Analytic regularity for solutions to sums of squares: an assessment

DOI: http://doi.org/10.1007/s40627-020-00055-8

ANALYTIC REGULARITY FOR SOLUTIONS TO SUMS OF SQUARES: AN ASSESSMENT

ANTONIO BOVE AND MARCO MUGHETTI

In memory of Nick Hanges

Abstract. We present a brief survey on the state of the theory of the real analytic regularity (real analytic hypoellipticity) for the solutions to sums of squares of vector fields satisfying the Hörmander condition.

Contents

Date: May 1, 2020.
2010 *Mathematics Subject Classification.* 35H10, 35H20 (primary), 35B65, 35A20, 35A27 (secondary).
Key words and phrases. Sums of squares of vector fields; Analytic hypoellipticity; Treves conjecture.

1. Introduction: the C^∞ hypoellipticity

The purpose of the present paper is to give an account of the actual status of the theory of the real analytic regularity for the solutions to sums of squares type equations.

While the problem of the C^∞ hypoellipticity of sums of squares has been settled from the very beginning by the famous paper of L. Hörmander, [34], the problem of the analytic hypoellipticity is still open and seems much more involved than the latter.

In this section we give a brief presentation of the results in the C^∞ category, since they have been the starting point of any further study. We tried to give all the references we are aware of, but by no means we claim completeness.

Consider an equation of the form

$$\sum_{i,j=1}^{n} a_{i,j}(x)\partial_i\partial_j u(x) + \sum_{j=1}^{n} b_j(x)u(x) + c(x)u(x) = f(x).$$

We say that it is a degenerate elliptic equation if the quadratic form corresponding to the principal symbol is non negative (or non positive, depending on the sign conventions):

$$\sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \geq 0.$$

Let us start by assuming that the coefficients of the above equation are smooth, i.e. C^∞ functions defined in an open subset $\Omega \subset \mathbb{R}^n$. Even then the problem of the regularity of the distribution solutions when the data are smooth seems too general. But if we assume that the matrix

$$A(x) = [a_{i,j}(x)]_{\substack{i=1,\dots,n\\ j=1,\dots,n}}$$

has constant rank near a point where its determinant vanishes, then, at least locally, we may find a finite number of vector fields

$$X_j(x, D_x) = \sum_{k=1}^{n} \alpha_{j,k}(x) D_k, \qquad j = 0, 1, \dots, r, \tag{1.1}$$

such that the above operator is written as

$$\sum_{j=1}^{r} X_j(x, D)^2 + X_0(x, D) + \alpha(x),$$

(see also the fundamental paper [34].)

In what follows we focus on operators of the form

$$P(x,D)=\sum_{j=1}^{r} X_j(x,D)^2, \tag{1.2}$$

where X_j denotes a vector field with smooth (or real analytic) coefficients, $a_{j,k}(x)$, with $a_{j,k}\in C^\infty(\Omega)$ or $a_{j,k}\in C^\omega(\Omega)$, the latter denoting the class of all real analytic functions on Ω.

In the paper [34] Hörmander proved for a slightly more general class of operators than the one in (1.2) the following

Proposition 1.1 ([34])**.** *If P is a second order differential operator and P is C^∞ hypoelliptic in the open subset Ω, then the principal symbol of P is semidefinite.*

Here is the famous result on the C^∞ hypoellipticity for operators of the form (1.2)

Theorem 1.1 ([34])**.** *Let P be given by* (1.2)*, where the vector fields have C^∞ coefficients in the open set $\Omega\subset\mathbb{R}^n$. Assume that among the operators X_{j_1}, $[X_{j_1},X_{j_2}]$, $\dots$, $[X_{j_1},[X_{j_2},[X_{j_3},\dots X_{j_k}]]]$, $\dots$, where $j_\ell=1,2,\dots,r$, there exist n which are linearly independent at any given point in Ω. Then P is C^∞ hypoelliptic.*

The condition on the vector fields appearing in Theorem 1.1 has been stated literally as Hörmander stated it, but it has a deep geometric meaning. In fact by $[X,Y]$ we denote the commutator of the vector fields: $[X,Y]u=XYu-YXu$. We easily see that $[X,Y]$ is a vector field and that

$$[X,Y]=\sum_{j,k=1}^{n}\left(a_j(x)\partial_j b_k(x)-b_j(x)\partial_j a_k(x)\right)\partial_k,$$

where a_j, b_k denote the (smooth) coefficients of X and Y, respectively.

The condition in Theorem 1.1 can then be rephrased as

Hörmander's Condition:
The Lie algebra over the open set Ω generated by the vector fields X_j and their brackets has dimension n, i.e. the dimension of the ambient space.

Derridj, in [22], proved that if the coefficients of the vector fields have real analytic regularity, then the Hörmander Condition (HC for short) is also necessary.

Theorem 1.1 has received a lot of attention over the years and we would like to mention the extensions that are particularly meaningful in the discussion of the real analytic hypoellipticity.

We first remark that the proof of the hypoellipticity of the operator P is done by establishing an a priori inequality showing the loss of derivatives of the operator P. The inequality with the optimal loss of derivatives is due to Rothschild and Stein, [51].

Theorem 1.2. *Let $x_0 \in \Omega$ and denote by U a neighborhood of x_0, $U \subset \Omega$. Assume that in U the Hörmander Condition is satisfied by taking iterated brackets involving at most m vector fields. Then for every $u \in C_0^\infty(U)$ there is a positive constant C such that*

$$\|u\|^2_{\frac{1}{m}} + \sum_{j=1}^{r} \|X_j(x,D)u\|^2 \leq C\left(\langle Pu, u\rangle + \|u\|^2\right). \tag{1.3}$$

Here $\|u\|_s$ denotes the norm of u in the Sobolev space H^s and the notation $\langle u, v\rangle$ denotes the L^2 scalar product.

A very important point of view when it comes to the problem of the real analytic hypoellipticity is the microlocal theory for sums of squares.

First of all we note that the symbol of the commutator of two vector fields is the Poisson bracket of the symbols. Let $X(x,D) = \sum_{j=1}^n a_j(x)D_j$, where $D_j = i^{-1}\partial_{x_j}$, then the symbol of X is

$$X(x,\xi) = \sum_{j=1}^{n} a_j(x)\xi_j.$$

Defining the Poisson bracket of two functions $f(x,\xi)$ and $g(x,\xi)$ as

$$\{f, g\} = \sum_{j=1}^{n} \left(\partial_{\xi_j} f \partial_{x_j} g - \partial_{x_j} f \partial_{\xi_j} g\right),$$

we have that

$$\sigma([X,Y]) = \frac{1}{i}\{X(x,\xi), Y(x,\xi)\}.$$

The Hörmander Condition can then be stated microlocally. In order to do this we define first the characteristic variety of the operator P in (1.2).

Definition 1.1. *Let P be as in* (1.2). *We define the set*

$$\mathrm{Char}(P) = \{(x,\xi) \mid (x,\xi) \in T^*\Omega \setminus \{0\},\ X_j(x,\xi) = 0,\ \textit{for}\ j = 1,\dots,r\}.$$

Here $T^*\Omega \setminus \{0\}$ denotes the cotangent bundle over Ω minus the zero section. We point out that, unless ad hoc assumptions are made this set in general is not a manifold.

The following is the microlocal statement of Hörmander's Condition; we refer to Bolley, Camus and Nourrigat, [7], and to Fefferman and Phong, [24], for a microlocal version of the results by Hörmander and Rothschild and Stein.

Microlocal Hörmander's Condition:

We may suppose that, instead of having vector fields we are dealing with (real valued) pseudodifferential operators of order 1. Let $(x_0, \xi_0) \in T^*\Omega \setminus \{0\}$. Then there exists an iterated commutator of length $m \geq 2$, i.e. an operator of the form

$$\operatorname{ad}(X_{i_1})(\operatorname{ad}(X_{i_2}(\cdots \operatorname{ad}(X_{i_{m-1}})(X_{i_m})\cdots)),$$

where $\operatorname{ad}(X)Y = XY - YX$, whose symbol is elliptic—i.e. non zero—at (x_0, ξ_0).

As an example we state Hörmander theorem in a microlocal context.

Theorem 1.3 ([7]). *Let $a_j(x, D)$, $j = 1, \ldots, r$, be real pseudodifferential operators of order 1 defined in Ω. Let $(x_0, \xi_0) \in T^*\Omega \setminus \{0\} \cap \operatorname{Char}(P)$, where $P(x, D) = \sum_{j=1}^{r} a_j(x, D)^2$. Assume further that the Microlocal Hörmander Condition holds at (x_0, ξ_0).*

Let U be a neighborhood of x_0 in Ω and $u, f \in \mathscr{D}'(U)$ such that $Pu = f$ in the distribution sense in U. Then if $(x_0, \xi_0) \notin WF(f)$, there is a neighborhood $U' \subset U$ of x_0 and a conic neighborhood Γ' of ξ_0, such that $WF(u) \cap U' \times \Gamma' = \varnothing$.

2. The real analytic case: a short history, examples and counterexamples

A natural question about the regularity of solutions to sums of squares is whether there is real analytic regularity provided the vector fields have real analytic coefficients and satisfy Hörmander Condition.

It is well known that in the non degenerate case, i.e. the elliptic case, the answer is in the affirmative.

The first example showing that the situation might be more involved is due to Baouendi and Goulaouic, [4], but before stating and discussing it let us introduce the definition of Gevrey class of functions.

Definition 2.1. *Let Ω be an open subset of $\mathbb{R}^n$. We say that the function $u \in C^\infty(\Omega)$ is in the Gevrey class $G^s(\Omega)$, with $s \geq 1$, real number, if for every compact set $K \subset \Omega$ there is a positive constant C_K such that*

$$|\partial^\alpha u(x)| \leq C_K^{|\alpha|+1} \alpha!^s, \qquad \text{for every} \quad x \in K,$$

and for every multiindex α.

It is straighforward that the class $G^1(\Omega) = C^\omega(\Omega)$ i.e. it coincides with the class of all real analytic functions in Ω.

Theorem 2.1 ([4])**.** *Consider the operator in* $\mathbb{R}^3$

$$P_{BG}(x, D_x) = D_1^2 + D_2^2 + x_1^2 D_3^2. \tag{2.1}$$

It obviously satisfies Hörmander Condition, but there exist solutions of $P_{BG}u = f$, with $f \in C^\omega(\mathbb{R}^3)$, belonging to G^2 and not to G^s with $1 \leq s < 2$.

Proof. The proof is the construction of a suitable solution of the equation $P_{BG}u = 0$. Define

$$u(x) = \int_0^{+\infty} e^{ix_3\rho^2 - \frac{x_1^2}{2}\rho^2 + zx_2\rho - \rho} d\rho,$$

where $z \in \mathbb{C}$ is suitable. The integral converges provided we keep x_2 in a small neighborhood of the origin. Now

$$D_1^2 u(x) = -\int_0^{+\infty} e^{ix_3\rho^2 - \frac{x_1^2}{2}\rho^2 + zx_2\rho - \rho} \left(-\rho^2 + x_1^2\rho^4\right) d\rho.$$

Moreover

$$x_1^2 D_3^2 u(x) = -\int_0^{+\infty} e^{ix_3\rho^2 - \frac{x_1^2}{2}\rho^2 + zx_2\rho - \rho} (-x_1^2\rho^4) d\rho$$

and finally

$$D_2^2 u(x) = -\int_0^{+\infty} e^{ix_3\rho^2 - \frac{x_1^2}{2}\rho^2 + zx_2\rho - \rho} z^2\rho^2 d\rho.$$

If we choose $z = \pm 1$ we see that $P_{BG}u = 0$ in a slab where x_2 is in a sufficiently small neighborhood of 0. Setting $z = 1$ then

$$u(x) = \int_0^{+\infty} e^{ix_3\rho^2 - \frac{x_1^2}{2}\rho^2 + x_2\rho - \rho} d\rho.$$

Compute now $\partial_3^k u(0)$:

$$\partial_3^k u(0) = \int_0^{+\infty} \rho^{2k} e^{-\rho} d\rho = (2k)! = \frac{(2k)!}{k!^2} k!^2 \geq 2^k k!^2.$$

It is also easy to see that

$$\partial_2^k u(0) = k!$$

Furthermore, taking k derivatives with respect to x_1 of u at zero we obtain a bunch of terms, among which the terms involving more factors

ρ—responsible for a higher Gevrey regularity—are those where the x_1 derivative lands on the exponential:

$$\int_0^{+\infty} e^{ix_3\rho^2 - \frac{x_1^2}{2}\rho^2 + x_2\rho - \rho}(x_1^2\rho^2)^{\frac{k}{2}}\rho^k d\rho.$$

The above quantity is bounded from above, when x_1 is near 0, by $C^{k+1}k!^{3/2}$. This shows that $u \in G^2$ and that its Gevrey regularity is not better than 2. $\square$

Another example was singled out by Oleĭnik, Oleĭnik and Radkevič in [49], [50]. Let p, q be positive integers and consider in $\mathbb{R}^3$ the following sum of squares

$$P_{OR}(x,\xi) = D_1^2 + x_1^{2(p-1)}D_2^2 + x_1^{2(q-1)}D_3^2, \tag{2.2}$$

where $p < q$. Then

Theorem 2.2 ([49], [50], [13], [31], [20]). *The operator in* (2.2) *is Gevrey hypoelliptic of order q/p and this number is optimal. Moreover if we define the "partial Gevrey regularity" of a solution in the variable x_j as s_j, where $|\partial_{x_j}^\alpha u(x)| \leq C^{\alpha+1}\alpha!^{s_j}$ for x in a compact set, we have that if $P_{OR}u = f \in G^{q/p}$ then u has partial Gevrey regularity*

$$\left(1 + \frac{1}{p} - \frac{1}{q}, 1, \frac{q}{p}\right).$$

The above results require some discussion. The characteristic variety of the operator in (2.1) is actually a real analytic submanifold of $T^*\mathbb{R}^3 \setminus \{0\}$ given by

$$\operatorname{Char}(P_{BG}) = \{(x,\xi) \in T^*\mathbb{R}^3 \setminus \{0\} \mid \xi_1 = \xi_2 = x_1 = 0, \xi_3 \neq 0\}. \tag{2.3}$$

For the operator in (2.2) we have

$$\operatorname{Char}(P_{OR}) = \{(x,\xi) \in T^*\mathbb{R}^3 \setminus \{0\} \mid \xi_1 = x_1 = 0, (\xi_2,\xi_3) \neq (0,0)\}. \tag{2.4}$$

In the first case $\operatorname{Char}(P_{BG})$ has codimension 3 while in the second case $\operatorname{codim}\operatorname{Char}(P_{OR}) = 2$.

We remark that in the first case $\operatorname{Char}(P_{BG})$ is a non symplectic submanifold of $T^*\mathbb{R}^3 \setminus \{0\}$, while in the second case $\operatorname{Char}(P_{OR})$ is symplectic. This means that the symplectic form $\sigma = d\xi \wedge dx$ is of maximal rank in the second case, while it has a kernel in the first case.

At the end of the seventies Tartakoff, [54], and Treves, [56], proved with different methods the following important result:

Theorem 2.3 ([54], [56]). *Consider a sum of squares operator*

$$P(x,D)=\sum_{j=1}^{r} X_j(x,D)^2,$$

where the vector fields X_j have real analytic coefficients defined in an open subset $\Omega \subset \mathbb{R}^n$ and satisfy Hörmander condition.

Assume further that

(a) - $\mathrm{Char}(P)$ *is a symplectic submanifold of* $T^*\mathbb{R}^n \setminus \{0\}$.

(b) - *The principal symbol of P, $p(x,\xi)=\sum_{j=1}^{r} X_j(x,\xi)^2$ vanishes exactly to the second order on* $\mathrm{Char}(P)$.

Then P is analytic hypoelliptic.

We clarify briefly what the expression "vanishes exactly to the second order" means.

Denote by $p(x,\xi)$ the (principal) symbol of P as defined above. Let $(x_0,\xi_0) \in \mathrm{Char}(P)$ and denote by Q the $2n \times 2n$ matrix $d^2 p(x_0,\xi_0)$. The Hamilton matrix of p at (x_0,ξ_0) is then defined as

$$\langle QX, Y\rangle = \sigma(X, F_p Y),$$

σ being the symplectic form. Here X, Y are vectors in $T_{(x_0,\xi_0)}T^*\Omega\setminus\{0\}$.

We say that $p(x,\xi)$ vanishes exactly to the second order at the point (x_0,ξ_0) if

$$\ker F_p(x_0,\xi_0) = T_{(x_0,\xi_0)}\,\mathrm{Char}(P).$$

Let us list a few examples of operators satisfying the assumptions of the theorem.

(a) The quadratic Grušin operator (also called the harmonic oscillator)

$$\sum_{j=1}^{n-1}(D_j^2 + x_j^2 D_n^2).$$

(b) The Heisenberg Laplacian

$$(D_1 - x_2 D_3)^2 + (D_2 + x_1 D_3)^2.$$

(c) The $\Box_b$ operator as well as the $\overline{\partial}_b$ operator in the context of CR-manifolds.

We remark that the operator P_{BG} does not satisfy the assumptions of the theorem because its characteristic manifold is not symplectic since its codimension is 3. On the other hand the operator P_{OR} does not vanish exactly at the second order, even though its characteristic manifold is symplectic.

Actually the result in Theorem 2.3 can be microlocalized. The statement in [56] was already microlocal, while the statement of [54] was formulated in a microlocal way in [55].

Theorem 2.4 ([56], [55])**.** *Let the same hypotheses of Theorem 2.3 be satisfied. Let u, f denote distributions for which the equation $Pu = f$ is satisfied. Then $WF_a(u) \subset WF_a(f)$.*

In [56] the author, regarding the Baouendi–Goulaouic model, writes the following words:

> if $\mathrm{Char}(P)$, assumed to be an analytic manifold, contains a smooth curve which is orthogonal for the fundamental symplectic form to the whole tangent plane to $\mathrm{Char}(P)$ at every point (of the curve), the operator P might not be analytic hypo-elliptic. Actually it is my belief that, in this case, P is necessarily not so.

This is what has been later called Treves curve conjecture, even though it has not been stated like a conjecture. It should also be said that no counter examples are known and no proof has been given so far. The above statement can be rephrased by saying that if $\mathrm{Char}(P)$ is not symplectic, denote by (x_0, ξ_0) a point in $\mathrm{Char}(P)$ and assume that

$$T_{(x_0,\xi_0)} \mathrm{Char}(P) \cap \left(T_{(x_0,\xi_0)} \mathrm{Char}(P)\right)^\sigma \neq \{0\}$$

then there is no analytic hypoellipticity. Here the notation E^σ, where E is a vector space, denotes the symplectic orthogonal to E.

In 1981 Métivier, in [44], proved that there is a lack of analytic hypoellipticity for the operator in $\mathbb{R}^2$

$$P_M(x, D) = D_1^2 + (x_1^2 + x_2^2) D_2^2. \tag{2.5}$$

Let us briefly see what are the Treves' curves in this case.

We have $\mathrm{Char}(P_M) = \{(0, 0; 0, \xi_2), \xi_2 \neq 0\}$. Since everything is flat we are allowed to confuse the manifold with its tangent space. Then

$$\mathrm{Char}(P_M)^\sigma = \{(y_1, 0; \eta_1, \eta_2)\},$$

so that when we take the intersection we have

$$\mathrm{Char}(P_M) \cap \mathrm{Char}(P_M)^\sigma = \{0, 0; 0, \xi_2)\},$$

which does not project injectively onto the base space. Hence the Treves curves are the ξ_2-lines along the fibers of the cotangent bundle.

This fact may let us surmise that the situation is very involved. We note in passing that Métivier proof of the non analytic hypoellipticity of P_M is much more difficult than that for the Baouendi–Goulaouic operator.

As a final remark of this section let us add that the case of the Oleĭnik and Radkevič operator is not explained, even though, clearly, it does not vanish of exact order 2, it still has a symplectic characteristic manifold.

One can also generalize the Métivier operator as

$$M_{p,q,a}(x,D) = D_1^2 + x_1^{2(q-1)} D_2^2 + x_1^{2(p-1)} x_2^{2a} D_2^2, \tag{2.6}$$

where a, p, q are integers, $p, q > 1$, $p < q$, $a > 0$. Its characteristic variety is the real analytic submanifold

$$\mathrm{Char}(M_{p,q,a}) = \{(0, x_2; 0, \xi_2),\ \xi_2 \neq 0\},$$

which is symplectic. In [12] it is proved that $M_{p,q,a}$ is Gevrey hypoelliptic of order s for any

$$s \geq \frac{aq}{aq - q + p}.$$

When $a = 1$, $q = 2$, $p = 1$ the above index gives 2, which is the value that Métivier proved to be optimal. It is worth to note that Métivier's proof is along the same lines of the proof of Theorem 2.1, but it is much more difficult. Moreover it uses the properties of the eigenfunctions of the harmonic oscillator operator in one variable. These properties are no longer true for the anharmonic oscillator: $D_t^2 + t^{2(q-1)}$ (see [28] for a proof of this fact.)

As a consequence there is no proof of the optimality of the above index for $M_{p,q,a}$, except for particular values of p, q, that is when $q-1 = 2k+1$ and $p-1 = k$. See Chinni, [18], for such a proof using the result [5], by Bender and Wang.

3. Geometry of the characteristic variety: stratifications and the Treves conjecture

In 1996, see the paper [58], F. Treves came up with a new idea for the study of the analytic hypoellipticity of sums of squares. In this section we are going to give a fairly precise description of his idea, because it is important for what follows.

Stimulated by the papers [30], [31] by N. Hanges and A. A. Himonas, who proved that the Oleĭnik and Radkevič operator for special values of p and q, is not analytic hypoelliptic, even though its characteristic manifold is a real analytic symplectic submanifold, F. Treves introduced the idea that in order to establish if there is analytic hypoellipticity or not one has to look at the strata of a stratification of the characteristic variety.

Hence he proposed a certain stratification that will be henceforth called the Poisson stratification and formulated the conjecture that an operator is analytic hypoelliptic if and only if all the strata in the stratification of its characteristic variety are symplectic real analytic submanifolds.

We now give a detailed description of the Poisson stratification as well as some examples. We shall follow the presentation in the paper [15].

Denote Σ the variety $\mathrm{Char}(P)$, where the symbols of all the vector fields are zero.

First of all let us define what we mean by the term stratification.

Definition 3.1 (see e.g. [60])**.** *By an analytic stratification of Σ in $T^*\mathbb{R}^n \setminus \{0\}$ we mean a partition of Σ*

$$\Sigma = \bigcup_{i\in I} S_i,$$

where the S_i are connected analytic submanifolds of $T^*\mathbb{R}^n \setminus \{0\}$ satisfying the conditions

(i) *Every compact subset of $T^*\mathbb{R}^n \setminus \{0\}$ intersects at most finitely many submanifolds S_i.*
(ii) *For any i, i' belonging to the index family I, $S_{i'} \cap \overline{S_i} \neq \varnothing$ implies $S_{i'} \subset \partial S_i$ and $\dim S_{i'} < \dim S_i$.*

The next is the definition of a (micro)local stratification. The definition is given in general terms, the adaptation to the homogeneous-on-the-fibers situation is straghtforward.

Definition 3.2 ([60])**.** *By a local analytic stratification of Σ we mean a system $(U, \{S_i\}_{i\in I})$, where U is an open set in $T^*\mathbb{R}^n \setminus \{0\}$, I is a finite index family, S_i is a connected analytic submanifold of U satisfying condition (ii) above and such that*

$$\Sigma \cap U = \bigcup_{i\in I} S_i.$$

Next we are going to describe how to construct a local analytic stratification. This can be accomplished in several ways, however we stick to the description of [15] to keep the content the least abstract and the most readable.

3.1. **The analytic stratification.** Let us denote by

$$X(x,\xi) = (X_1(x,\xi), \dots, X_r(x,\xi))$$

the map whose components are the symbols of the vector fields. Moreover let $\Sigma = X^{-1}(0) \cap T^*\Omega \setminus \{0\}$ the characteristic variety. Note that, since our maps are real valued, we might have used the function $p(x,\xi) = \sum_{j=1}^{r} X_j(x,\xi)^2$ to define Σ, but since in the following steps the minors of the Jacobian matrix of X are going to play a role, keeping the consistency of the notation would have been much more complicated. Thus we stick to the vector notation.

Define $\mathfrak{R}_0(\Sigma)$ as the subset of Σ whose points $z_0 = (x_0, \xi_0)$ have a neighborhood $U_{z_0} \subset V$, V open subset of $T^*\Omega \setminus \{0\}$, such that there are indices j_α, $\alpha = 1, \dots, m$, $1 \leq j_1 < \cdots < j_m \leq r$, for which

$$U_{z_0} \cap \Sigma = \{z \in U_{z_0} \mid X_{j_\alpha}(x,\xi) = 0, \alpha = 1, \dots, m\},$$

and the differentials $dX_{j_\alpha}(z_0)$ are all linearly independent. The latter is equivalent to saying that the minor

$$\frac{\partial(X_{j_1}, \dots, X_{j_m})}{\partial(z_{i_1}, \dots, z_{i_m})}(z_0),$$

where $1 \leq i_1 < \cdots < i_m \leq 2n$, is non zero. It is evident that $\mathfrak{R}_0(\Sigma)$ is a C^ω manifold of codimension m.

Next we define two subsets of Σ, Σ_1 and Σ_2. Let Σ_1 denote the subset of Σ in which all the $m \times m$ minors of the matrix $\frac{\partial X}{\partial z}$ vanish identically.

Define Σ_2 as the zero set in $V \setminus (\Sigma_1 \cup \mathfrak{R}_0(\Sigma))$ of all the $(m+1)\times(m+1)$ minors

$$\frac{\partial(X_{j_1}, \dots, X_{j_{m+1}})}{\partial(z_{i_1}, \dots, z_{i_{m+1}})},$$

$1 \leq i_1 < \cdots < i_{m+1} \leq 2n$.

We may now iterate for Σ_1, Σ_2 what has been done for Σ. For Σ_1 define the map

$$X^{(1)}(x,\xi) = \big(X(x,\xi), X^{j_1,\dots,j_m}_{i_1,\dots,i_m}\big) : V \to \mathbb{R}^{r_{1,1}}$$

with $X^{j_1,\dots,j_m}_{i_1,\dots,i_m}$ denoting the $m \times m$ minors and $r_{1,1} = r + r_1$, r_1 being the number of the $m \times m$ minors.

Analogously define

$$X^{(2)}(x,\xi) = \Big(X(x,\xi), X^{j_1,\dots,j_{m+1}}_{i_1,\dots,i_{m+1}}\Big) : V \to \mathbb{R}^{r_{1,2}}$$

with $X^{j_1,\dots,j_{m+1}}_{i_1,\dots,i_{m+1}}$ denoting the $(m+1)\times(m+1)$ minors and $r_{1,2} = r + r_2$, r_2 being the number of the $(m+1)\times(m+1)$ minors.

This leads to a local stratification of Σ: if V is a neighborhood of z_0 with a compact closure then

$$V \cap \Sigma = \bigcap_{\alpha=0}^{N_\Omega} \Lambda_\alpha, \tag{3.1}$$

where the Λ_α are C^ω manifolds. The Λ_α shall be called the analytic strata of Σ.

Example 1 (The Whitney umbrella)**.** *This example is not on a cotangent bundle. Let* $\Sigma = \{x \in \mathbb{R}^3 \mid x_1^2 - x_3x_2^2 = 0\}$. $\mathfrak{R}_0(\Sigma) = \{x \in \mathbb{R}^3 \mid x_1^2 - x_3x_2^2 = 0, x_1^2 + x_2^2 > 0\}$.

Then $X^{(1)}(x) = (x_1^2 - x_3x_2^2, x_1, x_2x_3, x_2^2)$. *Its differential is*

$$\begin{bmatrix} 2x_1 & -2x_2x_3 & -x_2^2 \\ 1 & 0 & 0 \\ 0 & x_3 & x_2 \\ 0 & 2x_2 & 0 \end{bmatrix}.$$

Its restriction to Σ_1 *has rank 2 if* $x_3 \neq 0$ *and rank 1 at the origin. The analytic stratification of* Σ *is composed of 5 strata.*

3.2. **The symplectic stratification.** Assuming we already have a stratified variety of the form (3.1), we denote by Σ one of the strata Λ_α in (3.1), i.e. a connected C^ω submanifold defined near a point $z_0 \in \mathrm{Char}(P)$, and let σ be the symplectic form in $\mathbb{R}^{2n}$.

Then there are functions $G_j(x,\xi)$, $j = 1,\dots,s$, and an open set $\Omega' \subset \Omega$ such that $\Sigma \cap \Omega' = \{z \in \Omega' \mid G_j(z) = 0, j = 1,\dots,s\}$. Moreover we may assume that the rank of the map $G = (G_1,\dots,G_s)$ is equal to $\operatorname{codim}\Sigma$ at each point of $\Sigma\cap\Omega'$. Thus if $d = \operatorname{codim}\Sigma$, each $z_0 \in \Sigma$ has a neighborhood $U_{z_0} \subset \Omega'$ in which there are indices $1 \le i_1 < \cdots < i_d \le s$ such that

(i) The differentials $dG_{i_k}(z_0)$ are linearly independent.
(ii) $\Sigma \cap U_{z_0} = \{z \in U_{z_0} \mid G_{i_1}(z) = \cdots = G_{i_d}(z) = 0\}$.

Consider the pull back of σ to Σ and denote it by $\sigma_{|\Sigma}$. Let $\sigma_{z|\Sigma}$, $z \in \Sigma$, denote the restriction of the symplectic form to $T_z\Sigma$. The rank of the linear map corresponding to the skew symmetric bilinear form $\sigma_{z|\Sigma}$ is called the rank of the symplectic form on Σ at the point z or the symplectic rank of Σ at the point z.

Denote by μ the maximum rank of Σ. Then the set Σ_0 of all the points z where the symplectic rank is equal to μ is a dense subset of Σ. Each connected component of Σ_0 is a C^ω submanifold of U_{z_0} whose symplectic rank at every point is equal to μ.

The subset $\Sigma \setminus \Sigma_0$ is an analytic variety that can be defined by the vanishing of the functions $G_1,\dots,G_s$, as well as of all the $\nu\times\nu$ minors of the matrix $[\{G_i,G_j\}]_{1\le i,j\le s}$, where $\nu = \mu + \operatorname{codim}\Sigma - \dim\Sigma$. Hence we can find an analytic stratification of this subset and the dimension of each analytic stratum of $\Sigma\setminus\Sigma_0$ is strictly less than the dimension of $\Sigma_0 = \dim\Sigma$.

This implies that we can decompose Σ so that

$$\Sigma \cap U = \bigcup_{\alpha=1}^{N_U} \Sigma_\alpha,$$

where each Σ_α is a connected C^ω submanifold with a constant symplectic rank.

3.3. **The Poisson stratification.** Again we start with the analytic set $\Sigma = \operatorname{Char}(P)$. For each multiindex $I = (i_1, \dots, i_\nu)$, $\nu \in \mathbb{N}$, we define

$$X_I(x,\xi) = \{X_{i_1}, \{X_{i_2}, \{\cdots \{X_{i_{\nu-1}}, X_{i_\nu}\} \cdots\}\}\}(x,\xi),$$

if $\nu \geq 2$ and $X_I = X_{i_1}$, if $I = (i_1)$. We also set $|I| = \nu$. Here $\{f, g\}$ denotes the Poisson bracket of the functions f and g:

$$\{f,g\}(x,\xi) = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial \xi_j} \right)(x,\xi).$$

Of course we *are assuming* that the vector fields X_i satisfy the microlocal Hörmander condition, i.e. that for every $(x,\xi) \in \operatorname{Char}(P)$ there exists a multiindex I such that $X_I(x,\xi) \neq 0$.

Let now U be a neighborhood of a point $z_0 = (x_0, \xi_0)$ and write as before $\Sigma = \operatorname{Char}(P)$. Then we may define a sequence of analytic subsets of U as

$$\Sigma^{(\nu)} = \{z \in U \mid \text{ for every multiindex } I, |I| \leq \nu, X_I(z) = 0\}.$$

We point out that the sequence $\Sigma^{(\nu)}$ is non increasing in ν and that in particular $\Sigma^{(1)} = \Sigma$. Furthermore, by the Hörmander condition, we have that

$$\bigcap_{\nu=1}^{\infty} \Sigma^{(\nu)} = \varnothing.$$

Now there is an increasing sequence of integers $1 = \nu_1 < \nu_2 < \cdots$ such that

(i) $\Sigma^{(\nu_{p+1})} \subsetneqq \Sigma^{(\nu_p)}$.
(ii) If $\nu_p < \nu_{p+1}$, then $\Sigma^{(\nu')} = \Sigma^{(\nu_p)}$, for every ν', $\nu_p \leq \nu' < \nu_{p+1}$.

Consider now for any integer p the symplectic stratification (in the open set U) of the analytic set $\Sigma^{(\nu_p)}$:

$$\Sigma^{(\nu_p)} = \bigcup_{\alpha=1}^{N_U} \Sigma_\alpha^{(\nu_p)}.$$

In each stratum $\Sigma_\alpha^{(\nu_p)}$ the set of points $z \in \Sigma^{(\nu_p)} \setminus \Sigma^{(\nu_{p+1})}$ is either empty or else an open and dense subset of $\Sigma_\alpha^{(\nu_p)}$. If it is not empty, denote by $\Sigma_{\alpha,\beta}^{(\nu_p)}$ its connected components. Thus we get the decomposition

$$\Sigma^{(\nu_p)} = \Sigma^{(\nu_{p+1})} \cup \bigcup_{\alpha=1}^{N_U} \bigcup_{\beta=1}^{M_U} \Sigma_{\alpha,\beta}^{(\nu_p)}.$$

Finally, letting p run over the integers we obtain a decomposition of the form

$$\Sigma = \bigcup_p \bigcup_j \Sigma_j^{(\nu_p)}, \tag{3.2}$$

where p, j have a finite range (in the open set U) and

(i) The C^ω manifolds $\Sigma_j^{(\nu_p)}$ are connected and pairwise disjoint.
(ii) The symplectic rank of $\Sigma_j^{(\nu_p)}$ is constant.
(iii) At every point of $\Sigma_j^{(\nu_p)}$ the Poisson brackets X_I, with $|I| < \nu_{p+1}$ vanish, but there is at least one bracket X_I with $|I| = \nu_{p+1}$ which does not vanish.

We may then give the following

Definition 3.3. *The partition* (3.2) *of* $\mathrm{Char}(P) = \Sigma$ *is called the (local) Poisson stratification corresponding to the vector fields* $X_1, \dots, X_r$*. Each submanifold* $\Sigma_j^{(\nu_p)}$ *is a Poisson stratum, or simply just a stratum, for* Σ*. We refer to the integer* ν_p *as the depth of the stratum* $\Sigma_j^{(\nu_p)}$*.*

Remark 3.1. *It follows immediately from the definition above that the stratification of* Σ *defined by the vector fields* X_j*,* $j = 1, \dots, r$*, is invariant under nonsingular* C^ω *linear substitutions, that means if we define*

$$\tilde{X}_j(x,\xi) = \sum_{k=1}^{r} a_{jk}(x,\xi) X_k(x,\xi),$$

for $j = 1, \dots, r$*, we obtain the same stratification.*

Assume that a stratum, say Σ', of the stratification (3.2) is not symplectic. Since the symplectic rank is constant we have that Σ' is foliated by C^ω submanifolds whose tangent space is isomorphic to $T_z\Sigma' \cap (T_z\Sigma')^\sigma$. We call these submanifolds the Hamilton leaves of the stratification. If $\mathrm{Char}(P)$ is a real analytic manifold and the symplectic form has constant rank on each connected component of $\mathrm{Char}(P)$, then there are curves (contained in the Hamilton leaves) satisfying the assumptions of the Treves curve conjecture.

It is also clear that the latter situation may occur at a deeper stratum.

We may then state the

Conjecture 3.1 (Treves conjecture, [58], [59], [15])**.** *The operator P is analytic hypoelliptic if and only if each stratum in its Poisson stratification is (microlocally) a symplectic C^ω submanifold.*

There are also other notions of analytic hypoellipticity, like global analytic hypoellipticity and germ analytic hypoellipticity. Moreover one might be interested in the analytic singular support of the solution, i.e. just the local theory. In this paper we stick to the microlocal point of view, since we think that it is the most basic and refer to the paper [59] for further details about the formulation of the conjecture in different, albeit related, situations.

4. Examples and counterexamples

In this section we discuss some model operators and examine their Poisson stratification as well as—when known—their hypoellipticity properties.

4.1. **Examples.** Consider the operator in (2.2), with $1 < p < q$. Then

$$\mathrm{Char}(P_{OR}) = \{(0, x_2, x_3; 0, \xi_2, \xi_3) \mid \xi_2^2 + \xi_3^2 > 0\}.$$

This is obviously a symplectic submanifold, so that the rank of the symplectic form restricted to $\mathrm{Char}(P_{OR})$ is constant and equal to 4.

As we said in Section 2, Theorem 2.2 holds, showing that it is not analytic hypoelliptic.

First of all this shows that the mere analytic and symplectic stratifications are not enough to imply analytic hypoellipticity.

The first Poisson strata are then

$$\Sigma_{1,\pm} = \{(x, \xi) \mid \xi_1 = x_1 = 0, \xi_2 \gtrless 0\}.$$

Points in $\Sigma_{1,\pm}$ are characteristic points and all Poisson brackets of length $k+1$ of the form $\mathrm{ad}(X_1)^k X_j$ are zero for $k < p-1$. It is evident that X_1 is the only field contributing to this computation since both X_2 and X_3 carry vanishing coefficients.

When we take brackets of length p we have that

$$\mathrm{ad}(X_1)^{p-1} X_2 = (p-1)!\xi_2.$$

This is zero if $\xi_2 = 0$, which is possible, provided $\xi_3 \neq 0$. Hence the strata of depth p are

$$\Sigma_{p,\pm} = \{(x, \xi) \mid \xi_1 = x_1 = 0 = \xi_2, \xi_3 \gtrless 0\}.$$

The latter is not symplectic since it has codimension 3. Note that the Baouendi–Goulaouic model is obtained for $p = 1$.

As a second example let us consider the operator

$$D_1^2 + \sum_{j=1}^{N} (p_j(x) D_2)^2, \qquad x \in \mathbb{R}^2, \tag{4.1}$$

where the polynomials p_j satisfy

$$p_j(\lambda x_1, \lambda^\theta x_2) = \lambda^{m_j} p_j(x_1, x_2), \qquad \lambda > 0, \tag{4.2}$$

θ, m_j being positive rational numbers. We may always assume that the labeling of the polynomials is such that

$$m_1 \leq m_2 \leq \cdots \leq m_N.$$

Then

Theorem 4.1 ([14])**.** *Consider the operator in* (4.1)*. Suppose that for a number r, $1 \leq r \leq N$, we have*

$$p_r(1,0) \neq 0, \; p_j(1,0) = 0, \qquad \textit{for } j < r.$$

Write

$$p_j(x) = \sum_{k=0}^{m_j} \alpha_{jk} x_1^k x_2^{q_{jk}},$$

where the q_{jk} are non–negative integers, $q_{jm_j} = 0$, and otherwise $q_{jk} \geq$ 1.

Then the operator in (4.1) *is G^s hypoelliptic for*

$$s \geq \frac{1}{1-\lambda},$$

where

$$\lambda = \frac{\theta}{m_r + 1} \max_{1 \leq j \leq r} \max_{\substack{0 \leq k < m_j \\ \alpha_{jk} \neq 0}} \frac{m_r - k}{m_j - k}.$$

Let us examine the stratification of (4.1). We consider only the case when $N = 1$; the more general case is quite similar. Thus let

$$P(x, D) = D_1^2 + (p(x) D_2)^2.$$

Since we are assuming that P satisfies Hörmander condition, we may assume, after application of Weierstraß preparation theorem, dropping for simplicity the non zero factor, that p has the form

$$p(x) = x_1^m + \sum_{k=0}^{m-1} \alpha_k x_1^k x_2^{q_k},$$

where the q_k are non negative integers such that the homogeneity hypothesis is satisfied.

The characteristic variety is then given by

$$\xi_1 = 0, p(x) = 0.$$

The zero set of p will be considered in more detail in Section 5. We mention here only the basic things necessary to understand the problem. One can show that $p^{-1}(0)$ has at most a finite number of branch points and in the complement of those points it is a C^ω submanifold of $\mathbb{R}^2$. Because of Hörmander condition we obtain that the characteristic variety is a symplectic submanifold of $T^*\mathbb{R}^2 \setminus \{0\}$ in the complement of the branch points.

Hence the stratification is essentially a stratification in the x-space of the form

$$\mathrm{Char}(P) = \bigcup_{i=1}^{L} M_i \cup \bigcup_{j=1}^{L_1} \{\rho_j\},$$

where

$$M_i = \{(x, \xi) \mid \xi_1 = 0, \xi_2 \neq 0, x \in \tilde{M}_i\},$$

$\tilde{M}_i$ denoting the C^ω connected components of $p^{-1}(0)$, while

$$\rho_j = (\tilde{\rho}_j; 0, \xi_2 \neq 0),$$

where the $\tilde{\rho}_j$ are the branch points in $p^{-1}(0)$.

In this case the only non symplectic strata are lines parallel to the fibers of the cotangent bundle and projecting onto a single point on the base space.

We point out that Theorem 4.1 gives Gevrey regularity that are known to be optimal only in particular cases, e.g. the Métivier operator, see (2.5). The optimality for a generic operator of that form is not proved.

Likewise the analog of Theorem 4.1 in a non homogenous case is not known. Proving optimality in a non homogenous case would amount to prove that Conjecture 3.1 holds true in two variables.

4.2. **Counterexamples.** Let r, p, $q \in \mathbb{N}$, $1 < r < p < q$, and $x \in \mathbb{R}^4$. Consider the operator

$$P_1(x, D) = D_1^2 + D_2^2 + x_1^{2(r-1)} \left(D_3^2 + D_4^2\right) + x_2^{2(p-1)} D_3^2 + x_2^{2(q-1)} D_4^2. \tag{4.3}$$

Evidently P_1 is a sum of squares operator verifying Hörmander condition, since $\mathrm{ad}(D_1)^{r-1} x_1^{r-1} D_i$ yields D_i, $i = 3, 4$.

The characteristic variety of P_1 is

$$\mathrm{Char}(P_1) = \left\{(x, \xi) \mid \xi_1 = \xi_2 = 0, x_1 = x_2 = 0, \xi_3^2 + \xi_4^2 > 0\right\}.$$

The stratification associated with P_1 is made up of a symplectic single stratum

$$\Sigma_1 = \left\{(0,0,x_3,x_4;0,0,\xi_3,\xi_4) \mid \xi_3^2+\xi_4^2>0\right\} = \mathrm{Char}(P_1).$$

Then we have

Theorem 4.2 ([3])**.** *Let*

$$\frac{1}{s_0} = \frac{1}{r} + \frac{r-1}{r}\frac{p-1}{q-1}.$$

Then P_1 in a neighborhood of the origin is locally Gevrey s_0 hypoelliptic and not better.

It is not difficult to show that Theorem 4.2 implies the following

Corollary 4.1. *The sufficient part of the Conjecture 3.1 does not hold in dimension n for $n \geq 4$.*

Next we give a sketchy idea of the proof of Theorem 4.2, since, in our opinion, it may help getting an idea about where and why analytic regularity fails in this model.

Idea of the proof of Theorem 4.2. First of all we note that the Hörmander hypothesis is satisfied at order r, meaning that the whole 4-dimensional Lie algebra is generated by taking iterated commutators of length at most r.

Using the subelliptic inequality it is not difficult to show that a distribution solution of $P_1u = f$, with f real analytic is in G^{s_0} near a characteristic point.

Hence we focus on the converse statement: There is a real analytic function f and a G^{s_0} function, u, such that $P_1u = f$ and moreover u is not better than G^{s_0}. To this end we must construct such a function u, basically doing the same as in Theorem 2.1, i.e. constructing some sort of inverse Fourier transform whose exponential decay at infinity prevents analyticity. Of course both the (complex) phase and the amplitude are more involved in this case. In particular the amplitude is obtained by studying the semiclassical eigenfunctions and eigenvalues of a certain Schrödinger operator with a double well potential with non degenerate minima blowing up at infinity.

We follow the proof in [3]. We look for a function u such that

$$P_1(x,D)A(u) = 0,$$

where

$$A(u)(x) = \int_{M_u}^{+\infty} e^{-i\rho x_4 + x_3 z(\rho)\rho^\theta - \rho^\theta} u(\rho^{\frac{1}{r}}x_1, \rho^\mu x_2, \rho)\,d\rho. \tag{4.4}$$

Here $\theta = s_0^{-1}$, $\mu > 0$, $z(\rho)$ and $M_u > 0$ are to be determined. We assume that x is in a suitable neighborhood of the origin whose size will ultimately depend on the upper estimate for $z(\rho)$.

Applying P_1 to $A(u)$ we obtain

$$
\begin{aligned}
&P_1(x,D)A(u)(x)\\
&\quad = \int_{M_u}^{+\infty} e^{-i\rho x_4 + x_3 z(\rho)\rho^\theta - \rho^\theta} \Big[-\rho^{\frac{2}{r}} \partial_{x_1}^2 u - x_1^{2(r-1)} (z(\rho))^2 \rho^{2\theta} u \\
&\qquad + x_1^{2(r-1)} \rho^2 u - \rho^{2\mu} \partial_{x_2}^2 u - x_2^{2(p-1)} (z(\rho))^2 \rho^{2\theta} u + x_2^{2(q-1)} \rho^2 u \Big] d\rho,
\end{aligned}
$$

which, in terms of the variables $y_1 = \rho^{\frac{1}{r}} x_1$, $y_2 = \rho^\mu x_2$, becomes

$$
\begin{aligned}
&P_1(x,D)A(u)(x)\\
&\quad = \int_{M_u}^{+\infty} e^{-i\rho x_4 + x_3 z(\rho)\rho^\theta - \rho^\theta} \Big[-\rho^{\frac{2}{r}} \partial_1^2 u - y_1^{2(r-1)} (z(\rho))^2 \rho^{2\theta - 2\frac{r-1}{r}} u \\
&\qquad + y_1^{2(r-1)} \rho^{2-2\frac{r-1}{r}} u - \rho^{2\mu} \partial_2^2 u - y_2^{2(p-1)} (z(\rho))^2 \rho^{2\theta - 2(p-1)\mu} u \\
&\qquad\qquad + y_2^{2(q-1)} \rho^{2-2(q-1)\mu} u \Big]_{\substack{y_1=\rho^{1/r}x_1 \\ y_2=\rho^\mu x_2}} d\rho.
\end{aligned}
$$

Choose $\mu = \frac{1}{q}$. Then

$$
\begin{aligned}
&P_1(x,D)A(u)(x)\\
&= \int_{M_u}^{+\infty} e^{-i\rho x_4 + x_3 z(\rho)\rho^\theta - \rho^\theta} \Big[-\rho^{\frac{2}{r}} \left(\partial_1^2 - y_1^{2(r-1)} \left(1 - (z(\rho))^2 \rho^{2(\theta-1)} \right) \right) u \\
&\qquad + \rho^{\frac{2}{q}} \left(-\partial_2^2 - y_2^{2(p-1)} (z(\rho))^2 \rho^{2\theta - 2\frac{p}{q}} + y_2^{2(q-1)} \right) u \Big]_{\substack{y_1=\rho^{1/r}x_1 \\ y_2=\rho^{\frac{1}{q}} x_2}} d\rho.
\end{aligned}
$$

We point out that $\theta - 1 < 0$. Make the Ansatz $|z(\rho)| < M_u^{1-\theta}$ and set $\tau(\rho) = \left(1 - (z(\rho))^2 \rho^{2(\theta-1)}\right)^{\frac{1}{2r}}$.

Choosing $u(y_1, y_2, \rho) = u_1(\tau(\rho) y_1) u_2(y_2, \rho)$, where

$$
\left(-\partial_1^2 + y_1^{2(r-1)} \tau(\rho)^{2r} \right) u_1(\tau(\rho) y_1) = \tau(\rho)^2 \lambda u_1(\tau(\rho) y_1), \tag{4.5}
$$

and $\lambda > 0$ is such that, for fixed $\rho > 0$, the factor in front of u_1 in the r.h.s. of the above equation is in the spectrum of the quantum anharmonic oscillator $\left(-\partial_1^2 + y_1^{2(r-1)} \left(1 - (z(\rho))^2 \rho^{2(\theta-1)} \right) \right)$, whose frequency depends on both ρ and $z(\rho)$. Then

$$
\begin{aligned}
&P_1(x, D)A(u)(x) \\
&= \int_{M_u}^{+\infty} e^{-i\rho x_4 + x_3 z(\rho)\rho^\theta - \rho^\theta} u_1(\tau(\rho)\rho^{\frac{1}{r}} x_1) \Big[\Big\{ \rho^{\frac{2}{r}} \left(1 - (z(\rho))^2 \rho^{2(\theta-1)}\right)^{\frac{1}{r}} \lambda \\
&+ \rho^{\frac{2}{q}} \left(-\partial_2^2 - y_2^{2(p-1)} (z(\rho))^2 \rho^{2\theta - 2\frac{p}{q}} + y_2^{2(q-1)} \right) \Big\} u_2(y_2, \rho) \Big]_{y_2 = \rho^{\frac{1}{q}} x_2} d\rho.
\end{aligned}
$$

Next we want to find u_2 as a solution to the differential equation

$$
\begin{aligned}
(4.6)\quad & \left(1 - (z(\rho))^2 \rho^{2(\theta-1)}\right)^{\frac{1}{r}} \lambda u_2 \\
&+ \rho^{\frac{2}{q} - \frac{2}{r}} \left(-\partial_2^2 - y_2^{2(p-1)} (z(\rho))^2 \rho^{2\theta - 2\frac{p}{q}} + y_2^{2(q-1)} \right) \Big) u_2 = 0.
\end{aligned}
$$

(4.6) above then may be written

$$
\begin{aligned}
&\left(1 - (z(\rho))^2 \rho^{2(\theta-1)}\right)^{\frac{1}{r}} \lambda u_2 \\
&+ \rho^{\frac{2}{q} - \frac{2}{r}} \left(-\partial_2^2 + y_2^{2(q-1)} \right) u_2 - (z(\rho))^2 \rho^{2\left(\theta - \frac{p-1}{q} - \frac{1}{r}\right)} y_2^{2(p-1)} u_2 = 0.
\end{aligned}
$$

Since

$$
\theta - \frac{p-1}{q} - \frac{1}{r} = \left(\frac{1}{q} - \frac{1}{r} \right) \frac{p-1}{q-1},
$$

we set

$$
(4.7) \qquad t = \rho^{\frac{1}{q} - \frac{1}{r}},
$$

so that the above equation becomes

$$
\begin{aligned}
&\left(1 - (z_1(t))^2 t^{2(r-1)\frac{q}{q-1}\,\frac{q-p}{q-r}}\right)^{\frac{1}{r}} \lambda u_2 \\
&+ t^2 \left(-\partial_2^2 + y_2^{2(q-1)} \right) u_2 - (z_1(t))^2 t^{2\frac{p-1}{q-1}} y_2^{2(p-1)} u_2 = 0,
\end{aligned}
$$

where $z_1(t) = z(\rho)$. The latter equation can be turned into a stationary semiclassical Schrödinger equation if we perform the canonical dilation

$$
y_2 = y t^{-\frac{1}{q-1}} :
$$

$$
\begin{aligned}
&\left(1 - (z_1(t))^2 t^{2(r-1)\frac{q}{q-1}\,\frac{q-p}{q-r}}\right)^{\frac{1}{r}} \lambda u_2 \\
&- t^{2\frac{q}{q-1}} \partial_y^2 u_2 + y^{2(q-1)} u_2 - (z_1(t))^2 y^{2(p-1)} u_2 = 0.
\end{aligned}
$$

Set

$$
(4.8) \qquad h = t^{\frac{q}{q-1}}.
$$

Note that t, h are small and positive for large ρ. Thus we may rewrite the above equation as
(4.9)
$$\Big[\Big(1-(z_2(h))^2h^{2(r-1)\frac{q-p}{q-r}}\Big)^{\frac{1}{r}}\lambda-h^2\partial_y^2+y^{2(q-1)}-(z_2(h))^2y^{2(p-1)}\Big]u_2=0,$$
where $z_2(h)=z_1(t)$. One can show that there are countably many choices for the function $z_2(h)$ in such a way that equation (4.9) has a non zero solution in $L^2(\mathbb{R})$, which is a smooth rapidly decreasing function.

First observe that the operator
$$-h^2\partial_y^2+y^{2(q-1)}-(z_2(h))^2y^{2(p-1)},$$
is a Schrödinger operator with a symmetric double well potential which is not positive. We obtain a positive potential just adding its minimum
$$\hat\gamma z_2^{2\frac{q-1}{q-p}},$$
where
$$\hat\gamma=-\frac{q-p}{q-1}\left(\frac{p-1}{q-1}\right)^{\frac{p-1}{q-p}}<0.$$
Equation (4.9) becomes
$$\begin{aligned}(4.10)\quad \Big[\Big(1-(z_2(h))^2h^{2(r-1)\frac{q-p}{q-r}}\Big)^{\frac{1}{r}}\lambda+\hat\gamma z_2(h)^{2\frac{q-1}{q-p}}\\ -h^2\partial_y^2+y^{2(q-1)}-(z_2(h))^2y^{2(p-1)}-\hat\gamma z_2(h)^{2\frac{q-1}{q-p}}\Big]u=0.\end{aligned}$$
Performing the canonical dilation $x\to xz_2^{\frac{1}{q-p}}$—we make here the Ansatz that z_2 is positive—(4.9) becomes
$$\begin{aligned}(4.11)\quad \Big[\Big(1-(z_2(h))^2h^{2(r-1)\frac{q-p}{q-r}}\Big)^{\frac{1}{r}}z_2(h)^{-2\frac{q-1}{q-p}}\lambda+\hat\gamma\\ -h^2z_2(h)^{-\frac{2q}{q-p}}\partial_x^2+x^{2(q-1)}-x^{2(p-1)}-\hat\gamma\Big]u=0.\end{aligned}$$
By [6] the operator in the second line above has a discrete simple spectrum depending in a real analytic way on the parameter $hz_2(h)^{-\frac{q}{q-p}}$, for $h>0$. Let
$$E\left(\frac{h}{z_2(h)^{\frac{q}{q-p}}}\right)$$

be one of its eigenvalues and $u = u(x, h)$ the corresponding eigenfunction. Equation (4.11) then becomes a scalar equation

$$(4.12)\quad \left(1 - (z_2(h))^2 h^{2(r-1)\frac{q-p}{q-r}}\right)^{\frac{1}{r}} z_2(h)^{-2\frac{q-1}{q-p}}\lambda + \hat{\gamma} + E\left(\frac{h}{z_2(h)^{\frac{q}{q-p}}}\right) = 0.$$

To solve the above equation, one proves that the function z_2 exists and is defined on a certain domain near the origin:

Proposition 4.1 ([3])**.** *There is $h_0 > 0$ such that equation* (4.12) *implicitly defines a function $z_2 \in C([0, h_0[) \cap C^\omega(]0, h_0[)$. In particular*

$$z_2(h) \to \tilde{z} = \left(-\frac{\lambda}{\hat{\gamma}}\right)^{\frac{q-p}{2(q-1)}} > 0$$

when $h \to 0+$. Therefore we may always assume that

$$(4.13)\quad z_2(h) \in \left[\frac{1}{2}\tilde{z}, \frac{3}{2}\tilde{z}\right],$$

for $h \in [0, h_0[$.

Let h_0 be the quantity define in Proposition 4.1. Set $h_0 = \rho_0^{\left(\frac{1}{q}-\frac{1}{r}\right)\frac{q}{q-1}}$. Choosing $M_u \geq \max\{\rho_0, (\frac{3}{2}\tilde{z})^{\frac{1}{1-\theta}}\}$ we have that the function z_2 is defined for $\rho \geq M_u$ and that $|z(\rho)| < M_u^{1-\theta}$ is satisfied, so that $1 - z(\rho)^2\rho^{2(\theta-1)} > 0$.

Using some *a priori* estimate for Schrödinger operators with a positive double well potential as well as an upper bound for the derivatives of the eigenfunctions (see [3]) one shows that the integral $A(u)$ is convergent and moreover

$$P_1(x, D)A(u) = 0.$$

Before concluding the proof of the sharpness of the Gevrey s_0 regularity for $A(u)$, we need to make sure that the function $u = u_1u_2$ does not have any effect on the convergence of the integral at infinity as well as on the Gevrey behavior of $A(u)$.

As far as u_1 is concerned, this is fairly obvious, since u_1 is a rapidly decreasing function of $\tau(\rho)\rho^{\frac{1}{r}}x_1$, where $\tau(\rho)$ is defined before equation (4.5), and, computing this function at the origin—as we need to do—will not affect the exponential in $A(u)$. We are thus left with $u_2 = u_2(\rho^{\frac{1}{q}}x_2, \rho)$. Even though u_2 is rapidly decreasing w.r.t. $\rho^{\frac{1}{q}}x_2$, we still need some estimate on u_2 allowing us to conclude that u_2 can be polynomially bounded in ρ, uniformly for x_2 in a neighborhood of the origin and moreover that $u_2(0, \rho)$ does not vanish for large ρ with so high a speed to compromise the Gevrey s_0 regularity.

To this end we need an estimate of u in a classically forbidden region, i.e. when $\hbar = \frac{h}{z_2(h)^{\frac{q}{q-p}}}$ is small (ρ is large) and x is in a neighborhood of the origin. This can be done by resorting to the following theorem providing a lower bound for the tunneling of the solution:

Theorem 4.3 (See [61], Theorem 7.7)**.** *Let U be a neighborhood of the origin in $\mathbb{R}$. There exist positive constants C, $\hbar_0$ such that*

$$\|u\|_{L^2(U)} \geq e^{-\frac{C}{\hbar}} \|u\|_{L^2(\mathbb{R})}, \tag{4.14}$$

for $0 < \hbar \leq \hbar_0$.

To finish the proof we argue for an even eigenfunction. A similar argument can be done for the odd eigenfunctions.

We may assume that

$$\|u\|_{L^2(\mathbb{R})} = 1, \quad u(0, \hbar) > 0,$$

since $u'(0, \hbar) = 0$ because of its parity and if $u(0, \hbar) = 0$ would imply that u, being a solution of a homogeneous differential equation, is identically zero.

Moreover, since u solves

$$Q_\hbar(x, \partial_x)u = E(\hbar)u, \tag{4.15}$$

we have $\partial_x^2 u(0, \hbar) > 0$.

Denote by $x_0 = x_0(\hbar)$ the first positive zero of $V(x) - E(\hbar) = x^{2(q-1)} - x^{2(p-1)} - \hat{\gamma} - E(\hbar)$. Note that u is strictly positive in the interval $0 \leq x \leq x_0$. In fact, by contradiction, denoting by $\bar{x}$ the first zero of u in $[0, x_0]$, by (4.15), we may conclude that $u'' > 0$ in $[0, \bar{x}[$ so that the same is true for u'. Hence $u(\bar{x}, \hbar) > u(0, \hbar) > 0$, which is absurd.

By (4.15), u is strictly convex for $0 \leq x \leq x_0$ and has its minimum at the origin and its maximum at x_0.

Define $y = \frac{\partial_x u}{u}$. We have $y > 0$ if $0 < x \leq x_0$. Then, writing y' for $\partial_x y$,

$$y' = \frac{V - E}{\hbar^2} - y^2.$$

The function y has a maximum in the interval $]0, x_0[$. In fact $y'(0) > 0$ and $y'(x_0) = -y^2(x_0) < 0$. Denote by $\bar{x}$ the point where where the maximum is attained: it lies in the interior of the interval $[0, x_0]$. Moreover we get

$$y(\bar{x}) = \frac{(V(\bar{x}) - E(\hbar))^{1/2}}{\hbar}.$$

From the definition of y we obtain

$$u(0) = e^{-\int_0^{x_0} y(s)ds} u(x_0) \geq e^{-x_0 y(\bar{x})} \frac{1}{\sqrt{2x_0}} \|u\|_{L^2([-x_0,x_0])}$$
$$\geq \frac{1}{\sqrt{2x_0}} e^{-\frac{(-\hat{\gamma})^{1/2}}{\hbar}} e^{-\frac{C}{\hbar}}.$$

Here we used Theorem 4.3, $x_0 < 1$, $E(\hbar) > 0$ and u is normalized. We remark that $x_0(\hbar) \to \hat{x}_0 > 0$ when $\hbar \to 0+$.

We are now in a position to conclude the proof of Theorem 4.2 for an even function u_2. We recall that

$$\hbar = \mathscr{O}\left(\rho^{\left(\frac{1}{q}-\frac{1}{r}\right)\frac{q}{q-1}}\right) = \mathscr{O}\left(\rho^{-\varkappa}\right).$$

Compute

$$(-D_{x_4})^k \partial_{x_1}^{\varepsilon} A(u)(0) = \int_{M_u}^{+\infty} e^{-\rho^\theta} \rho^{k+\frac{\varepsilon}{r}} \tau(\rho)^\varepsilon \partial^\varepsilon u_1(0) u_2(0,\rho) d\rho$$
$$\geq \partial^\varepsilon u_1(0) C \int_{M_u}^{+\infty} e^{-\rho^\theta - C_1\rho^\varkappa} \tau(\rho)^\varepsilon \rho^{k+\frac{\varepsilon}{r}} d\rho \geq C_2^{k+1} k!^{s_0},$$

where $\varepsilon = 0$ or 1 if u_1 is even or odd respectively and

$$\varkappa = \left(\frac{1}{r} - \frac{1}{q}\right) \frac{q}{q-1} < \theta.$$

The last inequality above holds since

$$\int_{M_u}^{+\infty} e^{-\rho^\theta - C_1\rho^\varkappa} \tau(\rho)^\varepsilon \rho^{k+\frac{\varepsilon}{r}} d\rho \geq C_\tau \int_{M_u}^{+\infty} e^{-c\rho^\theta} \rho^k d\rho$$
$$= -C_\tau \int_0^{M_u} e^{-c\rho^\theta} \rho^k d\rho + C_2^{k+1} k!^{s_0}$$
$$\geq C_2^{k+1} k!^{s_0} \left(1 - C_\tau C_2^{-(k+1)} M_u e^{-cM_u^\theta} \frac{M_u^k}{k!^{s_0}}\right) \geq C_3^{k+1} k!^{s_0},$$

if k is suitably large and C_3 is suitable.

We emphasize that in a global (or semiglobal) setting the operator P_1 may be analytic hypoelliptic, suggesting that analytic hypoellipticity might be a consequence of the spectral behavior of some operator. Concerning this we cite the following theorem by Chinni [17]:

Theorem 4.4 ([17]). *Let*

$$P_1(x, D) = D_1^2 + D_2^2 + a^2(x_1)\left(D_3^2 + D_4^2\right) + b_1^2(x_2) D_3^2 + b_2^2(x_2) D_4^2,$$

defined on $\mathbb{T}^4$, where a, b_1, b_2 are real valued real analytic functions not identically zero. Then, given any subinterval $I \subset \mathbb{T}^2_{x'}$, $x' = (x_1, x_2)$, and given any $u \in \mathscr{D}'(I \times \mathbb{T}^2_{x''})$, $x'' = (x_3, x_4)$, the condition $P_1 u \in C^\omega(I \times \mathbb{T}^2_{x''})$ implies $u \in C^\omega(I \times \mathbb{T}^2_{x''})$.

We note that the same phenomenon of Theorem 4.2 occurs when the Treves conjecture gives a more complicated stratification. Consider for example the following operator

$$
\begin{aligned}
(4.16)\quad P(x, D) = D_1^2 + x_1^{2(\ell+r-1)} \left(D_3^2 + D_4^2\right) \\
+ x_1^{2\ell} \left[D_2^2 + x_2^{2(p-1)} D_3^2 + x_2^{2(q-1)} D_4^2\right],
\end{aligned}
$$

where ℓ, r, p, $q \in \mathbb{N}$, $1 < r < p < q$, and $x = (x_1, \dots, x_4) \in \mathbb{R}^4$.

Hörmander's condition is satisfied by P and thus P is C^∞ hypoelliptic.

The characteristic manifold of P is the real analytic manifold

$$
\begin{aligned}
(4.17)\quad \mathrm{Char}(P) = \big\{ (x, \xi) \in T^*\mathbb{R}^4 \setminus \{0\} \mid \\
\xi_1 = 0, x_1 = 0,\ \xi_2^2 + \xi_3^2 + \xi_4^2 > 0 \big\}.
\end{aligned}
$$

According to Treves conjecture one has to look at the strata associated with P.

The stratification associated with P is made up of two symplectic strata:

a -

$$\Sigma_1 = \left\{ (0, x_2, x_3, x_4; 0, \xi_2, \xi_3, \xi_4) \mid \xi_2^2 + x_2^2 > 0,\ \sum_{j=2}^{4} \xi_j^2 > 0 \right\}.$$

This is a symplectic stratum and the restriction of the symplectic form to it has rank 6.

b -

$$\Sigma_2 = \big\{ (0, 0, x_3, x_4; 0, 0, \xi_3, \xi_4) \mid \xi_3^2 + \xi_4^2 > 0 \ \big\}.$$

This is also a symplectic stratum and the restriction of the symplectic form to it has rank 4.

According to the conjecture we would expect local real analyticity near the origin for the distribution solutions, u, of $Pu = f$, with a real analytic right hand side.

The following theorem holds

Theorem 4.5 ([10]). *Let*

$$\frac{1}{s_\ell} = \frac{\ell+1}{\ell+r} + \frac{r-1}{\ell+r}\, \frac{p-1}{q-1}.$$

Then P is locally Gevrey s_ℓ hypoelliptic and not better near the origin.

Note that for $\ell = 0$ s_ℓ coincides with s_0 of Theorem 4.2.

We also would like to mention the following result: let r, p, q and k be positive integers such that $r < p < q$. Consider the sum of squares operator in $\mathbb{R}^4$, obtained adding the square of the vector field $x_2^{p-1}x_3^k D_4$ to the operator in (4.3),

$$\begin{aligned} P(x, D) = {} & D_1^2 + D_2^2 + x_1^{2(r-1)}D_3^2 + x_1^{2(r-1)}D_4^2 + x_2^{2(p-1)}D_3^2 \\ & + x_2^{2(p-1)}x_3^{2k}D_4^2 + x_2^{2(q-1)}D_4^2 \end{aligned} \tag{4.18}$$

The characteristic variety of P is actually the real analytic manifold

$$\mathrm{Char}(P) = \{(x, \xi) \mid x_i = \xi_i = 0, i = 1, 2, \xi_3^2 + \xi_4^2 > 0\},$$

which is a symplectic manifold. Actually $\mathrm{Char}(P) = \mathrm{Char}(P_1)$.

We have

Theorem 4.6 ([11])**.** *The operator P in* (4.18) *is analytic hypoelliptic.*

The theorem above as well as the choice of the operator P are worth some explanation.

The operator P_1 in (4.3) is a counterexample to Treves conjecture. Actually the stratification associated to P_1 in the statement of the conjecture is made of the sole stratum

$$\mathrm{Char}(P_1) = \{(x, \xi) \mid x_i = \xi_i = 0, i = 1, 2, \xi_3^2 + \xi_4^2 > 0\} = \mathrm{Char}(P).$$

An inspection of the proof though, shows that the real analytic submanifold

$$\Sigma_1 = \{(x, \xi) \mid x_i = \xi_i = 0, i = 1, 2, \xi_3 = 0, \xi_4 \neq 0\}$$

is important for the Gevrey regularity of P_1 because of the presence of the vector field $x_2^{p-1}D_3$. This remark would lead us to consider the characteristic set $\mathrm{Char}(P_1)$ as the disjoint union of the following two analytic strata

$$\Sigma_0 = \{(x, \xi) \mid x_i = \xi_i = 0, i = 1, 2, \xi_3 \neq 0\},$$

$$\Sigma_1 = \{(x, \xi) \mid x_i = \xi_i = 0, i = 1, 2, \xi_3 = 0, \xi_4 \neq 0\}.$$

Actually Σ_1 is non symplectic and has Hamilton leaves which are the x_3 lines where the propagation of the Gevrey–s_0 wave front set occurs. Hence we might think of Σ_1 as a "non Treves stratum" where the existence of Hamilton leaves implies non analytic regularity.

We must make it clear though that, to our knowledge, there is neither a replacement conjecture nor an alternative definition of stratification.

The model operator P is such that, even though almost all the properties of P_1, as far as the Treves stratification is concerned, are retained, the manifold Σ_1 is replaced by

$$\Sigma_1 = \{(x, \xi) \mid x_i = \xi_i = 0, i = 1, 2, 3, \xi_4 \neq 0\}, \tag{4.19}$$

due to the presence in P of both vector fields $x_2^{p-1} D_3$ and $x_2^{p-1} x_3^k D_3$. We point out that in this case Σ_1 is a symplectic submanifold and hence has no Hamilton leaves.

In other words it seems that the analytic regularity of a sum of squares should depend on a suitable stratification of the characteristic variety of the operator and on the fact that its strata are analytic symplectic manifolds.

Unfortunately we cannot be more precise on this at the moment.

5. Open problems

5.1. **The 2 dimensional case.** Let us consider a sum of squares operator in $\mathbb{R}^2$. Denote by (x, y) the variables in $\mathbb{R}^2$:

$$P(x, y, D_x, D_y) = \sum_{j=1}^{N} X_j^2(x, y, D_x, D_y). \tag{5.1}$$

Without loss of generality we may suppose we are working in a neighborhood of the origin, Ω, and that $X_1 = D_x$.

Thus one of the equations of the characteristic variety is $\xi = 0$. For $j \geq 2$ we may then write $X_j(x, y, \xi, \eta) = a_j(x, y)\xi + b_j(x, y)\eta$. Since $\eta \neq 0$ we find that the other relations describing the characteristic variety are $b_j(x, y) = 0$, where the b_j are real analytic functions defined in Ω.

Since we are assuming that Hörmander condition is satisfied, we may suppose that $(0, 0)$ is a point of the characteristic variety and that, possibly shrinking Ω, there is an index j, $2 \leq j \leq N$, such that $\partial_x^m b_j(0, 0) \neq 0$; here m is minimal, i.e. $\partial_x^k b_j(0, 0) = 0$ when $2 \leq j \leq N$ and $k < m$. It is also evident that $X_1 = D_x$ is the only field that we can meaningfully use to form brackets of vector fields, i.e. we have to consider only brackets of the form $\mathrm{ad}(X_1)^k X_j$, since any other vector field has a vanishing coefficient in front (see also [14].)

Set

$$f(x, y) = \sum_{j=2}^{N} (b_j(x, y))^2.$$

The characteristic variety of P is then given by

$$\mathrm{Char}(P) = \{(x, y; 0, \eta) \mid \eta \neq 0, f(x, y) = 0\}.$$

We apply Weierstraß preparation theorem to f and write

$$f(x,y) = e(x,y)\left(x^{2m} + \sum_{\ell=1}^{2m} a_\ell(y)x^{2m-\ell}\right),$$

where $e(0,0) \neq 0$ is a C^ω function, $a_\ell(0) = 0$ for every $\ell = 1,\dots,2m$. Since e is different from zero, we may replace f by the Weierstraß polynomial above, because they define the same variety. Let us denote it by $q(x,y)$.

Definition 5.1 ([41], [60])**.** *We say that a polynomial of the form*

$$q(z', z_n) = z_n^m + \sum_{k=1}^{m} a_k(z')z_n^{m-k},$$

$z = (z', z_n) \in U$ *open subset of* $\mathbb{C}^n$*,* $0 \in U$*,* $a_k \in \mathscr{O}(U)$*, holomorphic functions on* U *such that* $a_k(0) = 0$ *for every* k *is a Weierstraß type polynomial of degree* m*.*

We have the following theorem

Theorem 5.1 ([41], [60])**.** *Let* f *be a holomorphic function defined in a neighborhood of the origin,* $U \subset \mathbb{C}^n$*. Suppose that* $f(0,\dots,0,z_n) \not\equiv 0$ *in* U*. Then there exists a Weierstraß type polynomial,* $q^\#$*, whose discriminant is not identically zero in* U *and such that* $f = 0$ *iff* $q^\# = 0$*.*
Same statement for a real analytic case.

Denote by $D_\#(y) = \operatorname{discr} q^\#$. We have that $D_\# \in C^\omega(\pi_2(U))$, where π_2 is the projection onto the y–axis.

As a consequence $D_\#^{-1}(0) = \{y_1,\dots,y_\nu\}$, for a certain $\nu \in \mathbb{N}$. Let $m^\# = \deg q^\#$ and denote by $\rho_1,\dots,\rho_{m^\#}$ the roots (real or complex) of $q^\#$. For every $j \in \{1,\dots,\nu\}$, there are at least two indices, i_1, i_2 in the range $\{1,\dots,m^\#\}$ such that $\rho_{i_1}(y_j) = \rho_{i_2}(y_j)$. We set

$$\tilde\rho_j = (x_{i_1}, y_j), \quad x_{i_1} = \rho_{i_1}(y_j), \quad j = 1,\dots,\nu. \tag{5.2}$$

Definition 5.2. *We call* $\tilde\rho_j$ *a branching point of* $f^{-1}(0)$*. Denote by* $\mathscr{B}(U)$ *the set of branching points in* U*.*

The above described facts determine the stratification. There are two cases:

(a) The set $\mathscr{B}(U)$ is empty. This means that the roots of $q^\#$ are simple and have the form $x = \rho_k(y)$, $k = 1,\dots,m^\#$. Since, according to our assumption, $(0,0) \in f^{-1}(0)$, we deduce that there is only one $k \in \{1,\dots,m^\#\}$ such that $\rho_k(0) = 0$. Possibly shrinking U we obtain that f has the form

$$f(x,y) = \tilde e(x,y)(x-\rho(y))^{2m'}, \qquad \tilde e(0,0) \neq 0, \qquad m' \leq m.$$

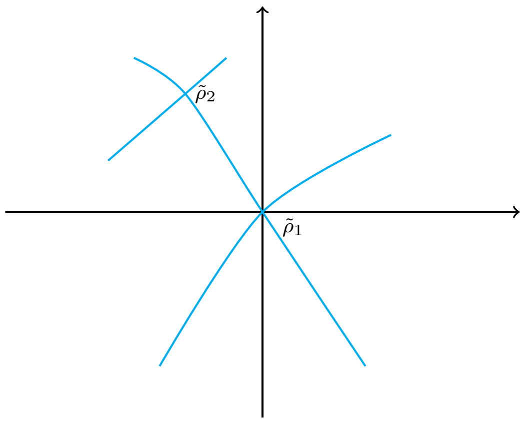

FIGURE 1. An example of $f^{-1}(0)$ near $(0,0) = \tilde\rho_1$

The characteristic variety of P is then symplectic and P is analytic hypoelliptic. This has been proved by Ōkaji, [48], and Cordaro and Hanges, [21], for operators where f has the above form.

(b) The set $\mathscr{B}(U)$ is not empty. Then we may always shrink the neighborhood U so that the origin—or $\tilde\rho_1$ is the only branching point in U. Then f has the form

$$f(x,y) = \tilde e(x,y) \prod_{j=1}^{m'} (x - \rho_j(y))^{m_j},$$

and $\rho_j(y) \neq \rho_k(y)$ if $y \neq 0$, but $\rho_j(0) = 0$ for every j, $m' \leq m^\#$, $\tilde e(0,0) \neq 0$.

The deeper stratum is

$$\Sigma_1 = \{(0,0;0,\eta) \mid \eta \neq 0\},$$

as we can see by taking derivatives of f with respect to x. $\mathrm{Char}(P) \setminus \Sigma_1$ is a union of disjoint arcs of C^ω curves of the form

$$\{(x,y,0,\eta) \mid \eta \neq 0, (x,y) \neq (0,0), x = \rho_j(y)\},$$

which gives simplectic strata at each point of which we get real analyticity.

Thus it seems that the Treves stratification completely describes all possible situations in two dimensions. The problem of the non analytic hypoellipticity of P in case (b) as well as the problem of its (optimal) Gevrey regularity are open (see [14] for a particular case.)

We explicitly note that proving that in case (b) there is no analytic hypoellipticity amounts to proving that the Treves conjecture holds in dimension two.

5.2. **The 3 dimensional case.** There are no known counterexamples to the Treves conjecture in dimension 3. However in [12] some examples have been proposed that should violate the conjecture. We briefly describe those models in this section.

Let $x \in \mathbb{R}^2$, $y \in \mathbb{R}$, a, p, q, r be positive integers. We shall specify later the relation between these integers. Define

$$Q(x, y, D_x, D_y) = D_1^2 + D_2^2 + x_2^{2(r-1)} D_y^2 + x_1^{2(q-1)} D_y^2 + x_1^{2(p-1)} y^{2a} D_y^2. \tag{5.3}$$

If we assume that $1 < p < q < r$, the Lie algebra is generated with brackets of length $m = q - 1$. The characteristic manifold is $\{(0, 0, y; 0, 0, \eta) \mid \eta \neq 0\}$.

Looking at the powers of the monomials in x, we can draw a (convex) Newton polygon in the x-plane; the precise definition of Newton polygon is given in [12], but, in three variables, it is just a segment—the red line in the figures below. When the powers of x having a possibly degenerate coefficient are added to the picture we obtain
where the dashed line has slope -1 and starts from the vertex closest to the origin, the triangle underneath the dashed line has points corresponding to monomials where the Treves stratification identifies a non symplectic stratum.

In [12] it is proved that

Theorem 5.2. *The operator Q in* (5.3) *is Gevrey s hypoelliptic for*

$$s \geq \left(1 - \frac{1}{a}\frac{p-1}{q}\right)^{-1}.$$

There is no proof of the optimality of the above index; we believe that it is optimal, due to the fact that Theorem 5.2 is a particular case of a result proved in [12], which, in the known cases, gives optimal values.

Let us now consider the operator Q in (5.3) when $1 < r < p < q$. If, as we did before, we draw the Newton polygon for Q and add to the picture the dots corresponding to degenerate monomials (i.e. monomials having coefficients containing powers of y) we obtain

In [12] it is proved that, in the latter case, Q is Gevrey s hypoelliptic for

$$s \geq \left(1 - \frac{1}{a} \cdot \frac{q-p}{q-1} \cdot \frac{r-1}{r}\right)^{-1}. \tag{5.4}$$

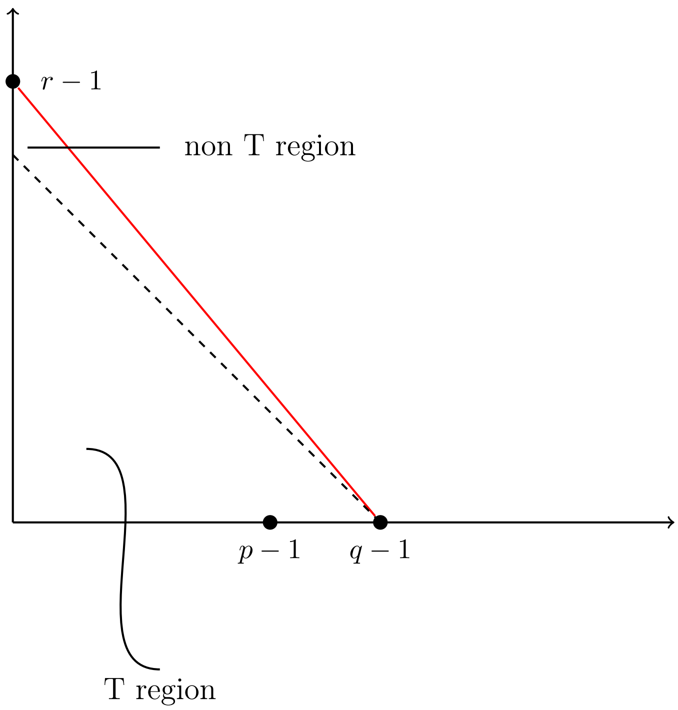

FIGURE 2. The Newton polygon for Q in (5.3) when $1 < p < q < r$

On the other hand Q has a symplectic characteristic manifold: $\mathrm{Char}(Q)$ $= \{x = \xi = 0, \eta \neq 0\}$ and no strata are found using the Poisson brackets of the fields, so that according to the conjecture it should be analytic hypoelliptic. We believe that the Gevrey regularity in (5.4) is optimal, based on the striking similarity of Q with the operator discussed in [3] which violates the conjecture. Actually the main difference between Q and the operator in [3] consists in the fact that the putative stratum is a non symplectic "stratum" whose Hamilton leaf lies on the fiber of the cotangent bundle.

At the moment we have no optimality proof for the Gevrey regularity (5.4) of Q both in the case of Figure 2 and of Figure 3. We also remark that the optimality of (5.4) would imply that the Treves conjecture does not hold in dimension 3.

Even though for the case considered in [12] the Newton polygon helps in identifying a (non symplectic) stratum in the three variables case, we would like to point out that this is not the case when the vector fields are not monomials. Here are two examples:

$$Q_1 = D_1^2 + D_2^2 + (x_1 - x_2^2)^2 D_y^2 + (y^2 x_1^3 + x_2^4)^2 D_y^2 \tag{5.5}$$

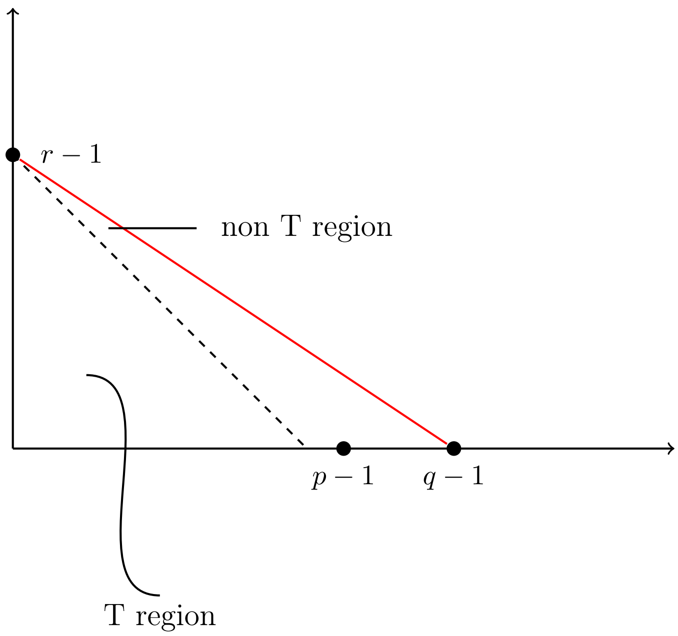

FIGURE 3. The Newton polygon for Q when $1 < r < p < q$

and

$$Q_2 = D_1^2 + D_2^2 + (x_1 - x_2^2)^2 D_y^2 + (x_1^3 + y^2 x_2^4)^2 D_y^2. \tag{5.6}$$

It is easy to show that

$$\mathrm{Char}(Q_j) = \{(0,0,y;0,0,\eta) \mid \eta \neq 0\},$$

i.e. a symplectic manifold.

One can prove, using the L^2 estimate , that Q_1 is analytic hypoelliptic. Unfortunately the same proof does not work for Q_2. We believe that Q_2 has a non symplectic non Treves stratum, and hence is not analytic hypoelliptic. No proof is known.

A similar model is

$$Q_3 = D_1^2 + D_2^2 + (yx_1^\ell + x_2^m)^2 D_y^2 + x_1^{2k} D_y^2. \tag{5.7}$$

We can show that

$$Q_3 \text{ is analytic hypoelliptic if } \begin{cases} m = 1, \\ \ell \geq k. \end{cases}$$

On the other hand we believe that Q_3 is not analytic hypoelliptic when $m > 1$ and $\ell < k$ even though there is no proof of this fact. Note that, depending on the relations between m and k we may or may not have a Treves non symplectic stratum.

5.3. **The case of dimension $n \geq 3$.** Finally let us consider the "general" case, i.e. the case of dimension n. Even though we have seen a number of examples where the Treves stratification does not identify a non symplectic stratum, while the operator is not analytic hypoelliptic, like those of Theorem 4.2 and 4.5, we think that the important idea in the formulation of the conjecture is the quest for a stratification.

Actually the stratification associated to P_1 in (4.3), is made of the sole stratum

$$\mathrm{Char}(P_1) = \{(x, \xi) \mid x_i = \xi_i = 0, i = 1, 2, \xi_3^2 + \xi_4^2 > 0\}.$$

An inspection of the proof though, shows that the real analytic submanifold

$$\Sigma_1 = \{(x, \xi) \mid x_i = \xi_i = 0, i = 1, 2, \xi_3 = 0, \xi_4 \neq 0\}$$

is important for the Gevrey regularity of P_1 because of the presence of the vector field $X_5 = x_2^{p-1} D_3$. This remark would lead us to consider the characteristic set $\mathrm{Char}(P_1)$ as the disjoint union of the following two analytic strata

$$\Sigma_0 = \{(x, \xi) \mid x_i = \xi_i = 0, i = 1, 2, \xi_3 \neq 0\},$$

$$\Sigma_1 = \{(x, \xi) \mid x_i = \xi_i = 0, i = 1, 2, \xi_3 = 0, \xi_4 \neq 0\}.$$

Σ_1 is non symplectic and has Hamilton leaves which are the x_3 lines where the propagation of the Gevrey–s_0 wave front set occurs. Hence we might think of Σ_1 as a "non Treves stratum" where the existence of Hamilton leaves implies non analytic regularity.

Somewhat symmetrically the analog of Σ_1 for the operator P of (4.18) turns out to be symplectic.

The following question has, to our knowledge, received no answer yet:

Problem 5.1. *Define a stratification of the characteristic variety in real analytic manifolds such that when each stratum is a symplectic manifold then the operator in analytic hypoelliptic.*

This would allow to reformulate, regardless of the local or microlocal aspect of the question, Treves conjecture as

Conjecture 5.2. *A sum of squares operator with real analytic coefficients is analytic hypoelliptic if and only if every stratum of the stratification is a symplectic real analytic manifold.*

References

[1] P. Albano, A. Bove and G. Chinni, *Minimal Microlocal Gevrey Regularity for "Sums of Squares"*, International Mathematics Research Notices, Vol. 2009, No. **12**, 2275–2302.

[2] P. Albano and A. Bove, *Wave front set of solutions to sums of squares of vector fields*, Mem. Amer. Math. Soc., **221** (2013), no. 1039.

[3] P. Albano, A. Bove and M. Mughetti, *Analytic Hypoellipticity for Sums of Squares and the Treves Conjecture*, J. Funct. Anal. **274** (2018), no. 10, 2725–2753.

[4] M. S. Baouendi and Ch. Goulaouic, *Nonanalytic-hypoellipticity for some degenerate operators*, Bull. A. M. S., **78** (1972), 483–486.

[5] C. M. Bender and Q. Wang, *A class of exactly solvable eigenvalue problems*, J. Physics A **34** (2001), 9835–9847.

[6] F.A. Berezin and M.A. Shubin, *The Schrödinger equation*, Mathematics and its Applications (Soviet Series), 66. Kluwer Academic Publishers Group, Dordrecht, 1991

[7] P. Bolley, J. Camus and J. Nourrigat, *La condition de Hörmander–Kohn pour les opérateurs pseudo-différentiels*, Comm. in Partial Differential Equations **7**(2) (1982), 197–221.

[8] A. Bove and M. Mughetti, *Analytic and Gevrey hypoellipticity for a class of pseudodifferential operators in one variable*, J. Differential Equations **255** (2013), no. 4, 728–758.

[9] A. Bove and M. Mughetti, *On a new method of proving Gevrey hypoellipticity for certain sums of squares*, Advances in Math. **293** (2016), 146–220.

[10] A. Bove and M. Mughetti, *Analytic Hypoellipticity for Sums of Squares and the Treves Conjecture, II*, Analysis and PDE **10** (7) (2017), 1613–1635.

[11] A. Bove and M. Mughetti, *Analytic Hypoellipticity for Sums of Squares in the Presence of Symplectic non Treves Strata* , J. Inst. Math. Jussieu, `doi:10.1017/S1474748018000580`.

[12] A. Bove and M. Mughetti, *Gevrey Regularity for a Class of Sums of Squares of Monomial Vector Fields*, preprint, 2019.

[13] A. Bove and D. S. Tartakoff, *Optimal non-isotropic Gevrey exponents for sums of squares of vector fields*, Comm. Partial Differential Equations **22** (1997), no. 7–8, 1263–1282.

[14] A. Bove and D. S. Tartakoff, *Gevrey hypoellipticity for sums of squares of vector fields in $\mathbb{R}^2$ with quasi–homogeneous polynomial vanishing*, Indiana Univ. Mathematics J. **64** (2015), 613–633.

[15] A. Bove and F. Treves, *On the Gevrey hypo-ellipticity of sums of squares of vector fields*, Ann. Inst. Fourier (Grenoble) **54**(2004), 1443-1475.

[16] S. Chanillo, B. Helffer and A. Laptev, *Nonlinear eigenvalues and analytic hypoellipticity*, J. Funct. Anal. **209** (2004), no. 2, 425–443.

[17] G. Chinni, *(Semi)-global analytic hypoellipticity for a class of "sums of squares" which fail to be locally analytic hypoelliptic*, to appear in Proc. Amer. Math. Soc., DOI: `https://doi.org/10.1090/proc/14464`.

[18] G. Chinni, *On the sharp Gevrey regularity for a generalization of the Métivier operator*, preprint, 2019.

[19] M. Christ, *Certain sums of squares of vector fields fail to be analytic hypoelliptic*, Comm. Partial Differential Equations **16** (1991), 1695–1707.

[20] M. Christ, *Intermediate optimal Gevrey exponents occur*, Comm. Partial Differential Equations **22** (1997), no. 3-4, 359âĂŞ-379.

[21] P. D. Cordaro and N. Hanges, *A New Proof of Okaji's Theorem for a Class of Sum of Squares Operators*, Ann. Inst. Fourier (Grenoble) **59** (2009), no. 2, 595–619.

[22] M. Derridj, *Un problème aux limites pour une classe d'opérateurs du second ordre hypoelliptiques*, Ann. Inst. Fourier (Grenoble) **21** (1971), no. 4, 99–148.

[23] L. Ehrenpreis, *Solutions of some problems of division IV* , Amer. J. Math **82** (1960), 522-588.

[24] C. Fefferman and D. H. Phong, *The uncertainty principle and sharp Gårding inequalities*, Comm. Pure Appl. Math. **34** (1981), 285–331.

[25] A. Grigis and J. Sjöstrand, *Front d'onde analytique et sommes de carrés de champs de vecteurs*, Duke Math. J. **52** (1985), no. 1, 35–51.

[26] V. V. Grušin, *On a class of elliptic pseudodifferential operators degenerating at a submanifold*, Mat. Sbornik **84 (2)** (1971), 163–195.

[27] V. V. Grušin, *On a class of Hypoelliptic operators*, Math. USSR Sb. **12** (1970), 458–476.

[28] G. G. Gundersen, *A class of anharmonic oscillators whose eigenfunctions have no recurrence relations*, Proc. Amer. Math. Soc. **58** (1976), 109–113.

[29] N. Hanges and A. A. Himonas, *Singular solutions for sums of squares of vector fields*, Comm. in Partial Differential Equations **16** (1991), 1503–1511.

[30] N. Hanges and A. A. Himonas, *Singular solutions for a class of Grusin type operators*, Proc. Amer. Math. Soc. **124** (1996), no. 5, 1549–1557.

[31] N. Hanges and A. A. Himonas, *Non-analytic hypoellipticity in the presence of symplecticity*, Proc. Amer. Math. Soc. **126** (1998), no. 2, 405–409.

[32] B. Helffer, *Sur l'hypoellipticité des opérateurs pseudodifférentiels à caractéristiques multiples (perte de 3/2 dérivées)*, Mémoires de la S. M. F. **51–52** (1977), 13–61.

[33] B. Helffer, *Théorie spectrale pour des opérateurs globalement elliptiques*, Astérisque, **112** (1984), ix+197 pp.

[34] L. Hörmander, *Hypoelliptic second order differential equations*, Acta Math. **119** (1967), 147–171.

[35] L. Hörmander, *Uniqueness Theorems and Wave Front Sets for Solutions of Linear Differential Equations with Analytic Coefficients*, Communications Pure Appl. Math. **24** (1971), 671–704.

[36] L. Hörmander, *The Analysis of Partial Differential Operators, I*, Springer Verlag, 1985.

[37] L. Hörmander, *The Analysis of Partial Differential Operators, III*, Springer Verlag, 1985.

[38] T. Hoshiro, *Failure of analytic hypoellipticity for some operators of $X^2 + Y^2$ type*, J. Math. Kyoto Univ. **35** (1995), no. 4, 569–581.

[39] J. J. Kohn, *Pseudo-differential operators and non-elliptic problems*, 1969 Pseudo-Diff. Operators (C.I.M.E., Stresa, 1968), 158–165, Edizioni Cremonese, Roma.

[40] J. J. Kohn and L. Nirenberg, *An algebra of pseudo-differential operators*, Comm. Pure Appl. Math. **18** (1965), 443–492.

[41] S. Łojasiewicz, *Ensembles semi–analytiques*, cours à Orsay, saisie en LaTeX par M. Coste, Juillet 1965.

[42] G. Métivier, *Une Classe d'Opérateurs Non Hypoelliptiques Analytiques*, Indiana Univ. Math. J. **29** (1980), 823–860.

[43] G. Métivier, *Hypoellipticité analitique sur des groupes nilpotents de rang 2*, Duke Math. J. **47** (1980), 195–221.

[44] G. Métivier, *Non-hypoellipticité Analytique pour* $D_x^2 + (x^2 + y^2)D_y^2$, C. R. Acad. Sci. Paris Sér. I Math. **292** (1981), no. 7, 401–404.

[45] G. Métivier, *Analytic hypoellipticity for operators with multiple characteristics*, Comm. Partial Differential Equations **6** (1981), no. 1, 1–90.

[46] M. Mughetti, *Regularity properties of a double characteristics differential operator with complex lower order terms*, J. Pseudo-Differ. Oper. Appl. **5** (2014), no. 3, 343–358.

[47] M. Mughetti, *On the spectrum of an anharmonic oscillator*, Trans. Amer. Math. Soc. **367** (2015), 835–865.

[48] T. Ōkaji, *Analytic hypoellipticity for operators with symplectic characteristics*, J. Math. Kyoto Univ. **25** (1985), no. 3, 489–514.

[49] O. A. Oleĭnik, *On the analyticity of solutions of partial differential equations and systems*, Colloque International CNRS sur les Équations aux Dérivées Partielles Linéaires (Univ. Paris- Sud, Orsay, 1972), 272–285. Astérisque, 2 et 3. Societé Mathématique de France, Paris, 1973.

[50] O. A. Oleĭnik and E. V. Radkevič, *The analyticity of the solutions of linear partial differential equations*, (Russian) Mat. Sb. (N.S.), **90(132)** (1973), 592–606.

[51] L. Rothschild and E. M. Stein, *Hypoelliptic differential operators and nilpotent groups*, Acta Math. **137** (3 - 4) (1976), 247–320.

[52] J. Sjöstrand, *Analytic wavefront set and operators with multiple characteristics*, Hokkaido Math. J. **12**(1983), 392–433.

[53] J. Sjöstrand, *Singularités analytiques microlocales*, Astérisque **95** (1982).

[54] D.S. Tartakoff, *On the Local Real Analyticity of Solutions to* $\Box_b$ *and the* $\bar\partial$*-Neumann Problem*, Acta Math. **145** (1980), 117–204.

[55] D.S. Tartakoff, *Elementary proofs of analytic hypoellipticity for* $\Box_b$ *and the* $\overline{\partial}$*-Neumann problem*, Analytic solutions of partial differential equations (Trento, 1981), 85–116; Astérisque, 89–90, Soc. Math. France, Paris, 1981.

[56] F. Trèves, *Analytic hypo-ellipticity of a class of pseudodifferential operators with double characteristics and applications to the* $\bar\partial$*-Neumann problem*, Comm. Partial Differential Equations **3** (1978), no. 6-7, 475–642.

[57] F. Treves, *Introduction to pseudodifferential and fourier integral operators ,* Vol. 1, Plenum Press, New York-London, 1980.

[58] F. Treves, *Symplectic geometry and analytic hypo-ellipticity, in Differential equations,* La Pietra 1996 (Florence), Proc. Sympos. Pure Math. **65**, Amer. Math. Soc., Providence, RI, 1999, 201-219.

[59] F. Treves, *On the analyticity of solutions of sums of squares of vector fields*, Phase space analysis of partial differential equations, Bove, Colombini, Del Santo ed.'s, 315-329, Progr. Nonlinear Differential Equations Appl., **69**, Birkhäuser Boston, Boston, MA, 2006.

[60] F. Treves, *Aspects of Analytic PDE*, book in preparation.

[61] M. Zworski, *Semiclassical Analysis,* Graduate Studies in Mathematics, Vol. 138, American Mathematical Society, Providence, R. I. 2012.

Dipartimento di Matematica, Università di Bologna, Piazza di Porta San Donato 5, Bologna Italy
E-mail address: bove@bo.infn.it

Dipartimento di Matematica, Università di Bologna, Piazza di Porta San Donato 5, Bologna Italy
E-mail address: marco.mughetti@unibo.it