



Available online at <http://scik.org>

Math. Finance Lett. 2020, 2020:1

<https://doi.org/10.28919/mfl/4547>

ISSN: 2051-2929

## NEW EXACT SOLUTIONS TO OPTIMAL CONSUMPTION-INVESTMENT PROBLEMS WITH EXPONENTIAL UTILITY

ROSSELLA AGLIARDI\*

Department of Mathematics, University of Bologna (Italy)

Copyright © 2020 the author(s). This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

**Abstract.** We revisit the seminal Merton's work on optimal portfolio-consumption problems in a more general framework where exact solutions are not available. We provide exact solutions in a few special cases which were not included in the original setting.

**Keywords:** HJB equation; optimal dynamic portfolio allocation; nonlinear ODEs; geometric Brownian process.

**2010 AMS Subject Classification:** 34L30, 49J15, 91G10.

### 1. INTRODUCTION

Optimal portfolio and consumption problems are a classical topic in mathematical finance since the seminal articles by Merton (1969, 1971), which established the framework for dynamic portfolio choices with stochastic variation in the prices of the securities in the portfolio. The problem is to dynamically manage a portfolio of securities in order to maximize the expected utility of consumption over time. A huge literature has been generated from the seminal works, dealing with various restrictions on the controls, different utility functions or alternative

---

\*Corresponding author

E-mail address: [rossella.agliardi@unibo.it](mailto:rossella.agliardi@unibo.it)

Received February 27, 2019

stochastic processes for the reference financial assets. Moreover, regarding the economic assumptions, some extensions have been provided, for example, introducing labour income and human capital (Basak, 1999), the possibility of bankruptcy risk (Sethi, 1997), various restrictions on the trading strategies (Bardhan, 1994), just to mention a few contributions in an extensive research field. In the original formulation, where the underlying risky asset follows a geometric Brownian motion, the stochastic optimal control problem leads to a Hamilton-Jacobi-Bellmann equation that is a fully nonlinear partial differential equation or a nonlinear ordinary differential equation. When the utility function has a power or a logarithmic shape an explicit solution is easily obtained, while in other situations the problem is hard to solve analytically, because the form of a solution to the HJB equation is difficult to guess. Usually the portfolio consists of a risk-free asset and one (or more) risky assets. In this paper, we focus the analysis on the infinite horizon problem, but we consider the general case of a portfolio of all risky assets following geometric Brownian motions. While most literature adopts the existence of a risk-free asset, critics contend that a riskless security is an idealization that is an acceptable approximation of real-world situations only when inflation, credit risk, liquidity risk and any other form of financial risk are truly tiny. Therefore we work in a more general framework: of course, the special case of a risk-free asset can be recovered just letting its volatility tend to zero. Even in the plain case of infinite horizon, when the utility function is of exponential form solving the nonlinear HJB differential equation is an open question. We point out that the alternative approach of Cox and Huang (1989) based on martingale methods is formulated only for the special case of a risk-free asset and does not apply to this problem. On the other hand, the natural guess of an exponential shape for the solution works only in the special case studied by Merton (1969).

The general problem is presented in Section 2 along with a brief review of the already solved cases. In Section 3 we derive a new closed-form solution for some special cases. In Section 4 we discuss the still open problem.

## 2. PORTFOLIO-CONSUMPTION PROBLEM WITH RISKY ASSETS

Consider a financial market with  $N$  assets being traded continuously on an infinite horizon. Assume that their prices,  $S_i(t)$ ,  $i = 1, \dots, N$ , follow a geometric Brownian motion:

$$(1) \quad dS_i(t) = \mu_i S_i(t) dt + \sigma_i S_i(t) dW_t^{(i)}$$

where the  $W_t^{(i)}$ 's are the components of an  $N$ -dimensional Wiener process with respect to a given filtration and  $E[dW_t^{(i)} dW_t^{(j)}] = \rho_{ij} dt$  with  $0 \leq \rho_{ij} \leq 1$ . Consider an investor that has an initial wealth endowment  $R_0 > 0$  and, at any time  $t$ , has to decide how to optimally split his/her wealth,  $R(t)$ , between consumption and investment, where the investment is made by optimally selecting the proportion of each financial asset in his/her portfolio. Let  $c(t)$  denote the consumption and  $x_i(t)$  the share of wealth invested in the  $i^{\text{th}}$  financial asset of the portfolio at time  $t$ . Then the array  $X = (x_1, \dots, x_N)$  is the investor's portfolio. Of course,  $\sum_{i=1}^N x_i(t) = 1$ . We allow for negative values of the  $x_i$ 's, which means that short selling is allowed in our setting. Usually it is assumed that the portfolio-consumption strategy  $(X(t), c(t))$  is self-financing, that is, there are no exogeneous injections of capital from outside and no withdrawal of money (i.e. one can finance any purchase only by selling assets which are in the portfolio). Then the wealth's dynamics take the form:

$$(2) \quad \begin{aligned} dR(t) = & \sum_{i=1}^{N-1} x_i(t) (\mu_i - \mu_N) R(t) dt + \mu_N R(t) dt - c(t) dt + \\ & \sum_{i=1}^{N-1} x_i(t) \sigma_i R(t) dW_t^{(i)} + (1 - \sum_{i=1}^{N-1} x_i(t)) \sigma_N R(t) dW_t^{(N)} \end{aligned}$$

The objective is to optimize the consumption policy under the budget equation (2), that is, to maximize

$$(3) \quad E \left[ \int_0^{\infty} e^{-\delta t} U(c(t)) dt \right]$$

where  $E$  is the expectation operator,  $U$  denotes the instantaneous utility function for consumption and  $\delta$  is a (subjective) discount rate. In other words, we have a stochastic control problem, where  $R$  is the state variable and  $(X, c)$  is the control process. Let  $J(R)$  denote the optimal value function.

In what follows we will consider the case  $N = 2$  to simplify the exposition and  $\rho_{12}$  will be denoted by  $\rho$ . Then the Hamilton-Jacobi-Bellman equation reads:

$$(4) \quad -\delta J + \sup_{c,x} [U(c) + [(x(\mu_1 - \mu_2) + \mu_2)R - c]J' + \frac{R^2}{2}[(x\sigma_1)^2 + (1-x)^2\sigma_2^2 + 2\rho\sigma_1\sigma_2x(1-x)]J'' = 0$$

Let  $\Sigma = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$ . By solving the static maximization problem w.r.t. the controls  $c$  and  $x$  the following first order conditions are obtained:

$$(5) \quad \begin{aligned} U'(c) &= J'(R) \\ x &= \frac{\mu_2 - \mu_1}{\Sigma} \frac{J'}{RJ''} + \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\Sigma} \end{aligned}$$

When the utility  $U$  is of the power type, the ODE can be easily solved by taking a power function of the same form for  $J$ . Here we focus on the case of an exponential utility, because a closed-form solution is not available in the general case.

Assume that  $U(c)$  is of the form  $-\frac{1}{\gamma}e^{-\gamma c}$  where  $\gamma > 0$  represents the relative risk aversion. Plugging the FOCs (5) into the equation (4) one gets the following non-linear ODE:

$$(6) \quad \frac{J'}{\gamma} + \delta J + \frac{1}{2} \frac{(\mu_1 - \mu_2)^2 (J')^2}{\Sigma J''} - \frac{J' \ln J'}{\gamma} - mRJ' - \frac{bR^2 J''}{2} = 0$$

where  $m = [\mu_1\sigma_2(\sigma_2 - \rho\sigma_1) + \mu_2\sigma_1(\sigma_1 - \rho\sigma_2)]/\Sigma$  and  $b = (\sigma_1\sigma_2)^2(1 - \rho^2)/\Sigma$ .

*Remark 1.* Consider all the portfolios  $(x, 1-x)$  obtained from linear combination of the assets with returns  $(\sigma_1, \mu_1)$  and  $(\sigma_2, \mu_2)$ . Then  $b$  turns out to be the minimum variance and  $m$  is the mean return of the minimum variance portfolio. In other word, the coefficients of (6) are linked to the static portfolio optimization.

In the sequel we confine the analysis to  $m \geq 0$ . When  $b = 0$  the corresponding value of  $m$  represents the return of the risk-free asset, which is usually denoted by  $r$ .

---

Here the usual convention of Markowitz's theory is adopted where portfolios are identifies by pairs  $(\sigma, \mu)$  with  $\sigma$  denoting the volatility and  $\mu$  the mean return.

The special case of one riskless asset was solved by Merton in explicit form. If  $\sigma_2 = 0$  and  $\mu_2 = r$  then the ODE (6) is written in the form:

$$\frac{J'}{\gamma} + \delta J + \frac{1}{2} \frac{(\mu_1 - r)^2 (J')^2}{\sigma_1^2 J^2} - \frac{J' \ln J'}{\gamma} - r R J' = 0$$

and a natural guess for the solution is  $J(R) = H \exp(hR)$  with negative  $H$  and  $h$ , as a concave function is needed. Substituting this function into the ODE one has

$$h = -\gamma r \text{ and } H = -\frac{1}{\gamma r} \exp\left(1 - \frac{\delta}{r} - \frac{1}{2r} \left(\frac{\mu_1}{\sigma_1}\right)^2\right).$$

Then the optimal portfolio is obtained with  $x = \frac{\mu_1 - r}{\sigma_1^2} \frac{1}{r\gamma}$ .

Apart from this well-known case, there is another special case that can be solved explicitly.

### 3. PERFECTLY CORRELATED ASSETS

In this section we assume that  $\rho = \pm 1$ . So the term with  $R^2$  in (6) vanishes and this ODE admits an elementary solution, as shown below.

*Proposition 1.* If  $\rho = \pm 1$  and  $m > 0$ , then the optimal value function is of the form  $J(R) = H \exp[-\gamma m R]$  with negative  $H$ , and for the optimal portfolio we have:

$$x = \frac{\mu_1 - \mu_2}{R\gamma[\sigma_1 \mp \sigma_2][\mu_1 \sigma_2 \mp \mu_2 \sigma_1]} + \frac{\sigma_2}{[\sigma_1 \mp \sigma_2]}.$$

*Proof.* If  $\rho = -1$  then  $\Sigma = (\sigma_1 + \sigma_2)^2$ . Trying an exponential solution of the form  $J(R) = H \exp[hR]$ , we get:

$$h = -\gamma m \text{ and } H = -\frac{1}{\gamma m} \exp\left(1 - \frac{\delta}{m} - \frac{1}{2m} \left(\frac{\mu_1 - \mu_2}{\sigma_1 + \sigma_2}\right)^2\right) \text{ with } m = [\mu_1 \sigma_2 + \mu_2 \sigma_1] / (\sigma_1 + \sigma_2).$$

As a result, for the optimal portfolio we take  $x = \frac{\mu_1 - \mu_2}{R\gamma[\sigma_1 + \sigma_2][\mu_1 \sigma_2 + \mu_2 \sigma_1]} + \frac{\sigma_2}{[\sigma_1 + \sigma_2]}$ .

Similarly, in the case  $\rho = 1$  a solution of the form  $J(R) = H \exp(hR)$  works and one gets:

$$h = -\gamma m \text{ and } H = -\frac{1}{\gamma m} \exp\left(1 - \frac{\delta}{m} - \frac{1}{2m} \left(\frac{\mu_1 - \mu_2}{\sigma_1 - \sigma_2}\right)^2\right) \text{ with } m = [\mu_1 \sigma_2 - \mu_2 \sigma_1] / (\sigma_2 - \sigma_1).$$

For the optimal portfolio one has:  $x = \frac{\mu_1 - \mu_2}{R\gamma(\sigma_2 - \sigma_1)[\mu_1 \sigma_2 - \mu_2 \sigma_1]} + \frac{\sigma_2}{[\sigma_2 - \sigma_1]}$ .

Note that, unfortunately, the procedure above does not work when  $m = 0$ . If in the case  $\rho = 1$  we rewrite  $m = \frac{\sigma_1 \sigma_2}{\sigma_2 - \sigma_1} [\lambda_1 - \lambda_2]$  with  $\lambda_i = \mu_i / \sigma_i$ ,  $i = 1, 2$ , then the interpretation is that  $m = 0$

occurs when both assets have the same Sharpe ratio. Note that  $\frac{\mu_1 - \mu_2}{\sigma_1 - \sigma_2}$  can be written as  $\lambda = \frac{\mu_i}{\sigma_i}$ .

In this situation, the nonlinear differential equation takes the form:

$$(7) \quad \frac{J'}{\gamma} + \delta J + \frac{\lambda^2 (J')^2}{2 J''} - \frac{J' \ln J'}{\gamma} = 0$$

and the solution method is not so trivial as above. Let us try with a function  $J(R)$  such that

$J'(R) = H \exp(h\sqrt{R})$  where  $H > 0$  and  $h < 0$ . Integrating by parts we get:

$J(R) = \frac{2H}{h} \exp(h\sqrt{R}) \left\{ \sqrt{R} - \frac{1}{h} \right\}$ . Plugging this function into (7) we find:

$$h = -\sqrt{\gamma(2\delta + \lambda^2)} \quad \text{and} \quad H = \exp[\lambda^2 / (2\delta + \lambda^2)].$$

Then we have the following

*Proposition 2.* If  $m = 0$ , then the optimal value function satisfying (7) is of the form

$$J(R) = \frac{2H}{h} \exp(h\sqrt{R}) \left\{ \sqrt{R} - \frac{1}{h} \right\}$$

with  $h = -\sqrt{\gamma(2\delta + \lambda^2)}$ ,  $\lambda = \mu_i / \sigma_i$  and  $H > 0$ . Finally, for the optimal portfolio we have:

$$x = \frac{2\lambda}{\sqrt{R}(\sigma_1 - \sigma_2)} \frac{1}{\sqrt{\gamma(2\delta + \lambda^2)}} + \frac{\sigma_2}{\sigma_2 - \sigma_1}.$$

Note that the optimal consumption  $c$  is positive whenever  $R > \frac{\lambda^2}{(\lambda^2 + 2\delta)\gamma}$ , that is, unless  $R$  is very small.

To our knowledge, this result is not found in the literature on optimal consumption and investment strategies.

*Remark 2.* In the case of one risky asset and a riskless asset with return  $r = 0$  Merton's solution does not work. However, following the suggestion of Proposition 2, a function on the form

$$J(R) = \frac{2H}{h} \exp(h\sqrt{R}) \left\{ \sqrt{R} - \frac{1}{h} \right\}$$

solves the ODE and thus the optimal share of the risky asset is:

$$x = \frac{2\mu_1}{\sqrt{\gamma R} \sigma_1^2 \sqrt{2\delta + (\mu_1 / \sigma_1)^2}}.$$

Note that Remark 2 complements the analysis of Merton (1971). A zero risk-free interest rate is the most common evidence in real-world markets. Therefore, our solution is of special interest in recent times, while it was of merely theoretical significance when Merton's article was published.

#### 4. THE GENERAL CASE

In the generic case, the nonlinear HJB equation is of the form:

$$(8) \quad \frac{J'}{\gamma} + \delta J + \frac{1}{2} \tilde{\mu}^2 \frac{(J')^2}{J''} - \frac{J' \ln J'}{\gamma} - mR J' - \frac{bR^2 J''}{2} = 0$$

with  $b > 0$ .

To our knowledge, a closed-form solution for this equation is not found in the literature. Even in the special case  $\delta = 0$  a non trivial solution is hard to find in an elementary form. In what follows we show how to reduce it to a 'named' class of equations. Let us look for a solution such that  $J' = e^{-V}$ . Then (8) with  $\delta = 0$  is transformed into the first order equation:

$$\frac{1+V}{\gamma} - \frac{1}{2} \tilde{\mu}^2 \frac{1}{V'} - mR + \frac{bR^2}{2} V' = 0$$

which can be solved for  $V'$ . Note that, being  $J$  a concave function,  $V'$  should be positive and thus we have:

$$V' = [mR - \frac{1+V}{\gamma} + \sqrt{(mR - \frac{1+V}{\gamma})^2 + b\tilde{\mu}^2 R^2}] / (bR^2).$$

Changing to variables

$$x = \ln R \text{ and } 1 + V(R) - m\gamma R = R \cdot Y(\ln R)$$

the equation becomes

$$[Y' + Y + m\gamma][Y + \sqrt{Y^2 + \mu_0}] = \gamma \tilde{\mu}^2 e^{-x}$$

with  $\mu_0 = b(\gamma \tilde{\mu})^2$ . We get a simpler expression by taking  $Z = \mu_0 + 2Y^2 + 2Y\sqrt{Y^2 + \mu_0}$ . In particular, for  $m = 0$  we obtain an Abel equation of the second kind of the form:

$$(9) \quad Z'[Z + \mu_0] + 2Z^2 - 2Z[\mu_0 + 2\gamma\tilde{\mu}^2 e^{-x}] = 0$$

This equation is not included in the list of Polyanin and Zaitsev (2003), p.10, and Cheb-Terrab and Roche (2003), providing non trivial solutions to Abel equations.

## 5. CONCLUSION

This paper revisits a long-standing problem of optimal portfolio-consumption allocation with an exponential utility. While the original problem with a risk-free asset is almost trivial, the general case is still unsolved in explicit form. Our contribution is twofold: we provide an interpretation for the coefficients of the related differential equation in terms of the static optimization problem and we find an explicit solution in a few special cases which occur in several financial strategies. Moreover, we find a solution to the case of zero risk-free interest rate that was left out in Merton's setting. At the time of publication of Merton's seminal paper only positive interest rates were meaningful, while nowadays strictly positive interest rates are achieved only for long maturities in real-world markets.

## REFERENCES

- [1] I. Bardhan, Consumption and investment under constraints, *J. Econ. Dyn. Control*, 18 (1994), 909-929.
- [2] S. Basak, On the fluctuations in consumption and market returns in the presence of labor and human capital: an equilibrium analysis, *J. Econ. Dyn. Control*, 23 (1999), 1029-1064.
- [3] E. S. Cheb-Terrab, A. D. Roche, An Abel ODE class generalizing known integrable classes, *Eur. J. Appl. Math.* 14 (2003), 217-229.
- [4] J. C. Cox, C. F. Huang, Optimum consumption and portfolio policies when asset prices follow a diffusion process, *J. Econ. Theory*, 49 (1989), 33-83.
- [5] R. C. Merton, Lifetime portfolio selection under uncertainty: the continuous-time case, *Rev. Econ. Stat.* 51 (1969), 247-257.
- [6] R. C. Merton, Optimum consumption and portfolio rules in a continuous-time model, *J. Econ. Theory*, 3 (1971), 373-413.
- [7] A. D. Polyanin, V. F. Zaitsev, *Handbook of Exact Solutions for Ordinary Differential Equations* (second edition), Chapman & Hall /CRC, USA, 2003