\underline{s}, \widehat{s})$. This level of the gross surplus is also the threshold above which full market coverage is viable under duopoly pricing, with firms playing p^* at $x_1 = L/4$ and $x_2 = 3L/4$.

One should recall that $s \in (\underline{s}, \widehat{s})$ may create an incentive for either firm to unilaterally undercut p^k in order to acquire full monopoly power. It is easily checked that the adoption of the undercutting price $p^{ku} = p^k - tL/2 = s - 3Lt/4$ is not appealing, as the resulting undercutting profits are $\pi^{ku} = dL\left(p^{ku} - cdL\right) > \pi^k$ for all $s > L\left(6cd + 5t\right)/4 > \widetilde{s}$, which amounts to saying that undercutting is unprofitable for all levels of the gross surplus such that $p^k \in \left(p^{SM}, p^*\right)$.

In case (ii), the relevant prices are $p_1^0 = s - tx_1$ and $p_2^L = s - t\left(L - x_2\right)$, and we may look at the firms' profit functions at the upstream stage,

$$\begin{aligned} \pi_1 &= \frac{d\left[L - 2\left(x_1 + x_2\right)\right]\left[cd\left(2\left(x_1 + x_2\right) - L\right) - 2\left(s - tx_1\right)\right]}{4} \\ \pi_2 &= \frac{d\left[L - 2\left(x_1 + x_2\right)\right]\left[cd\left(2\left(x_1 + x_2\right) - L\right) - 2\left(s - tx_1\right)\right]}{4} \end{aligned} \quad (35)$$

which are concave in x_1 and x_2, respectively, and generate the following FOCs:

$$\frac{\partial \pi_1}{\partial x_1} = \frac{d\left[t\left(L - 2\left(2x_1 + x_2\right)\right) + 2\left[s + cd\left(L - 2\left(x_1 + x_2\right)\right)\right]\right]}{2} = 0 \quad (36)$$

$$\frac{\partial \pi_2}{\partial x_2} = \frac{d\left[t\left(5L - 2\left(x_1 + 2x_2\right)\right) - 2\left[s - cd\left(3L - 2\left(x_1 + x_2\right)\right)\right]\right]}{2} = 0 \quad (37)$$

System (36-37) has a unique solution at

$$x_1^{0L} = \frac{2s - L\left(2cd + t\right)}{2t}\,;\; x_2^{0L} = \frac{L\left(2cd + 3t\right) - 2s}{2t} \quad (38)$$

with $\partial x_1^{0L}/\partial s = 1/t$ and $\partial x_2^{0L}/\partial s = -1/t$ and, obviously, $x_2^{0L} = L - x_1^{0L}$. At $\left(x_1^{0L}, x_2^{0L}\right)$, the mill price is $p^{0L} = L\,(2cd+t)\,/2$ and individual profits amount to $\pi^{0L} = dL^2\,(cd+t)\,/4$. Recalling that this regime is relevant for all $x_1 \in (L/4, L/2)$ and $x_2 \in (L/2, 3L/4)$, we may observe that $x_1^{0L} \geq L/4$ and $x_2^{0L} \leq 3L/4$ for all $s \geq \overline{s}$, a range in which we also have that $\pi^k \geq \pi^{0L}$. The reason is intuitive: in order to keep selling to consumers at the city boundaries, the price must decrease in order to exactly offset the linear increase in transportation costs, and indeed $p^{0L} \leq p^k$ for all $s \geq \overline{s}$: in correspondence of invariant and symmetric quantities at $y^{0L} = y^k = dL/2$, this implies that regime (i) dominates regime (ii) in the entire location range in which the latter is defined.

We are left with case (iii), defined for all $x_1 \in [0, L/4)$ and $x_2 \in (3L/4, L]$. This scenario can be quickly dealt with, as the relevant prices

$$p_1^{L/2} = s - t\left(\frac{L}{2} - x_1\right) \; ; \; p_2^{L/2} = s - t\left(x_2 - \frac{L}{2}\right) \tag{39}$$

imply that the profit functions at the first stage look as follows:

$$\begin{aligned} \pi_1 &= \frac{dL\,[2s - cdL + t\,(2x_1 - L)]}{4} \\ \pi_2 &= \frac{dL\,[2s - cdL + t\,(L - 2x_2)]}{4} \end{aligned} \tag{40}$$

with π_1 increasing in x_1 and π_2 decreasing in x_2 and consequently lower than or at most equal to π^k for all $x_1 \in [0, L/4]$ and $x_2 \in [3L/4, L]$.

To complete the spectrum of price-and-location regimes which are admissible at least in principle, we have to briefly tackle the scenario in which firms compete for a full neighbourhood of $L/2$ while pricing out in the right neighbourhood of 0 and the left neighbourhood of L. This case is considered by Economides (1984, figure 3, p. 354) and Hinloopen and van Marrewijk (1999, figure 2, p. 739).

The relevant demand functions are $y_1 = d\left(L/2 - x^l\right)$ and $y_2 = d\,(x^r - L/2)$ where

$$x^l = x_1 - \frac{s - p_1}{t} \; ; \; x^r = x_2 + \frac{s - p_2}{t} \tag{41}$$

are the locations of marginal consumers to the left (resp., right) of firm 1 (resp., 2). Proceeding by backward induction, the FOCs at the market stage deliver prices

$$p_1^l = \frac{(2cd+t)\left[2s+t\left(L-2x_1\right)\right]}{4\left(cd+t\right)}$$
$$p_2^r = \frac{(2cd+t)\left[2s+t\left(2x_2-L\right)\right]}{4\left(cd+t\right)} \tag{42}$$

Plugging these prices into the profit functions and taking the partial derivatives at the first stage,

$$\frac{\partial \pi_1^l}{\partial x_1} = -\frac{dt\left[2s+t\left(L-2x_1\right)\right]}{4\left(cd+t\right)} < 0$$
$$\frac{\partial \pi_2^r}{\partial x_2} = \frac{dt\left[2s+t\left(2x_2-L\right)\right]}{4\left(cd+t\right)} > 0 \tag{43}$$

we see that such derivatives imply that firms have incentives to move away from each other in order to become independent monopolists at the margin with both firms collapsing into the 'kink' equilibrium with full coverage, as illustrated above. To ascertain this, it suffices to observe that the net surplus of the median consumer when buying from, say, firm 1, becomes nil at

$$x_1 = \frac{L\left(4cd+3t\right)-2s}{2\left(4dc+3t\right)} = \frac{L}{2} - \frac{s}{4dc+3t} \tag{44}$$

which belongs to $[L/4, L/2)$ for all $s \in (0, \overline{s}]$ and it can be easily verified that the resulting profits

$$\pi_1^l = \frac{d\left[2s+t\left(L-2x_1\right)\right]^2}{16\left(cd+t\right)} \tag{45}$$

are lower than or equal to π^k for all $x_1 \in [L/4, L/2)$.

This discussion permits us to formulate

Proposition 5 *Consider $s \in (\underline{s}, \widehat{s})$. In this range, firms adopt two pricing regimes, depending on the level of the gross surplus:*

- *for all $s \in (\underline{s}, \widetilde{s})$, they locate at $x_1 = L/4$ and $x_2 = 3L/4$, cover the whole market and play $p_1 = p_2 = p^k$, driving to zero the net surplus of consumers at 0, $L/2$ and L;*

- *for all $s \in [\widetilde{s}, \widehat{s})$, duopolistic competition operates and, provided the cost function is sufficiently steep, i.e., $c \geq 2t/d \equiv \underline{c}$, firms play the subgame perfect prices $\{p_1^*, p_2^*\}$ and move themselves towards the median consumer. The resulting degree of differentiation lies between $L/2$ and zero and decreases monotonically in s.*

The second part of the above Proposition can be expanded to say that any s higher than $\underline{s}$ but possibly lower than $\widehat{s}$ may put into question the existence of a pure-strategy equilibrium in the price stage because of undercutting. If $c \geq \underline{c}$, firms will just move inwards as far as consumer's gross surplus allows them to go, without undercutting. As soon as s is at least equal to $\widehat{s}$, the prediction of the unconstrained duopoly model materialises and minimum differentiation appears. Conversely, any $c \in (0, \underline{c})$ and $s \in [\widetilde{s}, \widehat{s})$ will engender the incentive to undercut as soon as firms enter the interval in (23).

5 Extension: solving the duopoly game for a class of convex cost functions

Here we go back to the duopoly case and suppose now that the cost function of firm i is $C_i = cy_i^n$, with $n > 1$, so that the individual profit function is $\pi_i = \left(p_i - cy_i^{n-1}\right) y_i$. An analogous approach to a general transportation cost function in combination with linear production costs is in Economides (1986).

The relevant FOCs at the market stage are

$$\frac{\partial \pi_1}{\partial p_1} = \frac{d}{2t} \left\{ p_2 - 2p_1 + t(x_1 + x_2) + \frac{cn}{2^{n-1}} \left[\frac{d(p_2 - 2p_1 + t(x_1 + x_2))}{t} \right]^{n-1} \right\} = 0 \tag{46}$$

$$\frac{\partial \pi_2}{\partial p_2} = \frac{d}{2t} \left\{ p_1 - 2p_2 + t(2L - x_1 - x_2) + \frac{cn}{2^{n-1}} \left[\frac{p_1 - 2p_2 + t(2L - x_1 - x_2)}{t} \right]^{n-1} \right\} = 0 \tag{47}$$

Of course, solving (46-47) would require transcendental functions. However, the above system can be solved invoking symmetry between locations, $x_2 = L - x_1$, which also allows for symmetry in prices, $p_2 = p_1$.[3] As a result, we obtain the candidate equilibrium price:

$$p^* = Lt + cn\left(\frac{dL}{2}\right)^{n-1} \tag{48}$$

whereby candidate equilibrium profits are

$$\pi^* = \frac{c\,(n-1)\,d^n L^n + 2^{n-1} dL^2 t}{2^n} \tag{49}$$

with $y_i^* = dL/2$, with $i = 1, 2$.

Relying on full symmetry, the issue of partial undercutting can be treated as above with, say, firm 1 targeting the consumer at $x^u \in (L/2, x_2)$, who must be indifferent between firm 1 and firm 2 when facing prices p_1^u and p^*, in such a way that

$$s - p_1^u - t\,|\widehat{x} - x_1| = s - Lt - cn\left(\frac{dL}{2}\right)^{n-1} - t\,|x_2 - \widehat{x}| \tag{50}$$

with $\widehat{x} = (x_2 + x^u)/2$. The undercutting price solving (50),

$$p_1^u = nc\left(\frac{cL}{2}\right)^{n-1} + t\,(L + x_1 - x^u) \tag{51}$$

grants firm 1 the following profits:

$$\pi_1^u = \frac{d\left[nc\,(cL/2)^{n-1} + t\,(L + x_1 - x^u)\right](L - x_1 + x^u)}{2} - \frac{cd^n\,(L - x_1 + x^u)^n}{2^n} \tag{52}$$

whose properties imply that partial undercutting is never profitable. To ascertain this fact, one can proceed as follows. First, note that $\pi_1^u = \pi^*$ iff $x^u = x_1$. Secondly, observe that the partial derivative

$$\frac{\partial \pi_1^u}{\partial x^u} = dt\,(x_1 - x^u) - \frac{cd^n n}{2^n}\left[(L - x_1 + x^u)^{n-1} - L^{n-1}\right] \tag{53}$$

is nil at $x^u = x_1$ and negative for any $x^u > x_1$, and therefore also for all $x^u \in (L/2, x_2)$. This rules out partial undercutting for all $n > 1$.

Undercutting to monopoly, again by firm 1, requires setting $p_1^u = p_1^* - t\,|x_2 - x_1| - \varepsilon$, yielding profits $\pi_1^u = \left[p_1^u - c\,(dL)^{n-1}\right] dL$, provided ε is negligible. Undercutting is

[3] To this regard, we should also add that the explicit use of symmetry across locations and prices, which we adopt here for the first time in order to solve the price stage and investigate equilibrium existence, implies that we cannot analyse the first stage of the game to characterise location choices.

profitable iff $\pi_i^u > \pi_i^*$, this being true for all

$$x_1, x_2 \in \left(\frac{L\left[c\left(2^n - n - 1\right)d^{n-1}L^{n-2} + 2^{n-1}t\right]}{2^{n+1}t}, L - \frac{L\left[c\left(2^n - n - 1\right)d^{n-1}L^{n-2} + 2^{n-1}t\right]}{2^{n+1}t} \right) \tag{54}$$

which coincides with (23) if $n = 2$. The critical level of c beyond which the convex component of the cost function is so steep that $x_1^u \geq L/2$ is

$$\underline{c}\left(n\right) = \frac{2^{n-1}Lt}{\left(2^n - n - 1\right)d^{n-1}L^{n-1}} > 0 \text{ for all } n > 1 \tag{55}$$

with $\left.\underline{c}\left(n\right)\right|_{n=2} = 2t/d$ and $\partial \underline{c}\left(n\right)/\partial n < 0$; $\partial^2 \underline{c}\left(n\right)/\partial n^2 > 0$, i.e., $\underline{c}\left(n\right)$ is monotonically decreasing and convex in n. The $\lim_{n\to\infty} \underline{c}\left(n\right)$ is positive and can be arbitrarily low (while, for instance, if $L = 2$, $t = 1$ and $d = 1/2$, it is equal to one). Once again, this argument can be reformulated by saying that the existence of equilibrium requires a condition on the mark-up, which here is

$$mu_i^*\left(n\right) = \frac{2^{n-1}Lt}{cnd^{n-1}L^{n-1}} \tag{56}$$

and therefore the incentive to undercut vanishes for all

$$mu_i^*\left(n\right) \leq \overline{mu}\left(n\right) \equiv \frac{2^n - n - 1}{n} \tag{57}$$

with the expression on the r.h.s. of (57) collapsing to 1/2 at $n = 2$.

On the basis of the foregoing discussion, we may formulate the following, which includes Proposition 2 as a special case:

Proposition 6 *In presence of linear transportation costs and convex production costs, if the latter are steep enough to keep the mark-up at* $\overline{mu}\left(n\right)$ *or below, then there exists a unique subgame perfect pure-strategy equilibrium with firms locating at the midpoint of the linear city and pricing above marginal cost.*

A few additional words are in order. The shape of $c\left(n\right)$ illustrates the presence of a tradeoff along the frontier of the parameter region in which the price equilibrium exists in pure strategies: considering only the convex component of production costs, it is apparent that its curvature (or steepness) is determined by both c and n, and (55) says that if n is very large, then even a touch of convex costs, i.e., a very low value of c, may ensure the absence of undercutting incentives and therefore the existence of the pure-strategy price equilibrium at the second stage. Addressing the same issue from the opposite angle, one may note that (55) is a convex curve in the space (c, n) identifying the minimum value of the exponent of the cost function, say $\underline{n}(c)$,[4] such that any $n \geq \underline{n}(c)$ ensures the existence of the pure-strategy equilibrium. Seen under this light, (55) conveys much the same message as the condition identified by Economides (1986) concerning the degree of convexity of transportation costs. Hence, both settings assuming alternatively one cost function being linear and the other convex, we come away of this analysis and Economides's (1986) with a bottom line establishing that convexifying enough either one will indeed warrant equilibrium existence.

6 Concluding remarks

We have proposed a modified version of the Hotelling (1929) model in which linear transportation costs and uniform consumer distribution are accompanied by quadratic production costs, to show that, if the production cost function is are sufficiently steep (or the mark-up is sufficiently low), the minimum differentiation principle is restored, undercutting being unattractive. In a nutshell, this version of the Hotelling duopoly shows that, departing from a model lacking concavity because both production and transportation cost functions are linear, one may alternatively ensure the existence of a pure-strategy equilibrium by convexifying enough either one (with the additional proviso that production costs must be steep enough). However, the equilibrium configurations emerging from the two scenarios will substantively differ in that a convex disutility of transportation fosters product differentiation while a convex production cost restores the minimum differentiation principle.

The analysis of the unconstrained duopoly game has been complemented by a few extensions. We have singled out the parameter range in which partial coverage arises, with firms behaving as separate monopolists, as well as that in which they cover the whole market at the margin, producing a 'kink' equilibrium. The last step of our analysis has consisted in extending the proof of no undercutting to monopoly to a more general setting in which quadratic costs are substituted by a generic convex function.

[4] The expression of $\underline{n}(c)$ obtains by inverting $\underline{c}(n)$, or, equivalently, solving $c = \underline{c}(n)$ w.r.t. n. This requires the Lambert W *function*, also known as *omega function* or *product logarithm*.

All of the above has been done in the duopoly case. We may formulate one last remarks concerning the case of three or more firms (as in Cremer *et al.*, 1991; and Brenner, 2005) with convex production cost functions. The first is that, if production costs are quadratic, the presence of a third firm suffices to create undercutting incentives on the part of all firms alike, whereby the subgame perfect equilibrium ceases to exist in pure strategies once again.[5] If productions costs are strictly convex but not quadratic, we may not rely on full symmetry to solve first order conditions in the price space. This is already evident in the triopoly case, which becomes unmanageable once the quadratic case is abandoned. Consequently, we cannot formulate any educated guess about the arising of undercutting incentives (or the opposite) for a generic degrees of convexity of production costs and competition.

[5] This case has been omitted for brevity but it is available from the authors upon request.

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