Young–Capelli bitableaux, Capelli immanants in U(gl(n)) and the Okounkov quantum immanants

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Young-Capelli bitableaux,
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and
the Okounkov quantum immanants

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Abstract

The set of standard Capelli bitableaux and the set of standard Young-Capelli bitableaux are bases of $\mathbf{U}(gl(n))$, whose action on the Gordan-Capelli basis of polynomial algebra $\mathbb{C}[M_{n,n}]$ have remarkable properties (see, e.g. [5], [6], [7], [8]).

We introduce a new class of elements of $\mathbf{U}(gl(n))$, called *Capelli immanants*, that can be efficiently computed and provide a system of linear generators of $\mathbf{U}(gl(n))$. The Okounkov quantum immanants [40], [41] - *quantum immanants*, for short - are proved to be simple linear combinations of *diagonal* Capelli immanants, with explicit coefficients (Theorem 6.2, Eq. (33)). Quantum immanants can also be expressed as sums of *double Young-Capelli bitableaux* (Theorem 6.9, Eq. (40)). Since double Young-Capelli bitableaux uniquely expand into linear combinations of *standard* Young-Capelli bitableaux, Eq. (40) leads to *canonical* presentations of quantum immanants, and, furthermore, it doesn't involve the computation of the irreducible characters of symmetric groups.

Keyword: Young-Capelli bitableaux; Lie superalgebras; immanants; Capelli determinants; Capelli immanants; quantum immanants; central elements; combinatorial representation theory.

Contents

1 Introduction

The study of central elements in $\mathbf{U}(gl(n))$ is a classical subject of the theory of Lie algebras, see e.g. [21]; it is an old and actual one, since it may be regarded as an offspring of the celebrated Capelli identity ([12], [16], [26], [27], [46], [50], [53]), relates to its modern generalizations and applications ([1], [30], [31], [37], [38], [40], [41], [49]) as well as to the theory of *Yangians* (see, e.g. [35], [36], [39]).

The center $\boldsymbol{\zeta}(n)$ of $\mathbf{U}(gl(n))$ is isomorphic to the algebra $\Lambda^*(n)$ of shifted symmetric polynomials (e.g., *factorial symmetric functions*, [2], [17], [23]) via the Harish-Chandra isomorphism χ_n (see, e.g. [42]). The algebra $\Lambda^*(n)$ admits a quite relevant linear basis, the basis of the *shifted Schur polynomials* $s*_{\mu|n}$, $\widetilde{\mu}_1 \leq n$, discovered by Sahi [47], and extensively studied by Okounkov and Olshanski [42]. *Quantum immanants* are the preimages in $\boldsymbol{\zeta}(n)$ of the shifted Schur polynomials in $\Lambda^*(n)$ [40], [41] (see also [39]).

We define two set of linear generators in the enveloping algebra $\mathbf{U}(gl(n))$: the set of Young-Capelli bitableaux and the set of Capelli immanants. These two sets are obtained as the images of the corresponding set of generators in the polynomial algebra $\mathbb{C}[M_{n,n}]$, under the so-called *bitableaux correspondence isomomorphism*, which is an earlier result of the present authors [7], [8].

The action of the Capelli immanants on $\mathbb{C}[M_{n,n}]$ can be computed explicitly by using the method of *virtual variables* (Proposition 4.10 below). Using this computation, we are able to express Okounkov's quantum immanants as linear combinations of *diagonal* Capelli immanants with explicit coefficients.

Our method heavily relies upon the "Bitableax correspondence isomorphism/ Koszul map" Theorem (BCK Theorem, for short) [8] that describes a pair of mutually inverse vector space isomorphisms, the *Koszul map* ([32], see also [4] and [11])

$$\mathcal{K} : \mathbf{U}(gl(n)) \to \mathbb{C}[x_{ij}] \cong \mathbf{Sym}(gl(n)),$$

and the *bitableaux correspondence isomorphism* ([7], [8])

$$\mathcal{K}^{-1} : \mathbb{C}[x_{ij}] \cong \mathbf{Sym}(gl(n)) \to \mathbf{U}(gl(n)),$$

that deeply link the enveloping algebra $\mathbf{U}(gl(n))$ of the general linear Lie algebra $gl(n)$ and the polynomial algebra $\mathbb{C}[M_{n,n}]$ of polynomials in the entries of a "generic" square matrix of order n. The BCK Theorem has to be regarded as a sharpened version of the PBW Theorem for the enveloping algebra $\mathbf{U}(gl(n))$.

The isomorphism $\mathcal{K}^{-1}$ maps a *(determinantal) bitableau* $(S|T)$ in $\mathbb{C}[M_{n,n}]$ to the *Capelli bitableau* $[S|T]$ in $\mathbf{U}(gl(n))$ ([7], [8], [3]; see Section 3.3 below and Theorem 4.1). Since the *standard* bitableaux are a basis of $\mathbb{C}[M_{n,n}]$ ([22], [20], [19], [25]; see subsection 2.2.5 below, Theorem 2.1), then the *standard* Capelli bitableaux are a basis of $\mathbf{U}(gl(n))$ [7].

In the polynomial algebra $\mathbb{C}[M_{n,n}]$, *column bitableaux* are, up to a sign, *monomials*. Their images in $\mathbf{U}(gl(n))$ - under the isomorphism $\mathcal{K}^{-1}$ - are the *column Capelli bitableaux* (Section 4.3 below).

Therefore, column Capelli bitableaux play the same crucial role in $\mathbf{U}(gl(n))$ that monomials play in $\mathbb{C}[M_{n,n}]$. Capelli bitableaux and Young-Capelli bitableaux expand - up to a global sign - into column Capelli bitableaux just in the same way as bitableaux, right symmetrized bitableaux and immanants expand into the corresponding monomials in $\mathbb{C}[M_{n,n}]$.

The expressions of column Capelli bitableaux in $\mathbf{U}(gl(n))$ can be simply computed (Proposition 4.3.1 below). Furthermore, column Capelli bitableaux admit an elegant and meaningful interpretation as polynomial differential operators in the *Weyl algebra* associated to the polynomial algebra $\mathbb{C}[M_{n,d}]$ (Proposition 4.10 below).

The isomorphism $\mathcal{K}^{-1}$ leads to a natural definition of the *Capelli immanants*

$$Cimm_\lambda[i_1 i_2 \cdots i_h; j_1 j_2 \cdots j_h], \quad \lambda \vdash h$$

in $\mathbf{U}(gl(n))$ as images under $\mathcal{K}^{-1}$ of the classical *immanants*

$$imm_\lambda(i_1 i_2 \cdots i_h; j_1 j_2 \cdots j_h), \quad (i_1 i_2 \cdots i_h), (j_1 j_2 \cdots j_h) \in \underline{n}^h$$

in the polynomial algebra $\mathbb{C}[M_{n,n}]$ (Littlewood and Richardson [33], see also [34], [24]). Capelli immanants are generalizations of the famous Capelli determinant in $\mathbf{U}(gl(n))$, just as immanants are generalizations of the determinant in $\mathbb{C}[M_{n,n}]$.

The isomorphism $\mathcal{K}^{-1}$ maps a *right symmetrized bitableau* $(S|\boxed{T})$ in $\mathbb{C}[M_{n,n}]$ to the *Young-Capelli bitableau* $[S|\boxed{T}]$ in $\mathbf{U}(gl(n))$ (Section 3.3 below, and Theorem 4.2). Since the standard right symmetrized bitableaux $(S|\boxed{T})$ are the *Gordan-Capelli basis* of $\mathbb{C}[M_{n,n}]$ ([52], [5], [3]; see Subsection 2.3 below, Theorem 2.3), then the standard Young-Capelli bitableaux $[S|\boxed{T}]$ are a basis of $\mathbf{U}(gl(n))$.

Right symmetrized bitableaux $(S|\boxed{T})$ of shape $\lambda \vdash h$ expand into *immanants* defined by the irreducible character χ^λ of the symmetric group $\mathbf{S}_h$ associated to the *same* shape $\lambda \vdash h$, and *viceversa* (Propositions 2.15 and 2.12). Then, by applying the operator $\mathcal{K}^{-1}$, we obtain that any Capelli immanant

$$Cimm_\lambda[i_1 i_2 \cdots i_h; j_1 j_2 \cdots j_h]$$

can be written as a linear combination of standard Young-Capelli bitableaux $[U|\boxed{V}]$ in $\mathbf{U}(gl(n))$ of the **same** shape λ and *viceversa* (Theorems 5.1 and 5.3 below).

Quantum immanants ([40], [41]) are proved to be simple linear combinations of *diagonal* Capelli immanants with explicit coefficients (Theorem 6.2, Eq. (33)). This Theorem, in combination with Proposition 4.3.1, allows the computation of quantum immanants to be reduced to a fairly simple process (see, e.g. Example 38 below).

Quantum immanants can also be expressed as sums of *double Young-Capelli bitableaux* (Theorem 6.9, Eq. (40)). Since double Young-Capelli bitableaux uniquely expand into linear combinations of *standard* Young-Capelli bitableaux, Eq. (40) leads to *canonical* presentations of quantum immanants, and it doesn't involve the irreducible characters of symmetric groups. Furthermore, Eq. (40) is better suited to the study of the eigenvalues on irreducible $gl(n)-$modules, and of the duality in the algebra $\boldsymbol{\zeta}(n)$ (see our preliminary manuscript [10], Section 4).

2 The polynomial algebra $\mathbb{C}[M_{n,d}]$

2.1 Biproducts in $\mathbb{C}[M_{n,d}]$

As usual, the *algebra of algebraic forms in n vector variables of dimension d* is the polynomial algebra in $n \times d$ (commutative) variables:

$$\mathbb{C}[M_{n,d}] = \mathbb{C}[x_{ij}]_{i=1,\ldots,n;j=1,\ldots,d},$$

and $M_{n,d}$ denotes the matrix with n rows and d columns with "generic" entries x_{ij}:

$$M_{n,d} = [x_{ij}]_{i=1,\ldots,n;j=1,\ldots,d} = \begin{bmatrix} x_{11} & \ldots & x_{1d} \\ x_{21} & \ldots & x_{2d} \\ \vdots & & \vdots \\ x_{n1} & \ldots & x_{nd} \end{bmatrix}.$$

For the sake of readability, we will write $(i|j)$ in place of x_{ij}, and call the alphabets $L = \{1, 2, \ldots, n\}$ and $P = \{1, 2, \ldots, d\}$ the *letter* and the *place* alphabets, respectively; sometimes, we will consistently write $\mathbb{C}[(i|j)]_{i=1,2,\ldots,n;\ j=1,2,\ldots,d}$ in place of $\mathbb{C}[M_{n,d}]$.

Let $\omega = i_1 i_2 \cdots i_p$ be a word on the alphabet $L = \{1, 2, \ldots, n\}$, and $\varpi = j_1 j_1 \cdots j_q$ a word on the alphabet $P = \{1, 2, \ldots, d\}$.

Following [25] and [5], the *biproduct* of the two words ω and ϖ

$$(\omega|\varpi) = (i_1 i_2 \cdots i_p | j_1 j_2 \cdots j_q) \tag{1}$$

is the element of $\mathbb{C}[M_{n,d}]$ defined in the following way:

– If $p = q$, the biproduct $(\omega|\varpi)$ is the *signed minor*

$$(\omega|\varpi) = (-1)^{\binom{p}{2}}\ det\Big(\ (i_r|j_s)\ \Big)_{r,s=1,2,\ldots,p} \in \mathbb{C}[M_{n,d}].$$

– If $p \neq q$, the biproduct $(\omega|\varpi)$ is set to be zero.

2.2 Bitableaux in $\mathbb{C}[M_{n,d}]$

2.2.1 Young tableaux

Let $\lambda \vdash h$ be a partition, and label the boxes of its Ferrers diagram with the numbers $1, 2, \ldots, h$ in the following way:

$$\begin{array}{lllll} 1 & 2 & \cdots & \cdots & \lambda_1 \\ \lambda_1+1 & \lambda_1+2 & \cdots & \lambda_1+\lambda_2 & \\ \cdots & \cdots & \cdots & & \\ \cdots & \cdots & h & & \end{array}.$$

A *Young tableau* T of shape λ over a (finite) alphabet $\mathcal{A}$ is a map $T : \underline{h} = \{1, 2, \ldots, h\} \to \mathcal{A}$; the element $T(i)$ is the symbol in the cell i of the tableau T.

The sequences

$$\begin{array}{l} T(1)T(2)\cdots T(\lambda_1), \\ T(\lambda_1+1)T(\lambda_1+2)\cdots T(\lambda_1+\lambda_2), \\ \cdots\cdots \end{array}$$

are called the *row words* of the Young tableau T.

We will also denote a Young tableau by its sequence of rows words, that is $T = (\omega_1, \omega_2, \ldots, \omega_p)$. Furthermore, the *word of the tableau* T is the concatenation

$$w(T) = \omega_1\omega_2\cdots\omega_p. \tag{2}$$

The *content* of a tableau T is the function $c_T : \mathcal{A} \to \mathbb{N}$,

$$c_T(a) = \sharp\{i \in \underline{h};\ T(i) = a\}.$$

A Young tableau $\mathbf{T}$ is said to be *multilinear* if $\mathcal{A} = \underline{h}$ and the map $\mathbf{T}$ is a permutation of $\underline{h}$. As usual, $\mathbf{T}^{-1}$ denotes the inverse map. In the sequel, multilinear Young tableaux will be always denoted by bold symbols, and $\mathbf{T}_0$ will denote the "identity" tableau $\mathbf{T}_0(i) = i$, $i = 1, 2, \ldots, h$.

Note that, given any Young tableau S on an alphabet $\mathcal{A}$ and any multilinear Young tableau $\mathbf{T}$ on the alphabet $\underline{h}$ of the same shape $\lambda \vdash h$, there exists a unique (specialization) map $J : \underline{h} \to \mathcal{A}$ such that

$$S = J \circ \mathbf{T},$$

that is $S(i) = (J \circ \mathbf{T})(i),\ i = 1, 2, \ldots, h$.

To stress this relation between S and $\mathbf{T}$, we write

$$S = J_{\mathbf{T}}. \tag{3}$$

Given a linear order on the alphabet $\mathcal{A}$, a Young tableau over $\mathcal{A}$ is said to be *(semi)standard* whenever its rows are increasing from left to right and its columns are non-decreasing from top to bottom.

2.2.2 (determinantal) Young bitableaux

Let $S = (\omega_1, \omega_2, \ldots, \omega_p)$ and $T = (\varpi_1, \varpi_2, \ldots, \varpi_p)$ be Young tableaux on $L = \{x_1, x_2, \ldots, x_n\}$ and $P = \{1, 2, \ldots, d\}$ of shapes λ and μ, respectively.

Following again [25] and [5], the (determinantal) *Young bitableau*

$$(S|T) = \left(\begin{array}{c|c} \omega_1 & \varpi_1 \\ \omega_2 & \varpi_2 \\ \vdots & \vdots \\ \omega_p & \varpi_p \end{array} \right) \tag{4}$$

is the element of $\mathbb{C}[M_{n,d}]$ defined in the following way:

- If $\lambda = \mu$, the (determinantal) *Young bitableau* $(S|T)$ is the *signed* product of the biproducts of pairs of corresponding rows:

$$(S|T) = \pm\ (\omega_1|\varpi_1)(\omega_2|\varpi_2) \cdots (\omega_p|\varpi_p), \tag{5}$$

  where

$$\pm = (-1)^{\ell(\omega_2)\ell(\varpi_1)+\ell(\omega_3)(\ell(\varpi_1)+\ell(\varpi_2))+\cdots+\ell(\omega_p)(\ell(\varpi_1)+\ell(\varpi_2)+\cdots+\ell(\varpi_{p-1}))}, \tag{6}$$

  and the symbol $\ell(w)$ denotes the length of the word w.

- If $\lambda \neq \mu$, the *Young bitableau* $(S|T)$ is set to be zero.

2.2.3 Column bitableaux in $\mathbb{C}[M_{n,d}]$

A *column* tableau is a Young tableau of shape $\lambda = (1, 1, \ldots, 1) \vdash h$, and the number h of $1'$s is called the *depth*.

A column bitableau in $\mathbb{C}[M_{n,d}]$ is a (determinantal) bitableau $(S|T)$, where S and T are column Young tableaux of the same depth. A column bitableau of depth h equals, up to a sign, a *monomial* in $\mathbb{C}[M_{n,d}]$:

$$\left(\begin{array}{c|c} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array} \right) = (-1)^{\binom{h}{2}} (i_1|j_1)(i_2|j_2) \cdots (i_h|j_h). \tag{7}$$

Although the notion of column bitableaux may appear fairly obvious, it will play a crucial role in the passage from the polynomial algebra $\mathbb{C}[M_{n,d}]$ to the enveloping algebra $\mathbf{U}(gl(n))$ via the *bitableaux correspondence* isomorphism, Section 4 below.

2.2.4 Bitableaux expansion into column bitableaux

Recall that

$$(i_1 i_2 \cdots i_h | j_1 j_2 \cdots j_h) = (-1)^{\binom{h}{2}}\ det[(i_s|j_t)]_{s,t=1,2,\ldots,h} \in \mathbb{C}[M_{n,d}],$$

and, therefore, the biproduct $(i_1 i_2 \cdots i_h | j_1 j_2 \cdots j_h) \in \mathbb{C}[M_{n,d}]$ expands into column bitableaux as follows:

$$(i_1 i_2 \cdots i_h | j_1 j_2 \cdots j_h) = \sum_{\sigma \in \mathbf{S}_h} (-1)^{|\sigma|} \left(\begin{array}{c|c} i_{\sigma(1)} & j_1 \\ i_{\sigma(2)} & j_2 \\ \vdots & \vdots \\ i_{\sigma(h)} & j_h \end{array} \right) = \sum_{\sigma \in \mathbf{S}_h} (-1)^{|\sigma|} \left(\begin{array}{c|c} i_1 & j_{\sigma(1)} \\ i_2 & j_{\sigma(2)} \\ \vdots & \vdots \\ i_h & j_{\sigma(h)} \end{array} \right).$$

Notice that, in the passage from monomials to column bitableaux, the sign $(-1)^{\binom{h}{2}}$ disappears, due to Eq. (7).

The preceding arguments extend to bitableaux of any shape λ, $\lambda_1 \leq n$. Given a bitableau $(S|T) \in \mathbb{C}[M_{n,d}]$ of shape $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m) \vdash h$ with

$$S = \begin{pmatrix} i_{p_1} \ldots\ldots\ldots i_{p_{\lambda_1}} \\ i_{q_1} \ldots\ldots i_{q_{\lambda_2}} \\ \ldots\ldots \\ i_{r_1} \ldots i_{r_{\lambda_m}} \end{pmatrix}, \quad T = \begin{pmatrix} j_{s_1} \ldots\ldots\ldots j_{s_{\lambda_1}} \\ j_{t_1} \ldots\ldots j_{t_{\lambda_2}} \\ \ldots\ldots \\ j_{v_1} \ldots j_{v_{\lambda_m}} \end{pmatrix},$$

we have

$$(S|T) = \sum_{\sigma_1, \ldots, \sigma_m} (-1)^{\sum_{k=1}^{m} |\sigma_k|} \left(\begin{array}{c|c} i_{p_{\sigma_1(1)}} & j_{s_1} \\ \vdots & \vdots \\ i_{p_{\sigma_1(\lambda_1)}} & j_{s_{\lambda_1}} \\ \vdots & \vdots \\ \vdots & \vdots \\ i_{r_{\sigma_m(1)}} & j_{v_1} \\ \vdots & \vdots \\ i_{r_{\sigma_m(\lambda_m)}} & j_{v_{\lambda_m}} \end{array} \right)$$

$$= \sum_{\sigma_1, \ldots, \sigma_m} (-1)^{\sum_{k=1}^{m} |\sigma_k|} \left(\begin{array}{c|c} i_{p_1} & j_{s_{\sigma_1(1)}} \\ \vdots & \vdots \\ i_{p_{\lambda_1}} & j_{s_{\sigma_1(\lambda_1)}} \\ \vdots & \vdots \\ \vdots & \vdots \\ i_{r_1} & j_{v_{\sigma_m(1)}} \\ \vdots & \vdots \\ i_{r_{\lambda_m}} & j_{v_{\sigma_m(\lambda_m)}} \end{array} \right)$$

where the multiple sums range over all permutations $\sigma_1 \in \mathbf{S}_{\lambda_1}, \ldots, \sigma_m \in \mathbf{S}_{\lambda_m}$.

Notice that only the signs of permutations remain.

2.2.5 The straightening algorithm and the standard basis of $\mathbb{C}[M_{n,d}]$

,

Given a positive integer $h \in \mathbb{Z}^+$, let $\mathbb{C}_h[M_{n,d}]$ denote the $h-$th homogeneous component of $\mathbb{C}[M_{n,d}]$.

Consider the set of all bitableaux $(S|T) \in \mathbb{C}_h[M_{n,d}]$, where $sh(S) = sh(T) \vdash h$. In the following, let denote by $\leq$ the linear order on this set defined by the following two steps:

- $(S|T) < (S'|T')$ whenever $sh(S) <_l sh(S')$, where $<_l$ denotes the lexicographic order on partitions $\lambda \vdash h$.
- $(S|T) < (S'|T')$ whenever $sh(S) = sh(S')$, $w(S)w(T) >_l w(S')w(T')$.

where the shapes and the concatenated words $w(S)w(T), w(S')w(T')$ of the tableaux S, T and S', T' (see Eq. (2)) are compared in the lexicographic order.

The next Theorem is a well-known result for the polynomial algebra $\mathbb{C}[M_{n,d}]$ ([22], [20], [19], for the general theory of standard monomials see, e.g. [46], Chapt. 13).

Theorem 2.1. (The Standard basis theorem for $\mathbb{C}_h[M_{n,d}]$)

- *The set*

$$\{(S|T) \ standard; \ sh(S) = sh(T) = \lambda \vdash h, \lambda_1 \leq n, d \ \}.$$

is a basis of $\mathbb{C}_h[M_{n,d}]$.

- *Furthermore, a Young bitableau* $(P|Q) \in \mathbb{C}_h[M_{n,d}]$ *can be uniquely written as a linear combination*

$$(P|Q) = \sum_{S,T} a_{S,T} \ (S|T), \tag{8}$$

of standard bitableaux $(S|T)$*, where*

  - *the coefficient* $a_{S,T} = 0$ *whenever* $(S|T) \ngeq (P|Q)$*;*
  - *the contents of the tableaux are preserved, that is* $c_S = c_P$*,* $c_T = c_Q$*.*

For a proof, see e.g. [20], [19].

2.3 Right symmetrized bitableaux and the Gordan-Capelli basis of $\mathbb{C}[M_{n,d}]$

Given a Young tableau T, we say that another tableau $\overline{T}$ is a *column permuted* of T whenever each column of $\overline{T}$ can obtained by permuting the corresponding column of T.

A *right symmetrized bitableau* $(S|\boxed{T})$ is the element of the polynomial algebra $\mathbb{C}[M_{n,d}]$ defined as the following sum of bitableaux:

$$(S|\boxed{T}) = \sum_{\overline{T}} (S|\overline{T}),$$

where the sum is extended over *all* $\overline{T}$ column permuted of T (hence, repeated entries in a column give rise to multiplicities).

Example 2.2.

$$
\begin{aligned}
\left(\begin{array}{cc} 1 & 3 \\ 2 & 4 \end{array} \middle\| \boxed{\begin{array}{cc} 1 & 2 \\ 1 & 3 \end{array}}\right) &= \left(\begin{array}{cc|cc} 1 & 3 & 1 & 2 \\ 2 & 4 & 1 & 3 \end{array}\right) + \left(\begin{array}{cc|cc} 1 & 3 & 1 & 2 \\ 2 & 4 & 1 & 3 \end{array}\right) \\
&+ \left(\begin{array}{cc|cc} 1 & 3 & 1 & 3 \\ 2 & 4 & 1 & 2 \end{array}\right) + \left(\begin{array}{cc|cc} 1 & 3 & 1 & 3 \\ 2 & 4 & 1 & 2 \end{array}\right) \\
&= 2\left(\begin{array}{cc|cc} 1 & 3 & 1 & 2 \\ 2 & 4 & 1 & 3 \end{array}\right) + 2\left(\begin{array}{cc|cc} 1 & 3 & 1 & 3 \\ 2 & 4 & 1 & 2 \end{array}\right).
\end{aligned}
$$

□

We recall a fundamental result:

Theorem 2.3. (The Gordan-Capelli basis of $\mathbb{C}[M_{n,d}]$) *Let $h \in \mathbb{N}$.*

- *The set*

$$\{(S\|\boxed{T}); S, T \ \textit{standard}, \ sh(S) = sh(T) = \lambda \vdash h, \ \lambda_1 \leq n, d\}$$

  is a basis of $\mathbb{C}_h[M_{n,d}]$.

- *Any right symmetrized bitableau $(U\|\boxed{V})$, $sh(U) = sh(V) = \lambda \vdash h$, (uniquely) expands into a linear combination of right symmetrized bitableau $(S\|\boxed{T})$, S, T standard of the* ***same*** *shape $\lambda = sh(S) = sh(T)$.*

- *Let $(U\|\boxed{V})$, $sh(U) = sh(V) = \lambda \vdash h$, $\lambda_1 \not\leq n, d$. Then*

$$(U\|\boxed{V}) = 0.$$

Corollary 2.4. *The subspace $\mathbb{C}_h[M_{n,d}]$ decomposes as:*

$$\mathbb{C}_h[M_{n,d}] = \bigoplus_{\lambda \vdash h} \mathbb{C}_h^{\lambda}[M_{n,d}], \quad \lambda_1 \leq n, d, \tag{9}$$

where $\mathbb{C}_h^{\lambda}[M_{n,d}]$ is the subspace spanned by the right symmetrized bitableaux $(U\|\boxed{V})$ of shape $\lambda = sh(U) = sh(V)$.

Theorem 2.3 was proved, in a different language, by Wallace [52] in the classical commutative case. A superalgebraic version of this result was proved by the present authors in [5]; for a more detailed discussion, see [3].

2.4 Right symmetrized bitableaux in $\mathbb{C}_h[M_{n,d}]$, Young symmetrizers and the natural units in the group algebra $\mathbb{C}[\mathbf{S}_h]$

In this subsection, we summarize some basic notions from the representation theory of the symmetric group; furthermore, we provide useful descriptions of *right symmetrized determinantal bitableaux* in terms of *Young symmetrizers* and of the *natural units* in the group algebra $\mathbb{C}[\mathbf{S}_h]$.

2.4.1 The symmetric group $\mathbf{S}_h$

Our main reference here is the treatise of James and Kerber [28], Chapter 3, with the proviso that here the *role of rows and columns of a Young tableau are interchanged*.

Given a pair $\mathbf{S}, \mathbf{T}$ of multilinear tableaux of the same shape $sh(\mathbf{S}) = sh(\mathbf{T}) = \lambda \vdash h$, the *Young symmetrizer* $\mathbf{e}^\lambda_{\mathbf{ST}} \in \mathbb{C}[\mathbf{S}_h]$ is the element:

$$\mathbf{e}^\lambda_{\mathbf{ST}} = \sum_{\sigma \in R(\mathbf{S}), \tau \in C(\mathbf{T})} (-1)^{|\sigma|} \, \sigma \, \theta_{\mathbf{ST}} \, \tau, \tag{10}$$

where $\theta_{\mathbf{ST}}$ is the permutation of $\underline{h}$ such that $\theta_{\mathbf{ST}}(i) = (\mathbf{S} \circ \mathbf{T}^{-1})(i) = \mathbf{S}\big(\mathbf{T}^{-1}(i)\big)$ for every $i = 1, 2, \ldots, h$, and $R(\mathbf{S}),\ C(\mathbf{T}) \subseteq \mathbf{S}_h$ are the *row subgroup* of $\mathbf{S}$ and the *column subgroup* of $\mathbf{T}$, respectively.

Clearly,

$$\mathbf{e}^\lambda_{\mathbf{ST}} = \theta_{\mathbf{ST}} \, \mathbf{e}^\lambda_{\mathbf{TT}} = \mathbf{e}^\lambda_{\mathbf{SS}} \, \theta_{\mathbf{ST}}.$$

Remark 2.5. *By* Eq. (10)*, the* trivial representation *is* here *associated to the* column *shape* $\lambda = (1^h)$ *and the* sign representation *is* here *associated to the* row *shape* $\lambda = (h)$. □

We will denote by $\gamma^\lambda_{\mathbf{ST}}$ the *natural units* of the group algebra $\mathbb{C}[\mathbf{S}_h]$, $\lambda \vdash h$, $\mathbf{S}, \mathbf{T}$ multilinear standard tableaux of the same shape $sh(\mathbf{S}) = sh(\mathbf{T}) = \lambda \vdash h$.

Given $\lambda \vdash h$, recall that

$$\mathbb{C}[\mathbf{S}_h] = \bigoplus_{\lambda \vdash h} \mathbb{C}^\lambda[\mathbf{S}_h],$$

where $\mathbb{C}^\lambda[\mathbf{S}_h]$ denotes the *isotypic* (simple) component of $\mathbb{C}[\mathbf{S}_h]$ associated to λ.

Proposition 2.6.

1) The set

$$\Big\{\mathbf{e}^\lambda_{\mathbf{ST}}; \mathbf{S}, \mathbf{T} \text{ multilinear standard tableaux}, sh(\mathbf{S}) = sh(\mathbf{T}) = \lambda \vdash h\Big\}$$

is a basis of $\mathbb{C}^\lambda[\mathbf{S}_h]$.

2) The set

$$\Big\{\gamma^\lambda_{\mathbf{ST}}; \mathbf{S}, \mathbf{T} \text{ multilinear standard tableaux}, sh(\mathbf{S}) = sh(\mathbf{T}) = \lambda \vdash h\Big\}$$

is a basis of $\mathbb{C}^\lambda[\mathbf{S}_h]$.

3) Let $\mathbf{S}, \mathbf{S}', \mathbf{T}, \mathbf{T}'$ *be multilinear standard tableaux of shape* $\lambda \vdash h$*, then*

$$\gamma^\lambda_{\mathbf{ST}} \, \gamma^\lambda_{\mathbf{S}'\mathbf{T}'} = \delta_{\mathbf{T},\mathbf{S}'} \, \gamma^\lambda_{\mathbf{ST}'},$$

$$\gamma^\lambda_{\mathbf{ST}} \, \mathbf{e}^\lambda_{\mathbf{S}'\mathbf{T}'} = \delta_{\mathbf{T},\mathbf{S}'} \, \mathbf{e}^\lambda_{\mathbf{ST}'}.$$

Let $\lambda \vdash h$ be a partition and denote by χ^λ the *irreducible character* associated to the irreducible representation of shape λ of the symmetric group $\mathbf{S}_h$. Let

$$\overline{\chi}_\lambda = \sum_{\sigma \in \mathbf{S}_h} \chi^\lambda(\sigma)\sigma \in \mathbb{C}[\mathbf{S}_h]. \tag{11}$$

Proposition 2.7.

1) *The elements*

$$\frac{\chi^\lambda(I)}{n!}\,\overline{\chi}_\lambda, \quad \lambda \vdash h$$

are the primitive central idempotents *of* $\mathbb{C}[\mathbf{S}_h]$.

2) *We have:*

$$\frac{\chi^\lambda(I)}{n!}\,\overline{\chi}_\lambda = \sum_{\mathbf{T}} \gamma^\lambda_{\mathbf{T}\mathbf{T}},$$

where the sum ranges over all multilinear $\mathbf{T}$ *standard tableaux on* $\underline{h}$ *of shape* λ*, and*

$$\frac{\chi^\lambda(I)}{n!} = \frac{1}{H(\lambda)},$$

$H(\lambda)$ *the hook number of the partition* λ*.*

3) *The elements*

$$\frac{\chi^\lambda(I)}{n!}\,\overline{\chi}_\lambda, \quad \lambda \vdash h$$

are the projectors *from* $\mathbb{C}[\mathbf{S}_h]$ *to the isotypic (simple) components* $\mathbb{C}^\lambda[\mathbf{S}_h]$.

Proof. Assertion 1) is an instance of a standard fact of the representation theory of finite groups. Assertions 2) and 3) follow from assertion 1) and Proposition 2.6. □

2.4.2 Right symmetrized bitableaux and Young symmetrizers: the multilinear case

We consider the algebra $\mathbb{C}_h[M_{h,h}]$, that is $n = d = h$, the polynomial algebra generated by the variables $(i|j)$, $i, j = 1, 2, \ldots, h$.

We establish the following convention. Given an element

$$\mathbf{p} = \sum_s c_s \sigma_s \in \mathbb{C}[\mathbf{S}_h],$$

and a column tableau

$$\left(\begin{array}{c|c} I(1) & J(1) \\ \vdots & \vdots \\ I(h) & J(h) \end{array} \right),$$

we set

$$\begin{pmatrix} I\big(\mathbf{p}(1)\big) & J(1) \\ \vdots & \vdots \\ I\big(\mathbf{p}(h)\big) & J(h) \end{pmatrix} = \sum_s c_s \begin{pmatrix} I\big(\sigma_s(1)\big) & J(1) \\ \vdots & \vdots \\ I\big(\sigma_s(h)\big) & J(h) \end{pmatrix}. \tag{12}$$

Proposition 2.8. *Let* $\mathbf{S}, \mathbf{T}$ *be multilinear tableaux of the same shape* λ, *then*

$$(\mathbf{S}|\boxed{\mathbf{T}}) = \begin{pmatrix} \mathbf{e}^\lambda_{\mathbf{ST}}(1) & 1 \\ \vdots & \vdots \\ \mathbf{e}^\lambda_{\mathbf{ST}}(h) & h \end{pmatrix}. \tag{13}$$

Proof. Since

$$(\mathbf{S}|\mathbf{T}) = \sum_{\sigma \in R(\mathbf{S})} (-1)^{|\sigma|} \begin{pmatrix} \sigma\ \mathbf{S}(1) & \mathbf{T}(1) \\ \vdots & \vdots \\ \sigma\ \mathbf{S}(h) & \mathbf{T}(h) \end{pmatrix}$$

(see subsection 2.2.4), then

$$\begin{aligned} (\mathbf{S}|\boxed{\mathbf{T}}) &= \sum_{\sigma \in R(\mathbf{S}), \tau \in C(\mathbf{T})} (-1)^{|\sigma|} \begin{pmatrix} \sigma\ \mathbf{S}(1) & \tau\ \mathbf{T}(1) \\ \vdots & \vdots \\ \sigma\ \mathbf{S}(h) & \tau\ \mathbf{T}(h) \end{pmatrix} \\ &= \sum_{\sigma \in R(\mathbf{S}), \tau \in C(\mathbf{T})} (-1)^{|\sigma|} \begin{pmatrix} \sigma\ \theta_{\mathbf{ST}}\ \mathbf{T}(1) & \tau\ \mathbf{T}(1) \\ \vdots & \vdots \\ \sigma\ \theta_{\mathbf{ST}}\ \mathbf{T}(h) & \tau\ \mathbf{T}(h) \end{pmatrix} \\ &= \sum_{\sigma \in R(\mathbf{S}), \tau \in C(\mathbf{T})} (-1)^{|\sigma|} \begin{pmatrix} \sigma\ \theta_{\mathbf{ST}}\ \tau^{-1}\ \mathbf{T}(1) & \mathbf{T}(1) \\ \vdots & \vdots \\ \sigma\ \theta_{\mathbf{ST}}\ \tau^{-1}\ \mathbf{T}(h) & \mathbf{T}(h) \end{pmatrix} \\ &= \sum_{\sigma \in R(\mathbf{S}), \tau \in C(\mathbf{T})} (-1)^{|\sigma|} \begin{pmatrix} \sigma\ \theta_{\mathbf{ST}}\ \tau(1) & 1 \\ \vdots & \vdots \\ \sigma\ \theta_{\mathbf{ST}}\ \tau(h) & h \end{pmatrix}. \end{aligned}$$

Since

$$\mathbf{e}^\lambda_{\mathbf{ST}} = \sum_{\sigma \in R(\mathbf{S}), \tau \in C(\mathbf{T})} (-1)^{|\sigma|}\ \sigma\ \theta_{\mathbf{ST}}\ \tau,$$

then

$$(\mathbf{S}|\boxed{\mathbf{T}}) = \begin{pmatrix} \mathbf{e}^\lambda_{\mathbf{ST}}(1) & 1 \\ \vdots & \vdots \\ \mathbf{e}^\lambda_{\mathbf{ST}}(h) & h \end{pmatrix}.$$

□

2.4.3 Right symmetrized bitableaux and Young symmetrizers: the general case

Let U, V be Young tableaux on the alphabets $\underline{n}, \underline{d}$, and let $\mathbf{S}, \mathbf{T}$ be multilinear tableaux of the same shape $\lambda \vdash h$. There exists a unique pair of maps $I : \underline{h} \to \underline{n}$, $J : \underline{h} \to \underline{d}$, such that

$$U = I_{\mathbf{S}} \quad V = J_{\mathbf{T}}$$

(see Eq. (3)).

Proposition 2.9.

$$(U|\boxed{V}) = (I_{\mathbf{S}}|\boxed{J_{\mathbf{T}}}) = \left(\begin{array}{c|c} I\big(\mathbf{e}^{\lambda}_{\mathbf{ST}}(1)\big) & J(1) \\ \vdots & \vdots \\ I\big(\mathbf{e}^{\lambda}_{\mathbf{ST}}(h)\big) & J(h) \end{array} \right). \tag{14}$$

Proof. From Proposition 2.8, we get:

$$(I_{\mathbf{S}}|\boxed{J_{\mathbf{T}}}) = \sum_{\eta \in R(\mathbf{T}), \tau \in C(\mathbf{T})} (-1)^{|\eta|} \left(\begin{array}{c|c} I\big(\eta\ \theta_{\mathbf{ST}}\ \tau^{-1}\ (1)\big) & J(1) \\ \vdots & \vdots \\ I\big(\eta\ \theta_{\mathbf{ST}}\ \tau^{-1}(h)\big) & J(h) \end{array} \right).$$

□

2.5 Immanants in $\mathbb{C}_h[M_{n,d}]$

The immanant of a matrix was defined by D. E. Littlewood and A. R. Richardson as a generalization of the concepts of determinant and permanent [33] (see also [34], [24]).

Let $\lambda \vdash h$ be a partition and denote by χ^{λ} the *irreducible character* associated to the irreducible representation of shape λ of the symmetric group $\mathbf{S}_h$, and let

$$\overline{\chi}_{\lambda} = \sum_{\sigma \in \mathbf{S}_h} \chi^{\lambda}(\sigma)\sigma \in \mathbb{C}[\mathbf{S}_h].$$

Example 2.10. Let $n = 3$, $\lambda = (2, 1) \vdash 3$.

The irreducible character element of the group algebra $\mathbb{C}[\mathbf{S}_3]$ associated to the partition $\lambda = (2, 1)$ is

$$\overline{\chi}_{\lambda} = \sum_{\sigma \in \mathbf{S}_3} \chi^{\lambda}(\sigma)\sigma = 2I - (123) - (132) \in \mathbb{C}[\mathbf{S}_3].$$

□

The *(generalized) immanant*

$$imm_{\lambda}(i_1 i_2 \cdots i_h; j_1 j_2 \cdots j_h), \quad (i_1 i_2 \cdots i_h) \in \underline{n}^h,\ (j_1 j_2 \cdots j_h) \in \underline{d}^h$$

in the polynomial algebra in $\mathbb{C}_h[M_{n,d}]$ is the element:

$$imm_\lambda(i_1 i_2 \cdots i_h; j_1 j_2 \cdots j_h) = \sum_{\sigma \in \mathbf{S}_h} \chi^\lambda(\sigma) \left(\begin{array}{c|c} i_{\sigma(1)} & j_1 \\ i_{\sigma(2)} & j_2 \\ \vdots & \vdots \\ i_{\sigma(h)} & j_h \end{array} \right)$$

$$= \sum_{\sigma \in \mathbf{S}_h} \chi^\lambda(\sigma) \left(\begin{array}{c|c} i_1 & j_{\sigma(1)} \\ i_2 & j_{\sigma(2)} \\ \vdots & \vdots \\ i_h & j_{\sigma(h)} \end{array} \right).$$

Since the characters are *invariant on the conjugacy classes* of $\mathbf{S}_h$, it follows that

$$imm_\lambda(i_{\tau(1)} i_{\tau(2)} \cdots i_{\tau(h)}; j_{\tau(1)} j_{\tau(2)} \cdots j_{\tau(h)}) = imm_\lambda(i_1 i_2 \cdots i_h; j_1 j_2 \cdots j_h).$$

Hence,

Proposition 2.11. *The map*

$$IMM_\lambda : \left(\begin{array}{c|c} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array} \right) \mapsto imm_\lambda(i_1 i_2 \cdots i_h; j_1 j_2 \cdots j_h)$$

defines a linear map

$$IMM_\lambda : \mathbb{C}_h[M_{n,d}] \to \mathbb{C}_h[M_{n,d}].$$

Clearly, the immanant $imm_\lambda(i_1 i_2 \cdots i_h; j_1 j_2 \cdots j_h) \in \mathbb{C}_h[M_{n,d}]$ is the *natural* generalization of the biproducts (*signed minors*) $(i_1 i_2 \cdots i_h | j_1 j_2 \cdots j_h)$ in $\mathbb{C}[M_{n,d}]$.

It is obvious that the immanants $imm_\lambda(i_1 i_2 \cdots i_h; j_1 j_2 \cdots j_h)$ are homogeneous elements of degree $h \in \mathbb{N}$ of the polynomial algebra $\mathbb{C}[M_{n,d}]$; therefore, by Theorem 2.3, the immanants $imm_\lambda(i_1 i_2 \cdots i_h; j_1 j_2 \cdots j_h)$ expand into linear combination of standard right symmetrized bitableaux of shapes that are partitions of h.

Furthermore, the following stronger result holds.

Proposition 2.12. *Let $\lambda \vdash h$. Any immanant $imm_\lambda(i_1 i_2 \cdots i_h; j_1 j_2 \cdots j_h)$ can be written as a linear combination of standard right symmetrized bitableaux of the* ***same*** *shape λ:*

$$imm_\lambda(i_1 i_2 \cdots i_h; j_1 j_2 \cdots j_h) = \sum_{U,V} \varrho_{U,V} \ (U | \boxed{V}),$$

$$\varrho_{U,V} \in \mathbb{C}, \quad sh(U) = sh(V) = \lambda.$$

Proof. Let $I : \underline{h} \to \underline{n}$, $J : \underline{h} \to \underline{n}$, $I(s) = i_s$, $J(s) = j_s$, $\quad s = 1, 2, \ldots, h$.

Item 4) of Proposition 2.6 implies:

$$imm_\lambda(i_1 i_2 \cdots i_h; j_1 j_2 \cdots j_h) = \sum_{\sigma \in \mathbf{S}_h} \chi^\lambda(\sigma) \left(\begin{array}{c|c} I(\sigma(1)) & J(1) \\ I(\sigma(2)) & J(2) \\ \vdots & \vdots \\ I(\sigma(h)) & J(h) \end{array} \right)$$

$$= \left(\begin{array}{c|c} I(\chi_\lambda(1)) & J(1) \\ I(\chi_\lambda(2)) & J(2) \\ \vdots & \vdots \\ I(\chi_\lambda(h)) & J(h) \end{array} \right)$$

$$= H(\lambda) \sum_{\mathbf{T}} \left(\begin{array}{c|c} I(\gamma^\lambda_{\mathbf{TT}}(1)) & J(1) \\ I(\gamma^\lambda_{\mathbf{TT}}(2)) & J(2) \\ \vdots & \vdots \\ I(\gamma^\lambda_{\mathbf{TT}}(h)) & J(h) \end{array} \right)$$

Since the natural units $\gamma^\lambda_{\mathbf{TT}}$ expand into Young symmetrizers of the same shape:

$$\gamma^\lambda_{\mathbf{TT}} = \sum_{\mathbf{S_1}, \mathbf{S_2}} C_{\mathbf{TT}, \mathbf{S_1 S_2}} \cdot \mathbf{e}^\lambda_{\mathbf{S_1 S_2}}, \quad C_{\mathbf{TT}, \mathbf{S_1 S_2}} \in \mathbb{C},$$

then

$$imm_\lambda(i_1 i_2 \cdots i_h; j_1 j_2 \cdots j_h) = H(\lambda) \sum_{\mathbf{T}} \sum_{\mathbf{S_1}, \mathbf{S_2}} C_{\mathbf{TT}, \mathbf{S_1 S_2}} \left(\begin{array}{c|c} I(\mathbf{e}^\lambda_{\mathbf{S_1 S_2}}(1)) & J(1) \\ I(\mathbf{e}^\lambda_{\mathbf{S_1 S_2}}(2)) & J(2) \\ \vdots & \vdots \\ I(\mathbf{e}^\lambda_{\mathbf{S_1 S_2}}(h)) & J(h) \end{array} \right)$$

$$= H(\lambda) \sum_{\mathbf{T}} \sum_{\mathbf{S_1}, \mathbf{S_2}} C_{\mathbf{TT}, \mathbf{S_1 S_2}} (I_{S_1} \| \boxed{J_{S_2}}).$$

□

From Proposition 2.12 and Theorem 2.3, it follows:

Corollary 2.13. *Let $\lambda \vdash h$. If $\lambda_1 \nleq min\{n, d\}$, then*

$$imm_\lambda(i_1 i_2 \cdots i_h; j_1 j_2 \cdots j_h) = 0.$$

□

The scalar multiple $\frac{\chi^\lambda(I)}{n!}$ IMM_λ of the linear operator IMM_λ of Proposition 2.11 acts on $\mathbb{C}_h[M_{n,d}]$ as the *projector* on the direct summand $\mathbb{C}^\lambda_h[M_{n,d}]$ in the Gordan-Capelli direct sum decomposition (9) of Corollary 2.4.

Proposition 2.14. *Let U, V be Young tableaux of the same shape $sh(U) = sh(V) = \mu \vdash h$. We have:*

1. *if* $\mu = \lambda$, *then*

$$\frac{\chi^\lambda(I)}{n!}\ IMM_\lambda\Big((U|\boxed{V})\Big) = (U|\boxed{V}); \tag{15}$$

2. *if* $\mu \neq \lambda$, *then*

$$\frac{\chi^\lambda(I)}{n!}\ IMM_\lambda\Big((U|\boxed{V})\Big) = 0. \tag{16}$$

Proof. Set

$$U = I_{\mathbf{T_0}}, \quad V = J_{\mathbf{T_0}}, \quad sh(U) = sh(V) = sh(\mathbf{T_0}) = \mu.$$

Equation (14) implies

$$(U|\boxed{V}) = (I_{\mathbf{T_0}}|\boxed{J_{\mathbf{T_0}}}) = \left(\begin{array}{c|c} I\big(\mathbf{e}^\mu_{\mathbf{T_0T_0}}(1)\big) & J(1) \\ \vdots & \vdots \\ I\big(\mathbf{e}^\mu_{\mathbf{T_0T_0}}(h)\big) & J(h) \end{array}\right).$$

Item 4) of Proposition 2.6 implies

$$\begin{aligned} \frac{\chi^\lambda(I)}{n!}\ IMM_\lambda\Big((U|\boxed{V})\Big) &= (\sum_{\mathbf{T}}\ \gamma^\lambda_{\mathbf{TT}}) \left(\begin{array}{c|c} I\big(\mathbf{e}^\mu_{\mathbf{T_0T_0}}(1)\big) & J(1) \\ \vdots & \vdots \\ I\big(\mathbf{e}^\mu_{\mathbf{T_0T_0}}(h)\big) & J(h) \end{array}\right) \\ &= \left(\begin{array}{c|c} I\big(\sum_{\mathbf{T}}\ \gamma^\lambda_{\mathbf{TT}}\ \mathbf{e}^\mu_{\mathbf{T_0T_0}}(1)\big) & J(1) \\ \vdots & \vdots \\ I\big(\sum_{\mathbf{T}}\ \gamma^\lambda_{\mathbf{TT}}\ \mathbf{e}^\mu_{\mathbf{T_0T_0}}(h)\big) & J(h) \end{array}\right). \end{aligned}$$

If $\lambda \neq \mu$, the natural units $\gamma^\lambda_{\mathbf{TT}}$ and the Young symmetrizer $\mathbf{e}^\mu_{\mathbf{T_0T_0}}$ belong to different simple components of the semisimple algebra

$$\mathbb{C}[\mathbf{S}_h] = \bigoplus_{\nu \vdash h}\ \mathbb{C}^\nu[\mathbf{S}_h],$$

and are therefore orthogonal. This proves the second assertion.

If $\lambda = \mu$, since (Proposition 2.6, item 3))

$$\gamma^\lambda_{\mathbf{TT}}\ \mathbf{e}^\lambda_{\mathbf{T_0T_0}} = \delta_{\mathbf{T},\mathbf{T_0}}\ \mathbf{e}^\lambda_{\mathbf{T_0T_0}},$$

we get

$$\frac{\chi^\lambda(I)}{n!}\ IMM_\lambda\Big((U|\boxed{V})\Big) = \left(\begin{array}{c|c} I\big(\mathbf{e}^\lambda_{\mathbf{T_0T_0}}(1)\big) & J(1) \\ \vdots & \vdots \\ I\big(\mathbf{e}^\lambda_{\mathbf{T_0T_0}}(h)\big) & J(h) \end{array}\right) = (U|\boxed{V}),$$

and the first assertion is proved. □

Proposition 2.15. *Let $\lambda \vdash h$. Any right symmetrized bitableau* $(U\|\boxed{V})$ *of shape* $sh(U) = sh(V) = \lambda$ *can be written as a linear combination of immanants* $imm_\lambda(i_1 i_2 \cdots i_h; j_1 j_2 \cdots j_h)$ *associated to the **same** shape* λ.

Proof. Expand the right symmetrized bitableau $(U\|\boxed{V})$ into monomials and apply to each summand the linear operator IMM_λ. □

By combining Theorem 2.3 and Proposition 2.15, we get

Proposition 2.16. *The set of immanants*

$$imm_\lambda(i_1 i_2 \cdots i_h; j_1 j_2 \cdots j_h),$$

with

$$\lambda \vdash h,\ \lambda_1 \not\leq min\{n, d\},\ (i_1 i_2 \cdots i_h) \in \underline{n}^h,\ (j_1 j_2 \cdots j_h) \in \underline{d}^h,$$

is a spanning set of $\mathbb{C}_h[M_{n,d}]$.

3 The superalgebraic approach to the enveloping algebra $\mathbf{U}(gl(n))$

In this Section, we provide a synthetic presentation of the *superalgebraic method of virtual variables* for $gl(n)$.

This method was developed by the present authors for the general linear Lie superalgebras $gl(m|n)$, in the series of notes [3], [4], [5], [6], [7], [8], [9].

The technique of virtual variables is an extension of Capelli's method of *variabili ausilarie* (Capelli [16], see also Weyl [53]).

Capelli introduced the technique of *variabili ausilarie* in order to manage symmetrizer operators in terms of polarization operators and to simplify the study of some skew-symmetrizer operators (namely, the famous central Capelli operator).

Capelli's idea was well suited to treat symmetrization, but it did not work in the same efficient way while dealing with skew-symmetrization.

One had to wait the introduction of the notion of *superalgebras* (see,e.g. [48], [29]) to have the right conceptual framework to treat symmetry and skew-symmetry in one and the same way. To the best of our knowledge, the first mathematician who intuited the connection between Capelli's idea and superalgebras was Koszul in 1981 [32]; Koszul proved that the classical determinantal Capelli operator can be rewritten - in a much simpler way - by adding to the symbols to be dealt with an extra auxiliary symbol that obeys to different commutation relations.

The superalgebraic method of virtual variables allows us to express remarkable classes of elements in $\mathbf{U}(gl(n))$ as images - with respect to the *Capelli devirtualization epimorphism* (Subsection 3.2.1 below) - of simple *monomials* and to obtain transparent combinatorial descriptions of their actions on irreducible $gl(n)-$modules.

Among these classes, here we recall the classes of Capelli bitableaux $[S|T]$ and Young-Capelli bitableaux $[S|\boxed{T}]$ (see [6], [7], [3], and subsection 3.3.2 below), and introduce the *new* class of Capelli immanants

$$Cimm_{\lambda}[i_1 i_2 \cdots i_h; j_1 j_2 \cdots j_h]$$

(see Section 5 below).

Moreover, this method throws a bridge between the theory of $\mathbf{U}(gl(n))$ and the *(super)straightening techniques* in (super)symmetric algebras (see, e.g. [25], [7], [8], [3]).

3.1 The superalgebras $\mathbb{C}[M_{m_0|m_1+n,d}]$ and $gl(m_0|m_1+n)$

3.1.1 The general linear Lie super algebra $gl(m_0|m_1+n)$

Given a vector space V_n of dimension n, we will regard it as a subspace of a $\mathbb{Z}_2-$graded vector space $W = W_0 \oplus W_1$, where

$$W_0 = V_{m_0}, \qquad W_1 = V_{m_1} \oplus V_n.$$

The vector spaces V_{m_0} and V_{m_1} (informally, we assume that $dim(V_{m_0}) = m_0$ and $dim(V_{m_1}) = m_1$ are "sufficiently large") are called the *positive virtual (auxiliary) vector space*, the *negative virtual (auxiliary) vector space*, respectively, and V_n is called the *(negative) proper vector space*.

The inclusion $V_n \subset W$ induces a natural embedding of the ordinary general linear Lie algebra $gl(n)$ of V_n into the *auxiliary* general linear Lie *superalgebra* $gl(m_0|m_1+n)$ of $W = W_0 \oplus W_1$ (see, e.g. [29], [48]).

Let $A_0 = \{\alpha_1, \ldots, \alpha_{m_0}\}$, $A_1 = \{\beta_1, \ldots, \beta_{m_1}\}$, $L = \{x_1, \ldots, x_n\}$ denote *fixed bases* of V_{m_0}, V_{m_1} and V_n, respectively; therefore $|\alpha_s| = 0 \in \mathbb{Z}_2$, and $|\beta_t| = |x_i| = 1 \in \mathbb{Z}_2$.

Let

$$\{e_{a,b}; a, b \in A_0 \cup A_1 \cup L\}, \qquad |e_{a,b}| = |a| + |b| \in \mathbb{Z}_2$$

be the standard $\mathbb{Z}_2-$homogeneous basis of the Lie superalgebra $gl(m_0|m_1+n)$ provided by the elementary matrices. The elements $e_{a,b} \in gl(m_0|m_1+n)$ are $\mathbb{Z}_2-$homogeneous of $\mathbb{Z}_2-$degree $|e_{a,b}| = |a| + |b|$.

The superbracket of the Lie superalgebra $gl(m_0|m_1+n)$ has the following explicit form:

$$[e_{a,b}, e_{c,d}] = \delta_{bc}\ e_{a,d} - (-1)^{(|a|+|b|)(|c|+|d|)}\delta_{ad}\ e_{c,b},$$

$a, b, c, d \in A_0 \cup A_1 \cup L$.

Remark 3.1. *In the following, the elements of the sets A_0, A_1, L will be called* positive virtual symbols*,* negative virtual symbols *and* negative proper symbols*, respectively.*

3.1.2 The supersymmetric algebra $\mathbb{C}[M_{m_0|m_1+n,d}]$

As already said, we will write $(i|j)$ in place of x_{ij}, and regard the (commutative) algebra $\mathbb{C}[M_{n,d}]$ as a subalgebra of the *"auxiliary" supersymmetric algebra*

$$\mathbb{C}[M_{m_0|m_1+n,d}] = \mathbb{C}\big[(\alpha_s|j), (\beta_t|j), (i|j)\big]$$

generated by the ($\mathbb{Z}_2$-graded) variables $(\alpha_s|j), (\beta_t|j), (i|j)$, $j = 1, 2, \ldots, d$, where

$$|(\alpha_s|j)| = 1 \in \mathbb{Z}_2 \quad and \quad |(\beta_t|j)| = |(i|j)| = 0 \in \mathbb{Z}_2,$$

subject to the commutation relations:

$$(a|h)(b|k) = (-1)^{|(a|h)||(b|k)|}\ (b|k)(a|h),$$

for $a, b \in \{\alpha_1, \ldots, \alpha_{m_0}\} \cup \{\beta_1, \ldots, \beta_{m_1}\} \cup \{1, 2, \ldots, n\}$.

In plain words, all the variables commute each other, with the exception of pairs of variables $(\alpha_s|j), (\alpha_t|j)$ that skew-commute:

$$(\alpha_s|j)(\alpha_t|j) = -(\alpha_t|j)(\alpha_s|j).$$

In the standard notation of multilinear algebra, we have:

$$\begin{aligned}\mathbb{C}[M_{m_0|m_1+n,d}] &\cong \Lambda\left[W_0 \otimes P_d\right] \otimes \mathrm{Sym}\left[W_1 \otimes P_d\right] \\ &= \Lambda\left[V_{m_0} \otimes P_d\right] \otimes \mathrm{Sym}\left[(V_{m_1} \oplus V_n) \otimes P_d\right]\end{aligned}$$

where $P_d = (P_d)_1$ denotes the trivially (odd) $\mathbb{Z}_2-$graded vector space with distinguished basis $\{j;\ j = 1, 2, \ldots, d\}$.

The algebra $\mathbb{C}[M_{m_0|m_1+n,d}]$ is a supersymmetric $\mathbb{Z}_2-$*graded algebra* (superalgebra), whose $\mathbb{Z}_2-$graduation is inherited by the natural one in the exterior algebra.

3.1.3 Left superderivations and left superpolarizations

A *left superderivation* D ($\mathbb{Z}_2-$homogeneous of degree $|D|$) (see, e.g. [48], [29]) on $\mathbb{C}[M_{m_0|m_1+n,d}]$ is an element of the superalgebra $End_{\mathbb{C}}[\mathbb{C}[M_{m_0|m_1+n,d}]]$ that satisfies "Leibniz rule"

$$D(\mathbf{p} \cdot \mathbf{q}) = D(\mathbf{p}) \cdot \mathbf{q} + (-1)^{|D||\mathbf{p}|}\mathbf{p} \cdot D(\mathbf{q}),$$

for every $\mathbb{Z}_2-$homogeneous of degree $|\mathbf{p}|$ element $\mathbf{p} \in \mathbb{C}[M_{m_0|m_1+n,d}]$.

Given two symbols $a, b \in A_0 \cup A_1 \cup L$, the *superpolarization* $D_{a,b}$ of b to a is the unique *left* superderivation of $\mathbb{C}[M_{m_0|m_1+n,d}]$ of parity $|D_{a,b}| = |a| + |b| \in \mathbb{Z}_2$ such that

$$D_{a,b}\left((c|j)\right) = \delta_{bc}\ (a|j),\ \ c \in A_0 \cup A_1 \cup L,\ \ j = 1, \ldots, d. \tag{17}$$

Informally, we say that the operator $D_{a,b}$ *annihilates* the symbol b and *creates* the symbol a.

3.1.4 The superalgebra $\mathbb{C}[M_{m_0|m_1+n,d}]$ as a $\mathbf{U}(gl(m_0|m_1+n))$-module

Since

$$D_{a,b}D_{c,d} - (-1)^{(|a|+|b|)(|c|+|d|)}D_{c,d}D_{a,b} = \delta_{b,c}D_{a,d} - (-1)^{(|a|+|b|)(|c|+|d|)}\delta_{a,d}D_{c,b},$$

the map

$$e_{a,b} \to D_{a,b}, \qquad a,b \in A_0 \cup A_1 \cup L.$$

(that send the elementary matrices to the corresponding superpolarizations) is an (even) Lie superalgebra morphism from $gl(m_0|m_1+n)$ to $End_{\mathbb{C}}[\mathbb{C}[M_{m_0|m_1+n,d}]]$ and, hence, it uniquely defines a morphism (i.e. a representation):

$$\varrho : \mathbf{U}(gl(m_0|m_1+n)) \to End_{\mathbb{C}}[\mathbb{C}[M_{m_0|m_1+n,d}]].$$

In the following, we always regard the superalgebra $\mathbb{C}[M_{m_0|m_1+n,d}]$ as a $\mathbf{U}(gl(m_0|m_1+n))-$supermodule, with respect to the action induced by the representation ϱ:

$$e_{a,b} \cdot \mathbf{p} = D_{a,b}(\mathbf{p}),$$

for every $\mathbf{p} \in \mathbb{C}[M_{m_0|m_1+n,d}]$.

We recall that $\mathbb{C}[M_{m_0|m_1+n,d}]$ is a *semisimple* $\mathbf{U}(gl(m_0|m_1+n))-$supermodule, whose irreducible (simple) submodules are - up to isomorphism - *Schur supermodules* (see, e.g. [5], [6], [3]. For a more traditional presentation, see also [18]).

Clearly, $\mathbf{U}(gl(0|n)) = \mathbf{U}(gl(n))$ is a subalgebra of $\mathbf{U}(gl(m_0|m_1+n))$ and the subalgebra $\mathbb{C}[M_{n,d}]$ is a $\mathbf{U}(gl(n))-$submodule of $\mathbb{C}[M_{m_0|m_1+n,d}]$.

3.2 The virtual algebra $Virt(m_0+m_1,n)$ and the virtual presentations of elements in $\mathbf{U}(gl(n))$

3.2.1 The Capelli devirtualization epimorphism $\mathfrak{p} : Virt(m_0+m_1,n) \twoheadrightarrow \mathbf{U}(gl(n))$

We say that a product

$$e_{a_m b_m} \cdots e_{a_1 b_1} \in \mathbf{U}(gl(m_0|m_1+n)), \quad a_i, b_i \in A_0 \cup A_1 \cup L, \ i = 1, \ldots, m$$

is an *irregular expression* whenever there exists a right subword

$$e_{a_i,b_i} \cdots e_{a_2,b_2} e_{a_1,b_1},$$

$i \leq m$ and a virtual symbol $\gamma \in A_0 \cup A_1$ such that

$$\#\{j; b_j = \gamma, j \leq i\} > \#\{j; a_j = \gamma, j < i\}. \tag{18}$$

The meaning of an irregular expression in terms of the action of $\mathbf{U}(gl(m_0|m_1+n))$ on the algebra $\mathbb{C}[M_{m_0|m_1+n,d}]$ is that there exists a virtual symbol γ and a right subsequence in which the symbol γ is annihilated more times than it was already created.

Example 3.2. Let $\gamma \in A_0 \cup A_1$ and $x_i, x_j \in L$. The product

$$e_{\gamma,x_j} e_{x_i,\gamma} e_{x_j,\gamma} e_{\gamma,x_i}$$

is an irregular expression.

□

Let **Irr** be the *left ideal* of $\mathbf{U}(gl(m_0|m_1+n))$ generated by the set of irregular expressions.

Remark 3.3. *The action of any element of* **Irr** *on the subalgebra* $\mathbb{C}[M_{n,d}] \subset \mathbb{C}[M_{m_0|m_1+n,d}]$ *- via the representation* ϱ *- is identically zero.*

Proposition 3.4. *([7], [4]) The sum* $\mathbf{U}(gl(0|n)) + \mathbf{Irr}$ *is a direct sum of vector subspaces of* $\mathbf{U}(gl(m_0|m_1+n))$.

We come now to one of the main notions of the virtual method.

The *virtual algebra* $Virt(m_0 + m_1, n)$ is the subalgebra

$$Virt(m_0 + m_1, n) = \mathbf{U}(gl(0|n)) \oplus \mathbf{Irr} \subset \mathbf{U}(gl(m_0|m_1 + n)).$$

The proof of the following proposition is immediate from the definitions.

Proposition 3.5. *The left ideal* **Irr** *of* $\mathbf{U}(gl(m_0|m_1 + n))$ *is a two sided ideal of* $Virt(m_0 + m_1, n)$.

The *Capelli devirtualization epimorphism* is the projection

$$\mathfrak{p} : Virt(m_0 + m_1, n) = \mathbf{U}(gl(0|n)) \oplus \mathbf{Irr} \twoheadrightarrow \mathbf{U}(gl(0|n)) = \mathbf{U}(gl(n))$$

with $Ker(\mathfrak{p}) = \mathbf{Irr}$.

Example 3.6. Let $x \in L,\ \alpha \in A_0$. The element

$$e_{x,\alpha} e_{\alpha,x} = -e_{\alpha,x} e_{x,\alpha} + e_{x,x} + e_{\alpha,\alpha}$$

belongs to the virtual algebra $Virt(m_0 + m_1, n)$ and

$$\mathfrak{p}\big(e_{x,\alpha} e_{\alpha,x}\big) = e_{x,x} \in \mathbf{U}(gl(n)).$$

Example 3.7. Let $x, y \in L,\ \alpha \in A_0$. Then

$$\begin{aligned}
e_{y\alpha} e_{x\alpha} e_{\alpha x} e_{\alpha y} &= -e_{y\alpha} e_{\alpha x} e_{x\alpha} e_{\alpha y} + e_{y\alpha} e_{xx} e_{\alpha y} + e_{y\alpha} e_{\alpha\alpha} e_{\alpha y} \\
&= +e_{y\alpha} e_{\alpha x} e_{\alpha y} e_{x\alpha} - e_{y\alpha} e_{\alpha x} e_{xy} \\
&\quad - e_{xx} e_{\alpha y} e_{y\alpha} + e_{xx} e_{yy} - e_{xx} e_{\alpha\alpha} \\
&\quad + e_{y\alpha} e_{\alpha y} e_{\alpha\alpha} + e_{y\alpha} e_{\alpha y} \\
&= +e_{y\alpha} e_{\alpha x} e_{\alpha y} e_{x\alpha} - e_{\alpha x} e_{y\alpha} e_{xy} - e_{yx} e_{xy} \\
&\quad - e_{xx} e_{\alpha y} e_{y\alpha} + e_{xx} e_{yy} - e_{xx} e_{\alpha\alpha} \\
&\quad + e_{y\alpha} e_{\alpha y} e_{\alpha\alpha} + e_{\alpha y} e_{y\alpha} + e_{yy} + e_{\alpha\alpha} \in \mathbf{U}(gl(m_0|m_1 + n)).
\end{aligned}$$

Therefore

$$e_{y\alpha}e_{x\alpha}e_{\alpha x}e_{\alpha y} \in Virt(m_0 + m_1, n)$$

and

$$\mathfrak{p}\big(e_{y\alpha}e_{x\alpha}e_{\alpha x}e_{\alpha y}\big) = -e_{yx}e_{xy} + e_{xx}e_{yy} + e_{yy} \in \mathbf{U}(gl(n)).$$

□

Any element in $\mathbf{M} \in Virt(m_0 + m_1, n)$ defines an element in $\mathbf{m} \in \mathbf{U}(gl(n))$ - via the map $\mathfrak{p}$ - and $\mathbf{M}$ is called a *virtual presentation* of $\mathbf{m}$.

Since the map $\mathfrak{p}$ a surjection, any element $\mathbf{m} \in \mathbf{U}(gl(n))$ admits several virtual presentations. In the sequel, we even take virtual presentations as the *true definition* of special elements in $\mathbf{U}(gl(n))$, and this method will turn out to be quite effective.

Example 3.8. (A virtual presentation of the Capelli determinant) As a generalization of Example 3.7, we describe a "*monomial*" virtual presentation in $Virt(m_0 + m_1, n)$ of the classical Capelli determinant in $\mathbf{U}(gl(n))$.

Let $\alpha \in A_0$. The monomial element

$$C = e_{x_n,\alpha} \cdots e_{x_2,\alpha}e_{x_1,\alpha} \cdot e_{\alpha,x_1}e_{\alpha,x_2} \cdots e_{\alpha,x_n} \in \mathbf{U}(gl(m_0|m_1 + n)) \qquad (19)$$

belongs to the virtual algebra $Virt(m_0|m_1 + n)$. The image of the element C under the Capelli devirtualization epimorphism $\mathfrak{p}$ equals the *column determinant*[1]

$$\mathbf{H}_n(n) = \mathbf{cdet}\begin{pmatrix} e_{x_1,x_1} + (n-1) & e_{x_1,x_2} & \dots & e_{x_1,x_n} \\ e_{x_2,x_1} & e_{x_2,x_2} + (n-2) & \dots & e_{x_2,x_n} \\ \vdots & \vdots & \vdots & \\ e_{x_n,x_1} & e_{x_n,x_2} & \dots & e_{x_n,x_n} \end{pmatrix} \in \mathbf{U}(gl(n)).$$

This result is a special case of the result that we called the "Laplace expansion for Capelli rows" ([9] Theorem 2, [3] Theorem 6.3). A sketchy proof of it can also be found in Koszul [32]. □

The next results will play a crucial role in the study of *central* elements of $\mathbf{U}(gl(n))$.

Proposition 3.9. *For every $e_{x_i,x_j} \in gl(n) \subset gl(m_0|m_1 + n)$, let $ad(e_{x_i,x_j})$ denote its adjoint action on $Virt(m_0{+}m_1, n)$; the ideal* **Irr** *is $ad(e_{x_i,x_j})-$invariant. Then*

$$\mathfrak{p}\left(ad(e_{x_i,x_j})(\mathbf{m})\right) = ad(e_{x_i,x_j})\left(\mathfrak{p}(\mathbf{m})\right), \qquad \mathbf{m} \in Virt(m_0 + m_1, n). \qquad (20)$$

Corollary 3.10. *The Capelli epimorphism image of an element of $Virt(m_0|m_1{+}n)$ that is an invariant for the adjoint action of $gl(n)$ is in the center $\boldsymbol{\zeta}(n)$ of $\mathbf{U}(gl(n))$.*

[1]The symbol **cdet** denotes the column determinat of a matrix $A = [a_{ij}]$ with noncommutative entries: $\mathbf{cdet}(A) = \sum_\sigma \ (-1)^{|\sigma|} \ a_{\sigma(1),1}a_{\sigma(2),2} \cdots a_{\sigma(n),n}$.

Example 3.11. Recall that

$$ad(e_{x_i,x_j})\,(e_{x_h,\alpha}) = \delta_{jh} e_{x_i,\alpha},$$

$$ad(e_{x_i,x_j})\,(e_{\alpha,x_k}) = -\delta_{ki} e_{\alpha,x_j},$$

for every virtual symbol α, and that $ad(e_{x_i,x_j})$ acts as a derivation, for every $i, j = 1, 2, \ldots, n$.

The monomial C of Example 3.8, Eq.(19) is annihilated by $ad(e_{x_i,x_j}),\ i \neq j$, by skew-symmetry. Furthermore, $ad(e_{x_i,x_i})\,(C) = C - C = 0,\ i = 1, 2, \ldots, n$; hence, C is an invariant for the adjoint action of $gl(n)$.

Since $\mathfrak{p}\,(C) = \mathbf{H}_n(n)$, the Capelli determinant $\mathbf{H}_n(n)$ is central in $\mathbf{U}(gl(n))$, by Corollary 3.10. □

3.2.2 The action of $Virt(m_0 + m_1, n)$ on the subalgebra $\mathbb{C}[M_{n,d}]$

From the representation-theoretic point of view, the core of the *method of virtual variables* lies in the following result.

Theorem 3.12. *The action of $Virt(m_0+m_1, n)$ leaves invariant the subalgebra $\mathbb{C}[M_{n,d}] \subseteq \mathbb{C}[M_{m_0|m_1+n,d}]$, and, therefore, the action of $Virt(m_0 + m_1, n)$ on $\mathbb{C}[M_{n,d}]$ is well defined. Furthermore, for every $\mathbf{v} \in Virt(m_0+m_1, n)$, its action on $\mathbb{C}[M_{n,d}]$ equals the action of $\mathfrak{p}(\mathbf{v}) \in \mathbf{U}(gl(n))$.*

Therefore, instead of studying the action of an element in $\mathbf{U}(gl(n))$, one can study the action of a virtual presentation of it in $Virt(m_0|m_1 + n)$. The advantage of virtual presentations is that they are frequently of monomial form, admit quite transparent interpretations and are much easier to be dealt with (see, e.g. [5], [6], [9], [3], [4]).

A prototypical instance of this method is provided by the celebrated *Capelli identity* [12], [53], [26], [27], [50]. From Example 3.8, it follows that the action of the Capelli determinant $\mathbf{H}_n(n)$ on a form $f \in \mathbb{C}[M_{n,d}]$ is the same as the action of its monomial virtual presentation, and this leads to a few lines proof of the identity [9], [4].

3.2.3 Balanced monomials as elements of the virtual algebra $Virt(m_0+ m_1, n)$

In order to make the virtual variables method effective, we need to exhibit a class of nontrivial elements that belong to $Virt(m_0 + m_1, n)$.

A quite relevant class of such elements is provided by *balanced monomials*.

In plain words, a balanced monomial is product of two or more factors where the rightmost one *annihilates* the k proper symbols $x_{j_1}, \ldots, x_{j_k}$ and *creates* some virtual symbols; the leftmost one *annihilates* all the virtual symbols and *creates* the k proper symbols $x_{i_1}, \ldots, x_{i_k}$; between these two factors, there might be further factors that annihilate and create virtual symbols only.

In a formal way, balanced monomials are elements of the algebra $\mathbf{U}(gl(m_0|m_1+ n))$ of the forms:

- $e_{x_{i_1},\gamma_{p_1}} \cdots e_{x_{i_k},\gamma_{p_k}} \cdot e_{\gamma_{p_1},x_{j_1}} \cdots e_{\gamma_{p_k},x_{j_k}}$,
- $e_{x_{i_1},\theta_{q_1}} \cdots e_{x_{i_k},\theta_{q_k}} \cdot e_{\theta_{q_1},\gamma_{p_1}} \cdots e_{\theta_{q_k},\gamma_{p_k}} \cdot e_{\gamma_{p_1},x_{j_1}} \cdots e_{\gamma_{p_k},x_{j_k}}$,
- and so on,

where $x_{i_1}, \ldots, x_{i_k}, x_{j_1}, \ldots, x_{j_k} \in L$, i.e., the $x_{i_1}, \ldots, x_{i_k}, x_{j_1}, \ldots, x_{j_k}$ are k proper symbols.

The next result is the (superalgebraic) formalization of the argument developed by Capelli in [16], CAPITOLO I, §X.Metodo delle variabili ausiliarie, page 55 ff.

Proposition 3.13. ([5], [6], [9], [3], [4]) *Every balanced monomial belongs to $Virt(m_0+m_1,n)$. Hence its image under the Capelli epimorphism $\mathfrak{p}$ belongs to $\mathbf{U}(gl(n))$.*

In plain words, the action of a balanced monomial on the subalgebra $\mathbb{C}[M_{n,d}]$ equals the action of a suitable element of $\mathbf{U}(gl(n))$.

3.3 Two special classes of elements in $Virt(m_0+m_1,n)$ and their images in $\mathbf{U}(gl(n))$

We will introduce two classes of remarkable elements of the enveloping algebra $\mathbf{U}(gl(n))$, that we call *Capelli bitableaux* and *Young-Capelli bitableaux*, respectively.

Capelli bitableaux are the analogues in $\mathbf{U}(gl(n))$ of bitableaux in the polynomial algebra $\mathbb{C}[M_{n,d}]$, as well as Young-Capelli bitableaux are the analogues in $\mathbf{U}(gl(n))$ of right symmetrized bitableaux. Besides this analogy, their meaning lies deeper, as we shall see in Section 4.

3.3.1 Bitableaux monomials in $\mathbf{U}(gl(m_0+m_1,n))$

Let S and T be two Young tableaux of same shape $\lambda \vdash h$ on the alphabet $A_0 \cup A_1 \cup L$:

$$S = \left(\begin{array}{l} z_{i_1} \ldots\ldots\ldots z_{i_{\lambda_1}} \\ z_{j_1} \ldots\ldots z_{j_{\lambda_2}} \\ \ldots\ldots \\ z_{s_1} \ldots z_{s_{\lambda_p}} \end{array} \right), \qquad T = \left(\begin{array}{l} z_{h_1} \ldots\ldots\ldots z_{h_{\lambda_1}} \\ z_{k_1} \ldots\ldots z_{k_{\lambda_2}} \\ \ldots\ldots \\ z_{t_1} \ldots z_{t_{\lambda_p}} \end{array} \right). \tag{21}$$

To the pair (S,T), we associate the *bitableau monomial*:

$$e_{S,T} = e_{z_{i_1},z_{h_1}} \cdots e_{z_{i_{\lambda_1}},z_{h_{\lambda_1}}} e_{z_{j_1},z_{k_1}} \cdots e_{z_{j_{\lambda_2}},z_{k_{\lambda_2}}} \cdots\cdots e_{z_{s_1},z_{t_1}} \cdots e_{z_{s_{\lambda_p}},z_{t_{\lambda_p}}} \tag{22}$$

in $\mathbf{U}(gl(m_0|m_1+n))$.

By expressing the Young tableaux S,T in the functional form (see subsection 2.2.1):

$$S : \underline{h} \to A_0 \cup A_1 \cup L, \quad T : \underline{h} \to A_0 \cup A_1 \cup L,$$

the bitableau monomial $e_{S,T}$ of Eq. (22) becomes:

$$e_{S,T} = e_{S(1),T(1)} e_{S(2),T(2)} \cdots e_{S(h),T(h)}.$$

Let us denote by $\alpha_1, \ldots, \alpha_p \in A_0$, $\beta_1, \ldots, \beta_{\lambda_1} \in A_1$ two *arbitrary* families of *mutually distinct positive and negative virtual symbols*, respectively (see Remark 3.1). Set

$$D^*_\lambda = \begin{pmatrix} \beta_1 \ldots\ldots\ldots \beta_{\lambda_1} \\ \beta_1 \ldots\ldots \beta_{\lambda_2} \\ \ldots\ldots \\ \beta_1 \ldots \beta_{\lambda_p} \end{pmatrix}, \qquad C^*_\lambda = \begin{pmatrix} \alpha_1 \ldots\ldots\ldots \alpha_1 \\ \alpha_2 \ldots\ldots \alpha_2 \\ \ldots\ldots \\ \alpha_p \ldots \alpha_p \end{pmatrix}. \tag{23}$$

The tableaux of kind (23) are called *virtual Deruyts and Coderuyts* tableaux of shape λ, respectively.

3.3.2 Capelli bitableaux and Young-Capelli bitableaux

Given a pair of Young tableaux S, T of the same shape λ on the proper alphabet L, consider the elements

$$e_{S,C^*_\lambda}\ e_{C^*_\lambda,T} \in \mathbf{U}(gl(m_0|m_1+n)), \tag{24}$$

$$e_{S,C^*_\lambda}\ e_{C^*_\lambda,D^*_\lambda}\ e_{D^*_\lambda,T} \in \mathbf{U}(gl(m_0|m_1+n)). \tag{25}$$

Since elements (26) and (39) are balanced monomials in $\mathbf{U}(gl(m_0|m_1+n))$, then they belong to the subalgebra $Virt(m_0+m_1,n)$ (Section 3.2.3).

Hence, we can consider their images in $\mathbf{U}(gl(n))$ with respect to the Capelli epimorphism $\mathfrak{p}$.

We set

$$[S|T] = \mathfrak{p}\Big(e_{S,C^*_\lambda}\ e_{C^*_\lambda,T}\Big) \in \mathbf{U}(gl(n)), \tag{26}$$

and call the element $[S|T]$ a *Capelli bitableau.*

We set

$$[S|\boxed{T}] = \mathfrak{p}\Big(e_{S,C^*_\lambda}\ e_{C^*_\lambda,D^*_\lambda}\ e_{D^*_\lambda,T}\Big) \in \mathbf{U}(gl(n)). \tag{27}$$

and call the element $[S|\boxed{T}]$ a *Young-Capelli bitableau.*

Remark 3.14. *The elements defined in* (26) *and* (39) *do not depend on the choice of the virtual Deruyts and Coderuyts tableaux* D^*_λ *and* C^*_λ.

□

The next result will play a crucial role subsection 4.2 below. In plain words, it states that *Young-Capelli bitableaux* expand into *Capelli bitableaux* in the enveloping algebra $\mathbf{U}(gl(n))$ just in the same formal way as *right symmetrized bitableaux* expand into *bitableaux* in the polynomial algebra $\mathbb{C}[M_{n,d}]$ (subsection 2.3).

Proposition 3.15. *Let S, T be Young tableaux, $sh(S) = sh(T)$. The following identity holds in the enveloping algebra $\mathbf{U}(gl(n))$:*

$$[S|\boxed{T})] = \sum_{\overline{T}} [S|\overline{T}],$$

where the sum is extended over all $\overline{T}$ column permuted of T (hence, repeated entries in a column give rise to multiplicities).

The proof easily follows from the definitions, by applying the commutator identities in the superalgebra $\mathbf{U}(gl(m_0|m_1 + n))$.

Example 3.16. (cfr. Example 2.2)

$$\left[\begin{array}{cc} x_1 & x_3 \\ x_2 & x_4 \end{array}\middle|\boxed{\begin{array}{cc} x_1 & x_2 \\ x_1 & x_3 \end{array}}\right] = \left[\begin{array}{cc|cc} x_1 & x_3 & x_1 & x_2 \\ x_2 & x_4 & x_1 & x_3 \end{array}\right] + \left[\begin{array}{cc|cc} x_1 & x_3 & x_1 & x_2 \\ x_2 & x_4 & x_1 & x_3 \end{array}\right]$$
$$+ \left[\begin{array}{cc|cc} x_1 & x_3 & x_1 & x_3 \\ x_2 & x_4 & x_1 & x_2 \end{array}\right] + \left[\begin{array}{cc|cc} x_1 & x_3 & x_1 & x_3 \\ x_2 & x_4 & x_1 & x_2 \end{array}\right]$$
$$= 2\left[\begin{array}{cc|cc} x_1 & x_3 & x_1 & x_2 \\ x_2 & x_4 & x_1 & x_3 \end{array}\right] + 2\left[\begin{array}{cc|cc} x_1 & x_3 & x_1 & x_3 \\ x_2 & x_4 & x_1 & x_2 \end{array}\right].$$

□

4 The *bitableaux correspondence* isomorphism $\mathcal{K}^{-1}$ and the Koszul map $\mathcal{K}$

4.1 The BCK theorem

Our next aim is to describe an extremely relevant pair of (mutually inverse) vector space isomorphisms between the polynomial algebra of forms $\mathbb{C}[M_{n,n}]$ and the universal enveloping algebra $\mathbf{U}(gl(n))$.

In order to do this, it is worth to simplify the notation in the following way:

- we will write i in place of x_i and e_{ij} in place of e_{x_i,x_j};
- consistently, we set $L = P = \underline{n} = \{1, 2, \ldots, n\}$.

The main advantage of this convention is that it allows us to write bitableaux in $\mathbb{C}[M_{n,n}]$ and Capelli bitableaux in $\mathbf{U}(gl(n))$ as elements associated to pairs of Young tableaux on the *same* alphabet.

More specifically, given a shape (partition) λ with $\lambda_1 \leq n$, to any pair of Young tableaux S, T on the alphabet $\underline{n} = \{1, 2, \ldots, n\}$ and of the same shape $sh(S) = sh(T) = \lambda$, one associates the (determinantal) bitableau $(S|T) \in \mathbb{C}[M_{n,n}]$, and the Capelli bitableau $[S|T] \in \mathbf{U}(gl(n))$.

Theorem 4.1. (The BCK theorem) *The "bitableaux correspondence" map*

$$\mathcal{K}^{-1} : (S|T) \mapsto [S|T] \tag{28}$$

uniquely defines a linear isomorphism

$$\mathcal{K}^{-1} : \mathbb{C}[M_{n,n}] \cong \mathbf{Sym}(gl(n)) \to \mathbf{U}(gl(n)).$$

Furthermore, this isomorphism is the inverse of the Koszul map

$$\mathcal{K} : \mathbf{U}(gl(n)) \to \mathbb{C}[M_{n,n}] \cong \mathbf{Sym}(gl(n))$$

introduced by J.-L. Koszul in [32].

Eq. (28) indeed defines a *linear* operator since bitableaux in $\mathbb{C}[M_{n,n}]$ and Capelli bitableaux in $\mathbf{U}(gl(n))$ are ruled by the **same** *straightening laws* (see [7], Proposition 7).

The linear isomorphism $\mathcal{K}^{-1}$ was introduced in [8], Theorem 1. The fact that $\mathcal{K}^{-1}$ and $\mathcal{K}$ are inverse of each other was proved in [8], Theorem 2 (see also, [4]).

4.2 Right symmetrized bitableaux and Young-Capelli bitableaux

The "bitableaux correspondence" and the Koszul isomorphisms behave well with respect to right symmetrized bitableaux

$$(S|\boxed{T}) \in \mathbb{C}[M_{n,n}]$$

and *Young-Capelli bitableaux*

$$[S|\boxed{T}] = \mathfrak{p}\left(e_{SC^*_\lambda} e_{C^*_\lambda D^*_\lambda} e_{D^*_\lambda T} c\right) \in \mathbf{U}(gl(n)).$$

In plain words, any Young-Capelli bitableaux $[S|\boxed{T}]$ is the image - with respect to the linear operator $\mathcal{K}^{-1}$ - of the right symmetrized bitableaux $(S|\boxed{T})$.

Theorem 4.2. *We have:*

$$\mathcal{K}^{-1} : (S|\boxed{T}) \mapsto [S|\boxed{T}],$$
$$\mathcal{K} : [S|\boxed{T}] \mapsto (S|\boxed{T}).$$

Proof. Indeed, we have:

$$\mathcal{K}^{-1}\Big((S|\boxed{T})\Big) = \mathcal{K}^{-1}\Big(\sum_{\overline{T}} (S|\overline{T})\Big) = \sum_{\overline{T}} [S|\overline{T}],$$

where the sum is extended over *all* $\overline{T}$ column permuted of T.

By Proposition 3.15, the last summation equals the Young-Capelli bitableaux $[S|\boxed{T}]$. □

By Theorem 4.2 and Theorem 2.3, we have:

Theorem 4.3. *Let $h \in \mathbb{N}$. The set of Young-Capelli bitableaux*

$$\bigcup_{k=0}^{h} \Big\{ [S|\boxed{T}];\ S,T \ standard,\ sh(S) = sh(T) = \lambda \vdash k,\ \lambda_1 \leq n \Big\}$$

is a basis of the filtration element $\mathbf{U}(gl(n))^{(h)}$.

Remark 4.4. *The basis elements*

$$\Big\{ [S|\boxed{T}];\ S,T \ standard,\ sh(S) = sh(T) = \lambda \vdash k,\ \lambda_1 \leq n \Big\}$$

act in a quite remarkable way on Gordan-Capelli basis elements

$$\Big\{ (U|\boxed{V});\ U,V \ standard,\ sh(U) = sh(V) = \mu \vdash h,\ \mu_1 \leq n \Big\}.$$

Indeed, we have:

- *If $h < k$, the action is zero.*
- *If $h = k$ and $\lambda \neq \mu$, the action is zero.*
- *If $h = k$ and $\lambda = \mu$, the action is nondegenerate triangular (with respect to a suitable linear order on standard tableaux of the same shape).*

See [6] *and* [3], Theorem 10.1. □

4.3 Column Capelli bitableaux in $\mathbf{U}(gl(n))$

A column Capelli bitableau in $\mathbf{U}(gl(n))$ is a Capelli bitableau $[S|T]$, where S and T are column Young tableaux of the same depth.

Although column Capelli bitableaux are far from being "monomials" in $\mathbf{U}(gl(n))$, they play the same role that column bitableaux – signed monomials – play in the polynomial algebra $\mathbb{C}[M_{n,n}]$. Specifically, Capelli bitableaux and Young-Capelli bitableaux expand into column Capelli bitableaux just in the same way as bitableaux and right symmetrized bitableaux expand into column bitableaux in the polynomial algebra $\mathbb{C}[M_{n,n}]$.

Remark 4.5. *The column Capelli bitableau $[i|j]$ of depth $h = 1$ equals the generator $e_{i,j}$ of the algebra $\mathbf{U}(gl(n))$, $i,j = 1,2,\ldots,n$, $i,j = 1,2,\ldots,n$. Indeed*

$$[i|j] = \mathfrak{p}\,[e_{i,\alpha}e_{\alpha,j}] = \mathfrak{p}\,[-e_{\alpha,j}e_{i,\alpha} + e_{i,j} + \delta_{i,j}e_{\alpha,\alpha}] = e_{ij}.$$

Since column bitableaux in the polynomial algebra $\mathbb{C}[M_{n,n}]$ are signed *commutative* monomials, then column Capelli bitableaux are invariant with respect to permutations of their rows, that is

$$\left[\begin{array}{c|c} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array}\right] = \left[\begin{array}{c|c} i_{\sigma(1)} & j_{\sigma(1)} \\ i_{\sigma(2)} & j_{\sigma(2)} \\ \vdots & \vdots \\ i_{\sigma(h)} & j_{\sigma(h)} \end{array}\right]$$

for every $\sigma \in \mathbf{S}_h$.

Let us denote by $\mathbb{C}_h[M_{n,n}]$ the homogeneous component of degree $h \in \mathbb{N}$ of the polynomial algebra $\mathbb{C}[M_{n,n}]$ and denote $\mathbf{U}(gl(n))^{(h)}$ the $h-$th filtration element of the enveloping algebra $\mathbf{U}(gl(n))$.

Corollary 4.6. *The bitableaux correspondence isomorphism $\mathcal{K}^{-1}$ and the Koszul isomorphisms $\mathcal{K}$ induce, by restriction, a pair of mutually inverse isomorphisms*

$$\mathcal{K}^{-1} : \bigoplus_{k=0}^{h} \mathbb{C}_k[M_{n,n}] \ \rightarrow \ \mathbf{U}(gl(n))^{(h)}$$

and

$$\mathcal{K} : \mathbf{U}(gl(n))^{(h)} \ \rightarrow \ \bigoplus_{k=0}^{h} \mathbb{C}_k[M_{n,n}].$$

The preceding assertion can be regarded as a sharpened version of the PBW Theorem for $\mathbf{U}(gl(n))$.

4.3.1 Devirtualization of column Capelli bitableaux in $\mathbf{U}(gl(n))$

Given any column Capelli bitableau, *devirtualized expressions* of it as an element of $\mathbf{U}(gl(n))$ can be easily obtained by means of iterations of the following identities.

Proposition 4.7. *In the enveloping algebra $\mathbf{U}(gl(n))$, we have:*

$$\left[\begin{array}{c|c} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_{h-1} & j_{h-1} \\ i_h & j_h \end{array}\right] =$$

$$(-1)^{h-1} \ e_{i_1,j_1} \left[\begin{array}{c|c} i_2 & j_2 \\ \vdots & \vdots \\ i_{h-1} & j_{h-1} \\ i_h & j_h \end{array}\right] \ + \ (-1)^{h-2} \sum_{k=2}^{h} \delta_{i_k,j_1} \left[\begin{array}{c|c} i_1 & j_k \\ \vdots & \vdots \\ i_{k-1} & j_{k-1} \\ i_{k+1} & j_{k+1} \\ \vdots & \vdots \\ i_h & j_h \end{array}\right] =$$

$$(-1)^{h-1} \left[\begin{array}{c|c} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_{h-1} & j_{h-1} \end{array}\right] e_{i_h,j_h} \ + \ (-1)^{h-2} \sum_{k=1}^{h-1} \delta_{i_h,j_k} \left[\begin{array}{c|c} i_1 & j_1 \\ \vdots & \vdots \\ i_{k-1} & j_{k-1} \\ i_{k+1} & j_{k+1} \\ \vdots & \vdots \\ i_k & j_h \end{array}\right].$$

Proof. By definition,

$$\left[\begin{array}{c|c} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_{h-1} & j_{h-1} \\ i_h & j_h \end{array}\right] =$$

$$\begin{aligned}
&= \mathfrak{p}\big[e_{i_1,\alpha_1}e_{i_2,\alpha_2}\cdots e_{i_{h-1},\alpha_{h-1}}e_{i_h,\alpha_h}\cdot e_{\alpha_1,j_1}e_{\alpha_2,j_2}\cdots e_{\alpha_{h-1},j_{h-1}}e_{\alpha_h,j_h}\big] = \\
&= \mathfrak{p}\big[-e_{i_1,\alpha_1}e_{i_2,\alpha_2}\cdots e_{i_{h-1},\alpha_{h-1}}e_{\alpha_1,j_1}e_{i_h,\alpha_h}\cdot e_{\alpha_2,j_2}\cdots e_{\alpha_{h-1},j_{h-1}}e_{\alpha_h,j_h} \\
&\qquad + e_{i_1,\alpha_1}e_{i_2,\alpha_2}\cdots e_{i_{h-1},\alpha_{h-1}}\cdot \delta_{i_h,j_1}e_{\alpha_1,\alpha_h}e_{\alpha_2,j_2}\cdots e_{\alpha_{h-1},j_{h-1}}e_{\alpha_h,j_h}\big] = \\
&= \mathfrak{p}\big[-e_{i_1,\alpha_1}e_{i_2,\alpha_2}\cdots e_{i_{h-1},\alpha_{h-1}}e_{\alpha_1,j_1}e_{i_h,\alpha_h}\cdot e_{\alpha_2,j_2}\cdots e_{\alpha_{h-1},j_{h-1}}e_{\alpha_h,j_h} \\
&\qquad + e_{i_1,\alpha_1}e_{i_2,\alpha_2}\cdots e_{i_{h-1},\alpha_{h-1}}\cdot \delta_{i_h,j_1}e_{\alpha_2,j_2}\cdots e_{\alpha_{h-1},j_{h-1}}e_{\alpha_1,j_h}\big].
\end{aligned}$$

Notice that

$$\delta_{i_h,j_1}\ e_{i_1,\alpha_1}e_{i_2,\alpha_2}\cdots e_{i_{h-1},\alpha_{h-1}}\cdot e_{\alpha_2,j_2}\cdots e_{\alpha_{h-1},j_{h-1}}e_{\alpha_1,j_h} =$$

$$\delta_{i_h,j_1}\ (-1)^{h-2}\ e_{i_1,\alpha_1}e_{i_2,\alpha_2}\cdots e_{i_{h-1},\alpha_{h-1}}\cdot e_{\alpha_1,j_h}e_{\alpha_2,j_2}\cdots e_{\alpha_{h-1},j_{h-1}}$$

as elements of the algebra $\mathbf{U}(gl(m_0|m_1+n))$.

Therefore, the summand

$$\mathfrak{p}\big[e_{i_1,\alpha_1}e_{i_2,\alpha_2}\cdots e_{i_{h-1},\alpha_{h-1}}\cdot \delta_{i_h,j_1}e_{\alpha_2,j_2}\cdots e_{\alpha_{h-1},j_{h-1}}e_{\alpha_1,j_h}\big]$$

equals

$$(-1)^{h-2}\ \delta_{i_h,j_1}\left[\begin{array}{c|c} i_1 & j_h \\ i_2 & j_2 \\ \vdots & \vdots \\ i_{h-1} & j_{h-1} \end{array}\right].$$

By repeating the above procedure of moving left the element e_{α_1,j_1} - using the commutator identities in $\mathbf{U}(gl(m_0|m_1+n))$ - we finally get

$$\left[\begin{array}{c|c} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_{h-1} & j_{h-1} \\ i_h & j_h \end{array}\right] =$$

$$\begin{aligned}
&= \mathfrak{p}\big[(-1)^{h-1}e_{i_1,\alpha_1}e_{\alpha_1,j_1}e_{i_2,\alpha_2}\cdots e_{i_{h-1},\alpha_{h-1}}e_{i_h,\alpha_h}\cdot e_{\alpha_2,j_2}\cdots e_{\alpha_{h-1},j_{h-1}}e_{\alpha_h,j_h} \\
&\qquad + \sum_{i=0}^{h-2}\ (-1)^i\ e_{i_1,\alpha_1}\cdots \delta_{i_{h-i},j_1}\widehat{e_{i_{h-i},\alpha_{h-i}}}e_{\alpha_1,\alpha_{h-i}}\cdots e_{i_h,\alpha_h}\cdot e_{\alpha_2,j_2}\cdots e_{\alpha_h,j_h}\big] \\
&= \mathfrak{p}\big[(-1)^{h-1}e_{i_1,\alpha_1}e_{\alpha_1,j_1}e_{i_2,\alpha_2}\cdots e_{i_{h-1},\alpha_{h-1}}e_{i_h,\alpha_h}\cdot e_{\alpha_2,j_2}\cdots e_{\alpha_{h-1},j_{h-1}}e_{\alpha_h,j_h} \\
&\qquad + \sum_{i=0}^{h-2}\ (-1)^i\ e_{i_1,\alpha_1}\cdot\cdot\delta_{i_{h-i},j_1}\cdots e_{i_h,\alpha_h}\cdot e_{\alpha_2,j_2}\cdot\cdot e_{\alpha_1,j_{h-i}}\cdot\cdot e_{\alpha_h,j_h}\big].
\end{aligned}$$

Notice that the summand

$$(-1)^i \ \delta_{i_{h-i},j_1} \ e_{i_1,\alpha_1} \cdot \cdot \delta_{i_{h-i},j_1} \cdots e_{i_h,\alpha_h} \cdot e_{\alpha_2,j_2} \cdot \cdot e_{\alpha_1,j_{h-i}} \cdot \cdot e_{\alpha_h,j_h}$$

equals

$$(-1)^i \ \delta_{i_{h-i},j_1} \ (-1)^{h-i-2} \times$$
$$e_{i_1,\alpha_1} \cdots e_{\widehat{i_{h-i},\alpha_{h-i}}} \cdots e_{i_h,\alpha_h} \cdot e_{\alpha_1,j_{h-i}} e_{\alpha_2,j_2} \cdots e_{\widehat{\alpha_{h-i},j_{h-i}}} \cdots e_{\alpha_h,j_h}$$

as elements of the algebra $\mathbf{U}(gl(m_0|m_1+n))$.

Hence

$$\mathfrak{p}\left[(-1)^i \ \delta_{i_{h-i},j_1} \ e_{i_1,\alpha_1} \cdots e_{\widehat{i_{h-i},\alpha_{h-i}}} e_{\alpha_1,\alpha_{h-i}} \cdots e_{i_h,\alpha_h} \cdot e_{\alpha_2,j_2} \cdots e_{\alpha_h,j_h}\right]$$

equals

$$(-1)^{h-2} \ \delta_{i_{h-i},j_1} \left[\begin{array}{c|c} i_1 & j_{h-i} \\ i_2 & j_2 \\ \vdots & \vdots \\ i_{h-i-1} & j_{h-i-1} \\ \widehat{i_{h-i}} & \widehat{j_{h-i}} \\ i_{h-i+1} & j_{h-i+1} \\ i_h & j_h \end{array}\right].$$

Furthermore

$$\mathfrak{p}\left[(-1)^{h-1} e_{i_1,\alpha_1} e_{\alpha_1,j_1} e_{i_2,\alpha_2} \cdots e_{i_{h-1},\alpha_{h-1}} e_{i_h,\alpha_h} \cdot e_{\alpha_2,j_2} \cdots e_{\alpha_{h-1},j_{h-1}} e_{\alpha_h,j_h}\right] =$$
$$= (-1)^{h-1} \ e_{i_1,j_1} \left[\begin{array}{c|c} i_2 & j_2 \\ \vdots & \vdots \\ i_{h-1} & j_{h-1} \\ i_h & j_h \end{array}\right].$$

By setting $k = h - i$, we proved the first expansion identity. The second expansion identity can be proved in a similar way. □

Example 4.8.

$$\left[\begin{array}{c|c} 1 & 2 \\ 2 & 1 \\ 3 & 1 \end{array}\right] = [1|2] \left[\begin{array}{c|c} 2 & 1 \\ 3 & 1 \end{array}\right] - \left[\begin{array}{c|c} 1 & 1 \\ 3 & 1 \end{array}\right] = -e_{12}e_{21}e_{31} + e_{11}e_{31} \in \mathbf{U}(gl(n)).$$

Notice that

$$\left[\begin{array}{c|c} 1 & 2 \\ 2 & 1 \\ 3 & 1 \end{array}\right] = \left[\begin{array}{c|c} 3 & 1 \\ 2 & 1 \\ 1 & 2 \end{array}\right] = [3|1] \left[\begin{array}{c|c} 2 & 1 \\ 1 & 2 \end{array}\right] - \left[\begin{array}{c|c} 3 & 2 \\ 2 & 1 \end{array}\right] =$$
$$= -[3|1]([2|1][[1|2] - [2|2]) + [2|1][3|2] = -e_{31}e_{21}e_{12} + e_{31}e_{22} + e_{21}e_{32} =$$
$$= \left[\begin{array}{c|c} 1 & 2 \\ 2 & 1 \\ 3 & 1 \end{array}\right] = \left[\begin{array}{c|c} 1 & 2 \\ 2 & 1 \end{array}\right] [3|1] =$$
$$= (-[1|2][2|1] + [1|1])[3|1] = -e_{12}e_{21}e_{31} + e_{11}e_{31} \in \mathbf{U}(gl(n)).$$

$\square$

Remark 4.9. *Theorems* 4.1 *and* 4.2*, in combination with Proposition* 4.7*, allows the* explicit devirtualized *forms in* $\mathbf{U}(gl(n))$ *of Capelli bitableaux and of right Young-Capelli bitableaux to be easily computed. The process can be illustrated by an example. Let* $n \geq 2,\ h = 3,\ \lambda = (2,1)$*. Consider the Capelli bitableaux*

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 2 \\ 1 & & 2 & \end{array} \right] \in \mathbf{U}(gl(n)).$$

By Theorem 4.1*:*

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 2 \\ 1 & & 2 & \end{array} \right] = \mathcal{K}^{-1}\Big(\left(\begin{array}{cc|cc} 1 & 2 & 1 & 2 \\ 1 & & 2 & \end{array} \right) \Big)$$

$$= \mathcal{K}^{-1}\Big(\left(\begin{array}{c|c} 1 & 1 \\ 2 & 2 \\ 1 & 2 \end{array} \right) - \left(\begin{array}{c|c} 1 & 2 \\ 2 & 1 \\ 1 & 2 \end{array} \right) \Big)$$

$$= \left[\begin{array}{c|c} 1 & 1 \\ 2 & 2 \\ 1 & 2 \end{array} \right] - \left[\begin{array}{c|c} 1 & 2 \\ 2 & 1 \\ 1 & 2 \end{array} \right].$$

By Proposition 4.7*,*

$$\left[\begin{array}{c|c} 1 & 1 \\ 2 & 2 \\ 1 & 2 \end{array} \right] = -e_{11}e_{22}e_{12} + e_{12}e_{21} - e_{12} \in \mathbf{U}(gl(n)),$$

$$\left[\begin{array}{c|c} 1 & 2 \\ 2 & 1 \\ 1 & 2 \end{array} \right] = -e_{12}e_{21}e_{12} + e_{12}e_{22} + e_{11}e_{12} - e_{12} \in \mathbf{U}(gl(n)).$$

$\square$

4.3.2 Column Capelli bitableaux as polynomial differential operators on $\mathbb{C}[M_{n,d}]$

The next result will play a crucial role in Section 6. In the language of Procesi ([46], chapter 3), it describes the action of column Capelli bitableaux as elements of the *Weyl algebra* associated to the polynomial algebra $\mathbb{C}[M_{n,d}]$.

Proposition 4.10. *The action of the column Capelli bitableau*

$$\left[\begin{array}{c|c} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array} \right] \in \mathbf{U}(gl(n))$$

on the algebra $\mathbb{C}[M_{n,d}]$ *equals the action of the polynomial differential operator*

$$(-1)^{\binom{h}{2}} \sum_{(\varphi_1,\varphi_2,\ldots,\varphi_h) \in \underline{d}^h} (i_1|\varphi_1)(i_2|\varphi_2)\cdots(i_h|\varphi_h)\ \partial_{(j_1|\varphi_1)}\ \partial_{(j_2|\varphi_2)}\cdots\partial_{(j_h|\varphi_h)},$$

Proof. Consider a monomial $\mathbb{M} \in \mathbb{C}[M_{n,d}]$,

$$\mathbb{M} = \prod_{i=1}^{n} (i|1)^{s_{i1}} (i|2)^{s_{i2}} \cdots (i|d)^{s_{id}}$$

and let $\alpha \in A_0$ be a *positive virtual symbol*. Given $j_h = 1, 2, \ldots, n$, consider the action of the superpolarization D_{α, j_h} on the supersymmetric algebra $\mathbb{C}[M_{m_0|m_1+n,d}] \supseteq \mathbb{C}[M_{n,d}]$. A straightforward computation shows that

$$D_{\alpha,j_h}(\mathbb{M}) = \sum_{\varphi=1}^{d} \partial_{(j_1|\varphi)}(\mathbb{M})(\alpha|\varphi). \tag{29}$$

Furthermore, notice that:

$$D_{\alpha_s,j_k} D_{\alpha_t,j_h}(\mathbb{M}) = \sum_{\varphi=1}^{d} \Big(D_{\alpha_s,j_k}\big(\partial_{(j_1|\varphi)}(\mathbb{M})\big)\Big) (\alpha_t|\varphi),$$

that equals

$$\sum_{\varphi_1,\varphi_2=1,2,\ldots,d} \Big(\partial_{(j_k|\varphi_1)} \partial_{(j_h|\varphi_2)}(\mathbb{M})\Big) (\alpha_s|\varphi_1)(\alpha_t|\varphi_2). \tag{30}$$

Recall that the action of the column Capelli bitableau

$$\left[\begin{array}{c|c} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array}\right] \in \mathbf{U}(gl(n))$$

on the algebra $\mathbb{C}[M_{n,d}]$ is implemented by the product of superpolarizations

$$D_{i_1,\alpha_1} \cdots D_{i_{h-1},\alpha_{h-1}} D_{i_h,\alpha_h} D_{\alpha_1,j_1} \cdots D_{\alpha_{h-1},j_{h-1}} D_{\alpha_h,j_h},$$

where $\alpha_1, \ldots, \alpha_{h-1}, \alpha_h$ are distinct arbitrary positive virtual symbols. Note that $|D_{i_r,\alpha_r}| = |D_{\alpha_r,j_r}| = 1 \in \mathbb{Z}_2$, for every $r = 1, 2, \ldots, h$.

From Eqs. (29) and (30), it immediately follows:

$$D_{\alpha_1,j_1} \cdots D_{\alpha_h,j_h}(\mathbb{M}) = \sum_{(\varphi_1,\ldots,\varphi_h)\in \underline{d}^h} \partial_{(j_1|\varphi_1)} \cdots \partial_{(j_h|\varphi_h)}(\mathbb{M})(\alpha_1|\varphi_1)\cdots(\alpha_h|\varphi_h) \tag{31}$$

Since $|(\alpha_r|\varphi_r)| = 1 \in \mathbb{Z}_2$, for every $r = 1, 2, \ldots, h$, from Eq. (31), we infer:

$$D_{i_1,\alpha_1} \cdots D_{i_{h-1},\alpha_{h-1}} D_{i_h,\alpha_h}\Big(D_{\alpha_1,j_1} \cdots D_{\alpha_{h-1},j_{h-1}} D_{\alpha_h,j_h}(\mathbb{M})\Big)$$

equals

$$(-1)^{\binom{h}{2}} \sum_{(\varphi_1,\varphi_2,\ldots,\varphi_h)\in \underline{d}^h} (i_1|\varphi_1)(i_2|\varphi_2)\cdots(i_h|\varphi_h)\ \partial_{(j_1|\varphi_1)}\ \partial_{(j_2|\varphi_2)} \cdots \partial_{(j_h|\varphi_h)}(\mathbb{M}).$$

□

5 Capelli immanants and Young-Capelli bitableaux in $\mathbf{U}(gl(n))$

The *bitableaux correspondence* (linear) isomorphism

$$\mathcal{K}^{-1} : \mathbb{C}[M_{n,n}] \to \mathbf{U}(gl(n)),$$

leads to the following *natural* definition of *Capelli immanant*

$$Cimm_\lambda[i_1 i_2 \cdots i_h; j_1 j_2 \cdots j_h]$$

in the enveloping algebra in $\mathbf{U}(gl(n))$:

$$Cimm_\lambda[i_1 i_2 \cdots i_h; j_1 j_2 \cdots j_h] = \mathcal{K}^{-1}\Big(imm_\lambda(i_1 i_2 \cdots i_h; j_1 j_2 \cdots j_h)\Big).$$

By linearity of the operator $\mathcal{K}^{-1}$, we get

$$Cimm_\lambda[i_1 i_2 \cdots i_h; j_1 j_2 \cdots j_h] = \sum_{\sigma \in \mathbf{S}_h} \chi^\lambda(\sigma) \left[\begin{array}{c|c} i_{\sigma(1)} & j_1 \\ i_{\sigma(2)} & j_2 \\ \vdots & \vdots \\ i_{\sigma(h)} & j_h \end{array}\right]$$
$$= \sum_{\sigma \in \mathbf{S}_h} \chi^\lambda(\sigma) \left[\begin{array}{c|c} i_1 & j_{\sigma(1)} \\ i_2 & j_{\sigma(2)} \\ \vdots & \vdots \\ i_h & j_{\sigma(h)} \end{array}\right].$$

Clearly, the notion of Capelli immanants provides a natural generalization of the notion of Capelli determinant (see Example 3.8).

Since a Young-Capelli bitableau $[U|\boxed{V}] \in \mathbf{U}(gl(n))$ is the image of the right symmetrized bitableau $(U|\boxed{V}) \in \mathbb{C}[M_{n,n}]$ with respect to the isomorphism $\mathcal{K}^{-1}$, Proposition 2.12 implies

Theorem 5.1. *Let $\lambda \vdash h$. Any Capelli immanant $Cimm_\lambda[i_1 i_2 \cdots i_h; j_1 j_2 \cdots j_h]$ can be written as a linear combination of standard Young-Capelli bitableaux $[U|\boxed{V}]$ in $\mathbf{U}(gl(n))$ of the **same** shape λ:*

$$Cimm_\lambda[i_1 i_2 \cdots i_h; j_1 j_2 \cdots j_h] = \sum_{U,V} \varrho_{U,V} \ [U|\boxed{V}],$$
$$\varrho_{U,V} \in \mathbb{C}, \quad sh(U) = sh(V) = \lambda.$$

From Corollary 2.13, it follows:

Corollary 5.2. *Let $\lambda \vdash h$. If $\lambda_1 \nleq n$, then*

$$Cimm_\lambda(i_1 i_2 \cdots i_h; j_1 j_2 \cdots j_h) = 0.$$

Furthermore, Proposition 2.15 implies

Theorem 5.3. *Let $\lambda \vdash h$. Any Young-Capelli bitableau $[U|\boxed{V}]$ in $\mathbf{U}(gl(n))$ of shape $sh(U) = sh(V) = \lambda$ can be written as a linear combination of Capelli immanants $Cimm_\lambda[i_1 i_2 \cdots i_h; j_1 j_2 \cdots j_h]$ associated to the **same** shape λ.*

Proposition 2.16 implies

Theorem 5.4. *The set of Capelli immanants*

$$\bigcup_{k=0}^{h} \left\{Cimm_\lambda[i_1 i_2 \cdots i_k; j_1 j_2 \cdots j_k]; \lambda \vdash k, \lambda_1 \leq n,\ (i_1 i_2 \cdots i_k), (j_1 j_2 \cdots j_k) \in \underline{n}^k\right\}$$

is a spanning set of $\mathbf{U}(gl(n))^{(h)}$.

6 Quantum immanants

Our main result is a description of *quantum immanants* as simple linear combinations of *Capelli immanants*. This result – in combination with Proposition 4.7 – allows the computation of quantum immanants as elements of $\mathbf{U}(gl(n))$ to be reduced to a fairly simple process (see, e.g. Example 38 below).

Quantum immanants are the preimages of the shifted Schur polynomials [47], [42], with respect to the Harish-Chandra isomorphism.

We follow the notational conventions of Okounkov [40] and [41].

Remark 6.1. *Given a partition $\mu \vdash h$, V^μ denotes the irreducible representation associated to μ in the sense of James and Kerber [28]. We recall that the irreducible representation V^μ is the representation associated to the shape $\widetilde{\mu}$ in the notation of the previous sections of this work.*

Furthermore:

- $\mathbf{T}$ denotes a *multilinear standard* Young tableau of shape $sh(\mathbf{T}) = \mu \vdash h$.
- For every $s = 1, 2, \ldots, h$, let (i, j) be the pair of row and column indices of the cell of $\mathbf{T}$ that contains s. Set $\mathbf{c_T}(s) = j - i$ (the "Frobenius content" of the cell (i, j)).
- $v_\mathbf{T}$ denotes the element of the *seminormal Young basis* of V^μ associated to the multilinear standard tableau $\mathbf{T}$. Since each basis vector is defined only up to a scalar factor, we assume that $(v_\mathbf{T}, v_\mathbf{T}) = 1$ (see Okounkov and Vershik [43]; for a more traditional approach, see James and Kerber [28]).
- given the element

$$\Psi_\mathbf{T} = \sum_{\sigma \in \mathbf{S}_h} (\sigma \cdot v_\mathbf{T}, v_\mathbf{T}) \sigma^{-1} \in \mathbb{C}[\mathbf{S}_h],$$

  $\Psi^h_\mathbf{T}$ denotes the matrix that represents the element $\Psi_\mathbf{T}$ as a linear operator on the tensor space $(\mathbb{C}^n)^{\otimes h}$.

– Let $E = [e_{ij}]_{i,j=1,2,\ldots,n}$ be the matrix whose entries are the elements of the standard basis of $gl(n)$.

– Let

$$\mathbb{E}_{\mathbf{T}} = \big((E - \mathbf{c}_{\mathbf{T}}(1)) \otimes (E - \mathbf{c}_{\mathbf{T}}(2)) \otimes \cdots \otimes (E - \mathbf{c}_{\mathbf{T}}(h))\big)\ \Psi^h_{\mathbf{T}}$$

be the *fusion* matrix; the fusion matrix $\mathbb{E}_{\mathbf{T}}$ is a $(n^h \times n^h)-$matrix with entries in $\mathbf{U}(gl(n))$.

Following Okounkov ([41], [40]), the element

$$Tr\big(\mathbb{E}_{\mathbf{T}}\big) \in \mathbf{U}(gl(n))^{(h)}$$

is the *quantum immanant* associated to the multilinear standard tableau $\mathbf{T}$, $sh(\mathbf{T}) = \mu$.

The *higher Capelli identities* ([41], [40]), imply ([41], Eq. (5.1)) that the action of the quantum immanant

$$Tr\big(\mathbb{E}_{\mathbf{T}}\big), \quad sh(\mathbf{T}) = \mu$$

on the algebra $\mathbb{C}[M_{n,d}]$ equals the action of the polynomial differential operator

$$\frac{1}{dim(V^{\mu})}\ Tr\Big(X^{\otimes h}\ (D')^{\otimes h}\ \overline{\chi}^h_{\widetilde{\mu}}\Big), \tag{32}$$

where

– X denotes the matrix $\big[(i|\varphi)\big]_{i=1,\ldots,n;\varphi=1,\ldots,d}$

– D denotes the matrix $\big[\partial_{(i|\varphi)}\big]_{i=1,\ldots,n;\varphi=1,\ldots,d}$ of partial derivatives on the algebra $\mathbb{C}[M_{n,d}]$, and the prime stands for transposition.

– $\overline{\chi}^h_{\widetilde{\mu}}$ denotes the matrix that represents the element

$$\overline{\chi}_{\widetilde{\mu}} = \sum_{\sigma \in \mathbf{S}_h} \chi^{\widetilde{\mu}}(\sigma)\sigma \in \mathbb{C}[\mathbf{S}_h]$$

of Eq. (11) as a linear operator on the tensor space $(\mathbb{C}^n)^{\otimes h}$.

Since the action of $\mathbf{U}(gl(n))$ on the algebra $\mathbb{C}[M_{n,d}]$ is a faithful action whenever $n \leq d$, and the differential operator of Eq. (32) is independent from the choice of the multilinear standard tableau $\mathbf{T}$, the quantum immanant $Tr\big(\mathbb{E}_{\mathbf{T}}\big)$ only depends on the shape μ.

Theorem 6.2. *The quantum immanant*

$$Tr\big(\mathbb{E}_{\mathbf{T}}\big), \quad sh(\mathbf{T}) = \mu$$

equals the linear combination of Capelli immanants:

$$(-1)^{\binom{h}{2}} \sum_{h_1+h_2+\cdots+h_n=h} \frac{H(\mu)}{h_1!h_2!\cdots h_n!}\ Cimm_{\widetilde{\mu}}[1^{h_1}2^{h_2}\ldots n^{h_n}; 1^{h_1}2^{h_2}\ldots n^{h_n}], \tag{33}$$

where $1^{h_1}2^{h_2}\ldots n^{h_n}$ *is a short notation for the non decreasing sequence* $i_1 i_2 \cdots i_h$ *with*

$$h_p = \sharp\{i_q = p;\ q = 1, 2, \ldots, h\}, \quad p = 1, 2, \ldots, n.$$

Proof. For every $\sigma \in \mathbf{S}_h$, $\overline{i} = (i_1, \ldots, i_h) \in \underline{n}^h$, $\overline{\varphi} = (\varphi_1, \ldots, \varphi_h) \in \underline{d}^h$, we set

$$P_\sigma[\overline{i}; \overline{\varphi}] = (i_1|\varphi_1)\cdots(i_h|\varphi_h)\ \partial_{(i_{\sigma(1)}|\varphi_1)}\cdots\partial_{(i_{\sigma(h)}|\varphi_h)}.$$

By straightforward computation, the right-hand side of Eq. (32) equals

$$\frac{1}{dim(V^\mu)} \sum_{\overline{i}=(i_1,\ldots,i_h)\in\underline{n}^h} \Big(\sum_{\sigma\in\mathbf{S}_h} \chi^{\widetilde{\mu}}(\sigma)\ \big(\sum_{\overline{\varphi}=(\varphi_1,\ldots,\varphi_h)\in\underline{d}^h} P_\sigma[\overline{i};\overline{\varphi}]\big)\Big). \tag{34}$$

By Proposition 4.10, the action of the Capelli immanant

$$Cimm_{\widetilde{\mu}}[i_1 i_2\cdots i_h; i_1 i_2\cdots i_h] \in \mathbf{U}(gl(n))$$

on the algebra $\mathbb{C}[M_{n,d}]$ equals the action of the polynomial differential operator

$$(-1)^{\binom{h}{2}} \sum_{\sigma\in\mathbf{S}_h} \chi^{\widetilde{\mu}}(\sigma)\ \big(\sum_{\overline{\varphi}=(\varphi_1,\ldots,\varphi_h)\in\underline{d}^h} P_\sigma[\overline{i};\overline{\varphi}]\big),$$

for every $\overline{i} = (i_1, \ldots, i_h) \in \underline{n}^h$.

Since the action of $\mathbf{U}(gl(n))$ on the algebra $\mathbb{C}[M_{n,d}]$ is a faithful action whenever $n \leq d$, it immediately follows that any *quantum immanant* equals - up to a scalar factor - the following linear combinatio of *Capelli immanants*:

$$Tr\big(\mathbb{E}_{\mathbf{T}}\big) = (-1)^{\binom{h}{2}} \frac{1}{dim(V^\mu)} \sum_{(i_1,\ldots,i_h)\in\underline{n}^h} Cimm_{\widetilde{\mu}}[i_1 i_2\cdots i_h; i_1 i_2\cdots i_h] \in \mathbf{U}(gl(n)). \tag{35}$$

Since

$$Cimm_{\widetilde{\mu}}[i_1 i_2\cdots i_h; i_1 i_2\cdots i_h] = Cimm_{\widetilde{\mu}}[i_{\tau(1)} i_{\tau(2)}\cdots i_{\tau(h)}; i_{\tau(1)} i_{\tau(2)}\cdots i_{\tau(h)}],$$

for every $\tau \in \mathbb{C}[\mathbf{S}_h]$, the right-hand side of Eq. (35) equals

$$(-1)^{\binom{h}{2}} \sum_{h_1+h_2+\cdots+h_n=h} \frac{H(\mu)}{h_1!h_2!\cdots h_n!}\ Cimm_{\widetilde{\mu}}[1^{h_1}2^{h_2}\ldots n^{h_n}; 1^{h_1}2^{h_2}\ldots n^{h_n}].$$

□

From Theorem 6.2 and Corollary 5.2, it follows:

Corollary 6.3. *Let* $\mathbf{T}$ *be a multilinear standard tableau,* $sh(\mathbf{T}) = \mu$.

If $\widetilde{\mu}_1 \not\leq n$, *then*

$$Tr\big(\mathbb{E}_{\mathbf{T}}\big) = 0.$$

□

Let μ with $\widetilde{\mu}_1 \leq n$, and let recall that $\boldsymbol{\zeta}(n)$ is the center of $\mathbf{U}(gl(n))$. According with Okoukov [41], [40], the *Schur element* $\mathbf{S}_\mu(n) \in \boldsymbol{\zeta}(n)$ is defined by setting

$$\mathbf{S}_\mu(n) = \frac{dim(V^\mu)}{h!}\ Tr\big(\mathbb{E}_{\mathbf{T}}\big).$$

Since $dim(V^\mu) = \frac{h!}{H(\mu)}$, Theorem 6.2 implies:

Corollary 6.4.

$$\mathbf{S}_\mu(n) = (-1)^{\binom{h}{2}} \sum_{h_1+\cdots+h_n=h} \frac{1}{h_1!\cdots h_n!}\ Cimm_{\widetilde{\mu}}[1^{h_1}\ldots n^{h_n}; 1^{h_1}\ldots n^{h_n}]. \tag{36}$$

Remark 6.5. *If* $\mu = (1^h)$, $h \leq n$, *is the column shape of length* h, *then* $\mathbf{S}_{(1^h)}(n)$ *is immediately recognized as the* $h-th$ determinantal Capelli generator $\mathbf{H}_h$ *(see, e.g.* [8], [4]*; see also Capelli* [12], [14], [15] *and* [16]*, Howe and Umeda* [27]*).*

If $\mu = (h)$ *is the row shape of length* h, *then* $\mathbf{S}_{(h)}(n)$ *is immediately recognized as the* $h-th$ permanental Nazarov–Umeda generator $\mathbf{I}_h$ *(see, e.g.* [4]*, Nazarov* [38]*, Umeda* [51]*).*

Example 6.6. Let $h = 3$, $\mu = (2,1) = \widetilde{\mu}$, $n = 2$. Recall that $H(\mu) = 3$. Then

$$\mathbf{S}_{(2,1)}(2) = -\frac{1}{2}\left(Cimm_{(2,1)}[112;112] + Cimm_{(2,1)}[122;122]\right) \tag{37}$$

$$= -\left[\begin{array}{c|c} 1 & 1 \\ 1 & 1 \\ 2 & 2 \end{array}\right] + \left[\begin{array}{c|c} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{array}\right] - \left[\begin{array}{c|c} 1 & 1 \\ 2 & 2 \\ 2 & 2 \end{array}\right] + \left[\begin{array}{c|c} 1 & 2 \\ 2 & 1 \\ 2 & 2 \end{array}\right]. \tag{38}$$

By Eq. (38) and Proposition 4.7, we have:

$$\begin{aligned}\mathbf{S}_{(2,1)}(2) &= + e_{11}^2 e_{22} - e_{11}e_{22} - e_{11}e_{12}e_{21} + e_{11}^2 + e_{12}e_{21} - e_{11} \\ &\quad + e_{11}e_{22}^2 - e_{11}e_{22} - e_{12}e_{21}e_{22} + e_{11}e_{22} + e_{12}e_{21} - e_{11} \\ &= + e_{11}^2 e_{22} - e_{11}e_{12}e_{21} + e_{11}e_{22}^2 - e_{12}e_{21}e_{22} \\ &\quad - e_{11}e_{22} + e_{11}^2 + 2e_{12}e_{21} - 2e_{11} \in \mathbf{U}(gl(2)).\end{aligned}$$

□

Remark 6.7. *According to* Theorem 5.1*, the central element* $\mathbf{S}_{(2,1)}(2)$ *also equals, in turn, a linear combination of Young-Capelli bitableaux. Indeed, we* *have*

$$
-\frac{1}{2}\left[\begin{array}{ll} 1 & 2 \\ 1 & \end{array}\middle|\boxed{\begin{array}{ll} 1 & 2 \\ 1 & \end{array}}\right] - \left[\begin{array}{ll} 1 & 2 \\ 2 & \end{array}\middle|\boxed{\begin{array}{ll} 1 & 2 \\ 2 & \end{array}}\right]
$$

$$
= -\left[\begin{array}{c|c} 1 & 1 \\ 1 & 1 \\ 2 & 2 \end{array}\right] + \left[\begin{array}{c|c} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{array}\right] - \left[\begin{array}{c|c} 1 & 1 \\ 2 & 2 \\ 2 & 2 \end{array}\right] + \left[\begin{array}{c|c} 1 & 2 \\ 2 & 1 \\ 2 & 2 \end{array}\right] = \mathbf{S}_{(2,1)}(2).
$$

The previous identity is an instance of an alternative presentation (see our preliminary manuscript [10]*,* Subsection 4.5.1*) of the Schur element* $\mathbf{S}_\mu(n) \in \boldsymbol{\zeta}(n)$, $\widetilde{\mu}_1 \leq n$ *of* Corollary 6.4. $\square$

Given a pair of row (strictly) increasing tableaux S and T of shape $sh(S) = sh(T) = \widetilde{\mu} \vdash h$ on the proper alphabet $L = \{1, 2, \ldots, n\}$, consider the element

$$
e_{S, C^*_{\widetilde{\mu}}} \cdot e_{C^*_{\widetilde{\mu}}, D^*_{\widetilde{\mu}}} \cdot e_{D^*_{\widetilde{\mu}}, C^*_{\widetilde{\mu}}} \cdot e_{C^*_{\widetilde{\mu}}, T} \in Virt(m_0 + m_1, n) \subset \mathbf{U}(gl(m_0|m_1 + n)).
$$

We set

$$
[\,\boxed{S \mid T}\,] = \mathfrak{p}\Big(e_{S, C^*_{\widetilde{\mu}}} \cdot e_{C^*_{\widetilde{\mu}}, D^*_{\widetilde{\mu}}} \cdot e_{D^*_{\widetilde{\mu}}, C^*_{\widetilde{\mu}}} \cdot e_{C^*_{\widetilde{\mu}}, T}\Big) \in \mathbf{U}(gl(n)). \tag{39}
$$

and call the element $[\,\boxed{S \mid T}\,]$ a *double Young-Capelli bitableau.*

Proposition 6.8. *Any* double Young-Capelli bitableau *equals a sum of Young-Capelli bitableaux:*

$$
[\,\boxed{S \mid T}\,] = (-1)^{\binom{h}{2}} \sum_{\sigma} (-1)^{|\sigma|}\ [S|\boxed{T^\sigma}],
$$

where the sum is extended to all Young tableaux T^σ obtained from T by permutations of the elements of each row, and $(-1)^{|\sigma|}$ is the product of the signatures of row permutations.

Proof. See our preliminary manuscript [10]. $\square$

Theorem 6.9. *We have*

$$
\mathbf{S}_\mu(n) = \frac{1}{H(\mu)} \sum_{S} [\,\boxed{S \mid S}\,] \in \mathbf{U}(gl(n)) \tag{40}
$$

where the sum is extended to all row (strictly) increasing tableaux S of shape $sh(S) = \widetilde{\mu} \vdash h$ on the proper alphabet $L = \{1, 2, \ldots, n\}$.

Proof. (Sketch) Let χ_n be the Harish-Chandra isomorphism

$$
\chi_n : \boldsymbol{\zeta}(n) \longrightarrow \Lambda^*(n),
$$

where $\boldsymbol{\zeta}(n)$ is the center $\boldsymbol{\zeta}(n)$ of $\mathbf{U}(gl(n))$, and $\Lambda^*(n)$ is the algebra of *shifted symmetric polynomials* in n variables (see, e.g. [42]).

The technique is to show that both sides of Eq. (40) have the same image under the isomorphism χ_n.

The right-hand side of Eq. (40) is easily proved to be an element of the center $\boldsymbol{\zeta}(n)$, and its image via the Harish-Chandra isomorphism satisfies the hypotheses (see Theorem 6.64 of [10]) of the *Sahi/Okounkov Characterization Theorem* (Theorem 1 of [47] and Theorem 3.3 of [42], see also [40]) for the *Schur shifted symmetric polynomial* $s^*_{\mu|n}$ of [42]. Then

$$\chi_n\left(\frac{1}{H(\mu)}\sum_S [\,\boxed{S \mid S}\,]\right) = s^*_{\mu|n}.$$

Since

$$\chi_n\left(\frac{dim(V^\mu)}{h!}\,Tr\big(\mathbb{E}_{\mathbf{T}}\big)\right) = s^*_{\mu|n}$$

(see [41], [40]), the assertion follows (for details, see our preliminary manuscript [10]). $\square$

Example 6.10. We have

$$\begin{aligned}\mathbf{S}_{(2,1)}(2) &= \frac{1}{3}\Big(\left[\boxed{\begin{array}{cc|cc}1&2&1&2\\1&&1&\end{array}}\right] + \left[\boxed{\begin{array}{cc|cc}1&2&1&2\\2&&2&\end{array}}\right]\Big)\\ &= \frac{1}{3}\Big(-\left[\begin{array}{cc}1&2\\1&\end{array}\middle\|\boxed{\begin{array}{cc}1&2\\1&\end{array}}\right] + \left[\begin{array}{cc}1&2\\1&\end{array}\middle\|\boxed{\begin{array}{cc}2&1\\1&\end{array}}\right]\\ &\qquad -\left[\begin{array}{cc}1&2\\2&\end{array}\middle\|\boxed{\begin{array}{cc}1&2\\2&\end{array}}\right] + \left[\begin{array}{cc}1&2\\2&\end{array}\middle\|\boxed{\begin{array}{cc}2&1\\2&\end{array}}\right]\Big).\end{aligned}$$

Since

$$-\left[\begin{array}{cc}1&2\\1&\end{array}\middle\|\boxed{\begin{array}{cc}1&2\\1&\end{array}}\right] + \left[\begin{array}{cc}1&2\\1&\end{array}\middle\|\boxed{\begin{array}{cc}2&1\\1&\end{array}}\right] = -\frac{3}{2}\left[\begin{array}{cc}1&2\\1&\end{array}\middle\|\boxed{\begin{array}{cc}1&2\\1&\end{array}}\right]$$

and

$$-\left[\begin{array}{cc}1&2\\2&\end{array}\middle\|\boxed{\begin{array}{cc}1&2\\2&\end{array}}\right] + \left[\begin{array}{cc}1&2\\2&\end{array}\middle\|\boxed{\begin{array}{cc}2&1\\2&\end{array}}\right] = -3\left[\begin{array}{cc}1&2\\2&\end{array}\middle\|\boxed{\begin{array}{cc}1&2\\2&\end{array}}\right],$$

then

$$\begin{aligned}\mathbf{S}_{(2,1)}(2) &= \frac{1}{3}\Big(\left[\boxed{\begin{array}{cc|cc}1&2&1&2\\1&&1&\end{array}}\right] + \left[\boxed{\begin{array}{cc|cc}1&2&1&2\\2&&2&\end{array}}\right]\Big)\\ &= -\frac{1}{2}\left[\begin{array}{cc}1&2\\1&\end{array}\middle\|\boxed{\begin{array}{cc}1&2\\1&\end{array}}\right] - \left[\begin{array}{cc}1&2\\2&\end{array}\middle\|\boxed{\begin{array}{cc}1&2\\2&\end{array}}\right],\end{aligned}$$

as in Remark 6.7. $\square$

Presentation (40) is more supple and effective than presentation (36). Indeed:

- Presentation (40) doesn't involve the irreducible characters of symmetric groups.
- Presentation (40) is better suited to the study of the eigenvalues on irreducible $gl(n)-$modules, and of the duality in the algebra $\boldsymbol{\zeta}(n)$ (see our preliminary manuscript [10], Section 4).
- Presentation (40) is better suited to the study of the limit $n \to \infty$, via the *Olshanski decomposition*, see our preliminary manuscript [10], Section 5, Olshanski [44], [45] and Molev [35], pp. 928 ff.

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