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On Linear Existential Graphs

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Abstract

Peirce's linear versions of the language of his Existential Graphs (EGs), presented in 1902, are examined. Differences between standard three-dimensional and linear languages are explained by permutational invariance and type- vs. occurrence-referentiality: Standard EGs are permutationally invariant with respect to linear EGs, while the Beta part of the system, which corresponds to the first-order quantificational theory with identity, is occurrence-referential. This explanation contrasts with the proposal for linear Beta graphs that are type-referential. However, occurrence-referentiality of Beta graphs constitutes a defect of expressivity: since the meaning of a quantifier is inextricably connected to that of the meaning of the sign of identity, certain complex assertions arguably cannot be expressed in the language of Beta graphs without a new extension of its standard notation.

Keywords: · Existential Graphs · Linear Notations · Type vs. Occurrence-Referentiality · Quantification and Identity ·

1 Introduction

The linearisation problem of Peirce's logic of Existential Graphs (EGs) has not received much attention in the literature. Drawing up a linear version of what otherwise is the three-dimensional diagrammatic syntax of EGs has attracted some commentators, most notably Hammer (1995, 2011), to propose a design for a linear notation for these graphs. Hammer's focus in particular was the theory of Alpha graphs, which corresponds to classical propositional calculus (two-element Boolean algebra, see Ma & Pietarinen 2018c).

An examination of the problem of linearising Alpha and Beta systems has been undertaken in a couple of articles. By defining the cuts and juxtaposition as a set of generalised "either not . . . or not (i.e. not both)" of classical propositional logic, that is, such that "[$PR(Q)$]" means "either not-P or not-R or not-not-Q," Liu (2005) presented a Hilbert-style system for linear Alpha graphs. The most important advantage of this system is that both the decision procedure and the proof of the completeness theorem via counter-model construction are immediate. Dau (2006) surveyed the work on proofs with Alpha graphs. After revisiting his own previous work (Hammer 1995), Hammer also suggested that "equivalent systems could be constructed by defining the cut as the merge of parenthetical grouping with a different unary logical operator such as 'neither ... nor' or 'not both' " (Hammer 2011: 132). However, Hammer left the implementation of this suggestion open. His focus was on finding a linear notation for that Beta portion of EGs that would be equivalent to a fragment of first-order logic with identity, and to illustrate how Peirce's transformation rules for the Beta graphs could function in the light of that linear notation.

Hammer's aim was to provide a detailed analysis of the syntax, semantics and proof theory of the Alpha system, and to address some logical features of the system such as strong completeness that had not been fully examined before (see also Hammer 2008). Hammer felt, however, that such linearised graphs might lose some of the peculiar notational advantages of Peirce's own notation. He also noted that the theory of Beta graphs presents a more complicated case: "[C]ompared to the alpha system, the syntax, semantics, and rules of inference of Peirce's beta system are extremely complex. While some of the logic of beta is studied in Roberts 1973, Sowa 1984, and Zeman 1964, further investigation of its logic would certainly be worthwhile" (Hammer 1995: 97, fn1).

The present paper responds to these sentiments (i) by precisely defining the correct linear versions of the Alpha and Beta graphs, and (ii) by providing the necessary conceptual and notational tools for an analysis of the difference between linear and non-linear languages for (fragments of) first-order logic with identity, with a focus on Beta graphs. It also provides a systematic review of the central historical, notational, logical and conceptual issues involved in the problem of finding an adequate linearisation of Peirce's many-dimensional graphical or diagrammatic syntax of the logic of EGs (Peirce 1897, 1911a,b), including Peirce's own attempts at linear notations, some of which have not been noticed in the literature before (Peirce 1903a). The paper is organised as follows. Section 2 outlines the details of the standard syntax the Alpha and Beta parts of EGs, following Peirce's own definitions and remarks. Two analytic notions are then introduced, namely *permutational invariance* and the *type-referential vs. occurrence-referential* distinction. Sections 3 and 4 apply these notions to the historical and systematic treatment of the linearisation problem of both the Alpha and the Beta parts, respectively. In particular, Section 4 discusses how Hammer's suggestion for linearisation of EGs could be better implemented. Section 5 then shows how in the occurrence-referential notation of Beta graphs, the entanglement of the graphical sign for quantifiers (the line of identity) with that of identity implies a defect of expressivity, which is due to the manner in which the line of identity (the sign that stands both for identity and quantification) interacts with the cuts (the sign that stands both for negation and scope). Details of the proposed system to correctly define linear Beta graphs, termed linear-Beta*, together with its complete axiomatisation, are given in the Appendix.

2 Alpha and Beta Graphs: The Diagrammatic Syntax

By late 1896, Peirce had invented a new system and notation for logic, first entitled "Positive Logical Graphs" (Peirce 1896b) and soon afterwards renamed as "Existential Graphs" (Peirce 1896a,c). He estimated that this new method of graphs is his "*chef d'œuvre*" (Peirce 2019). A good number of systems and variants of logical graphs are found in his manuscripts and have mostly remained unpublished to date (but see Peirce 2019). Peirce's goal was to show the relevance of the graphical method to a number of topics in logic and in philosophy, including modality, the theory of signs, the doctrine of categories, the philosophy of mathematics, logic of science, and pragmaticism.[1] A brief description of this method was published in his 1906 paper, "Prolegomena to an Apology for Pragmaticism" (Peirce 1906a). Only some rather disorganised selections and fragments concerning EGs were reprinted in Volume 4, Book 2 of the *Collected Papers of Charles S. Peirce* (Peirce 1933; hereafter CP). EGs were examined in Roberts (1973), Zeman (1964) and Shin (2002), among others, but overall the amount of relevant studies has remained modest. Yet Peirce considered EGs to be the best method to carry out a logical analysis of meanings, confidence that may be attributed to the unique notational and geometrical apparatus of the proposed systems. In a letter to William James, Peirce writes that the system of logical graphs "ought to be the Logic of the Future" (25 December 1909; Peirce 2019).

By 1903, Peirce had divided EGs into Alpha, Beta and Gamma parts, which roughly correspond to propositional logic, fragments of first-order logic with identity, and modal and higher-order logics, respectively (Peirce 2020a). These systems exploit two or three-dimensional notations, in which formulas are considered as projections or images of graphs that in reality are higher-dimensional, onto the surface termed the *sheet of assertion* (SA). *Conjunction* is indicated by *juxtaposing* graphs as connected by the blank space of the sheet of assertion. *Negation* is typically indicated by enclosing a graph within an oval, which is a continuous simple closed curve and which Peirce since 1903 calls a "cut".

[1] See e.g. Bellucci (2017), Bellucci & Pietarinen (2016a,b,c); Ma & Pietarinen (2017a,b); Bellucci & Pietarinen 2017; Bellucci, Moktefi & Pietarinen (2018); Pietarinen (2014, 2016, 2020).

Cuts have two functions: their primary function is to *group* elements that occur in the interior area together (and as such they function in the same way as parentheses, dots, vincula, etc. in other logical notations). Peirce call those signs that represent such *notational function* and can have distinct notational realisations "collectional signs". The second function of the cuts is to *negate* the graphs that occur in that interior area. Cuts thus combine the truth-functional role (as negations) with the notational role (as groupings). In standard, linear notation, these two functions are typically represented by distinct notational realisations, such as "$\neg$" and "(", ")".

The SA is, topologically speaking, an *isotropic* space, i.e. an open-compact manifold unordered in all directions. The ordering is introduced into the sheet by the cuts and their nesting. As collectional signs, cuts divide the SA into *areas* that may be nested but may not partially overlap. In Fig. 1, for example, the SA is divided into three areas: the area that is outside the outermost cut, the area that is inside the outermost cut and outside the innermost cut (the one that has P on it), and the area that is inside the innermost cut (the one that has Q on it).

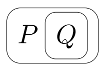


Figure 1.

It is commonplace to take the graph of Fig. 1 to represent a conditional structure. Indeed the structure in which there is one cut inside the other—often drawn with one continuous line that has one intersection point:  —was termed by Peirce the *scroll*.[2]

Unlike ordinary natural or logical languages, the graphs are not interpreted from left-to-right or from right-to-left, but what Peirce terms *endoporeutically*: the interpretation proceeds inwardly; so that a nest draws in the meaning from without inwards unto its center, "as a sponge absorbs water" (Peirce 1910). The endoporeutic interpretation prescribes that the outermost area is interpreted first, proceeding inwardly to the areas enclosed in whatever cut is placed on the first, and so on in a recursive fashion. In other words, any enclosed area depends on, or is in the scope of, the area on which its enclosing cut lies. Thus the graph in Fig. 1 may be read, in the standard language of sentential calculus, as "$\neg(P \& \neg Q)$."

[2]One manuscript sheet exists (R 1010, undated) in which Peirce proposes scrolls to be unfolded so that the inloop of the scroll is moved to the outside to form a connected loop with the outloop, distinguished from the outloop with a dashed boundary.

Since the SA and any area on it are isotropic spaces, no continuous transformation that preserves the topological property of "lying within the same area" produces syntactically distinct graphs. Any such transformation only produces syntactical variants, or *tokens*, of one and the same graph *type*. In other words, only the juxtaposition of graphs on the same area counts as a syntactically relevant fact, while their position and orientation within an area is syntactically irrelevant. All positions within an area are equivalent. Thus, each of the Alpha graphs in Figs. 2(a)–(d) is a distinct token of the same graph-type. By contrast, formulas in (2a*) and (2b*) are not distinct sentence-tokens of the same sentence-type but distinct sentence-types (that is, syntactically distinct sentences), which in ordinary sentential logic are logically equivalent.

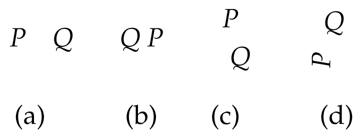


Figure 2.

(2a*) $P \,\&\, Q$
(2b*) $Q \,\&\, P$

The reason for this divergence is that while the notation in which (2a*) and (2b*) are written is linear, and thus any permutation of elements in a string of characters results in a syntactically distinct string (whether logically equivalent to the permuted string or not), the Alpha graphs are *non-linear* in that any mutation of the position and orientation of graphs lying on the same area (since all such areas are isotropic spaces) results in distinct graph tokens of the same graph type. So the Alpha graphs in Fig. 2 do not differ in the manner in which (2a*) and (2b*) differ, but only in the trivial, typographical manner in which two or more notational variants of the same sentence, say, $P \& Q$ and $P \;\;\&\;\; Q$, may be taken to differ.

Another way of stating the same thing is that while any statement of equivalence of the kind '(2a*) is equal to (2b*)', is a statement of a *logical* equivalence (that is, two distinct sentence types are said to have the same truth-conditions), any statement of equivalence of the kind 'The graph in Fig. 2(a) is equal to the graph in Fig. 2(b)' (or 'is the same as') is a statement of *syntactical* equivalence (that is, a statement in which two distinct graphs tokens are said to be tokens of one and the same graph type). [3]

[3]See Bellucci & Pietarinen (2020).

The property of permutational invariance distinguishes EGs from customary logical languages. In particular, the fact that a graph can freely be scribed at any position in an area makes the inferential system of the graphs a true manifestation of what in the proof-theoretic literature is known as *deep inference* (Brünnler 2004, Schütte 1977). Indeed Peirce made the following observation, in 1901, concerning permutational invariance:

> Operations of commutation, like $xy \therefore yx$, may be dispensed with by not recognizing any order of arrangement as significant. Associative transformations, such as $(xy)z \therefore x(yz)$, which is a species of commutation, will be dispensed with in the same way; that is, by recognizing an equiparant as what it is, a symbol of an unordered set. (CP 4.374; DPP: 640).

Indeed associativity follows from a complete commutativity. The property of permutational invariance of Alpha graphs has been remarked upon in Dipert (1996) and Hammer (1996).

The system of Beta graphs adds to the syntax of Alpha the device of the *line of identity* (LI). LIs represent individuals occurring in the universe of discourse. But LIs also represent co-reference, which in first-order logic is represented by an equality sign, as in "$x = y$.' By drawing a line from one position on SA to another position, we assert the *identity* of the individuals denoted by the two extremities of the line. In standard notation, this would be expressed by "$\exists x \exists y(x = y)$". Thus the line also asserts the existence of the individual objects denoted by its extremities, branches, or outermost portions. So the line functions as an existential quantifier. A LI written on the SA unattached to any spot is a well-formed graph: Fig. 3 expresses what in standard notation, given its linear ordering, would be expressed by any of "$\exists x \exists y(x = y)$", "$\exists y \exists x(x = y)$", "$\exists x \exists y(y = x)$" or "$\exists y \exists x(y = x)$".

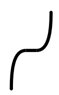
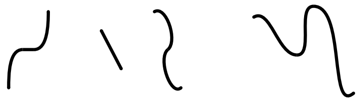


Figure 3.

Since the SA and any area marked by the cuts are isotropic, no continuous transformation of a graph-type produces a syntactically distinct graph-type. Continuous transformations of a graph-token only produce distinct graph-tokens of the same type. The same applies to LIs: no shortening, stretching, or curving of a LI changes its syntactical identity. Thus the LI in Fig. 3 is the same LI-type (homotopy) as any of (a–d) in Fig. 4. None of (a–d) is a distinct LI-type; there is only one LI-type, of which (a–d) are distinct tokens.

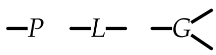


(a) (b) (c) (d)

Figure 4.

A *spot* is Peirce's term for the graphical representation of what in the standard notation of first-order predicate logic is a predicate term followed by a list of variables as its arguments. Instead of variables, a spot has *hooks*, namely connectors at which LIs can be anchored. Thus a spot with a certain number of hooks corresponds to a predicate with that number of variable-places. The arity of a spot is the number of its hooks. When a LI is attached to a hook of a spot, this attachment represents existential quantification that relates specifically to that hook of the spot. No hook can be occupied by more than one end of a line. A complex line (a branching line, line that abuts a cut) is termed by Peirce a *ligature*. A ligature that crosses a cut is not a well-formed graph.

The Beta graph in Fig. 5(a) is obtained by attaching a LI to the single hook of the monadic spot "P", thus existentially quantifying in that position ("$\exists x Px$"). The Beta graph in Fig. 5(b) is obtained by attaching two distinct LIs to the two hooks of the dyadic spot "L", thus existentially quantifying in both of those positions ("$\exists x \exists y Lxy$"). The Beta graph in Fig. 5(c) is obtained by attaching three distinct LIs to the three hooks of the triadic spot "G", thus existentially quantifying in those three positions ("$\exists x \exists y \exists z Gxyz$"). It is clear that the LI is a sign of identity, existence, and predication, at once. It is not possible to scribe a well-formed spot on the SA without attaching a dot or a LI to it. Any well-formed Beta graph is a closed formula, or sentence.



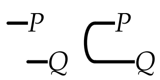
(a) (b) (c)

Figure 5.

This notational invention of LIs makes Beta graphs an occurrence-referential notation, as opposed to the more commonplace type-referential notations. We define these two properties as follows. A TYPE-REFERENTIAL notation is a language in which the identity of individuals is represented by the identity of the variable-type, each occurrence of the (unrenameable) variable-type referring to the *same* individual.[4] Thus, for example, in (6a) the fact that the variable x in "Fx" refers to the same entity as the variable x in "Gx" is expressed by the sameness of the sign for that variable, namely "x", which cannot be renamed without a change in the truth-conditions of the formula in question. One the other hand, in (6b) the possible difference between the individuals referred to by the two variables in "Lxy" is expressed by the difference of the sign for those variables, namely"x" and "y".

(6a) $\forall x(Fx \,\&\, Gx)$
(6b) $\forall x \exists y Lxy$

Beta graphs are not type-referential, because in each graph it is the *occurrence*, and not the type, of a LI that represents an individual. In the graph of Fig. 7(a) the two occurrences of LI may represent two distinct individuals ("$\exists x \exists y(Px \& Qy)$"), while in the graph of Fig. 7(b) the two occurrences are joined, thus becoming one single LI, which refers to one individual ("$\exists x(Px \& Qx)$"). Thus, let an occurrence-referential notation mean a language in which the identity of individuals is represented by the identity of the occurrence of the variable-type, and in which each occurrence of the type refers to a possibly *distinct* individual.[5]



(a) (b)

Figure 7.

Given the double role of the cuts (their truth-functional and collection-functional roles), in Beta graphs cuts acquire one more role: they show the logical priority between different ligatures. Consider the following two sentences:

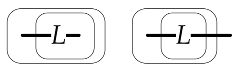
(8a) $\forall y \exists x Lxy$
(8b) $\exists x \forall y Lxy$

[4]Or a name, individual constant, etc. We forego such semantic differences in the interpretation of quantifiers for the moment. By "unrenameable" it is meant that one cannot rename a variable in a formula because it is bound by a quantifier that also has another variable in its binding scope with the same name. The formula is called *rectified* when no such renaming of variables can take place in it, that is, no variable occurs both bound and free and all quantifiers refer to different variables. The type- vs. occurrence-referential distinction also combines with the distinction between the *inclusive* and *exclusive* interpretations of variables; see Bellucci & Burton (2020).

[5]An *occurrence* has to be conceptually distinguished from a *token*: two tokens of one and the same formula are not two occurrences of it; an occurrence is such in reference to a sentential context (cf. Wetzel 1993); the sype- vs. occurrence-referentiality distinction was adumbrated in Stenning (2000).

The formulas (8a) and (8b) are two distinct sentences with different truth-conditions. Their difference is a difference in the logical dependence of the quantifiers, which is expressed by the linear ordering of them from left-to-right. Since EGs have abandoned linearity, they must find another means of expressing such relations of dependence. In the propositional language of Alpha graphs, the cuts are both signs of negation and collectional signs. When in the quantificational Beta part cuts interact with LIs, they also express the logical dependence relations between the ligatures composed of those LIs. The only syntactically relevant fact about the logical dependence of ligatures is the topological property of "lying within the same area" of their outermost extremities. In other words, a ligature is considered to be as much enclosed within cuts as its outermost portion is.

For example, in Fig. 9(a) the outermost portion of the ligature on the left of the spot is enclosed within one cut, while the outermost portion of the ligature on the right of the spot is enclosed within two cuts; thus, given the endoporeutic principle of interpretation, the first ligature has logical precedence over the second. The graph in Fig. 9(a) corresponds to (8a), that in Fig. 9(b) to (8b). The difference in quantificational dependencies, which in linear notation is expressed by the linear ordering of the quantifiers, in Beta graphs is expressed by means of the interaction of ligatures and cuts.



(a) (b)

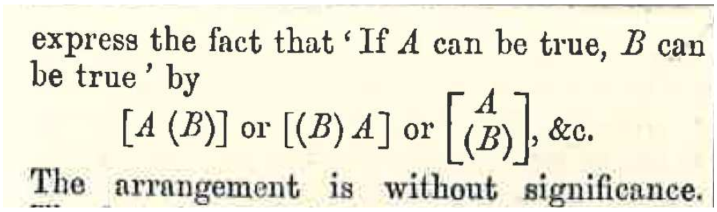
Figure 9.

In graphs involving somewhat more complex quantificational patterns, this interaction gives rise to an unexpected feature which is discussed in Section 4.

3 Linear Alpha Graphs

In 1902, Peirce published an article "Symbolic Logic" in the *Dictionary of Philosophy and Psychology* (Peirce 1902a, 2020b). There he uses, instead of the two-dimensional sheet and graphs projected on it, what looks like a linear, one-dimensional notation. Peirce intended this alternative notation to carry out the same tasks as two-dimensional graphs do, intending the two kinds of representations, linear and non-linear, to amount to expressively equivalent systems.

A snippet from the published version shows what Peirce's notation and the idea of the translation of Alpha graphs into this linear format look like (Peirce 1902a: 647):


express the fact that 'If A can be true, B can be true' by

$$[A\ (B)] \text{ or } [(B)\ A] \text{ or } \begin{bmatrix} A \\ (B) \end{bmatrix}, \text{ \&c.}$$

The arrangement is without significance.

Figure 10.

The qualification that "the arrangement is without significance" means that the order of representing graphs in different positions in the same area is immaterial. This, we have seen, is a crucial feature of the standard diagrammatic syntax of EGs. Peirce meant this feature to be preserved in the version of the notation of the graphs published in this dictionary entry.

Motivation for creating some simplified, linearised versions of graphs was largely dictated by the need to control the costs that the setting of new types would incur. The editor of the dictionary, James Mark Baldwin, had commissioned Peirce and his former student Christine Ladd-Franklin to write an entry on symbolic logic, but Baldwin was reluctant to include graphs in the entry, the reason being the complexity of printing curved lines and the space-consuming, two-dimensional format. Fabian Franklin (Christine's husband and also one of Peirce's former students and colleagues from the Johns Hopkins University in the early 1880s), whom Peirce had earlier sent a draft entry on "Exact Logic" for comment, sided with Baldwin in instructing Peirce that dictionary articles should not be used to promote new ideas and topics with no established role in the extant literature. The irony of his advice is that this entry is not only the first but in fact the only widely published article that ever emanated from Peirce's hand that presents the essentials of EGs, both the Alpha and Beta parts, in a sufficiently detailed and accessible manner.[6]

[6]The 1906 "Prolegomena" paper of *The Monist*, in contrast, charges EGs with many additional and tangential tasks, including the promised "proof of pragmaticism", and is much less successful in introducing the theory of logical graphs than the Dictionary entry. But getting the ideas across even in the Dictionary article was somewhat of an ordeal. Peirce revealed the editorial discussions that went on in the background to Royce on January 19, 1902: "As for my article on *Symbolic Logic*, Baldwin would allow no notation to be used not approved by Mrs. Franklin, and the rules laid down have had the effect of effectually preventing the expression of my doctrines. In fact, I could not go into the algebra at all, but only speak generally concerning it. I was not allowed to give my reasons for preferring one notation over another. I was not allowed even to introduce Euler's diagrams. I offered to supply the types for a few formulæ of my own; but this was not permitted" (R L 386, 1902). On the other hand, the 23-page pamphlet *Syllabus of Certain Topics of Logic* that Peirce wrote for his 1903 Lowell Lectures contained the basic conventions and definitions of EGs, but it was printed in at most 100 copies (Peirce 2020a).

In the linearised EG notation, cuts are indicated by matching pairs of parentheses, with brackets and braces added for convenience and readability. In Peirce's words, the common "habit is to cut [a graph] off from the main sheet by enclosing it within an oval line; but in order to facilitate the printing, we will here enclose it in square brackets" (Peirce 1902a: 646). For example, the Alpha graph presented in Fig. 1 is linearised as "$[P(Q)]$", which represents the conditional sentence "If P then Q".

Regarding the seemingly straightforward idea of linearising Alpha graphs in the fashion that the above snippet suggests, Hammer (2011) writes:

> For the alpha portion, cuts will be indicated in linear notation by matching parentheses. So "(**P**)" is equivalent to "not **P**," and "(**P** (**Q**))" is equivalent to "if **P** then **Q**" (rewritten as always into a combination of conjunction and negation). Observe that this is the same graph as "((**Q**) **P**)" because order is syntactically irrelevant. The two graphs are different tokens of the same graph. Juxtaposition and enclosure are the only relevant syntactic operations, and graphs are equivalent up to those two operations. (Hammer 2011: 130)

In this explanation, the graphs "$[P(Q)]$" and "$[(Q)P]$" are not different (i.e., syntactically distinct) graphs. They are, in our terminology, graph-*tokens* of the same graph-*type*. As noticed, this idea comes directly from Peirce—that "the arrangement is without significance" (Peirce 1902a: 647).

However, is such "syntactical irrelevance" of the arrangement really possible in a linear notation? Hammer—and the Peirce of the linear EGs—must think that it is. One could argue that even in a linear language, the possibility remains of stipulating that, while permutation of the elements of a string *generally* yields distinct string-types, yet in some cases, e.g. in the case of symmetric relations or operations (i.e., in those cases in which permutation would not alter the logical value), permutation only yields distinct string-tokens of the same type. In the case of linear Alpha graphs, one could say that while permutation of the elements of a linear graph *generally* yields distinct graph-types (for example, permuting 'P' and 'Q' in '$[P(Q)]$'), yet when the permuted elements are enclosed within the same number of parentheses (for example, 'P' and 'Q' in '$[P\,Q(R)]$') the permutation only yields distinct graph-tokens of the same type. In this way, '$[P(Q)]$' and '$[Q(P)]$' would *not* be the same graph-type, as is required by their logical difference, while '$[PQ(R)]$' would be the same graph-type as '$[QP(R)]$', as required by their logical equivalence.

Hammer's proposal for linear Alpha graphs is implicitly based on this arrangement. However, it is open to the following objection. Let us construct a language with a symmetric relation or operator (♠) and another, anti-symmetric relation or operation (♥). Thus, the idea is that one could stipulate that in such a language (11a) and (11b) are distinct formula-tokens of the same formula-type, while (12a) and (12b) are distinct formula-types.

(11a) ξ ♠ ζ
(11b) ζ ♠ ξ

(12a) ξ ♥ ζ
(12b) ζ ♥ ξ

Such stipulation would yield a notation with a dis-homogeneous syntax: one and the same syntactical operation (permutation of elements in a string) would *not* invariably yield the same syntactical result: with ♠, permutation would yield distinct tokens of the same type, while with ♥, it would yield distinct types. The difference is semantic: the outcome of the permutation of the elements flanking ♠ and the outcome of the permutation of the elements flanking ♥ differ because of the *meaning* of ♠ and ♥ (the former is symmetric and the latter anti-symmetric).

The proposed notation would be the outcome of a *systematic conflation* between syntax and semantics, and in particular a conflation of the well-defined concepts of syntactic equivalence and logical equivalence. In order to avoid this conflation, one only needs to recognise that in any linear notation whatever, *permutation always yields distinct types*, and that such rules of logical equivalence are introduced that distinguish the outcome of the permutation of the elements flanking ♠ (distinct types which are logically equivalent) from the outcome of the permutation of the elements flanking ♥ (distinct types which are not logically equivalent). To say that in a linear notation permutation *sometimes* yields only distinct tokens is to say that rules of commutation are introduced into the system (which are rules of logical equivalence).

If this objection is sound, then Peirce's and *ipso facto* Hammer's linear Alpha graphs are linear in the full sense of the term, because the permutation of elements in a string always yields distinct string-types and never distinct string-tokens of the same string-type. To say that in linear Alpha graphs the arrangement is without significance does not make those graphs less linear than the standard notation does. It only amounts to the implicit adoption of a generalised commutation rule, which is precisely what is done explicitly in linear languages with respect to symmetric relations or operations.

Let us consider two further remarks on the syntax of the linearised logical graphs by Hammer (2011):

> One problem of using parenthetical grouping to also serve as a unary logical operator concerns definitions. [. . .] The problem with trying to define a new logical operator such as disjunction within the alpha system is that a second parenthetical grouping syntactic mechanism becomes necessary, and the result is a step backwards in terms of visual power and elegance. Lost is the simplicity and power of odd and even areas of the graph. Also lost is the original iconicity of the system. (Hammer 2011: 132)
>
> Another issue raised by the linear notation is what makes Peirce's graphical notation diagrammatic/iconic, and whether all or some of that iconicity is lost in the linear notation. I believe that part of the iconic, graphical aspect of the existential graphs, the thing that gives them the visual power they have, is that typically separate syntactic elements are combined into a single dual-purpose syntactic element, resulting in substantial notational simplicity. (Hammer 2011: 139)

What is the kind of iconicity referred to in these passages which is presumably lost in linear Alpha graphs? Hammer is correct to observe that introducing a new logical operator by *fiat* in order to represent disjunction in linear Alpha would require the introduction of a new collectional sign as well. This is because parentheses, being already invested with the meaning of negation, cannot perform this collectional office alone, without at once also bearing the meaning of a truth-operation. It is evident, however, that no more than *one single kind* of collectional signs can be adopted in any system whatever, for otherwise one should stipulate the relations of dependence or interaction between the two or more kinds of collectional signs, which would amount to nothing else than the adoption of a *single* yet more complex system of collectional signs.

What Hammer says of linear Alpha graphs is also true of standard Alpha graphs: the cut functions both as a collectional sign and as a sign of negation. It would be impossible to introduce a new logical operator to represent disjunction, because this would require the introduction of a distinct collectional sign, and this is impossible. But then, the introduction of a new logical operator in Alpha graphs is not a loss in iconicity; it is no iconicity at all: due to the nature of the merging of collectional function and truth-function, no such introduction is possible at all. Nor is the presumed loss in iconicity due, as Hammer thinks, to the merging in one single syntactic device of a collectional function and a truth-function. This is precisely what happens in the linearised version of the graphs: just like the cuts of standard EGs, the parentheses fulfill both these functions to perfection.

What is actually lost in the linearised version of Alpha graphs is the permutational invariance that characterises the standard diagrammatic syntax of the graphs. But Hammer cannot see this because he considers linear Alpha graphs as permutation-invariant relative to elements enclosed within the same number of parentheses. This, we have argued, would conflate syntactical and logical equivalence and has to be rejected: Hammer's linear Alpha graphs are linear just as any other linear notation is. Therefore, the *only* feature that distinguishes the standard Alpha graphs from Hammer's linearised version is that in linear Alpha graphs the permutational invariance that characterises standard Alpha graphs is lost.

This difference can also be expressed by saying that the translation function from linear Alpha graphs to Alpha graphs is a many-one (non-injective, surjective) function: every Alpha graph-type corresponds to a class of logically equivalent linear Alpha graph-types which merely differ in the linear ordering of those elements that in the corresponding Alpha graph-type lie on the same area. For example, to each of the linear Alpha graph-types (13a–f) there corresponds the unique Alpha graph type in Fig. 14:

(13a) $[PQ(R)]$
(13b) $[QP(R)]$
(13c) $[(R)PQ]$
(13d) $[(R)QP]$
(13e) $[P(R)Q]$
(13f) $[Q(R)P]$

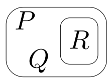


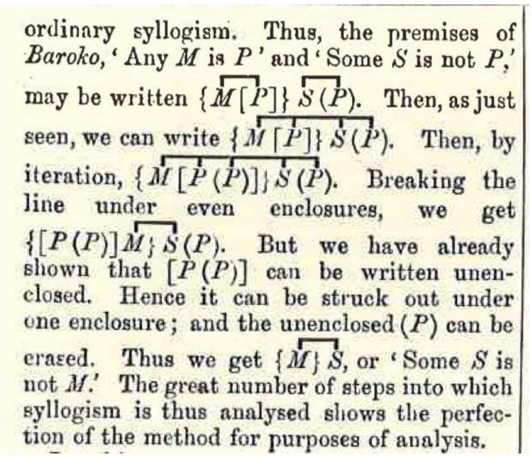
Figure 14.

Of this type, infinitely many distinct tokens may be produced.

4 Linear Beta Graphs

In this section, we compare Hammer's linearisation of Beta graphs with a linearisation in which quantification is merged with the cut operator. The linearisation of Beta graphs builds upon the principles of the linearisation of Alpha graphs. LIs and ligatures are connectors drawn both above and below one-dimensional, concatenated strings of punctuation marks and letters. Again, Peirce's motivation in proposing such compressed format for Beta expressions was to facilitate printing graphs in a more economical fashion, only requiring oversetting multiple lines over character types.

The proposal for the linear Beta notation is presented in the same Dictionary article (Fig. 15, from Peirce 1902a: 649):



ordinary syllogism. Thus, the premises of *Baroko*, 'Any M is P' and 'Some S is not P,' may be written $\{M[P]\}\ S(P)$. Then, as just seen, we can write $\{M[P]\}\ S(P)$. Then, by iteration, $\{M[P(P)]\}\ S(P)$. Breaking the line under even enclosures, we get $\{[P(P)]M\}\ S(P)$. But we have already shown that $[P(P)]$ can be written unenclosed. Hence it can be struck out under one enclosure; and the unenclosed (P) can be erased. Thus we get $\{M\}\ S$, or 'Some S is not M.' The great number of steps into which syllogism is thus analysed shows the perfection of the method for purposes of analysis.

Figure 15.

LIs are indicated by heavy lines; according to Peirce this "very iconoidal way of representing that there is one quasi-instant at which both A and B are true" is A—B (Peirce 1902a: 648). For example, the linearised Beta graphs [A—$\leftarrow$—B]and [$(\overline{A)(B}$)] express "whatever A there may be is B" and "everything is either A or B", respectively. "In taking account of relations", he then states, "let l be taken in such a sense that X—l—Y means 'X loves Y',... if m— means something is a man, and —w means something is a woman, m—l—w will mean 'Some man loves some woman'; m [(l) w] means 'Some man loves all women'; [(m–l) w] means 'Every woman is loved by some man', etc." (Peirce 1902a: 649).

In order to incorporate these two basic elements of Beta graphs into linear notation, namely to match variables with the LIs and to mark the area with the outmost portion of LIs in order to indicate quantifier scope, Hammer introduces the standard existential quantifier '$\exists x$', together with a modification of the idea that is derived from Peirce's use of matching variables which he calls "selectives", similar to proper names (Pietarinen 2010) and substituting characters for the LIs and ligatures:

> [T]here is an unavoidable intersection of two lines of identity. In such a case, and indeed in any case in which the lines of identity become too intricate to be perspicuous, it is advantageous to replace some of them by signs of a sort that in this system are called *selectives*. A selective is very much of the same nature as a proper name; for it denotes an individual and its outermost occurrence denotes a wholly indesignate individual of a certain category (generally a thing) existing in the universe, just as a proper name, on the first occasion of hearing it, conveys no more. (CP 4.460)

Accordingly, the Beta graph in Fig. 16 is represented in Hammer's linear Beta as "$\exists x(Ax)$":

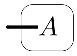


Figure 16.

In contexts that give rise to no ambiguities, that is, in contexts in which there is no need for a separate symbol for universal quantification (as may be required in intuitionistic Beta graphs, for example, see Bellucci, Chiffi & Pietarinen 2020), there is no reason to use any symbol such as "$\exists$" to express existential quantification. Since the only office of "$\exists x$" is to mark the binding scope of quantification, this is perfectly achieved by writing "x" rather than "$\exists x$". Thus, Hammer's Beta formula "$\exists x(Ax)$" may be simplified to "$x(Ax)$" without loss of expressivity (one should notice that Geach 1981 and Soames 1983 exploit similar techniques in their version of a Tractarian first-order logic). The crux of the matter is that once there is a means to represent the identity of individuals, the only other thing is to spell out how to represent scope.

Hammer provides some definitions for linear notation:

> The definition of an open graph and subgraph is as follows. (1) '$\exists x$' is an open graph for any variable 'x'. (2) $Rx_1 \ldots x_i$ is an open graph for any relation R and i variables. (3) Given graphs 'G' and 'H' their concatenation/juxtaposition 'GH' is an open graph and 'G' and 'H' are subgraphs of it. (4) Given any open graph 'G', its enclosure within a cut, '(G)' is an open graph and 'G' is a subgraph of it. (5) Given any open graph 'G' and variable 'x' the graph '$\exists xG$' is an open graph and 'G' is a subgraph of it.
>
> The definition of a graph, scope and free are as follows. (i) The scope of a quantifier '$\exists x$' is any subgraph contained within all graphs in which '$\exists x$' falls. (ii) A variable x occurs free within a graph if it does not fall within the scope of a quantifier '$\exists x$'. (iii) A graph or closed graph is any open graph in which no variable occurs free. (Hammer 2011: 134)

However, Hammer does not reveal how to represent the following graph in his linear notation, which corresponds to "$\neg\exists x(Px \wedge Qx)$" in first-order logic:

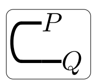


Figure 17.

The correct representation is "$(xPxQx)$", or just simply "$(PxQx)$", because the scope of a quantifier is defined *only* in relation to the cut.

The graph of Fig. 17 is the only example in Hammer (2011) that involves a simultaneous application of the LI and juxtaposition. By it, Hammer intends to shows how variables can be substituted for LIs, linking them by the same variable name use:

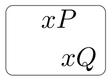


Figure 18.

But what does the graph in Fig. 18 say? Does it say that "it is not the case that something is both Px and Qx" or that "it is not the case that there exists both an x such that Px and an x such that Qx"? Hammer does not provide an explanation. This imperfection also obscures the definition of an open graph. According to his definition, the result obtained from items (3) and (5) cannot be well-formed: for how could we combine the juxtaposition "$G\,H$" of two open graphs with the quantifier "$\exists x$" with the matching parentheses used as both negation and actually as the scope of their juxtaposition? "$\exists x G\,H$" in Hammer's linear notation means the classical first-order formula "$(\exists x Gx) \wedge H$", but "$\exists x(GH)$" means "$\exists x \neg(Gx \wedge Hx)$". The trouble comes from his introducing "other types of definitions well within the system, such as collapsing a subgraph into an abbreviation 'G'" (Hammer 2011: 132), namely the subgraph "$G\,H$" abbreviated as a single letter which could attach to the quantifier "$\exists x$" according to the item (5) of the definition.

Adding the double-cut around "$G\,H$" will repair the problem. We will thus make such a modification to Hammer's notation. The details and an axiomatic system for the notation resulting from this change is presented in the Appendix. To see how this improvement works, let us begin with the following Beta graph:

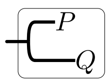


Figure 19.

In Hammer's notation, the graph in Fig. 19 is represented as "$\exists x(PxQx)$", which in the ordinary language of predicate logic is "$\exists x\neg(Px \wedge Qx)$". We will delete the unnecessary symbol "$\exists$" in "$\exists x(PxQx)$" and move x into the parentheses. Thus we will get

$$(x;\, PxQx).$$

Figure 20.

The semicolon indicates that in the same pair of parentheses the scope of the quantified x to the left of the semicolon is the sequence of graphs to the right of it.

How do we now distinguish between "$\exists x \neg (Px \wedge Qx)$" and "$\neg \exists x (Px \wedge Qx)$"? In Hammer's linear Beta, these are, respectively, "$\exists x (PxQx)$" (or better, "$x(PxQx)$") and "$(\exists x PxQx)$" (or better, "$(xPxQx)$"). Moving "$\exists x$" or "x" inside the parentheses changes the scope of quantification, and for that reason we denote the difference between the two formulas "$\exists x \neg (Px \wedge Qx)$" and "$\neg \exists x (Px \wedge Qx)$" as the difference between "$x(PxQx)$" and "$(xPxQx)$".

Hammer suggests that "equivalent systems could be constructed by defining the cut as the merge of parenthetical grouping with a different unary logical operator such as 'neither…nor' or 'not both' " (Hammer 2011: 132). In fact, we can go further and take the "cut plus juxtaposition" as a *single* operator, namely as the joint denial of an n-ary conjunction. The resulting Alpha graphs are equivalent to the propositional N-operator that Wittgenstein presented in the *Tractatus* (see Cheung 1999, Fogelin 1982, Geach 1981, Rogers & Wehrmeier 2012, Soames 1983). In fact, these Alpha graphs fulfill Wittgenstein's "notational ideal" to have just one operator to represent all possible truth-functions of elementary propositions.

As far as the linearisation of Beta graphs is concerned, we can take the "not both" approach of Hammer and go one step further so as to merge the quantifier "$\exists x$" into this unary operator. This is significant, since the resulting unary operator is what Bimbò (2011) has presented as the first of the four Schönfinkel–Bimbò (S–B) operators (S–B1 below). The general result is that any one of these four operators is expressive enough alone to build up a first-order language.

In the language of Beta graphs, these four operators have a particularly convenient, symmetrical representation, shown in Fig. 21:

(a) (S-B1) $\neg \exists x (Px \wedge Qx)$

(b) (S-B2) $\exists x (\neg Px \wedge \neg Qx)$

(c) (S-B3) $\exists x \neg (Px \wedge Qx)$

(d) (S-B4) $\neg \exists x (\neg Px \wedge \neg Qx)$

Figure 21.

Moreover, we can generalise the corresponding graph of Fig. 21(c), namely "$(x; PxQx)$", in linear notation. Details of such formal syntax, semantics, and an axiomatic treatment of such new version of linear Beta, termed linear-Beta*, are presented in the Appendix.

REMARK 1 *A note on Peirce's "line-feed" notation of R 510.* Peirce experimented also on another, a kind of quasi-linear notation, for his planned second edition of the *Syllabus* of his 1903 Lowell Lectures (R 478; Peirce 2020a). Among the worksheets of R 510 (with a couple of additional loose sheets in R 278), Peirce is seen to suggest an algorithm to rewrite Beta graphs to a format that has gotten rid of the junctures (LIs) by substituting selectives for them and using line skips, indentations and other common types to denote enclosure nesting etc. In the resulting "line-feed" notation, graphs grouped within different juxtaposed cuts are sorted in parallel columns and polarities between positive and negative areas denoted by single large parentheses and braces, respectively, with indentation to ease the recognition of which graphs rest on positive and which on negative areas:

> [i] A selective, or capital letter, is to be substituted for each least enclosed juncture, a juncture not being within a cut unless it is wholly within it. [A]nd this is to be repeated until all the junctures are abolished. [ii] Junctures evenly enclosed are to be replaced [by] early letters of the alphabet A to L, [iii] junctures oddly enclosed by late letters Z to M.
>
> The entire graph is to be transcribed, [iv] more enclosed spots being scribed lower down in the same columns and [v] spots enclosed in cuts within the same cut to be in parallel columns, the columns being split by braces.
>
> [vi] In place of each evenly enclosed cut is to be placed a single large parenthesis mark to the left and [vii] in place of each oddly enclosed cut is to be placed a single large square bracket to the left. [viii] Oddly enclosed spots are to be put a little further to the right than evenly enclosed spots in the same column. (Peirce 1903a)

The result of an application of this transcription method is a juncture-free notation which in 1903 would have been possible to be typeset without the compositor having to draw complex diagrams with curved lines or to set up some entirely new and expensive types. Peirce provides a couple of examples of this translation, but other than that, the method seems to have remained an incomplete and under-utilised suggestion on these worksheets. As those quasi-linear graphs were never printed anywhere, he seems to have left the idea aside after 1903.

Coming back to the 1902 linear notation of Peirce's Dictionary article, consider next Hammer's remark:

> Peirce has merged syntactic elements in at least two different cases. First, the mechanism for negation has been merged with the mechanism for grouping as discussed previously—by enclosing a graph within a cut one is also thereby grouping the contents. Second, the mechanism for associating variables, the line of identity, also serves as the quantifier, so identity is merged with quantification. (Hammer 2011: 134)

The first kind of merging is the merging of the collectional function with negation in the cut, as discussed in Section 2. The second kind of merging is the merging of the sign of identity of reference ("associating variables") and the sign of quantification (quantifier). Since only the existential quantifier is used in this system of linearised Beta, there is no need to separately represent which of the two quantifiers is meant. Thus the sign of quantification is simply the sign that denotes the *scope* of quantification.

Hammer imparts that the merging of these two functions is lost in his linear version of Beta, given that the "linear notation presented forces the separation of correlating variables from quantifying over the variables, and so this case of merging logical concepts is lost when moving to linear notation" (*ibidem*). This is true, but it is not the only loss that linear Beta graphs have with respect to standard Beta graphs. Hammer's Beta graphs not only force the separation of correlating variables from quantification; they also force the separation of variables themselves. That is, they represent, by distinct occurrences of the same variable type, the *identity* of the object denoted by that type. In our terminology, Hammer's linear Beta graphs are type-referential, while standard Beta graphs, as well as the "linearised" Beta graphs of Peirce's Dictionary entry, are occurrence-referential.

The next section shows how this second kind of merging, which is lacking in Hammer's account, is brought to its perfection in the theory of Beta graphs. But now a hitherto unnoticed aspect of Beta graphs is seen to arise, which has to be taken into consideration in analysing meanings of certain complex sentences and their quantificational structures. The final question to be resolved thus concerns the precise workings of the interaction between lines and cuts. We next argue that one needs to extend the notation of ordinary Beta graphs by introducing "bridges".

5 Identity and Quantification in the Diagrammatic Syntax of Beta Graphs

A closer look at the implications of the assimilation of quantification and identity under one and the same notational device reveals a previously unnoticed aspect of EGs. That aspect comes to be masked in standard linear notations, but can be explained by appealing to the distinction between notations, type- and occurrence-referential. It is only in occurrence-referential systems that distinct logical operations, such as existence and identity, may be subsumed under one and the same logical sign.

The job description of the sign for the identity of reference (such as Hammer's "associating variables") is not only to reuse the same variable names and then binding their tokens. One also needs to be able to use the LI as the true sign of equality, in the sense in which linear notations express it by the identity sign in clauses such as "$x = x$", "$x = y$", and so on. We will now observe that the meaning of quantification that is connected to the meaning of identity requires a change in the usual definition of the language of Beta graphs.

The question is: Can all first-order formulas with identity[7] in fact be represented in the language of Beta graphs, as it is commonly defined in the literature? In the literature Beta graphs have been taken to be expressively equivalent to a fragment of first-order logic with identity, with a (non-bijective) translation between them. However, a suspicion arises from a closer scrutiny of the proposed subsumption of the functions of identity and that of quantification under the behavior of one and the same logical sign, namely the LI.

An example that prompts such a suspicion was originally contemplated by Peirce himself, and is found in an unpublished manuscript written for his *Minute Logic* book proposal (Peirce 1902b) in the same year as the publication of his Dictionary article "Symbolic Logic". In fact, Peirce not only noticed the general nature of the emerging problem, but also outlined some original proposals to fix it.

Peirce's version of this quantification *cum* identity problem is, in brief, as follows. Peirce asked how the following sentence is to be represented in the language of Beta graphs:[8]

(22) Any man there may be is born of something, X; and any man there may be is coexistent with a woman who is that X. (Peirce 1902b, 2019; R 430)

In the standard first-order notation with identity, we could try to represent this sentence as follows (M_1, M_2: "is a man"; Bxy: "is born of"; Czu: "coexists with"; Wu: "is a woman"):

(23) $\forall x \exists y \forall z \exists u ((M_1 x \to Bxy) \land (M_2 z \to (Wu \land Czu)) \land (y = u))$.

[7]For simplicity, we assume a restriction of formulas to a fragment of FOL with only symmetric relations and without function and constant symbols in the signature of that fragment.

[8]This question of the limitations of the expressivity of Beta graphs takes place in R 430 within the larger context of the question of the operation of rules of transformations in Beta graphs and the enlisting of a number of different functions ("offices") that the ovals (the cuts) have in the system.

But how to represent this sentence in the language of Beta graphs? In R 430, Peirce seems to have recognised the troubles that arise from the application of ordinary Beta graphs to the cases that involve logical analysis of complex quantificational natural-language statements with anaphora.[9] First, Peirce notices that the graph in Fig. 24(a) does not do, since it lacks the continuous lines that would be needed in order to represent the identities of the values of the two tokens of Xs in (22) (namely the identity of that X which any man is being born of and the X which is a woman).

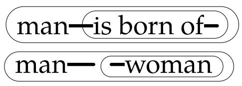


Figure 24(a). (Peirce 1902b)

On the other hand, the graph in Fig. 24(b), which now does represent the desired identity by a continuous line, fails in another respect: that line has to pass through the space outside the cuts (the sheet of assertion), but in doing so the line's outermost extremity rests on that positive space (the sheet of assertion) and is not within the scope of any other line. According to the Beta conventions, this means that the individual identity (the existential quantifier) denoted by that line has wider scope than the two universal quantifiers. But this is not what is meant by (22), in which the relative dependence between the two pairs of quantifiers is the other way round. Indeed the trouble is the separate expression of equality of the two existentially quantified variables, which cannot be directly represented in standard Beta graphs.

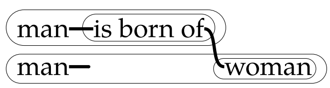


Figure 24(b). (Peirce 1902b)

The sentence in (22) mentions the relation of coexistence. As Peirce repeatedly argues (and proves), the relation of coexistence is represented by the blank. Two loose ends of the LI occupying positions in the same area, but which do not

[9] Similar hunches antedate R 430: see e.g. Peirce's *The Peripatetic Talks* of 1898 (Peirce 2019) on sentences that gave him pause when he attempted to represent them in the two-dimensional language of graphs. All of these are examples of the complications that arise from the attempted logical analyses of the meaning of anaphora in English sentences, and resemble complex Donkey anaphora and cumulative quantification, among others (Peirce 2019; Pietarinen 2015).

otherwise connect to each other, are in such relation of coexistence with each other. In 1903, Peirce explains this as follows: “Of dyadic relations of second intention four are prominent: [First,] [t]he relation which everything bears to everything else, expressed by [. . .] the blank” (R 492, Alt.). In Beta graphs, the relation of coexistence can be, instead of the blank, explicitly expressed at the first-order level by the spot “____ is co-existent with____ ”.

In the graphical representation of the conditional *de inesse* (the scroll), there is no such blank between the antecedent (the outloop) and the consequent (the inloop) areas. Coexistent are only those propositions that are “scribed simultaneously on the sheet” (R 430). Hence a separate dyadic relation “____ is co-existent with____ ”, which expresses the co-existence of the man and the woman and which is not represented in Peirce’s examples of graphs such as Figs. 24(a) and 24(b), needs to be included in the graphical representation of (22).

In other words, a conditional alone does not suffice to represent coexistence. To say that “If something is a man, then something is a woman” does not involve the meaning of the relation of coexistence between the two predicates. This is shown in the scroll notation (namely ) not by the absence of a continuous blank between the two quantified lines but, more pronouncedly, as the disruption of any blank that would otherwise constitute coexistence (or non-coexistence) by the innermost cut encircled around the expression “something that is a woman”. There is no blank that could connect the two loose ends of the line in the scrolls of these graphs. Thus the relation of coexistence needs to be explicitly graphically represented, in the same manner as in the formalisation of (22) as (23), by the sign of identity.

Peirce provides four more attempts to solve the puzzles of anaphora and coreference in manuscript R 430 (Peirce 1902b; Peirce 2019). The first is to add two more cuts around the original graph, which would extend the lines (universal quantification) so that the wide-scope portion of the line (existential quantification) in the graph of Fig. 24(b) would now remain within the scope of the top-most universal quantifier (the graph of Fig. 24(c) below). This proposal also fails, however, and as Peirce rightly notices, the resulting graph now changes the meaning of the sentence to one in which the indefinite (“some woman that is coexistent with that man”) is within the scope of both universal LI that connect to the predicate “man”, that is, is also within the scope of the top universal line (as it is nested within the context created by these additional cuts between which there is the loose end of the universally quantifier line). But this is not what the original sentence in (22) means. It does not say that “a woman who is that X” varies with (or is functionally dependent on) “any man there may be” of the first clause of (22).

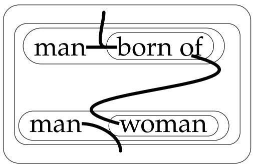


Figure 24(c). (Peirce 1902b)

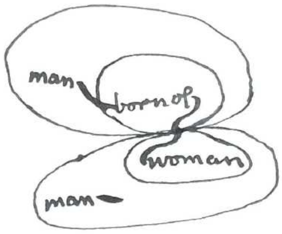


Figure 24(d). (Peirce 1902b, in Peirce's hand)

One might nevertheless attempt to argue that the graph Peirce gave in Fig. 24(c) is in fact the right or intended representation of (22), and that, consequently, there exists an elementary, first-order formula corresponding to it which is expressible in the received syntax of Beta graphs. For such argument to hold, however, one must assume that the relation of coexistence is captured by the implicational structure, and that any relation of coexistence can accordingly be dispensed with. If that were to be the case, one could represent the graph of Fig. 24(c) in terms of the first-order formula (24c′):

(24c′) $\forall x \exists y \forall z \exists u ((M_1 x \rightarrow\ Bxy) \wedge (M_2 z \rightarrow Wu) \wedge (y = u))$.

The question is: does the implicational structure "$M_2 z\ \rightarrow\ Wu$" analyse the relation of coexistence? When this structure is quantified (as it needs to be), "$\forall z \exists u (M_2 z\ \rightarrow\ Wu)$" is equivalent to "$\exists z M_2 z\ \rightarrow\ \exists u Wu$". This asserts an implicational relation between being a man and being a woman. That is, if the former has a value that makes the antecedent true, then the latter has a value that makes the consequent true.

Since now—given the absence of the binding relation of coexistence between the two variables z and u—both existential quantifiers can reside within the priority scopes of both universal quantifiers, the translation of the graph of Fig. 24(c) to a first-order logic can take the form of (24c″):

(24c″) $\forall x \forall z \exists y \exists u ((M_1 x \rightarrow\ Bxy) \wedge (M_2 z \rightarrow Wu) \wedge (y = u))$.

One can then move the quantifier for z inside in the formula to precede the antecedent of the second implicational structure, because z does appear in any relations with any other variable. This yields the following equivalent formula:

(24c‴) $\forall x \exists y \exists u((M_1 x \rightarrow\ Bxy) \wedge (\exists z M_2 z \rightarrow Wu) \wedge (y = u))$.

From the point of view of Peirce's graph in Fig. 24(c), this equivalence means (from (24c″) to (24c‴) direction) that the bottom-most line can be cut on where it trespasses the positive area, and the remaining loose segment of the line can be erased (by an axiom of the Beta system). Conversely, a piece of line may be inserted in the outermost negative area and then iterated inwards until it meets the line connecting to the man in the negative area. After that, the branch that had a loose end in the positive area may be retracted.

There is an objection to this recasting of Peirce's graph and the reading of (22) along the lines of the above argument, however. As already noted, the implicational structure is not tantamount to the creation of the appropriate relation of coexistence. Now it is true that an explicit spot that would represent such relation is missing from Peirce's own example (it transpires in the linguistic example (22) only, as "is coexistent with"). But the reason for this is that in that context Peirce is not concerned with the expressivity in Beta graphs in general, or the expressivity of sentence (22) in Beta graphs in particular. In order to express the relation of coexistence, either a continuous blank or an explicit relation of coexistence need to be present. Since there is no continuous blank that could connect the two lines of the predicate "man" and the predicate "woman" in an implicational structure, a genuine relation of coexistence is needed. But the above argument (24c′)–(24c‴) dispenses with any such relation. In conclusion, then, since a relation of coexistence is needed, when added to the graph of Fig. 24(c) (namely, added to the consequent area of the lower implicational structure and connected to the line of the predicate 'man' on the antecedent of that structure and to the line of the predicate 'woman' on the consequent of that structure), it blocks the possibility of reading it in terms of standard first-order formulas with identity, namely any of (24c′)–(24c‴).

Peirce then proposes, secondly, a modification to the syntax that would allow cuts to touch each other from the outside, without overlapping. There is only one example of this among the Peirce papers (see Fig. 24(d)), the one in which the inner line is taken to traverse between the two spots and presumably without making contact with the sheet of assertion. If this could be done, then the existentially quantified line would no longer have wider scope than the universal lines: the former line would not have its outermost portion on the SA.

However, this is an *ad hoc* modification to the diagrammatic syntax, and moreover appears to be accompanied with some unintended consequences.

When the cuts connect from the outside, according to Convention 9 of the theory of Beta graphs (see e.g. Roberts 1973: 54; Peirce 1903b), the line that meets the boundary of the cut from the inside is interpreted to be outside that cut (the cut is "outside its own close", CP 4.501). The extremal point of the line that terminates at the boundary would then be interpreted as belonging to the area outside the enclosure formed by the cut and its area. That is, the polarity of such a line would be that of the polarity of the area immediately outside the cut. But that area would be the area immediately inside the other cut which the previous cut meets, and vice versa. When the two boundaries meet as in Fig. 24(d), the two ends of the two segments of the ligature that abut at the same boundary point at which the two cuts make contact are, according to Convention 9, both outside their respective enclosures. Thus not only the line retains its continuity and hence its signification as an assertion of identity, but also its outermost portion can no longer lie inside the cuts. The intersection point of the two colliding cuts would behave like a double negation and cancel the effect of the cut as a disruption of the continuity of the ligature. In sum, the proposed second solution would necessitate changing Convention 9, which in turn would fundamentally change the way the lines of identity are interpreted in the theory of Beta graphs.

As to the last two solutions, Peirce's proposals are equally innovative. First, he proposed to add a "nodule" to the lines to create *direction*. This is shown in the graph depicted in Fig. 24(e) in Peirce's hand, where the nodule added towards one extremity of the line is intended to cancel the customary, endoporeutic interpretation. Hence the intended scope of that line would not derive from the outermost portion of that line but from the extremity on which that nodule is placed. One might well accept this as an appropriate modification. (It also has some further merits such as adding to the expressive power of logical graphs, see Pietarinen 2015.) This new piece of notation necessitates redefining the language of Beta graphs, however, as in Beta EGs ligatures have direction only as prescribed by the ordering of the SA by the nesting of cuts.

The last solution is depicted in Fig. 24(f). Peirce now proposes a special kind of *selective*, the Virgo, to carry out the job by mediating the information between the two ends of the lines, without any sign having to pass through the intermediate space between the lines connected to these selectives. This is what selectives in graphical notation and variables flagged to identity symbols in linear notations do. The drawback is that the occurrence-referential character of the notation is now lost, since selectives introduce an identity based on types and not on occurrences: two occurrences of the same selective denote the *same* individual.

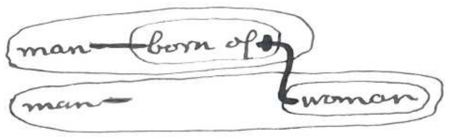


Figure 24(e) (Peirce 1902b. In Peirce's hand).

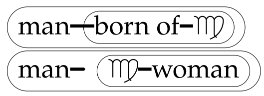


Figure 24(f) (Peirce 1902b).

How then one is to scribe the graph to logically analyse the sentence (22) with an explicit relation of coexistence? Representing quantification *cum* identity requires the ability to represent complex strings of quantifiers and identities that depend on other quantifiers in their context. Just as the notation of first-order logic might be made more expressive in terms of, say, non-linear, partially-ordered quantificational structures (Enderton 1970, Henkin 1961, Pietarinen 2004), so do Beta graphs, when scribed in two dimensions, face geometrical limitations.[10]

As these limitations concern the planar structure of the SA, they can be overcome by taking the sheet to be a many-dimensional manifold. This is a natural solution: the SA in logical graphs serves the same role as the ambient space does in topology.

The resolution for the problem of expressing the sentence (22) and similar ones involving complex quantification and phenomenon such as cross-pronominal anaphora in the language of Beta graphs can take various forms. Our solution follows the previous lines of thought but is really nothing else than a generalisation of the features that are already present in Beta graphs. For in Beta graphs, LI may have to depart from the planar surface and escape to the third dimension. A minimal example is three spots with three hooks and three branching lines that connect all three spots: in FOL, "$\exists x \exists y \exists z (P_1 xyz \wedge P_2 xyz \wedge P_3 xyz)$" (with non-symmetric relations, one quaternary spot will do: "$\exists x \exists y Pxyxy$"). Beta graphs—just as their pseudo-linear forms as presented in Peirce's Dictionary article—are projections of graph-instances that live on in three dimensions. Now for such situations, Peirce had introduced the convention of *bridges*,

[10]On some other, non-classical forms of EGs, see e.g. Bellucci, Chiffi & Pietarinen 2017; Chiffi & Pietarinen 2018, 2019 on Assertive Graphs (AGs); Ma & Pietarinen 2017b on modal graphs; Ma & Pietarinen 2018a on sub-propositional Alpha graphs; and Ma & Pietarinen 2018b on intuitionistic Alpha graphs.

which are routinely resorted to in order to represent lines that cross each other without joining:

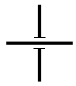


Figure 25. A bridge for two lines of identity (Convention 12, R 450, R 492).

The presence of bridges means that the standard diagrammatic syntax for Beta graphs is readily three-dimensional. The general modification required to the language of Beta graphs is then only the following. Just as bridges are applied to prevent lines that have to cross each other to join into one another, one can apply bridges *to prevent lines that cross cuts to be cut by them*. That is, lines can overpass (or underpass) the cuts as well. This can be notated by applying the bridge as in Fig. 26.

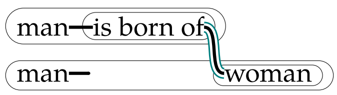


Figure 26. A bridge for the line that overpasses the cuts.

This extension calls for no new signs to be added to the language of Beta graphs. The blue (or grey in B/W) tracks in Fig. 26 represent a bridge not only for the lines to cross other lines but also for the lines to cross cuts without being cut by them (for the sake of perspicuity, one can add another kind of bridge for the latter cases). Nor does this extension of the applicability of the bridge result in anything more expressive than what the relevant fragment of first-order logic with identity already is; it merely designates a necessary qualification to an already extant convention, which is needed in order to match the expressive power of Beta graphs with that of a fragment of first-order logic with identity. This qualification is thus nothing else than an extension of the application of an already existing piece of notation, the bridge, which now applies it not only to the crossing of lines but also to the crossing of lines *and* cuts.[11]

[11] One potential hazard is that as soon as the line leapfrogging cuts escapes into the third dimension, in which it will be undisturbed by those cuts and hence by the polarity counters that those cuts effect. What is the polarity signature of those lines then? What if the two extremities of the line that rest on the sheet of assertion are of opposite polarities? What is the polarity of the line in that case? The answer is that the signification of cut-leapfrogging lines is defined by the endoporeutic

This extension of the applicability of the bridge is required to express certain sentences in the language of graphs that can readily be expressed and analysed in linear languages that possess the equality sign. The graph in Fig. 26 is an example of such graphical representation of a sentence (22), which in terms of the representation in the language of first-order logic with the equality sign can be given in terms of the sentence (23).[12]

There is a wider lesson to be learned from such an expanded notation. The full meaning of a quantifier has to also include the meaning of identity, because there obviously can and must be such crossings of cuts by lines unaffected by those cuts. Yet no one between Peirce and the present paper has pointed out the real need for what is nothing but a modest expansion of Beta graphs. It merely takes the point that elementary graphs are not two- but three-dimensional objects into its inevitable consequences.[13]

Another point concerns the nature of the lines that cross the cuts with these bridges: cut-crossing lines remain lines, not ligatures. They are not ligatures (complexes of LIs), which without bridges they would be, because these lines do not abut the cuts and do not interact with them.

In sum, when quantification and identity are both represented in an occurrence-referential notation, some non-trivial consequences emerge regarding the notion of scope. One office of the cuts, analogous to parentheses in linear notation, is that they impose ordering on the sheet of assertion. This ordering shows, among others, the logical scope of quantifiers: the less-enclosed extremity of a ligature is logically prior to the ligature whose outermost extremity resides deeper in the nest of the graph, that is, is enclosed within a greater number of

interpretation, *excluding* the portion of the line that rests in the third dimension. The part of the line that is in the air is not part of the signification of the line. Then the polarity of these leapfrogging lines is unambiguously defined in terms of the least-enclosed outermost portions of the lines; that is, the 'outermost portion' refers to the collection of all those non-loose end-points and portions of the line that rest on the two-dimensional sheet of assertion, taking those points and portions to be connected by the line as usual. We thank the reviewer for raising this issue, which points at an important distinction between the meaning, on the one hand, of the line as a *continuity* between its connections and loose extremities and, on the other, of the line as a *type* of quantification inferred from its interaction with the enclosures in occurrence-referential notations. This distinction reflects that of the binding and priority scopes of quantifiers in type-referential notations.

[12]The graph in Fig. 26, just as in Peirce's own proposals for the graphical representation of (22) given above, is assumed to contain the relation of coexistence "___ is coexistent with ___ " within its lower implicational structure. This poses no complications and is omitted merely to avoid cluttering the graph.

[13]A reviewer raises the following point: Are we sure that adding a third dimension will completely contain the problem? That is, might there be (admittedly convoluted) cases which would demand adding still further dimensions? Right now we are inclined to answer No and Yes, respectively. We know how complex anaphoric meanings can be (take Geach–Kaplan sentences, for example). And we know that non-linear formulas are needed to analyse certain sentences involving branching quantification, for example. Then take a graphical representation of such branching quantification (Pietarinen 2004), and imagine that there is a complex cross-pronominal anaphora in which it becomes necessary that not only one-dimensional lines but also two-dimensional *planes* of identity would have to leapfrog cuts. Then we are at once talking about those planes escaping into the fourth dimension. And if cuts are in fact projections of spheres, lines and planes that escape them might require the ambient space having *five* dimensions.

cuts in the same nest.But the two offices of the lines, the binding of variables as denoted by the continuity of the line, and the nestedness of their outermost portions, need not go hand in hand. Since in the ordinary, linear notation these two offices need not be considered separately, the real logical differences that they bear may be easily lost in occurrence-referential languages such as EGs.

6 Conclusions

The occurrence-referential logical graphs scribed in three-dimensions testifies to the inseparability of the meanings of quantification and identity. This phenomenon is easily masked in linear, type-referential notations. Alternative and non-standard logical notations are thus important to be developed and analysed as they suggest that this inseparability is not only inevitable but also that it is only through assimilating the two notions also at the notational level that we see how the cuts (as expressing logical scopes) and lines (as expressing quantification and identity) interact.

As far as linear notations are concerned, it is not obvious what is gained and what is lost by moving from multi-dimensional to one-dimensional linear notation. To examine this question, a linear Beta* system is precisely defined, correcting the defects of Hammer (2011), who proposed a linear version of Peirce's Beta graphs. In the present paper, it was shown that the distinctions between permutational invariance and the type- vs. occurrence-referentiality can explain the principal, if not the only, difference between EGs and their linearised versions, as presented in the Alpha and Beta parts of EGs in Peirce's Dictionary entry as well as in Hammer (2011). First, standard EGs are permutationally invariant with respect to linear EGs. Second, the Beta component of the system is occurrence-referential while Hammer's linear Beta graphs are type-referential. When occurrence-referentiality constitutes a defect in expressivity (such as when quantification and identity are represented by the same sign and when some complex assertions cannot be expressed in the standard notation of Beta graphs), the standard notation of logical graphs, as has been argued in the present paper, has to be modified by expanding the application of Peirce's bridge-notation.[14]

[14]The distinctions of occurrence-referential vs. type-referential notations and non-linear vs. linear notations, together with permutational invariance and the necessary widening of the coverage of the extant notations of EGs, are useful analytical devices by which one could gain new insights into Wittgenstein's conclusion that "the identity sign is not an essential part of logical notation" (Tr. 5.533). That is, "Identity of the object I express by identity of the sign and not by means of a sign of identity" and "Difference of the objects I express by difference of the signs" (Tr. 5.53). We leave a closer analysis of this Wittgensteinian tracing of the Peircean paths for a future occasion.

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Appendix: Linear Beta*

A.1 Syntax and Semantics

The definitions of a graph and a subgraph of the language of linear Beta graphs, termed linear-Beta*, is as follows:

(1) "$Rx_1 \dots x_k$" is an (atomic) graph for any relation "R" and variables "x_i".

(2) Given n ($n \geq 0$) graphs "G_0"…"G_{n-1}" and m ($m \geq 0$) variables "x_0"…"x_{m-1}", we get the graph "$(x_0 \dots x_{m-1}; G_0 \dots G_{n-1})$", of which G_i are subgraphs.

Thus, when $m = n = 0$, we get the *blank cut* "( )" as a graph (Peirce's "pseudo-graph"). When $m = 0$ and $n = 1$, we get the *negation* "(G_0)" of the graph "G_0". And when $m = 0$, we get the *Alpha graph* "$(G_0 \dots G_{n-1})$" as well as "G_0"…"G_{n-1}" as direct subgraphs of it, of which we highlight a special kind of graphs called *atomic* NAND *graphs*: "G_i" is "( )", some atom, or its negation.

Sometimes we need individual parameters standing at the place of the *hooks* at the periphery of the spots of the Beta graphs: the graph "$G\{x/a\}$" means the result of substituting the individual parameter a for the individual variable x in the graph "G". Similarly we have the graph "$G\{x_0/a_0, \dots, x_{m-1}/a_{m-1}\}$".

Hammer (1998) defined a Tarskian semantics for EGs, and the following semantics is of this kind.[15] A model for the Beta graphs (and similarly for the Alpha graphs) is a pair $U = \langle D, \mathfrak{I} \rangle$, in which the non-empty set D is the domain of U, and the function $\mathfrak{I}$ interprets every individual variable x_i as some individual in the domain. The only complex item in the semantics is about the Beta graph "$(x_0 \dots x_{m-1}; G_0 \dots G_{n-1})$": the model U satisfies "$(x_0 \dots x_{m-1}; G_0 \dots G_{n-1})$", if and only if there exist $u_0, \dots, u_{m-1} \in U$ such that the interpretation $\mathfrak{I}[x_0 - u_0, \dots x_{m-1} - u_{m-1}]$ makes some graph "G_i" false. Truth-in-a-model, valid graph, satisfiability of a set of graphs, and logical consequence are all defined as usual.

A.2 Axiomatics

Two axiom-schemata are as follows:

Axiom 1 Atomic NAND-graphs with ( ) as its direct subgraph.

Axiom 2 Atomic NAND-graphs with "$Rx_1 \dots x_k$" and "$(Rx_1 \dots x_k)$" as its direct subgraphs.

For notational convenience, we introduce a new set of variables $\{a_0, \dots, a_{m-1}, \dots\}$ to be called *parameters*, distinct from the variables of quantification $\{x_0, \dots, x_{m-1}, \dots\}$. The rules of inference for the Beta graphs of the language of linear-Beta* are called the *rules of the tree construction*. Informally, a tree consists of branches that grow upward from the conclusion of the rules to the premise(s). The graph that

[15] An alternative is Game-Theoretic Semantics (GTS), which comes close—closer than the Tarski-type semantics—to Peirce's own intended, "endoporeutic interpretation" of EGs.

all of the branches begin with is called the *root* of the tree. A branch stops growing whenever it meets an application of an axiom, otherwise the tree continues to grow. The tree with the graph "G" as its root is called its *deduction tree*. The two rules of the tree construction are as follows:

Rule 1 From n graphs

$$\text{"}((G_0\{x_0/a_0,\ldots,x_{m-1}/a_{m-1}\})H)\text{"},\ldots,$$
$$\text{"}((G_{n-1}\{x_0/a_0,\ldots,x_{m-1}/a_{m-1}\})H)\text{"}$$

we construct the graph "$((x_0\ldots x_{m-1};G_0/G_{n-1})H)$", in which $a_0,\ldots,a_{m-1}$ are new in the tree, and $m\geq 0, n\geq 1$.

Rule 2 From the graph

$$\text{"}(G_0\{x_0/a_0,\ldots,x_{m-1}/a_{m-1}\}\ldots G_{n-1}\{x_0/a_0,\ldots,x_{m-1}/a_{m-1}\}H)\text{"}$$

we construct the graph "$(((x_0\ldots x_{m-1};G_0/G_{n-1}))H)$", in which $a_0,\ldots,a_{m-1}$ are not used in the branch yet, and $m,n\geq 0$.

The rule of *Modus Ponens* does not hold in general: in the rules of the tree construction, the conclusion is longer than its premise(s).

A.3 Soundness and Completeness

The soundness and completeness theorem together assert the equivalence of provability with validity. The following proof method is in the spirit of Kanger (1957).

All the axioms are valid graphs and the rules of the tree construction preserve validity. Therefore, if every branch terminates in an axiom, the graphs proven at the root are valid. For the completeness, we call the branch in the deduction tree that does not terminate in an axiom a *bad branch*. Subgraphs of any graph in one bad branch might all be true. Hence, if some branch does not terminate in an axiom, we can use that branch as a source of a counterexample for the candidate, namely as a source of an assignment of values to the parameters which make it come out false.

Call a branch B of the deduction tree of graph "G" a *full normal branch*, only if: "G" is the first member "G_1" of the branch B, and for the rest of the graphs "G_i" $(i\geq 1)$ of the branch B:

(1) If the graph "G_i" is an atomic NAND graph, then "G_i" is the terminal graph of B;

(2) If "G_i" is of the form "$(G_k H)$", in which "G_k" is not "( )", "$Rx_1 \ldots x_k$", or "$(Rx_1 \ldots x_k)$",

then:

(i) If "G_k" is "$(x_0 \ldots x_{m-1}; G_0 \ldots G_{n-1})$", then for $0 \geq j \geq n-1$, G_{k+1} is

$$((G_j\{x_0/a_0, \ldots, x_{m-1}/a_{m-1}\})H),$$

and

(ii) If "G_k" is "$(x_0 \ldots x_{m-1}; G_0 \ldots G_{n-1})$", then G_{k+1} is

$$(G_0\{x_0/a_0, \ldots, x_{m-1}/a_{m-1}\} \ldots G_{n-1}\{x_0/a_0, \ldots, x_{m-1}/a_{m-1}\}H).$$

That a graph "G" is a direct subgraph of a branch B means that "G" is a direct subgraph of some graph in B.

If a branch B is a full normal branch of the deduction tree of a graph "G", and B does not terminate in an axiom, then, if the negation "(A)" of an atomic graph "A" is a direct subgraph of B then "A" is not a direct subgraph of B. The reason is as follows. The branch B is a bad branch since it does not terminate in an axiom. No step growing upward in a bad branch from "G" leaves out atom or its negation, and in every graph "$(G_l H)$" of the bad branch they are "G_l", or the subgraph of "H". Thus, if "(A)" occurs as a direct subgraph in some graph of the branch B, and in some subsequent graph of the branch B "A" occurs as a direct subgraph, then graphs "(A)" and "A" will occur as direct subgraphs in the termination graph of the branch B so that the termination graph is an axiom. This is a contradiction.

Moreover, let B be a full normal branch that does not terminate in an axiom. Then from the definition of full normal branch we have:

(i) If "$(x_0 \ldots x_{m-1}; G_0 \ldots G_{n-1})$" is a direct subgraph of the branch B, then for some i, $0 \geq i \geq n-1$, "$(G_i\{x_0/a_0, \ldots, x_{m-1}/a_{m-1}\})$" is also a direct subgraph of B.

(ii) If "$((x_0 \ldots x_{m-1}; G_0 \ldots G_{n-1}))$" is a direct subgraph of the branch B, then for some i, $0 \geq i \geq n-1$, "$G_i\{x_0/a_0, \ldots, x_{m-1}/a_{m-1}\}$" is also a direct subgraph of B.

Suppose that there is a bad branch B in the deduction tree of the graph "G". Based on B we can construct a model $U = \langle D, \mathfrak{I} \rangle$ as follows:

(1) $D = \{0, 1, 2, \ldots, n, \ldots\}$.

(2) Interpretation function $\mathfrak{I}$ assigns natural numbers i $(i = 0, 1, \dots \omega)$ to every parameter a_i.

(3) For every relation R, we assign the following natural function:

   (i) Mapping $\langle i_1, \dots, i_n \rangle$ to the value *true* only if "$Ra_0 \dots Ra_{n-1}$" is a direct subgraph of B.

   (ii) Mapping $\langle i_1, \dots, i_n \rangle$ to the value *false* only if "$(Ra_0 \dots Ra_{n-1})$" is a direct subgraph of B.

In order to show the completeness we observe that if a branch B of the deduction tree of graph "G" is fully normal and does not terminate in an axiom, then the above model will make every direct subgraph of B true, and thus it will make "G" false. The proof of this is straightforward. Suppose that all the direct subgraphs with length less than n of B are true. Let us prove that all the direct subgraphs are true as follows:

(1) If "$Ra_0 \dots Ra_{n-1}$" is a direct subgraph of B, then it is assigned the value *true*. If "$(Ra_0 \dots Ra_{n-1})$" is a direct subgraph of B, then "$Ra_0 \dots Ra_{n-1}$" is not a direct subgraph of B and thus is assigned the value *false*, which means that "$(Ra_0 \dots Ra_{n-1})$" is true.

(2) If "$(x_0 \dots x_{m-1}; G_0 \dots G_{n-1})$" is a direct subgraph of B, then for some i, "$(G_i\{x_0/a_0, \dots, x_{m-1}/a_{m-1}\})$" is false. And so "$(x_0 \dots x_{m-1}; G_0 \dots G_{n-1})$" is true.

(3) If "$((x_0 \dots x_{m-1}; G_0 \dots G_{n-1}))$" is a direct subgraph of B, then for all i, "$G_i\{x_0/a_0, \dots, x_{m-1}/a_{m-1}\}$" is true. So "$((x_0 \dots x_{m-1}; G_0 \dots G_{n-1}))$" is true. $\square$