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Operational total space theory of principal 2–bundles II: 2–connections and 1– and 2–gauge transformations

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Abstract: The geometry of the total space of a principal bundle with regard to the action of the bundle's structure group is elegantly described by the bundle's operation, a collection of derivations consisting of the de Rham differential and the contraction and Lie derivatives of all vertical vector fields and satisfying the six Cartan relations. Connections and gauge transformations are defined by the way they behave under the action of the operation's derivations. In the first paper of a series of two extending the ordinary theory, we constructed an operational total space theory of strict principal 2–bundles with reference to the action of the structure strict 2–group. Expressing this latter through a crossed module (E, G), the operation is based on the derived Lie group $\mathfrak{e}[1] \rtimes \mathsf{G}$. In this paper, the second of the series, an original formulation of the theory of 2–connections and 1– and 2–gauge transformations of principal 2–bundles based on the operational framework is provided.

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Contents

1 Introduction

Principal 2–bundle theory is a topic of higher geometry important, among other reasons, for its relevance in higher gauge theory (see e. g. [1] for a review). Various approaches to this subject have been developed so far constituting a large body of literature [2–18].

This is the second of a series of two papers aimed at the construction of an operational total space theory of principal 2–bundles. In a companion paper, henceforth referred to as I [19], we laid the foundations of the operational total space framework [20]. In this paper, referred to as II, based on the operational setup worked out in I, we provide an original formulation of the theory of 2–connections and 1– and 2–gauge transformations.

1.1 Operational theory of principal 2–bundles

Before proceeding to illustrating the plan of II, we review briefly the content of I to privide the reader with a general overview of the matter.

A principal 2–bundle consists of a morphism manifold $\hat{P}$ with an object submanifold $\hat{P}_0$ forming a groupoid, a base manifold M, compatible projection maps $\hat{\pi} : \hat{P} \to M$ and $\hat{\pi}_0 : \hat{P}_0 \to M$ describing a functor, a morphism group $\hat{\mathsf{K}}$ with an object subgroup $\hat{\mathsf{K}}_0$ organized as a strict Lie 2-group and compatible right actions $\hat{R} : \hat{P} \times \hat{\mathsf{K}} \to \hat{P}$ and $\hat{R}_0 : \hat{P}_0 \times \hat{\mathsf{K}}_0 \to \hat{P}_0$ constituting a functor and respecting $\hat{\pi}$ and $\hat{\pi}_0$. The 2–bundle is also locally trivializable, that is on any sufficiently small neighborhood U of M the groupoid $(\hat{P}|_U, \hat{P}_0|_U)$ is equivariantly projection preservingly equivalent to the groupoid $(U \times \hat{\mathsf{K}}, U \times \hat{\mathsf{K}}_0)$ with the obvious projection and right action structures.

In I, we showed that there exists a synthetic structure adjoined to a principal 2–bundle as above consisting of morphism and object manifolds P and P_0, the base manifold M, projections π and π_0, morphism and object groups K and K_0 and right K– and K_0– actions R and R_0 on P and P_0. The synthetic setup is formally obtained from the original non synthetic one as follows. Describe the strict Lie 2–group $(\hat{\mathsf{K}}, \hat{\mathsf{K}}_0)$ by its associated Lie group crossed module (E, G) so that $\hat{\mathsf{K}} = \mathsf{E} \rtimes \mathsf{G}$ and $\hat{\mathsf{K}}_0 = \mathsf{G}$. Then, $\mathsf{K} = \mathfrak{e}[1] \rtimes \mathsf{G}$ and $\mathsf{K}_0 = \mathsf{G}$. Formally extend further the $\hat{\mathsf{K}}$–action $\hat{R}$ to a K–action. Then, P is the K–action image of $\hat{P}_0$ and $P_0 = \hat{P}_0$, R is the restriction of $\hat{R}$ to P and $R_0 = \hat{R}$. Above, K and P must be thought of as certain spaces of functions from $\mathbb{R}[-1]$ to $\mathsf{E} \rtimes \mathsf{G}$ and P, respectively, in the spirit of synthetic smooth geometry. Although the synthetic structure shares many of the properties of the underlying principal 2–bundle, it is not one because neither pairs $(\mathsf{K}, \mathsf{K}_0)$ and (P, P_0) have a groupoid structure.

With any Lie group crossed module such as (E, G), there are associated the derived Lie group $\mathrm{D}\mathsf{M} = \mathfrak{e}[1] \rtimes \mathsf{G}$ and its subgroup $\mathrm{D}\mathsf{M}_0 = \mathsf{G}$ whose rich properties were exhaustively studied in I. When expressing $\hat{\mathsf{K}}$, $\hat{\mathsf{K}}_0$ in terms of the crossed module encoding their underlying 2–group, one has $\mathsf{K} = \mathrm{D}\mathsf{M}$ and $\mathsf{K}_0 = \mathrm{D}\mathsf{M}_0$. The K– and K_0–actions on P and P_0 can in this way be described in terms of $\mathrm{D}\mathsf{M}$ and $\mathrm{D}\mathsf{M}_0$, respectively.

As explained at length in I, the right $\mathrm{D}\mathsf{M}$–action on P is codified in an operation $\mathrm{O{\scriptstyle P}}S_P$. This is the geometrical structure consisting of the graded algebra $\mathrm{F{\scriptstyle UN}}(T[1]P)$ of internal functions of $T[1]P$ and the collection of graded derivations of $\mathrm{F{\scriptstyle UN}}(T[1]P)$ comprising the de Rham vector field d_P and the contraction and Lie vector fields j_{PZ}, l_{PZ}, $Z \in \mathrm{D}\mathfrak{m}$, describing the action infinitesimally, where $\mathrm{D}\mathfrak{m}$ is the Lie algebra of $\mathrm{D}\mathsf{M}$. The derivations obey the six Cartan relations,

$$[d_P, d_P] = 0, \tag{1.1.1}$$

$$[d_P, j_{PZ}] = l_{PZ}, \tag{1.1.2}$$

$$[d_P, l_{PZ}] = 0, \tag{1.1.3}$$

$$[j_{PZ}, j_{PW}] = 0, \tag{1.1.4}$$

$$[l_{PZ}, j_{PW}] = j_{[Z,W]}, \tag{1.1.5}$$

$$[l_{PZ}, l_{PW}] = l_{[Z,W]}. \tag{1.1.6}$$

It is possible to similarly construct an operation $\mathrm{O{\scriptstyle P}}S_{P0}$ codifying the right $\mathrm{D}\mathsf{M}_0$–action on P_0 consisting of the internal function algebra $\mathrm{F{\scriptstyle UN}}(T[1]P_0)$ acted upon by the de Rham vector field d_{P_0} and the contraction and Lie vector fields $j_{P_0Z_0}$, $l_{P_0Z_0}$, $Z_0 \in \mathrm{D}\mathfrak{m}_0$.

1.2 2–connections and 1– and 2–gauge transformations

The operational framework of I just reviewed is the geometric setup on which the theory of 2–connections and 1– and 2–gauge transformations presented in this paper rests.

In sect. 2, we review the ordinary total space theory of principal bundle connections and gauge transformations concentrating on the two aspects of it which are most relevant for us, the operational description (cf. subsect. 2.1) and and the basic theory (cf. subsect. 2.2). This will furnish a prototypical model inspiring the construction of the corresponding higher theory.

In sect. 3, synthetic definitions of 2–connections and 1– and 2–gauge transformations are given in the operational framework (cf. subsect. 3.1). A 2–connection A is a degree 1 $\mathrm{D}\mathfrak{m}$–valued internal function on $T[1]P$ behaving in a prescribed way under the action of the vector fields d_P, j_{PZ}, l_{PZ} of the operation $\mathrm{O{\scriptstyle P}}\,S_P$ (cf. subsect. 3.2). The grading of $\mathrm{D}\mathfrak{m}$ ensures that A has a degree 1 $\mathfrak{g}$–valued component ω and a degree 2 $\mathfrak{e}$–valued component Ω which directly correspond to and have properties closely related to those of the familiar components of a 2–connection in strict higher gauge theory. Similarly, a 1–gauge transformation Ψ is a degree 0 $\mathrm{D}\mathsf{M}$–valued internal function on $T[1]P$ acted upon in a certain way by d_P, j_{PZ}, l_{PZ}, which by the grading of $\mathrm{D}\mathsf{M}$ has a degree 0 G–valued component g and a degree 1 $\mathfrak{e}$–valued component J directly corresponding to and with properties closely related to those of the components of a 1–gauge transformation in strict higher gauge theory (cf. subsect. 3.3). The action of a 1–gauge transformation Ψ on a 2–connection A can be defined and has the expected properties. 2–gauge transformations and their action on 1–gauge transformations can be incorporated into this operational framework as well (cf. subsect. 3.4).

A 2–connection A can be pulled back from $T[1]P$ to $T[1]P_0$ using the inclusion map $I : P_0 \to P$. The pull–back I^*A behaves under the action of the vector fields d_{P_0}, $j_{P_0Z_0}$, $l_{P_0Z_0}$ of the operation $\mathrm{O{\scriptstyle P}}\,S_{P0}$ in a way determined by the behaviour of A under the vector fields d_P, j_{PZ}, l_{PZ} of $\mathrm{O{\scriptstyle P}}\,S_P$. It is possible to consistently impose the condition that the degree 2 component $I^*\Omega$ of I^*A vanishes. Upon doing so, the degree 1 component $I^*\omega$ of I^*A formally functions in $\mathrm{O{\scriptstyle P}}\,S_{P0}$ as a connection of an ordinary principal bundle P_0 with structure group $\mathrm{D}\mathsf{M}_0$, though P_0 is not one in general. Similarly, a 1–gauge transformation Ψ can be pulled back from $T[1]P$ to $T[1]P_0$ using I. The pull–back $I^*\Psi$ behaves under the vector fields d_{P_0}, $j_{P_0 Z_0}$, $l_{P_0 Z_0}$ of $\mathrm{O{\scriptstyle P}}\,S_{P0}$ in a way determined by the behaviour of Ψ under the vector fields d_P, j_{PZ}, l_{PZ} of $\mathrm{O{\scriptstyle P}}\,S_P$. It is possible to consistently impose the condition that the degree 1 component I^*J of $I^*\Psi$ vanishes. The degree 0 component I^*g of $I^*\Psi$ then formally functions in $\mathrm{O{\scriptstyle P}}\,S_{P0}$ as if it were a gauge transformation of P_0 as a would–be ordinary principal bundle.

The internal functions of $T[1]P$ annihilated by all vector fields j_{PZ}, l_{PZ} with $Z \in \mathrm{D}\mathfrak{m}$ constitute the basic subalgebra $\mathrm{F{\scriptstyle UN}}_{\mathrm{b}}(T[1]P)$ of $\mathrm{F{\scriptstyle UN}}(T[1]P)$. Unlike for ordinary principal bundles, $\mathrm{F{\scriptstyle UN}}_{\mathrm{b}}(T[1]P)$ cannot be identified with $\mathrm{F{\scriptstyle UN}}(T[1]M)$, as the $\mathrm{D}\mathsf{M}$–action of P is free but generally not fiberwise transitive. In the case of a trivial principal 2–bundle, however, $P = M \times \mathrm{D}\mathsf{M}$, the $\mathrm{D}\mathsf{M}$–action is both free and fiberwise transitive and $\mathrm{F{\scriptstyle UN}}_{\mathrm{b}}(T[1]P)$ is isomorphic to $\mathrm{F{\scriptstyle UN}}(T[1]M)$. So, since a principal 2–bundle is locally weakly isomorphic to a trivial 2–bundle with the same structure 2–group by definition, the basic internal functions of $T[1]P$ can still be identified with the internal functions of $T[1]M$ locally in a weak sense. By this feature, the basic theory of the higher case is definitely unlike that of the ordinary one. Appropriate notions are so required for its formulation and construction. It is possible in principle to work out the basic theory also for the internal functions of $T[1]P_0$ and similar considerations apply. However, there apparently are no relevant applications of it.

On a trivializing neighborhood $U \subset M$ of the principal 2–bundle, 2–connections and 1– and 2–gauge transformations are described by basic Lie valued data on the portion of $T[1]P$ above $T[1]U$ (cf. subsects. 3.5, 3.6). More specifically a 2–connection A is characterized by a local basic degree 1 $\mathrm{D}\mathfrak{m}$–valued internal function A_{b} comprising a degree 1 $\mathfrak{g}$–valued function ω_{b} and a degree 2 $\mathfrak{e}$–valued function Ω_{b}. Similarly, a 1–gauge transformation Ψ is characterized by a local basic degree 0 $\mathrm{D}\mathsf{M}$–valued internal function Ψ_{b} comprising a degree 0 G–valued function g_{b} and a degree 1 $\mathfrak{e}$–valued function J_{b}. 2–gauge transformations too have a basic representation. Local 2–connection and 1– and 2–gauge transformation data relative to distinct overlapping trivializing neighborhoods of $U, U' \subset M$ match trough a local basic degree 0 DM–valued internal function D_{b} decomposable in a degree 0 G–valued function f_{b} and a degree 1 $\mathfrak{e}$–valued function F_{b}.

The local basic data mentioned in the previous paragraph can be constructed for a full open covering of M made of trivializing neighborhoods (cf. 3.7). Under certain conditions, among which fake flatness, the local 2–connection and matching data fit into a structure called a differential paracocycle having formal properties analogous to those of a (trivial) differential cocycle but defined on the total space morphism manifold P rather than the base manifold M. The paracocycle data are then expressed through the pull–back of the bundle's projection map in terms of local Lie valued data defined on M constituting a genuine differential cocycle. Similarly, in the presence of a suitable differential paracocycle, the local 1–gauge transformation data fit into a structure called a gauge paraequivalence subordinated to it. The paraequivalence data are then expressed through the projection map's pull–back in terms of local Lie valued data defined on M. Further, the gauge transform of the paracocycle is defined.

In sect. 4, we evaluate the results of the total space synthetic theory of 2–connections and 1– and 2–gauge transformations illustrated above by comparing it with other approaches to the topic (cf. subsect. 4.3) and outlining a more geometric interpretation of it (cf. subscet. 4.2).

1.3 Outlook

Our work is an attempt to formulate principal 2–bundle geometry in a total space perspective, while remaining committed as much as possible to the language and the techniques of graded differential geometry which have shown their usefulness in gauge theory. The operational formulation we propose enriches and completes the range of approaches to and descriptions of principal 2–bundle geometry. It may provide, it is our hope, alternative more elegant proofs of known facts and point to new hitherto unknown developments.

The operational framework has shown its power in the study of the differential topology, in particular the characteristic classes, of ordinary principal bundles

[20]. It has thus the potential of being useful in the study of the corresponding problems for strict principal 2–bundles.

More specific applications may include a strict Lie 2–algebraic extension of the classic theory of coadjoint orbits [21] and the attendant Borel–Bott–Weil theory [22], which at key points invoke a total space description of principal bundles. Coadjoint orbit and Borel–Bott–Weil play an important role in the one–dimensional path integral representation of Wilson lines (see ref. [23] for a nice review of this topic). It is conceivable that their higher counterparts may enter prominently in a two–dimensional path integral representation of Wilson surfaces [24–26].

The operational framework has also some non standard features which call for further investigation. The use of external function algebras introduces internal multiplicities and endows 2–connections and 1– and 2–gauge transformations with ghostlike partners rendering the whole geometrical framework akin to that used in the AKSZ formulation of BV theory [27] (see also [28]). These are absent though could be added in the ordinary operational framework. In higher one, they are instead unavoidable.

2 Connections and gauge transformations

In this section, we review the total space theory of principal bundle connections and gauge transformations from an operational perspective. This will furnish a guiding model for the construction of the corresponding higher theory carried out later in sect. 3. For a comprehensive treatment, we refer the reader to [20].

2.1 Operational theory

The operational total space theory of principal bundles, expounded in this subsection, relies on the operational setup of subsect. 2.1 of I. As shown in subsect. 2.2 of I, with a principal G–bundle P there is associated the Lie group space $S_P = (P, \mathsf{G}, R)$ and with this the operation $\mathrm{Op}S_P = (\mathrm{Fun}(T[1]P), \mathfrak{g})$. $\mathrm{Op}S_P$ provides a powerful graded differential geometric framework for the study of connections and gauge transformations. Following the customary point of view, the ordinary function algebra $\mathrm{Fun}(T[1]P)$ is considered here. Much of the theory presented below could be formulated also assuming the internal function algebra $\mathrm{F{\scriptsize UN}}(T[1]P)$. In higher gauge theory, the latter turns out to be the only available option, as we shall see in due course.

Definition **2.1.** *A connection of P is a pair of Lie algebra valued functions $\omega \in \mathrm{Map}(T[1]P, \mathfrak{g}[1])$ and $\theta \in \mathrm{Map}(T[1]P, \mathfrak{g}[2])$, called respectively connection and curvature component, on which the operation derivations act as*

$$d_P\omega = -\frac{1}{2}[\omega, \omega] + \theta, \tag{2.1.1}$$

$$d_P\theta = -[\omega, \theta], \tag{2.1.2}$$

$$j_{Px}\omega = x, \tag{2.1.3}$$

$$j_{Px}\theta = 0, \tag{2.1.4}$$

$$l_{Px}\omega = -[x, \omega], \tag{2.1.5}$$

$$l_{Px}\theta = -[x, \theta] \tag{2.1.6}$$

with $x \in \mathfrak{g}$.

(2.1.1) is just the expression of the curvature component θ in terms of the connection one ω. (2.1.2) is the Bianchi identity obeyed by the curvature. The connection is said flat if $\theta = 0$. The definition of connection we gave in subsect. 2.2 of I is essentially the same as the one provided here. Indeed, it can be shown that the horizontal G–invariant distribution H in term of which former definition is formulated corresponds to the annihilator of ω of the latter one and the flatness conditions of the two notions are equivalent.

Definition **2.2.** *A gauge transformation of P is a pair of Lie group and algebra valued functions $g \in \mathrm{Map}(T[1]P, \mathsf{G})$ and $h \in \mathrm{Map}(T[1]P, \mathfrak{g}[1])$, called respectively transformation and shift component, on which the operation derivations act as*

$$d_P g g^{-1} = -h, \tag{2.1.7}$$

$$d_P h = -\frac{1}{2}[h, h], \tag{2.1.8}$$

$$j_{Px} g g^{-1} = 0, \tag{2.1.9}$$

$$j_{Px} h = x - \mathrm{Ad}\, g(x), \tag{2.1.10}$$

$$l_{Px} g g^{-1} = -x + \mathrm{Ad}\, g(x), \tag{2.1.11}$$

$$l_{Px} h = -[x, h] \tag{2.1.12}$$

with $x \in \mathfrak{g}$.

Relations (2.1.7) effectively defines the shift component h in terms of the transformation one g. (2.1.8) is the associated Maurer–Cartan equation. The definition of gauge transformation we gave in subsect. 2.2 of I coincides with the one provided here. The G–equivariant fiber preserving diffeomorphism $\varPhi$ in the former definition corresponds to the transformation component g in the latter one.

As well–known, gauge transformations act on connections of P.

Definition **2.3.** *The gauge transform of a connection of components ω, θ by a gauge transformation of components g, h is given by*

$${}^{g,h}\omega = \mathrm{Ad}\, g(\omega) + h, \tag{2.1.13}$$

$${}^{g,h}\theta = \mathrm{Ad}\, g(\theta). \tag{2.1.14}$$

Substituting above identity (2.1.7) expressing h in terms of g, these relations are formally identical to the familiar ones of standard gauge theory.

Proposition 2.1. *${}^{g,h}\omega$, ${}^{g,h}\theta$ are the components of a connection. Flatness is gauge invariant.*

Indeed, the action (2.1.1)–(2.1.6) and (2.1.7)–(2.1.12) of the operation derivations on the components ω, θ and g, h ensures that the action of those derivations on the transformed components ${}^{g,h}\omega$, ${}^{g,h}\theta$ also obey to (2.1.1)–(2.1.6).

Since the shift component of a gauge transformation is determined by the transformation one by (2.1.7), a gauge transformation is fully specified by these latter. As $\mathrm{Map}(T[1]P, \mathsf{G}) = \mathrm{Map}(P, \mathsf{G})$, gauge transformations can be viewed as elements of the group $\mathrm{Map}(P, G)$ of G–valued maps. They form indeed a distinguished subgroup of this latter, the gauge group of P. Gauge transformation is a left action of the gauge group on connection space. As expected, the definitions of gauge group and the gauge transformation action on connection space we gave in subsect. 2.2 of I precisely correlate to the operational theoretic definitions of the same notions provided here.

2.2 Basic theory

Every principal G–bundle P is trivializable on any sufficiently small neighborhood U of the base M, that is $\pi^{-1}(U)$ is projection preservingly, G–equivariantly isomorphic to the trivial G–bundle $U \times \mathsf{G}$. The existence of a trivializing isomorphism $\Phi_U : \pi^{-1}(U) \to U \times \mathsf{G}$ provides structural information about the operation $\mathrm{Op}\, S_{\pi^{-1}(U)}$ of $\pi^{-1}(U)$. It entails the existence of coordinates of $\pi^{-1}(U)$ modelled on $U \times \mathsf{G}$ with special properties under the action of the operation's derivations. In this way, an operational description of the local fibered geometry of P can be furnished.

Proposition 2.2. *There are coordinates of $\pi^{-1}(U)$ adapted to $U \times \mathsf{G}$, namely functions $u \in \mathrm{Map}(T[1]\pi^{-1}(U), \mathbb{R}^{\dim M})$, $v \in \mathrm{Map}(T[1]\pi^{-1}(U), \mathbb{R}^{\dim M}[1])$ for U and*

$\gamma \in \mathrm{Map}(T[1]\pi^{-1}(U), \mathsf{G})$, $\sigma \in \mathrm{Map}(T[1]\pi^{-1}(U), \mathfrak{g}[1])$ *for* G *on which the operation derivations act as follows. For* u, v, *one has*

$$d_{\pi^{-1}(U)}u = v, \qquad d_{\pi^{-1}(U)}v = 0 \tag{2.2.1}$$

with trivial action of all operation derivations $j_{\pi^{-1}(U)x}$, $l_{\pi^{-1}(U)x}$ *with* $x \in \mathfrak{g}$. *For* γ, σ, *the structure equations take the form*

$$\gamma^{-1}d_{\pi^{-1}(U)}\gamma = \sigma, \tag{2.2.2}$$

$$d_{\pi^{-1}(U)}\sigma = -\frac{1}{2}[\sigma, \sigma], \tag{2.2.3}$$

$$\gamma^{-1}j_{\pi^{-1}(U)x}\gamma = 0, \tag{2.2.4}$$

$$j_{\pi^{-1}(U)x}\sigma = x, \tag{2.2.5}$$

$$\gamma^{-1}l_{\pi^{-1}(U)x}\gamma = x, \tag{2.2.6}$$

$$l_{\pi^{-1}(U)x}\sigma = -[x, \sigma] \tag{2.2.7}$$

with $x \in \mathfrak{g}$.

Relation (2.2.1) can be viewed as the definition of the generator v. Relation (2.2.2) can similarly be viewed as the definition of the generator σ. Eq. (2.2.3) states that σ is a fiberwise Maurer–Cartan form and (2.2.3) itself is the classic Maurer–Cartan equation it satisfies.

We can use the explicit description of the operation $\mathrm{Op}S_{\pi^{-1}(U)}$ we detailed above to analyze such structures as connections and gauge transformations of the principal bundle P in terms of data defined locally on U in the base M. This will lead to basic theory.

Consider a connection of P of connection and curvature components ω, θ.

Definition 2.4. *The basic components of the connection on* U *are defined as*

$$\omega_{\mathrm{b}} = \mathrm{Ad}\,\gamma(\omega - \sigma), \tag{2.2.8}$$

$$\theta_{\mathrm{b}} = \mathrm{Ad}\,\gamma(\theta). \tag{2.2.9}$$

Above, restriction of ω, θ to $T[1]\pi^{-1}(U)$ is tacitly understood. The name given to ω_{b}, θ_{b} is motivated by the fact that, by construction, they are annihilated by all derivations $j_{\pi^{-1}(U)x}$ and $l_{\pi^{-1}(U)x}$ with $x \in \mathfrak{g}$.

Proposition 2.3. *ω_{b}, θ_{b} are basic elements of the operation* $\mathrm{Op}\, S_{\pi^{-1}(U)}$.

Hence, ω_{b}, θ_{b} can be identified with certain functions $\omega_{\mathrm{b}} \in \mathrm{Map}(T[1]U, \mathfrak{g}[1])$, $\theta_{\mathrm{b}} \in \mathrm{Map}(T[1]U, \mathfrak{g}[2])$.

Proposition 2.4. *ω_{b}, θ_{b} satisfy the relations*

$$d_{\pi^{-1}(U)}\omega_{\mathrm{b}} = -\frac{1}{2}[\omega_{\mathrm{b}}, \omega_{\mathrm{b}}] + \theta_{\mathrm{b}}, \tag{2.2.10}$$

$$d_{\pi^{-1}(U)}\theta_{\mathrm{b}} = -[\omega_{\mathrm{b}}, \theta_{\mathrm{b}}]. \tag{2.2.11}$$

These are formally identical to relations (2.1.1), (2.1.2). We recover in this way the familiar local base space description of connections used in the space–time formulation of gauge theory.

Next, consider a gauge transformation of P of transformation and shift components g, h.

Definition 2.5. *The basic components of the gauge transformation on U are*

$$g_{\mathrm{b}} = \gamma g \gamma^{-1}, \tag{2.2.12}$$

$$h_{\mathrm{b}} = \mathrm{Ad}\,\gamma(h - \sigma + \mathrm{Ad}\, g(\sigma)). \tag{2.2.13}$$

Above, again, restriction of g, h to $T[1]\pi^{-1}(U)$ is understood. The name given to g_{b}, h_{b} is motivated by the fact that, by construction, they are annihilated by all derivations $j_{\pi^{-1}(U)x}$ and $l_{\pi^{-1}(U)x}$ with $x \in \mathfrak{g}$.

Proposition 2.5. *g_{b}, h_{b} are basic elements of the operation* $\mathrm{Op}\, S_{\pi^{-1}(U)}$.

Therefore, again, g_{b}, h_{b} can be identified with functions $g_{\mathrm{b}} \in \mathrm{Map}(T[1]U, \mathsf{G})$, $h_{\mathrm{b}} \in \mathrm{Map}(T[1]U, \mathfrak{g}[1])$.

Proposition 2.6. *g_{b}, h_{b} satisfy the relations*

$$d_{\pi^{-1}(U)} g_{\mathrm{b}} g_{\mathrm{b}}{}^{-1} = -h_{\mathrm{b}}, \tag{2.2.14}$$

$$d_{\pi^{-1}(U)} h_{\mathrm{b}} = -\frac{1}{2}[h_{\mathrm{b}}, h_{\mathrm{b}}]. \tag{2.2.15}$$

These are formally identical to relations (2.1.7), (2.1.8). We recognize here the familiar local base space description of gauge transformations of standard gauge theory.

Next, consider the gauge transformed connection ${}^{g,h}\omega$, ${}^{g,h}\theta$. A simple calculation yields the following result.

Proposition 2.7. *The basic components* ${}^{g,h}\omega_{\rm b}$, ${}^{g,h}\theta_{\rm b}$ *of the gauge transformed connection are given by*

$$({}^{g,h}\omega)_{\rm b} = \operatorname{Ad} g_{\rm b}(\omega_{\rm b}) + h_{\rm b}, \tag{2.2.16}$$

$$({}^{g,h}\theta)_{\rm b} = \operatorname{Ad} g_{\rm b}(\theta_{\rm b}). \tag{2.2.17}$$

These have the same form as relations (2.1.13), (2.1.14). If, with an abuse of notation, we read the above expressions as ${}^{g,h}\omega_{\rm b} = {}^{g_{\rm b},h_{\rm b}}\omega_{\rm b}$, ${}^{g,h}\theta_{\rm b} = {}^{g_{\rm b},h_{\rm b}}\theta_{\rm b}$, we recover the familiar local base space description of gauge transformations in gauge theory.

For a given trivializing neighborhood $U \subset M$, the basic components of connections and gauge transformations are Lie valued functions on $T[1]U$, so they are only locally defined. The problem arises of matching the local data pertaining to distinct but overlapping trivializing neighborhoods U, $U' \subset M$. Below, we denote by u, v, γ, σ and u', v', γ', σ' the standard adapted coordinates of $\pi^{-1}(U)$ $\pi^{-1}(U')$, respectively.

Definition 2.6. *The local basic matching transformation and shift components are the Lie group and algebra valued functions* $f_{\rm b} \in \operatorname{Map}(T[1]\pi^{-1}(U \cap U'), \mathsf{G})$ *and* $s_{\rm b} \in \operatorname{Map}(T[1]\pi^{-1}(U \cap U'), \mathfrak{g}[1])$ *defined by*

$$f_{\rm b} = \gamma'\gamma^{-1}, \tag{2.2.18}$$

$$s_{\rm b} = \operatorname{Ad}\gamma(\sigma' - \sigma). \tag{2.2.19}$$

Above, γ, σ and γ', σ' are tacitly restricted to $T[1]\pi^{-1}(U \cap U')$.

Proposition 2.8. $f_{\rm b}$, $s_{\rm b}$ *are basic elements of the operation* $\operatorname{Op} S_{\pi^{-1}(U \cap U')}$.

Therefore, $f_{\rm b}$, $s_{\rm b}$ can be identified with functions $f_{\rm b} \in \operatorname{Map}(T[1](U \cap U'), \mathsf{G})$, $s_{\rm b} \in \operatorname{Map}(T[1](U \cap U'), \mathfrak{g}[1])$.

Proposition **2.9.** *The local basic components ω_{b}, θ_{b} and ${\omega'}_{\mathrm{b}}$, ${\theta'}_{\mathrm{b}}$ of a connection ω, θ are related on $T[1](U \cap U')$ as*

$$
{\omega'}_{\mathrm{b}} = \mathrm{Ad}\, f_{\mathrm{b}}(\omega_{\mathrm{b}} - s_{\mathrm{b}}), \tag{2.2.20}
$$

$$
{\theta'}_{\mathrm{b}} = \mathrm{Ad}\, f_{\mathrm{b}}(\theta_{\mathrm{b}}). \tag{2.2.21}
$$

Upon observing that $s_{\mathrm{b}} = {f_{\mathrm{b}}}^{-1} d_{\pi^{-1}(U \cap U')} f_{\mathrm{b}}$, one recognizes above the well–known matching relations of local connection data.

Proposition **2.10.** *The local basic components g_{b}, h_{b} and ${g'}_{\mathrm{b}}$, ${h'}_{\mathrm{b}}$ of a gauge transformation g, h are related on $T[1](U \cap U')$ as*

$$
{g'}_{\mathrm{b}} = f_{\mathrm{b}} g_{\mathrm{b}} {f_{\mathrm{b}}}^{-1}, \tag{2.2.22}
$$

$$
{h'}_{\mathrm{b}} = \mathrm{Ad}\, f_{\mathrm{b}}(h_{\mathrm{b}} - s_{\mathrm{b}} + \mathrm{Ad}\, g_{\mathrm{b}}(s_{\mathrm{b}})). \tag{2.2.23}
$$

The above are the matching relations of local gauge transformation data.

Upon choosing an open covering $\{U_i\}$ of M and for each set U_i adapted coordinates u_i, v_i, γ_i, σ_i, one can describe a connection, respectively a gauge transformation, by means of the collection $\{\omega_{\mathrm{b}i}, \theta_{\mathrm{b}i}\}$, respectively $\{g_{\mathrm{b}i}, h_{\mathrm{b}i}\}$, of its local basic data defined according (2.2.8), (2.2.9), respectively (2.2.12), (2.2.13) on the U_i. The matching of the local connection and gauge transformation data is controlled through the rules (2.2.20), (2.2.21) and (2.2.22), (2.2.23) by the local basic matching data $\{f_{\mathrm{b}ij}, s_{\mathrm{b}ij}\}$ defined according to (2.2.18), (2.2.19) on the non empty intersections $U_i \cap U_j$, respectively. This yields the familiar differential cocycle theory of connections and gauge transformations.

3 2–connections and 1– and 2–gauge transformations

In this section, we construct the synthetic operational total space theory of 2–connections and 1– and 2–gauge transformations of a principal 2–bundle taking the standard connection and gauge transformation reviewed in theory of subsect. 2.1 as a model. We also show that, just as in the ordinary case, a basic framework can be worked out pointing in this way to a more conventional base space theory. Finally, an explanation of the eventual relation of the formulation presented to the theory of non Abelian differential cocycles is put forward.

3.1 General remarks on the operational setup

In what follows, we systematically refer to the synthetic apparatus of principal 2–bundle theory of subsect. 3.2 of I. The basic geometrical datum is so a principal $\hat{\mathscr{K}}$–2–bundle $\hat{\mathcal{P}}$. Its associated synthetic setup comprises the synthetic morphism and object Lie groups K, K_0 of $\hat{\mathscr{K}}$, the synthetic morphism and object manifolds P, P_0 of $\hat{\mathcal{P}}$ together with their projections π, π_0 on the base manifold M, the synthetic right K–, K_0–actions R, R_0 of P, P_0 and for any small open neighborhood $U \subset M$ synthetic K–, K_0–equivariant trivializing maps $\varPhi_U$, $\varPhi_{U0}$, respectively.

In the synthetic theory, 2–connections and 1– and 2–gauge transformations of $\hat{\mathcal{P}}$ are Lie valued graded differential forms on P suitably transforming under the K–action R. These notions are best formulated by describing K as the derived Lie group $\mathrm{D}\mathsf{M}$ of the Lie group crossed module $\mathsf{M} = (\mathsf{E}, \mathsf{G})$ underlying $\hat{\mathscr{K}}$ on one hand and the graded differential form algebra of P as the internal function algebra of $T[1]P$ on the other (cf. subsect. 3.8 of I). Because of the role of the $\mathrm{D}\mathsf{M}$–action R of P, the natural setting for studying 2–connections and 1– and 2–gauge transformations is provided then by the morphism space $S_P = (P, \mathsf{M}, R)$ of P and the associated operation $\mathrm{O{\scriptstyle P}}\, S_P = (\mathrm{F{\scriptstyle UN}}(T[1]P), \mathfrak{m})$.

The action of the derivations j_{PZ}, l_{PZ} with $Z \in \mathrm{D}\mathfrak{m}$ of $\mathrm{O{\scriptstyle P}}\, S_P$ on the internal function algebra $\mathrm{F{\scriptstyle UN}}(T[1]P)$ is expressed as a rule through the image $\zeta_{\mathfrak{m}} Z$ of Z under the isomorphism $\zeta_{\mathfrak{m}} : \mathrm{D}\mathfrak{m} \xrightarrow{\simeq} \mathrm{D}\mathfrak{m}^+$ (cf. def. 3.19 and prop. 3.26 of I). When decomposing Z in its components $x \in \mathfrak{g}$, $X \in \mathfrak{e}[1]$ according to 3.4.6

of I, the action is correspondingly expressed through $x \in \mathfrak{g}$, $\zeta_{\mathfrak{e},1}X \in \mathfrak{e}[1]^+$. The reason for this is slightly technical. The action of the vertical vector fields of P on $\textrm{F}\textsc{un}(T[1]P)$ is necessarily expressed in terms of constant $\mathrm{D}\mathfrak{m}$–valued internal functions, i. e. functions of the space $\textrm{M}\textsc{ap}(T[1]P, \mathrm{D}\mathfrak{m})$ arising by pull–back by the map $T[1]P \to *$ of functions of the space $\textrm{M}\textsc{ap}(*, \mathrm{D}\mathfrak{m}) = \mathrm{D}\mathfrak{m}^+$, the cross modality of $\mathrm{D}\mathfrak{m}$ (cf. subsect. 3.6 of I). In an ungraded setting, this careful distinction would make no difference. In a graded one, it is demanded by overall consistency. However, to simplify the notation, we tacitly shall not distinguish notationally between Z and $\zeta_{\mathfrak{m}}Z$ and similarly X and $\zeta_{\mathfrak{e},1}X$ in the following.

The study of the properties of a 2–connections and 1–gauge transformations on the object manifold P_0 as a submanifold of the morphism manifold P can also be performed. As the right $\mathrm{D}\mathsf{M}$–action R of P restricts to the the right $\mathrm{D}\mathsf{M}_0$–action R_0 of P_0, the appropriate framework for this analysis is the object space $S_{P0} = (P_0, \mathsf{M}_0, R_0)$ of P and the associated operation $\textrm{O}\textsc{p}\, S_{P0} = (\textrm{F}\textsc{un}(T[1]P_0), \mathfrak{m}_0)$ (cf. subsect. 3.8 of I). The action of the derivations of $\textrm{O}\textsc{p}\, S_{P0}$ fits with the restriction operation morphism $\textrm{O}\textsc{p}\, L : \textrm{O}\textsc{p}\, S_P \to \textrm{O}\textsc{p}\, S_{P0}$.

3.2 2–connections

In the synthetic formulation, a 2–connection of the $\hat{\mathscr{K}}$–2–bundle $\hat{\mathcal{P}}$ is a degree 1 $\mathfrak{k}$–valued graded differential form over P suitably transforming under the K–action R. Proceeding along the lines described in subsect. 3.1, a 2–connection is most naturally defined making reference to the operation $\textrm{O}\textsc{p}\, S_P$ of P.

Definition 3.1. *A 2–connection of P is a pair of Lie algebra valued internal functions $A \in \textrm{M}\textsc{ap}(T[1]P, \mathrm{D}\mathfrak{m}[1])$ and $B \in \textrm{M}\textsc{ap}(T[1]P, \mathrm{D}\mathfrak{m}[2])$, called respectively connection and curvature component, on which the action of the derivations of the operation $\textrm{O}\textsc{p}\, S_P$ is given by*

$$d_P A = -\frac{1}{2}[A, A] - d_{\dot{\tau}} A + B, \tag{3.2.1}$$

$$d_P B = -[A, B] - d_{\dot{\tau}} B, \tag{3.2.2}$$

$$j_{PZ} A = Z, \tag{3.2.3}$$

$$j_{PZ}B = 0, \tag{3.2.4}$$

$$l_{PZ}A = -[Z, A] + d_{\dot{\tau}}Z, \tag{3.2.5}$$

$$l_{PZ}B = -[Z, B] \tag{3.2.6}$$

with $Z \in \mathrm{D}\mathfrak{m}$.

Above, $[-,-]$ and $d_{\dot{\tau}}$ are the Lie bracket and the coboundary of the virtual Lie algebra $\mathrm{M{\scriptstyle AP}}(T[1]P, \mathrm{ZD}\mathfrak{m})$ (cf. eqs. 3.5.13, 3.5.15 of I). Z is tacitly viewed as an element of $\mathrm{D}\mathfrak{m}^+$ as explained in subsect. 3.1. (3.2.1)–(3.2.6) are by design formally analogous to relations (2.1.1)–(2.1.6) defining an ordinary connection, once one assumes $d_P + d_{\dot{\tau}}$ as relevant differential. (3.2.1) is just the expression of the curvature component B in terms of the connection component A. (3.2.2) is the Bianchi identity obeyed by the curvature component. The 2-connection is said flat if $B = 0$.

Lemma 3.1. (3.2.1)–(3.2.6) *respect the operation commutation relations 2.1.1–2.1.6 of I.*

Proof. One has to show that the six derivation commutators in the left hand sides of eqs. 2.1.1–2.1.6 of I act as the corresponding derivations in the right hand sides when they are applied to the functions A, B and the (3.2.1)–(3.2.6) are used. The graded commutativity of $d_{\dot{\tau}}$ with all derivations must be taken into account. This is a straightforward verification. □

By 3.5.12 of I, we can express the components A, B of a 2–connection as

$$A(\alpha) = \omega - \alpha\Omega, \tag{3.2.7}$$

$$B(\alpha) = \theta + \alpha\Theta, \quad \alpha \in \mathbb{R}[1], \tag{3.2.8}$$

through projected connection and curvature components $\omega \in \mathrm{M{\scriptstyle AP}}(T[1]P, \mathfrak{g}[1])$, $\Omega \in \mathrm{M{\scriptstyle AP}}(T[1]P, \mathfrak{e}[2])$ and $\theta \in \mathrm{M{\scriptstyle AP}}(T[1]P, \mathfrak{g}[2])$, $\Theta \in \mathrm{M{\scriptstyle AP}}(T[1]P, \mathfrak{e}[3])$. We further write $Z \in \mathrm{D}\mathfrak{m}$ as $Z(\bar{\alpha}) = x + \bar{\alpha}X$, $\bar{\alpha} \in \mathbb{R}[-1]$, with $x \in \mathfrak{g}$ and $X \in \mathfrak{e}[1]$ as in 3.4.6 of I.

Proposition 3.1. *In terms of projected components, the operation relations* (3.2.1)–

(3.2.6) *take the more explicit form*

$$d_P\omega = -\frac{1}{2}[\omega,\omega] + \dot{\tau}(\varOmega) + \theta, \tag{3.2.9}$$

$$d_P\varOmega = -\dot{\mu}(\omega,\varOmega) + \varTheta, \tag{3.2.10}$$

$$d_P\theta = -[\omega,\theta] - \dot{\tau}(\varTheta), \tag{3.2.11}$$

$$d_P\varTheta = -\dot{\mu}(\omega,\varTheta) + \dot{\mu}(\theta,\varOmega), \tag{3.2.12}$$

$$j_{PZ}\omega = x, \tag{3.2.13}$$

$$j_{PZ}\varOmega = X, \tag{3.2.14}$$

$$j_{PZ}\theta = 0, \tag{3.2.15}$$

$$j_{PZ}\varTheta = 0, \tag{3.2.16}$$

$$l_{PZ}\omega = -[x,\omega] + \dot{\tau}(X), \tag{3.2.17}$$

$$l_{PZ}\varOmega = -\dot{\mu}(x,\varOmega) + \dot{\mu}(\omega,X), \tag{3.2.18}$$

$$l_{PZ}\theta = -[x,\theta], \tag{3.2.19}$$

$$l_{PZ}\varTheta = -\dot{\mu}(x,\varTheta) + \dot{\mu}(\theta,X). \tag{3.2.20}$$

Above, X is tacitly viewed as an element of $\mathfrak{e}[1]^+$ (cf. subsect. 3.1).

Proof. To get these relations, we substitute the expressions of A, B in terms of ω, $\varOmega$, θ, $\varTheta$ of eqs. (3.2.7), (3.2.8) and that of Z in terms of x, X into (3.2.1)–(3.2.6) and use relations 3.5.13 and 3.5.15 of I. The calculations are elementary. □

(3.2.9), (3.2.10) are just the expressions of the curvature components θ, $\varTheta$ in terms of the connection components ω, $\varOmega$ and (3.2.11), (3.2.12) are the Bianchi identities obeyed by θ, $\varTheta$ familiar in strict higher gauge theory. The 2–connection is flat if $\theta = 0$, $\varTheta = 0$ and it is said fake flat if $\theta = 0$ only.

The study of the properties of a 2–connection on P_0 as a submanifold of P can also be performed. Following the lines of subsect. 3.1, the appropriate way of doing this is by making reference to the operation $\mathrm{O{\scriptstyle P}}\, S_{P0}$.

The restriction operation morphism $\mathrm{O{\scriptstyle P}}\, L : \mathrm{O{\scriptstyle P}}\, S_P \to \mathrm{O{\scriptstyle P}}\, S_{P0}$ of subsect. 3.8 of I maps the components ω, $\varOmega$, θ, $\varTheta$ of a 2–connection of P into

$$\omega_0 = I^*\omega, \tag{3.2.21}$$

$$\varOmega_0 = I^*\varOmega, \tag{3.2.22}$$

$$\theta_0 = I^*\theta, \tag{3.2.23}$$

$$\varTheta_0 = I^*\varTheta, \tag{3.2.24}$$

where $I^* : \mathrm{Fun}(T[1]P) \to \mathrm{Fun}(T[1]P_0)$ is the restriction morphism associated to the inclusion map $I : P_0 \to P$. The action of the derivations of the operation $\mathrm{Op}\,S_{P0}$ on ω_0, $\varOmega_0$, θ_0, $\varTheta_0$ is given by the right hand side of eqs. (3.2.9)–(3.2.20) with ω, $\varOmega$, θ, $\varTheta$ replaced by ω_0, $\varOmega_0$, θ_0, $\varTheta_0$ and X set to 0. By inspecting the resulting expressions, it appears that one can consistently impose the conditions

$$\varOmega_0 = 0, \tag{3.2.25}$$

$$\varTheta_0 = 0. \tag{3.2.26}$$

Upon doing so, the surviving components ω_0, θ_0 satisfy relations formally identical to (2.1.1)–(2.1.6). In spite of the seeming similarities to a connection of a principal G–bundle there are two basic differences. First, P_0 is not a principal G–bundle, as the G–action on P_0 is free but fiberwise transitive only up to isomorphism of P. Second, in the customary definition of connection the ordinary function algebras $\mathrm{Map}(T[1]P_0, \mathfrak{g}[p])$ appears.

In certain cases, it may be appropriate to restrict the range of 2–connections to those enjoying (3.2.25), (3.2.26)

Definition 3.2. *A 2–connection ω, $\varOmega$, θ, $\varTheta$ is special if (3.2.25), (3.2.26) are satisfied.*

3.3 1–gauge transformations

In the synthetic formulation, of higher gauge theory of subsect. 3.2 of I, a 1–gauge transformation of the $\hat{\mathscr{K}}$–2–bundle $\hat{\mathcal{P}}$ is a degree 0 K–valued graded differential form on P suitably transforming under the K–action R. Proceeding along the lines described in subsect. 3.1, a 1–gauge transformation is most naturally defined making reference to the operation $\mathrm{Op}\,S_P$, as for a 2–connection.

Definition 3.3. *A* 1*–gauge transformation of* P *is a pair of a Lie group valued internal function* $\varPsi \in \mathrm{MAP}(T[1]P, \mathrm{D}\mathsf{M})$ *and a Lie algebra valued internal function* $\varUpsilon \in \mathrm{MAP}(T[1]P, \mathrm{D}\mathfrak{m}[1])$*, called respectively transformation and shift component, on which the action of the derivations of the operation* $\mathrm{OP}\,S_P$ *reads as*

$$d_P\varPsi\varPsi^{-1} = -d_{\dot{\tau}}\varPsi\varPsi^{-1} - \varUpsilon, \tag{3.3.1}$$

$$d_P\varUpsilon = -\frac{1}{2}[\varUpsilon, \varUpsilon] - d_{\dot{\tau}}\varUpsilon, \tag{3.3.2}$$

$$j_{PZ}\varPsi\varPsi^{-1} = 0, \tag{3.3.3}$$

$$j_{PZ}\varUpsilon = Z - \mathrm{Ad}\,\varPsi(Z), \tag{3.3.4}$$

$$l_{PZ}\varPsi\varPsi^{-1} = -Z + \mathrm{Ad}\,\varPsi(Z), \tag{3.3.5}$$

$$l_{PZ}\varUpsilon = -[Z, \varUpsilon] + d_{\dot{\tau}}Z - \mathrm{Ad}\,\varPsi(d_{\dot{\tau}}Z) \tag{3.3.6}$$

with $Z \in \mathrm{D}\mathfrak{m}$*.*

The above relations involve several algebraic constructs studied in subsect. 3.5 of I. $[-,-]$ and $d_{\dot{\tau}}$ are respectively the Lie bracket and the coboundary of the virtual Lie algebra $\mathrm{MAP}(T[1]P, \mathrm{ZD}\mathfrak{m})$ defined in eqs. 3.5.13, 3.5.15) of I. Ad is the adjoint action of the virtual Lie group $\mathrm{MAP}(T[1]P, \mathrm{D}\mathsf{M})$ on $\mathrm{MAP}(T[1]P, \mathrm{ZD}\mathfrak{m})$ given in eq. 3.5.18 of I. The terms $D\varPsi\varPsi^{-1}$ with $D = d_P, j_{PZ}, l_{PZ}$ are the pull–back of the first Maurer–Cartan element of $\mathrm{D}\mathsf{M}$ by $\varPsi$ followed by contraction with D seen as a vector field on $T[1]P$, see eq. 3.5.22 of I. The term $d_{\dot{\tau}}\varPsi\varPsi^{-1}$ is similarly given by eq. 3.5.24 of I. Z is tacitly viewed as an element of $\mathrm{D}\mathfrak{m}^+$ as explained in subsect. 3.1. Again, upon considering $d_P + d_{\dot{\tau}}$ as relevant differential, the (3.3.1)–(3.3.6) are formally analogous to relations (2.1.7)–(2.1.12) defining an ordinary gauge transformation. Relation (3.3.1) effectively defines the shift component $\varUpsilon$ in terms of the transformation component $\varPsi$. (3.3.2) is the associated Maurer–Cartan equation.

Lemma 3.2. (3.3.1)–(3.3.6) *respect the operation commutation relations 2.1.1–2.1.6 of I.*

Proof. One has to verify that the six derivation commutators in the left hand sides of eqs. 2.1.1–2.1.6 of I act as the corresponding derivations in the right hand sides when they are applied to the functions $\varPsi$, $\varUpsilon$ and the (3.3.1)–(3.3.6) are used. In the case of $\varPsi$, one must employ systematically the basic relation $[D, D']GG^{-1} = D(D'GG^{-1}) - (-1)^{|D||D'|}D'(DGG^{-1}) - [DGG^{-1}, D'GG^{-1}]$ holding for two graded derivations D, D' and a Lie group valued function G. The graded commutativity of $d_{\dot{\tau}}$ with all derivations must further be taken into account. The verification is straightforward. ☐

Since by (3.3.1) the shift component $\varUpsilon$ of a 1–gauge transformation can be expressed in terms of the transformation component $\varPsi$, a 1–gauge transformation is effectively specified by this latter. 1–gauge transformations can thus be viewed as elements of the virtual Lie group $\mathrm{MAP}(T[1]P, \mathrm{D}\mathsf{M})$ of $\mathrm{D}\mathsf{M}$–valued internal functions of $T[1]P$. As (3.3.3), (3.3.5) are evidently preserved under the group operations of $\mathrm{MAP}(T[1]P, \mathrm{D}\mathsf{M})$, 1–gauge transformations form in fact a distinguished subgroup of this latter, the 1–gauge group in the present formulation.

1–gauge transformations act on 2–connections of P (cf. subsect. 3.2, def. 3.1) compatibly with the K–action on both types of items.

Proposition 3.2. *If A, B and $\varPsi$, $\varUpsilon$ are the components of a 2–connection and a 1–gauge transformation, respectively, then*

$$A' = \operatorname{Ad}\varPsi(A) + \varUpsilon, \tag{3.3.7}$$

$$B' = \operatorname{Ad}\varPsi(B) \tag{3.3.8}$$

are the components of a 2–connection.

Proof. To show that A', B' are the components of a 2–connection, we have to check that the action of the derivations of the operation on A', B' conforms to (3.2.1)–(3.2.6) using that the action of those derivations on A, B and $\varPsi$, $\varUpsilon$ is given by (3.2.1)–(3.2.6) and (3.3.1)–(3.3.6), respectively. This is a matter of a simple calculation. ☐

Definition 3.4. *The gauge transform of a 2–connection of components A, B by a 1–gauge transformation of components $\varPsi$, $\varUpsilon$ is the 2–connection of components*

$$ {}^{\Psi,\Upsilon}A = \mathrm{Ad}\,\Psi(A) + \Upsilon, \tag{3.3.9} $$

$$ {}^{\Psi,\Upsilon}B = \mathrm{Ad}\,\Psi(B). \tag{3.3.10} $$

(3.3.9), (3.3.10) are formally identical to relations (2.1.13), (2.1.14) defining the gauge transform of a connection in ordinary principal bundle theory. Notice that flatness of a 2–connections is a 1–gauge invariant property. (3.3.9), (3.3.10) yield a left action of the 1–gauge transformation group on the 2–connection space, as it is readily verified.

Making use of 3.5.1, 3.5.12 of I, we can express the components Ψ, Υ of a 1–gauge transformation as

$$ \Psi(\alpha) = \mathrm{e}^{\alpha J}\, g, \tag{3.3.11} $$

$$ \Upsilon(\alpha) = h - \alpha K, \quad \alpha \in \mathbb{R}[1], \tag{3.3.12} $$

by means of projected transformation and shift components $g \in \mathrm{MAP}(T[1]P, \mathsf{G})$, $J \in \mathrm{MAP}(T[1]P, \mathfrak{e}[1])$ and $h \in \mathrm{MAP}(T[1]P, \mathfrak{g}[1])$, $K \in \mathrm{MAP}(T[1]P, \mathfrak{e}[2])$ (cf. subsect. 3.5 of I). We further write $Z \in \mathrm{D}\mathfrak{m}$ as $Z(\bar{\alpha}) = x + \bar{\alpha}X$, $\bar{\alpha} \in \mathbb{R}[-1]$, with $x \in \mathfrak{g}$ and $X \in \mathfrak{e}[1]$ as in 3.4.6 of I.

Proposition 3.3. *In terms of projected components, the operation relations (3.3.1)–(3.3.6) take the explicit form*

$$ d_P g g^{-1} = -h - \dot{\tau}(J), \tag{3.3.13} $$

$$ d_P J = -K - \frac{1}{2}[J, J] - \dot{\mu}\dot{}(h, J), \tag{3.3.14} $$

$$ d_P h = -\frac{1}{2}[h, h] + \dot{\tau}(K), \tag{3.3.15} $$

$$ d_P K = -\dot{\mu}\dot{}(h, K), \tag{3.3.16} $$

$$ j_{PZ} g g^{-1} = 0, \tag{3.3.17} $$

$$ j_{PZ} J = 0, \tag{3.3.18} $$

$$ j_{PZ} h = x - \mathrm{Ad}\, g(x), \tag{3.3.19} $$

$$ j_{PZ} K = \dot{\mu}\dot{}(\mathrm{Ad}\, g(x), J) + X - \mu\dot{}(g, X), \tag{3.3.20} $$

$$ l_{PZ} g g^{-1} = -x + \mathrm{Ad}\, g(x), \tag{3.3.21} $$

$$l_{PZ}J = -\dot{\mu}\dot{}(x, J) - X + \mu\dot{}(g, X), \tag{3.3.22}$$

$$l_{PZ}h = -[x, h] + \dot{\tau}(X) - \operatorname{Ad} g(\dot{\tau}(X)), \tag{3.3.23}$$

$$l_{PZ}K = -\dot{\mu}\dot{}(x, K) + \dot{\mu}\dot{}(h, X) + [\mu\dot{}(g, X), J]. \tag{3.3.24}$$

Above, X is tacitly viewed as an element of $\mathfrak{e}[1]^+$ (cf. subsect. 3.1).

Proof. To obtain the above relations, we substitute the expressions of $\varPsi$, $\varUpsilon$ in terms of g J, h, K of eqs. (3.3.11), (3.3.12) and that of Z in terms of x, X into (3.3.1)–(3.3.6) and use systematically relations 3.5.13, 3.5.15 and 3.5.18 as well as expressions 3.5.22 and 3.5.24 of I. This is again a straightforward though a bit length calculation. □

In the projected framework we are using, so, the 1–gauge group is the subgroup of $\mathrm{MAP}(T[1]P, \mathfrak{e}[1] \rtimes_\mu \mathsf{G})$ formed by the pairs g, J satisfying (3.3.17), (3.3.18) and (3.3.21), (3.3.22). g, J are indeed the data of a 1–gauge transformation familiar in strict higher gauge theory.

Expressing the components A, B of a 2–connection and $\varPsi$, $\varUpsilon$ of a 1–gauge transformation in terms of projected components ω, $\varOmega$, θ, $\varTheta$ and g, J, h, K using (3.2.7), (3.2.8) and (3.3.11), (3.3.12) respectively, we obtain projected component expressions of the transformation relations (3.3.9), (3.3.10).

Proposition 3.4. *In terms of projected components, the transformation relations (3.3.9), (3.3.10) take the explicit form*

$${}^{g,J,h,K}\omega = \operatorname{Ad} g(\omega) + h, \tag{3.3.25}$$

$${}^{g,J,h,K}\varOmega = \mu\dot{}(g, \varOmega) - \dot{\mu}\dot{}(\operatorname{Ad} g(\omega), J) + K, \tag{3.3.26}$$

$${}^{g,J,h,K}\theta = \operatorname{Ad} g(\theta), \tag{3.3.27}$$

$${}^{g,J,h,K}\varTheta = \mu\dot{}(g, \varTheta) - \dot{\mu}\dot{}(\operatorname{Ad} g(\theta), J). \tag{3.3.28}$$

One recognizes here the standard expressions of the gauge transform of a 2–connection of strict higher gauge theory.

Proof. To obtain the above relations, we substitute the expressions of A, B in terms of ω, Ω, θ, Θ and Ψ, Υ in terms of g J, h, K of eqs. (3.2.7), (3.2.8) and (3.3.11), (3.3.12) respectively, as anticipated above, and use 3.5.18 of I. □

The study of the properties of a 1–gauge transformation on P_0 as a submanifold of P can also be carried out. Following the lines of subsect. 3.1, this is most naturally done making reference to the operation $\mathrm{O{\scriptstyle P}}\, S_{P0}$, just as for a 2–connection.

Under the restriction operation morphism $\mathrm{O{\scriptstyle P}}\, L : \mathrm{O{\scriptstyle P}}\, S_P \to \mathrm{O{\scriptstyle P}}\, S_{P0}$, of subsect. 3.8 of I, the components g, J, h, K of a 1–gauge transformation of P get

$$g_0 = I^* g, \tag{3.3.29}$$

$$J_0 = I^* J, \tag{3.3.30}$$

$$h_0 = I^* h, \tag{3.3.31}$$

$$K_0 = I^* K, \tag{3.3.32}$$

where $I^* : \mathrm{F{\scriptstyle UN}}(T[1]P) \to \mathrm{F{\scriptstyle UN}}(T[1]P_0)$ is the restriction morphism associated to the inclusion map $I : P_0 \to P$. The action of the derivations of the operation $\mathrm{O{\scriptstyle P}}\, S_{P0}$ on g_0, J_0, h_0, K_0 is given by the right hand side of eqs. (3.3.13)–(3.3.24) with g, J, h, K replaced by g_0, J_0, h_0, K_0 and X set to 0. From the resulting expressions, it appears that one can consistently impose the conditions

$$J_0 = 0, \tag{3.3.33}$$

$$K_0 = 0. \tag{3.3.34}$$

Upon doing so, the surviving components g_0, h_0 satisfy relations formally identical to (2.1.10)–(2.1.12). Again, in spite of similarities to a gauge transformation of an ordinary principal G–bundle, the differences recalled below eq. (3.2.26) hold and should be kept in mind.

In certain cases, it may be befitting to restrict the range of 1–gauge transformations so as to allow only those enjoying the above property.

Definition 3.5. *A 1–gauge transformation g, J, h, K is special if (3.3.33), (3.3.34) are met.*

Special 1–gauge transformations form a subgroup of the 1–gauge group.

In subsect. 3.2, def. 3.2, we introduced the notion of special 2–connection which pairs with that of special 1–gauge transformation put forward above. It turns out that the action of 1–gauge transformations on 2–connections is compatible with specialty in the following sense.

Proposition 3.5. *If a 2–connection ω, $\varOmega$, θ, $\varTheta$ and a 1–gauge transformation g, J, h, K are both special then the 1–gauge transformed 2–connection ${}^{g,J,h,K}\omega$, ${}^{g,J,h,K}\varOmega$, ${}^{g,J,h,K}\theta$, ${}^{g,J,h,K}\varTheta$ also is.*

Proof. Inspection of (3.3.26), (3.3.28) shows that when the ω, $\varOmega$, θ, $\varTheta$ and g, J, h, K satisfy respectively (3.2.25), (3.2.26) and (3.3.33), (3.3.34), then ${}^{g,J,h,K}\omega$, ${}^{g,J,h,K}\varOmega$, ${}^{g,J,h,K}\theta$, ${}^{g,J,h,K}\varTheta$ also satisfy (3.2.25), (3.2.26) as well, as required. □

Comparison of (3.3.25), (3.3.27) and (2.1.13), (2.1.14) shows further that

$$ {}^{g,J,h,K}\omega_0 = {}^{g_0,h_0}\omega_0, \tag{3.3.35}$$

$$ {}^{g,J,h,K}\theta_0 = {}^{g_0,h_0}\theta_0, \tag{3.3.36}$$

where ω_0, θ_0 and g_0, h_0 are formally treated as an ordinary connection and gauge transformation, respectively. In this sense, one recovers in this way the well–known expressions of the gauge transform of a connection.

3.4 2–gauge transformations

In the synthetic formulation, a 2–gauge transformation of the $\hat{\mathscr{K}}$–2–bundle $\hat{\mathcal{P}}$ is a degree 0 E–valued graded differential form on P suitably transforming under the K–action R. Hence, E instead of $D\mathsf{M}$ is the relevant target group in this case. The operational framework remains however perfectly adequate. In this way, a 2–gauge transformation is most naturally defined by making again reference to the operation $\mathrm{O}\textsc{p}\, S_P$.

To make contact with the standard higher gauge theoretic treatment of 2–gauge transformations, it is necessary to express the action of the operation derivations with reference to a given 2–connection of $\hat{\mathcal{P}}$ (cf. subsect. 3.2). We assume so that a 2–connection of projected components ω, $\varOmega$, θ, $\varTheta$ is assigned.

Definition **3.6.** *We define a 2–gauge transformation as a pair of a Lie group valued internal function $E \in \mathrm{MAP}(T[1]P, \mathsf{E})$ and a Lie algebra valued internal function $C \in \mathrm{MAP}(T[1]P, \mathfrak{e}[1])$, called respectively modification and variation component, which are acted upon by the operation derivations as*

$$d_P E E^{-1} = -C - \dot{\mu}(\omega, E), \tag{3.4.1}$$

$$d_P C = -\frac{1}{2}[C, C] - \dot{\mu}\dot{}(\omega, C) - \dot{\mu}(\theta, E) - \varOmega + \mathrm{Ad}\, E(\varOmega), \tag{3.4.2}$$

$$j_{PZ} E E^{-1} = 0, \tag{3.4.3}$$

$$j_{PZ} C = 0, \tag{3.4.4}$$

$$l_{PZ} E E^{-1} = -\dot{\mu}(x, E), \tag{3.4.5}$$

$$l_{PZ} C = -\dot{\mu}\dot{}(x, C) - X + \mathrm{Ad}\, E(X) \tag{3.4.6}$$

with $Z \in \mathrm{D}\mathfrak{m}$ written in terms of its projected components $x \in \mathfrak{g}$, $X \in \mathfrak{e}[1]$.

Above, X is tacitly viewed as an element of $\mathfrak{e}[1]^+$ as in earlier instances. Relations (3.4.1) effectively defines the variation component C in terms of the modification component E and the reference 2–connection ω, $\varOmega$, θ, $\varTheta$. (3.4.2) is the corresponding Bianchi type identity.

Lemma **3.3.** (3.4.1)–(3.4.6) *respect the operation commutation relations 2.1.1–2.1.6 of I.*

Proof. The proof consists in checking that the six derivation commutators in the left hand sides of eqs. 2.1.1–2.1.6 act as the corresponding derivations in the right hand sides when they are applied to the functions E, C and the (3.4.1)–(3.4.6) are used. In the case of E, it is necessary to use the basic relation $[D, D']GG^{-1} = D(D'GG^{-1}) - (-1)^{|D||D'|}D'(DGG^{-1}) - [DGG^{-1}, D'GG^{-1}]$ holding for two graded derivations D, D' and a Lie group valued function G. The verification is straightforward. □

2–gauge transformations act on 1–gauge transformations (cf. subsect. 3.3) and do so in a proper way depending on the reference 2–connection and compatibly with the K action on all these items.

Proposition 3.6. *If g, J, h, K and C, E are the components of a 1– and a 2–gauge transformation, respectively, then*

$$g' = \tau(E)g, \tag{3.4.7}$$

$$J' = \operatorname{Ad} E(J) + {}^{\cdot}\mu(\omega - \operatorname{Ad} g(\omega) - h, E) + C, \tag{3.4.8}$$

$$h' = \operatorname{Ad} \tau(E)(h) + \dot{\tau}({}^{\cdot}\mu(\operatorname{Ad} g(\omega) + h, E)), \tag{3.4.9}$$

$$\begin{aligned} K' = \operatorname{Ad} E(K) &+ {}^{\cdot}\mu^{\cdot}(\operatorname{Ad}(\tau(E)g)(\omega), {}^{\cdot}\mu(\omega - \operatorname{Ad} g(\omega) - h, E) + C) \\ &+ {}^{\cdot}\mu(\operatorname{Ad} g(\theta) + \dot{\tau}(\mu^{\cdot}(g, \varOmega) - {}^{\cdot}\mu^{\cdot}(\operatorname{Ad} g(\omega), J) + K), E) \end{aligned} \tag{3.4.10}$$

are the components of a 1–gauge transformation.

Proof. A straightforward algebraic calculation shows that the action (3.2.9)–(3.2.20), (3.3.13)–(3.3.24) and (3.4.1)-(3.4.6) of the operation derivations on the components ω, $\varOmega$, θ, $\varTheta$, g, J, h, K and E, C ensures that the action of those derivations on the transformed components g', J', h', K' satisfies (3.3.13)–(3.3.24) as well and that consequently g', J', h', K' are also the components of a 1–gauge transformation as claimed. □

Definition 3.7. *The 2–gauge transform of a 1–gauge transformation of components g, J, h, K by a 2–gauge transformation of components E, C is given by*

$${}^{E,C}g = \tau(E)g, \tag{3.4.11}$$

$${}^{E,C}J = \operatorname{Ad} E(J) + {}^{\cdot}\mu(\omega - \operatorname{Ad} g(\omega) - h, E) + C, \tag{3.4.12}$$

$${}^{E,C}h = \operatorname{Ad} \tau(E)(h) + \dot{\tau}({}^{\cdot}\mu(\operatorname{Ad} g(\omega) + h, E)), \tag{3.4.13}$$

$$\begin{aligned} {}^{E,C}K = \operatorname{Ad} E(K) &+ {}^{\cdot}\mu^{\cdot}(\operatorname{Ad}(\tau(E)g)(\omega), {}^{\cdot}\mu(\omega - \operatorname{Ad} g(\omega) - h, E) + C) \\ &+ {}^{\cdot}\mu(\operatorname{Ad} g(\theta) + \dot{\tau}(\mu^{\cdot}(g, \varOmega) - {}^{\cdot}\mu^{\cdot}(\operatorname{Ad} g(\omega), J) + K), E). \end{aligned} \tag{3.4.14}$$

Inserting above relations (3.3.13), (3.3.14) expressing h, K in terms of g, J, and relation (3.4.1) expressing C in terms of E, these expressions are formally identical to the standard ones of strict higher gauge theory.

Since for an assigned reference 2–connection the variation component of a 2–gauge transformation can be expressed in terms of the modification component by

(3.4.1), a 2–gauge transformation is effectively specified by this latter. 2–gauge transformations can hence be viewed as elements of the group $\mathrm{MAP}(T[1]P,\mathsf{E})$ of E–valued internal functions. They form indeed a distinguished subgroup of this latter, the 2–gauge group, as (3.4.3), (3.4.5) are preserved under the group operations of $\mathrm{MAP}(T[1]P,\mathsf{E})$ (see eq. B.0.8 of I). 2–gauge transformation is a left action of the 2–gauge group on 1–gauge transformations.

2–gauge transformation action has the further relevant property.

Proposition 3.7. *For a 1– and a 2–gauge transformation of components g, J, h, K and E, C, respectively, one has*

$$^{E,C}g,{}^{E,C}J,{}^{E,C}h,{}^{E,C}K\,\omega = {}^{g,J,h,K}\omega, \tag{3.4.15}$$

$$^{E,C}g,{}^{E,C}J,{}^{E,C}h,{}^{E,C}K\,\varOmega = {}^{g,J,h,K}\varOmega + \dot{}\mu(\mathrm{Ad}\,g(\theta),E), \tag{3.4.16}$$

$$^{E,C}g,{}^{E,C}J,{}^{E,C}h,{}^{E,C}K\,\theta = {}^{g,J,h,K}\theta - \dot\tau(\dot{}\mu(\mathrm{Ad}\,g(\theta),E)), \tag{3.4.17}$$

$$\begin{aligned}&^{E,C}g,{}^{E,C}J,{}^{E,C}h,{}^{E,C}K\,\varTheta \\ &\qquad = {}^{g,J,h,K}\varTheta + d_P\dot{}\mu(\mathrm{Ad}\,g(\theta),E) + \dot{}\mu\dot{}\,({}^{g,J,h,K}\omega,\dot{}\mu(\mathrm{Ad}\,g(\theta),E)).\end{aligned} \tag{3.4.18}$$

Above, the 1–gauge transformed connection components are given by (3.3.25)–(3.3.28).

Proof. The proof is a matter of evaluating the right hand sides of relations (3.3.25)–(3.3.28) with g, J, h, K replaced by ${}^{E,C}g$, ${}^{E,C}J$, ${}^{E,C}h$, ${}^{E,C}K$. The calculations are straightforward. □

Hence, the gauge transformation action of g, J, h, K and ${}^{E,C}g, {}^{E,C}J, {}^{E,C}h, {}^{E,C}K$ is the same on fake flat 2–connections. With these qualifications, 2–gauge transformation corresponds to gauge for gauge symmetry.

3.5 Local operational description of a principal 2–bundle

The local trivializability of the relevant principal $\hat{\mathscr{K}}$–2–bundle $\hat{\mathcal{P}}$ implies that of the associated synthetic manifold P (cf. subsect. 3.2 of I). On any sufficiently small neighborhood U of the base M, there exists so a projection preserving K–

equivariant map $\Phi_U \in \mathrm{Map}(\pi^{-1}(U), U \times \mathsf{K})$ (cf. def. 3.12 of I). Φ_U provides a set of coordinates of $\pi^{-1}(U)$ modelled on $U \times \mathsf{K}$. These are in many ways analogous to the standard adapted coordinates of an ordinary principal bundle. One must keep in mind however that they are not anything like genuine coordinates, because they arise from a local trivializing functor $\hat{\Phi}_U$ of $\hat{\mathcal{P}}$ that is only weakly invertible (cf. subsect. 3.1 of I).

2–connections and 1– and 2– gauge transformations are Lie valued internal functions on $T[1]P$ rather than ordinary functions on P (cf. subsects. 3.2–3.4). For this reason, their local description on U presumably requires a set of coordinates modelled on $U \times \mathrm{D}\mathsf{M}$ which are internal functions on $T[1]\pi^{-1}(U)$ rather than ordinary functions on $\pi^{-1}(U)$. (Recalll that $\mathsf{K} = \mathrm{D}\mathsf{M}$ by 3.8.1 of I.) The coordinates furnished by the trivializing map Φ_U are thus not general enough to serve for our purposes. A more general and weaker notion of coordinates is necessary here.

By the general philosophy of our operational framework, the natural setup for studying the desired kind of internal adapted coordinates is the operation $\mathrm{O}\textsc{p}\,S_{\pi^{-1}(U)} = (\textsc{Fun}(T[1]\pi^{-1}(U)), \mathfrak{m})$, since the synthetic morphism manifold of the $\hat{\mathscr{K}}$–2–bundle $\hat{\mathcal{P}}|_U$ is precisely $\pi^{-1}(U)$.

A full set of internal coordinates of $\pi^{-1}(U)$ modelled on $U \times \mathrm{D}\mathsf{M}$ comprises two subsets of coordinates modelled on U and $\mathrm{D}\mathsf{M}$ respectively. These require separate consideration.

By virtue of prop. 3.4 of I, the internal coordinates of $\pi^{-1}(U)$ modelled on U are yielded by the synthetic projection π (cf. def. 3.10 of I). They are so ordinary functions. In the operational setup, they can be characterized as follows.

Proposition 3.8. *A set of internal coordinates of $\pi^{-1}(U)$ modelled on U is described by vector–valued ordinary functions $u \in \mathrm{Map}(T[1]\pi^{-1}(U), \mathbb{R}^{\dim M})$ and $v \in \mathrm{Map}(T[1]\pi^{-1}(U), \mathbb{R}^{\dim M}[1])$ on which the operation derivations act as*

$$d_{\pi^{-1}(U)}u = v, \qquad d_{\pi^{-1}(U)}v = 0 \tag{3.5.1}$$

with trivial action of all derivations $j_{\pi^{-1}(U)Z}$, $l_{\pi^{-1}(U)Z}$ for all $Z \in \mathrm{D}\mathfrak{m}$.

Above, u, v are treated as special cases of internal functions and as such are acted upon by the derivations of the operation.

Proof. Upon composing the factor π with a set of ordinary coordinates of U, we obtain an ordinary function $u \in \mathrm{Map}(\pi^{-1}(U), \mathbb{R}^{\dim M}) \subset \mathrm{Map}(T[1]\pi^{-1}(U), \mathbb{R}^{\dim M})$. The scalar nature and $\mathsf{D}\mathsf{M}$–invariance of π (cf. prop. 3.3 of I) entail that u is annihilated by all the derivations $j_{\pi^{-1}(U)Z}$, $l_{\pi^{-1}(U)Z}$. With u there is associated a further ordinary function $v \in \mathrm{Map}(T[1]\pi^{-1}(U), \mathbb{R}^{\dim M}[1])$ defined through (3.5.1). The action of $d_{\pi^{-1}(U)}$ and the $j_{\pi^{-1}(U)Z}$, $l_{\pi^{-1}(U)Z}$ on v follows from that on u and the operation relations 2.1.1–2.1.6 of I. □

The internal coordinates of $\pi^{-1}(U)$ modelled on $\mathsf{D}\mathsf{M}$ are a novelty requiring a precise definition. The one provided here is generic and may require some tuning at a later stage, but it is enough for our purposes for the time being.

Definition 3.8. *A set of internal coordinates of $\pi^{-1}(U)$ modelled on $\mathsf{D}\mathsf{M}$ is constituted by a Lie group valued internal function $\varLambda \in \mathrm{M{\scriptsize AP}}(T[1]\pi^{-1}(U), \mathsf{D}\mathsf{M})$ and a Lie algebra valued internal function $\varDelta \in \mathrm{M{\scriptsize AP}}(T[1]\pi^{-1}(U), \mathsf{D}\mathfrak{m}[1])$ acted upon by the operation derivations as*

$$\varLambda^{-1} d_{\pi^{-1}(U)} \varLambda = -\varLambda^{-1} d_{\dot{\tau}} \varLambda + \varDelta, \tag{3.5.2}$$

$$d_{\pi^{-1}(U)} \varDelta = -\frac{1}{2}[\varDelta, \varDelta] - d_{\dot{\tau}} \varDelta, \tag{3.5.3}$$

$$\varLambda^{-1} j_{\pi^{-1}(U)Z} \varLambda = 0, \tag{3.5.4}$$

$$j_{\pi^{-1}(U)Z} \varDelta = Z, \tag{3.5.5}$$

$$\varLambda^{-1} l_{\pi^{-1}(U)Z} \varLambda = Z, \tag{3.5.6}$$

$$l_{\pi^{-1}(U)Z} \varDelta = -[Z, \varDelta] + d_{\dot{\tau}} Z \tag{3.5.7}$$

with $Z \in \mathsf{D}\mathfrak{m}$. It is further required that $\varLambda|_{\pi^{-1}(U)} \in \mathrm{Map}(\pi^{-1}(U), \mathsf{D}\mathsf{M})$ and that $\varLambda|_{\pi^{-1}(U)} = \mathrm{pr}_{\mathsf{K}} \circ \varPhi_U$, where $\pi^{-1}(U)$ is embedded in $T[1]\pi^{-1}(U)$ as its zero section.

The notational remarks stated below eqs. (3.3.1)–(3.3.6) apply here as well with obvious changes and will not be repeated. As in similar cases considered earlier, upon considering $d_{\pi^{-1}(U)} + d_{\dot{\tau}}$ as relevant differential (3.5.2)–(3.5.7) are formally analogous to relations (2.2.2)–(2.2.7) holding for the adapted coordinates of ordinary principal bundles. Eq. (3.5.2) defines the coordinate $\varDelta$ in terms of its partner $\varLambda$. (3.5.3) is the associated Maurer–Cartan–like equation. Upon comparing eqs. (3.5.3), (3.5.5), (3.5.7) with (3.2.1), (3.2.3), (3.2.5), it emerges also that $\varDelta$ is the connection component of a flat 2–connection of the principal 2–bundle $\hat{\mathcal{P}}|_U$ (cf. subsect. 3.2).

Lemma 3.4. *The operation commutation relations 2.1.1–2.1.6 of I are respected by (3.5.2)–(3.5.7).*

Proof. This is shown by checking that the six derivation commutators in the left hand sides of eqs. 2.1.1–2.1.6 of I act as the corresponding derivations in the right hand sides when they are applied to the functions $\varLambda$, $\varDelta$ and the (3.5.2)–(3.5.7) are used. □

The requirement on $\varLambda|_{\pi^{-1}(U)}$ is added in order to render the definition of coordinates modelled on DM provided above geometrically meaningful, though it plays no direct role in the basic theory of subsects. 3.6, 3.7. Note also that the condition that $\varLambda|_{\pi^{-1}(U)}$ be an ordinary rather than internal function is *not* preserved by the derivations $l_{\pi^{-1}(U)Z}$ by (3.5.6). This is expected on general grounds, since by the graded nature of $\mathsf{D}\mathfrak{m}$ $l_{\pi^{-1}(U)Z}$ turns ordinary functions on $\pi^{-1}(U)$ into internal ones.

Employing 3.5.1, 3.5.12 of I, we can expand the fiber coordinates $\varLambda$, $\varDelta$ as

$$\varLambda(\alpha) = \mathrm{e}^{\alpha\varGamma}\gamma, \tag{3.5.8}$$

$$\varDelta(\alpha) = \sigma - \alpha\varSigma, \quad \alpha \in \mathbb{R}[1]. \tag{3.5.9}$$

In the above relations, $\gamma \in \mathrm{M{\scriptstyle AP}}(T[1]\pi^{-1}(U), \mathsf{G})$, $\varGamma \in \mathrm{M{\scriptstyle AP}}(T[1]\pi^{-1}(U), \mathfrak{e}[1])$ and $\sigma \in \mathrm{M{\scriptstyle AP}}(T[1]\pi^{-1}(U), \mathfrak{g}[1])$, $\varSigma \in \mathrm{M{\scriptstyle AP}}(T[1]\pi^{-1}(U), \mathfrak{e}[2])$ are the projected internal coordinates modelled on $\mathsf{DM} = \mathfrak{e}[1] \rtimes_{\dot{\mu}} \mathsf{G}$. We also write $Z \in \mathsf{D}\mathfrak{m}$ as $Z(\bar{\alpha}) = x + \bar{\alpha}X$, $\bar{\alpha} \in \mathbb{R}[-1]$, with $x \in \mathfrak{g}$ and $X \in \mathfrak{e}[1]$ as in 3.4.6 of I.

Proposition 3.9. *Expressed in terms of projected internal adapted coordinates, the operation relations (3.5.2)–(3.5.7) take the form*

$$\gamma^{-1} d_{\pi^{-1}(U)}\gamma = \sigma - \dot{\tau}(\dot{\mu}(\gamma^{-1}, \Gamma)), \tag{3.5.10}$$

$$\dot{\mu}(\gamma^{-1}, d_{\pi^{-1}(U)}\Gamma) = \Sigma - \frac{1}{2}\dot{\mu}(\gamma^{-1}, [\Gamma, \Gamma]), \tag{3.5.11}$$

$$d_{\pi^{-1}(U)}\sigma = -\frac{1}{2}[\sigma, \sigma] + \dot{\tau}(\Sigma), \tag{3.5.12}$$

$$d_{\pi^{-1}(U)}\Sigma = -\dot{\mu}(\sigma, \Sigma), \tag{3.5.13}$$

$$\gamma^{-1} j_{\pi^{-1}(U)Z}\gamma = 0, \tag{3.5.14}$$

$$\dot{\mu}(\gamma^{-1}, j_{\pi^{-1}(U)Z}\Gamma) = 0, \tag{3.5.15}$$

$$j_{\pi^{-1}(U)Z}\sigma = x, \tag{3.5.16}$$

$$j_{\pi^{-1}(U)Z}\Sigma = X, \tag{3.5.17}$$

$$\gamma^{-1} l_{\pi^{-1}(U)Z}\gamma = x, \tag{3.5.18}$$

$$\dot{\mu}(\gamma^{-1}, l_{\pi^{-1}(U)Z}\Gamma) = X, \tag{3.5.19}$$

$$l_{\pi^{-1}(U)Z}\sigma = -[x, \sigma] + \dot{\tau}(X), \tag{3.5.20}$$

$$l_{\pi^{-1}(U)Z}\Sigma = -\dot{\mu}(x, \Sigma) + \dot{\mu}(\sigma, X). \tag{3.5.21}$$

Moreover, $\gamma|_{\pi^{-1}(U)} \in \mathrm{Map}(\pi^{-1}(U), \mathsf{G})$, $\Gamma|_{\pi^{-1}(U)} \in \mathrm{Map}(\pi^{-1}(U), \mathfrak{e}[1])$ *and* $\gamma|_{\pi^{-1}(U)}$, $\Gamma|_{\pi^{-1}(U)}$, *in the combination* (3.5.8), *yield* $\mathrm{pr}_{\mathsf{K}} \circ \Phi_U$.

Proof. The proof is a matter of a straightforward albeit lengthy calculation. We substitute the expressions of Λ, Δ in terms of γ Γ, σ, Σ of eqs. (3.5.8), (3.5.9) and that of Z in terms of x, X into (3.5.2)– (3.5.7) and use relations 3.5.13, 3.5.15 and 3.5.18 as well as expressions 3.5.23 and 3.5.25 of I. □

Under the restriction operation morphism $\mathrm{O}\mathrm{P}\, L_U : \mathrm{O}\mathrm{P}\, S_{\pi^{-1}(U)} \to \mathrm{O}\mathrm{P}\, S_{\pi^{-1}(U)0}$ of subsect. 3.8 of I, the projected internal coordinates γ, Γ, σ, Σ of $\pi^{-1}(U)$ get

$$\gamma_0 = I_U{}^*\gamma, \tag{3.5.22}$$

$$\Gamma_0 = I_U{}^*\Gamma, \tag{3.5.23}$$

$$\sigma_0 = I_U{}^*\sigma, \tag{3.5.24}$$

$$\Sigma_0 = I_U{}^*\Sigma, \tag{3.5.25}$$

where ${I_U}^* : \mathrm{F{\scriptstyle UN}}(T[1]\pi^{-1}(U)) \to \mathrm{F{\scriptstyle UN}}(T[1]{\pi_0}^{-1}(U))$ is the restriction morphism associated with the inclusion map $I_U : {\pi_0}^{-1}(U) \to \pi^{-1}(U)$.

Definition **3.9.** *The internal coordinates $\pi^{-1}(U)$ modelled on* $\mathrm{D}\mathsf{M}$ *are special if*

$$\mathit{\Gamma}_0 = 0, \tag{3.5.26}$$

$$\mathit{\Sigma}_0 = 0. \tag{3.5.27}$$

γ_0, σ_0 are then a subset of internal coordinates of ${\pi_0}^{-1}(U)$ adapted to $\mathrm{D}\mathsf{M}_0$.

Proposition **3.10.** *For special adapted coordinates, the action of the operation derivations on γ_0, σ_0 is given by the right hand side of eqs. (3.5.10)–(3.5.21) with γ, σ replaced by γ_0, σ_0 and $\mathit{\Gamma}$, $\mathit{\Sigma}$ and X set to* 0.

Proof. The action of the derivations of the operation $\mathrm{O{\scriptstyle P}}\,S_{\pi^{-1}(U)0}$ on γ_0, $\mathit{\Gamma}_0$, σ_0, $\mathit{\Sigma}_0$ is given by the right hand side of eqs. (3.5.10)–(3.5.21) with γ, $\mathit{\Gamma}$, σ, $\mathit{\Sigma}$ replaced by γ_0, $\mathit{\Gamma}_0$, σ_0, $\mathit{\Sigma}_0$ and X set to 0. Taking (3.5.26), (3.5.27) into account, the action on γ_0, σ_0 has the properties stated. ☐

3.6 Basic formulation of principal 2–bundle theory

As recalled in subsect. 3.5, for the $\hat{\mathscr{K}}$–2–bundle $\hat{\mathcal{P}}$, on any sufficiently small neighborhood $U \subset M$ there exists a projection preserving K–equivariant trivializing map $\mathit{\Phi}_U \in \mathrm{Map}(\pi^{-1}(U), U \times \mathsf{K})$. We saw further that it is possible to attach to $\mathit{\Phi}_U$ a special set of internal coordinates of $\pi^{-1}(U)$ modelled on $U \times \mathrm{D}\mathsf{M}$ the adapted coordinates u, v and $\mathit{\Lambda}$, $\mathit{\Delta}$, or γ, $\mathit{\Gamma}$, σ, $\mathit{\Sigma}$ in projected form, for the factors U and $\mathrm{D}\mathsf{M}$, respectively. These are internal functions on $T[1]\pi^{-1}(U)$ with special properties in the operation $\mathrm{O{\scriptstyle P}}\,S_{\pi^{-1}(U)}$ of the morphism space $S_{\pi^{-1}(U)}$.

In this subsection, we shall use these coordinates to analyze 2–connections and 1– and 2–gauge transformations of $\hat{\mathcal{P}}$ in terms of basic Lie valued function data on $T[1]\pi^{-1}(U)$. Remember that a function $F_{\mathrm{b}} \in \mathrm{F{\scriptstyle UN}}(T[1]\pi^{-1}(U))$ is basic if it is annihilated by all derivations $j_{\pi^{-1}(U)Z}$, $l_{\pi^{-1}(U)Z}$ (cf. subsect. 2.1 of I).

Before proceeding further, we note that the inclusion map $N_U : \pi^{-1}(U) \to P$ yield a morphisms $Q_U : S_{\pi^{-1}(U)} \to S_P$ of the morphism spaces of $\pi^{-1}(U)$ and

P and through this a morphism $\mathrm{O{\small P}}\,Q_U : \mathrm{O{\small P}}\,S_P \to \mathrm{O{\small P}}\,S_{\pi^{-1}(U)}$ of the associated operations (cf. subsect. 3.7 of I). Therefore, if a function $F \in \mathrm{F{\small UN}}(T[1]P)$ obeys certain relations under the actions of the derivations j_{PZ}, l_{PZ} of $\mathrm{O{\small P}}\,S_P$, its restriction $F|_{T[1]\pi^{-1}(U)} = N_U{}^*F \in \mathrm{F{\small UN}}(T[1]\pi^{-1}(U))$ obeys formally identical relations under the actions of the derivations $j_{\pi^{-1}(U)Z}$, $l_{\pi^{-1}(U)Z}$ of $\mathrm{O{\small P}}\,S_{\pi^{-1}(U)}$.

Consider a 2–connection of $\hat{\mathcal{P}}$ with connection and curvature components A, B (cf. subsect. 3.2, def. 3.1).

Definition 3.10. *The basic connection and curvature components of the 2–connection are the Lie algebra valued internal functions $A_{\mathrm{b}} \in \mathrm{M{\small AP}}(T[1]\pi^{-1}(U), \mathrm{D}\mathfrak{m}[1])$ and $B_{\mathrm{b}} \in \mathrm{M{\small AP}}(T[1]\pi^{-1}(U), \mathrm{D}\mathfrak{m}[2])$ defined by*

$$A_{\mathrm{b}} = \mathrm{Ad}\,\varLambda(A - \varDelta), \tag{3.6.1}$$

$$B_{\mathrm{b}} = \mathrm{Ad}\,\varLambda(B). \tag{3.6.2}$$

Above, restriction of A, B to $T[1]\pi^{-1}(U)$ is tacitly understood in order not to clutter the notation. The names given to A_{b}, B_{b} are justified by the following proposition.

Proposition 3.11. *A_{b}, B_{b} are basic elements of the operation $\mathrm{O{\small P}}\,S_{\pi^{-1}(U)}$.*

Proof. One has to show that A_{b}, B_{b} are annihilated by all derivations $j_{\pi^{-1}(U)Z}$ and $l_{\pi^{-1}(U)Z}$ with $Z \in \mathrm{D}\mathfrak{m}$. This can be verified using relations (3.2.3)–(3.2.6) and (3.5.4)–(3.5.7). □

Proposition 3.12. *A_{b}, B_{b} obey the relations*

$$d_{\pi^{-1}(U)}A_{\mathrm{b}} = -\frac{1}{2}[A_{\mathrm{b}}, A_{\mathrm{b}}] - d_{\dot{\tau}}A_{\mathrm{b}} + B_{\mathrm{b}}, \tag{3.6.3}$$

$$d_{\pi^{-1}(U)}B_{\mathrm{b}} = -[A_{\mathrm{b}}, B_{\mathrm{b}}] - d_{\dot{\tau}}B_{\mathrm{b}}. \tag{3.6.4}$$

These are formally analogous to (3.2.1), (3.2.2).

Proof. Relations (3.6.3), (3.6.4) to be proven follow from (3.2.1), (3.2.2) and (3.5.2), (3.5.3) through a simple calculation. □

Just as the connection and curvature components A, B can be expressed in terms of the projected connection and curvature components ω, $\varOmega$, θ, $\varTheta$ according to (3.2.7), (3.2.8), so the basic components A_{b}, B_{b} can be expressed in terms of basic projected components ω_{b}, $\varOmega_{\mathrm{b}}$, θ_{b}, $\varTheta_{\mathrm{b}}$ as

$$A_{\mathrm{b}}(\alpha) = \omega_{\mathrm{b}} - \alpha\varOmega_{\mathrm{b}}, \tag{3.6.5}$$

$$B_{\mathrm{b}}(\alpha) = \theta_{\mathrm{b}} + \alpha\varTheta_{\mathrm{b}}, \quad \alpha \in \mathbb{R}[1]. \tag{3.6.6}$$

In the above relations, $\omega_{\mathrm{b}} \in \mathrm{M{\scriptstyle AP}}(T[1]\pi^{-1}(U), \mathfrak{g}[1])$, $\varOmega_{\mathrm{b}} \in \mathrm{M{\scriptstyle AP}}(T[1]\pi^{-1}(U), \mathfrak{e}[2])$, $\theta_{\mathrm{b}} \in \mathrm{M{\scriptstyle AP}}(T[1]\pi^{-1}(U), \mathfrak{g}[2])$, $\varTheta_{\mathrm{b}} \in \mathrm{M{\scriptstyle AP}}(T[1]\pi^{-1}(U), \mathfrak{e}[3])$.

Proposition 3.13. *ω_{b}, $\varOmega_{\mathrm{b}}$, θ_{b}, $\varTheta_{\mathrm{b}}$ are related to ω, $\varOmega$, θ, $\varTheta$ by*

$$\omega_{\mathrm{b}} = \mathrm{Ad}\,\gamma(\omega - \sigma), \tag{3.6.7}$$

$$\varOmega_{\mathrm{b}} = \mu\dot{}\,(\gamma, \varOmega - \varSigma) - \dot{}\mu\dot{}\,(\mathrm{Ad}\,\gamma(\omega - \sigma), \varGamma), \tag{3.6.8}$$

$$\theta_{\mathrm{b}} = \mathrm{Ad}\,\gamma(\theta), \tag{3.6.9}$$

$$\varTheta_{\mathrm{b}} = \mu\dot{}\,(\gamma, \varTheta) - \dot{}\mu\dot{}\,(\mathrm{Ad}\,\gamma(\theta), \varGamma). \tag{3.6.10}$$

Restriction of ω, $\varOmega$, θ, $\varTheta$ to $T[1]\pi^{-1}(U)$ is here also tacitly understood.

Proof. Inserting (3.2.7), (3.2.8), (3.5.8), (3.5.9) and (3.6.5), (3.6.6) into (3.6.1), (3.6.2) and using 3.5.18 of I, one gets (3.6.7)–(3.6.10) by simple calculations. □

Proposition 3.14. *ω_{b}, $\varOmega_{\mathrm{b}}$, θ_{b}, $\varTheta_{\mathrm{b}}$ satisfy*

$$d_{\pi^{-1}(U)}\omega_{\mathrm{b}} = -\frac{1}{2}[\omega_{\mathrm{b}}, \omega_{\mathrm{b}}] + \dot{\tau}(\varOmega_{\mathrm{b}}) + \theta_{\mathrm{b}}, \tag{3.6.11}$$

$$d_{\pi^{-1}(U)}\varOmega_{\mathrm{b}} = -\dot{}\mu\dot{}\,(\omega_{\mathrm{b}}, \varOmega_{\mathrm{b}}) + \varTheta_{\mathrm{b}}, \tag{3.6.12}$$

$$d_{\pi^{-1}(U)}\theta_{\mathrm{b}} = -[\omega_{\mathrm{b}}, \theta_{\mathrm{b}}] - \dot{\tau}(\varTheta_{\mathrm{b}}), \tag{3.6.13}$$

$$d_{\pi^{-1}(U)}\varTheta_{\mathrm{b}} = -\dot{}\mu\dot{}\,(\omega_{\mathrm{b}}, \varTheta_{\mathrm{b}}) + \dot{}\mu\dot{}\,(\theta_{\mathrm{b}}, \varOmega_{\mathrm{b}}). \tag{3.6.14}$$

These relations are formally identical to (3.2.9)–(3.2.12). Our basic formulation has so reproduced the familiar local description of 2–connections of strict higher gauge theory. This statement will be qualified more precisely in subsect. 3.7.

Proof. One demonstrates (3.6.11)–(3.6.14) by substituting (3.6.5), (3.6.6) into (3.6.3), (3.6.4) and using 3.5.13 and 3.5.15 of I. □

The basic components of the 2–connection behave as expected when the 2–connection is special.

Proposition 3.15. *If the 2–connection and the adapted coordinates are both special (cf. defs. 3.2, 3.9), then one has*

$$I_U{}^*\varOmega_{\mathrm{b}} = 0, \tag{3.6.15}$$

$$I_U{}^*\varTheta_{\mathrm{b}} = 0, \tag{3.6.16}$$

where $I_U : \pi_0{}^{-1}(U) \to \pi^{-1}(U)$ is the inclusion map.

Proof. This follows from (3.6.8), (3.6.10) upon substituting (3.2.25), (3.2.26) and (3.5.26), (3.5.27) □

Next, consider a 1–gauge transformation of $\hat{\mathcal{P}}$ with transformation and shift components $\varPsi$, $\varUpsilon$ (cf. subsect. 3.3, def. 3.3).

Definition 3.11. *The basic transformation and shift components of the 1–gauge transformation are the Lie group and algebra valued internal functions $\varPsi_{\mathrm{b}} \in$ $\mathrm{MAP}(T[1]\pi^{-1}(U),\,\mathsf{D}\mathsf{M})$ and $\varUpsilon_{\mathrm{b}} \in \mathrm{MAP}(T[1]\pi^{-1}(U), \mathrm{D}\mathfrak{m}[1])$ defined by*

$$\varPsi_{\mathrm{b}} = \varLambda\varPsi\varLambda^{-1}, \tag{3.6.17}$$

$$\varUpsilon_{\mathrm{b}} = \mathrm{Ad}\,\varLambda(\varUpsilon - \varDelta + \mathrm{Ad}\,\varPsi(\varDelta)). \tag{3.6.18}$$

Above, restriction of $\varPsi$, $\varUpsilon$ to $T[1]\pi^{-1}(U)$ is understood. The name given to $\varPsi_{\mathrm{b}}$, $\varUpsilon_{\mathrm{b}}$ are justified by the following proposition.

Proposition 3.16. *$\varPsi_{\mathrm{b}}$, $\varUpsilon_{\mathrm{b}}$ are basic elements of the operation $\mathrm{OP}\,S_{\pi^{-1}(U)}$.*

Proof. One has to show that $\varPsi_{\mathrm{b}}$, $\varUpsilon_{\mathrm{b}}$ are annihilated by all derivations $j_{\pi^{-1}(U)Z}$ and $l_{\pi^{-1}(U)Z}$ with $Z \in \mathrm{D}\mathfrak{m}$. This can be verified using relations (3.3.3)–(3.3.6) and (3.5.4)–(3.5.7). □

Proposition 3.17. *$\varPsi_{\mathrm{b}}$, $\varUpsilon_{\mathrm{b}}$ satisfy the relations*

$$d_{\pi^{-1}(U)}\varPsi_{\mathrm{b}}\varPsi_{\mathrm{b}}{}^{-1} = -d_{\dot{\tau}}\varPsi_{\mathrm{b}}\varPsi_{\mathrm{b}}{}^{-1} - \varUpsilon_{\mathrm{b}}, \tag{3.6.19}$$

$$d_{\pi^{-1}(U)}\varUpsilon_{\mathrm{b}} = -\frac{1}{2}[\varUpsilon_{\mathrm{b}}, \varUpsilon_{\mathrm{b}}] - d_{\dot{\tau}}\varUpsilon_{\mathrm{b}}. \tag{3.6.20}$$

These are analogous in form to (3.3.1), (3.3.2).

Proof. Relations (3.6.19), (3.6.20) to be shown follow from eqs. (3.3.1), (3.3.2) and (3.5.2), (3.5.3) through a simple calculation. □

Next, consider the 1–gauge transform ${}^{\varPsi,\varUpsilon}A$, ${}^{\varPsi,\varUpsilon}B$ of a 2–connection A, B (cf. subsect. 3.3, def. 3.4).

Proposition 3.18. *The basic components ${}^{\varPsi,\varUpsilon}A_{\mathrm{b}}$, ${}^{\varPsi,\varUpsilon}B_{\mathrm{b}}$ of the 1–gauge transformed transformed 2–connection are given in terms of A_{b}, B_{b} and $\varPsi_{\mathrm{b}}$, $\varUpsilon_{\mathrm{b}}$ by*

$$ {}^{\varPsi,\varUpsilon}A_{\mathrm{b}} = \mathrm{Ad}\,\varPsi_{\mathrm{b}}(A_{\mathrm{b}}) + \varUpsilon_{\mathrm{b}}, \tag{3.6.21}$$

$$ {}^{\varPsi,\varUpsilon}B_{\mathrm{b}} = \mathrm{Ad}\,\varPsi_{\mathrm{b}}(B_{\mathrm{b}}). \tag{3.6.22}$$

Proof. Relations (3.6.21), (3.6.22) can be straightforwardly verified combining (3.6.1), (3.6.2) and (3.6.17), (3.6.18) with (3.3.9), (3.3.10). □

Eqs. (3.6.21), (3.6.22) suggest defining the basic component gauge transforms ${}^{\varPsi_{\mathrm{b}},\varUpsilon_{\mathrm{b}}}A_{\mathrm{b}}$, ${}^{\varPsi_{\mathrm{b}},\varUpsilon_{\mathrm{b}}}B_{\mathrm{b}}$ to be given by the right hand sides of (3.6.21), (3.6.22) themselves. By doing so, ${}^{\varPsi_{\mathrm{b}},\varUpsilon_{\mathrm{b}}}A_{\mathrm{b}}$, ${}^{\varPsi_{\mathrm{b}},\varUpsilon_{\mathrm{b}}}B_{\mathrm{b}}$ are given be expressions formally analogous to those holding for the ordinary components, viz (3.3.9), (3.3.10).

Again, in the same way as the transformation and shift components $\varPsi$, $\varUpsilon$ can be expanded in their projected transformation and shift components g, J, h, K according to (3.3.11), (3.3.12), so their basic counterparts $\varPsi_{\mathrm{b}}$, $\varUpsilon_{\mathrm{b}}$ can be expanded in basic projected components g_{b}, J_{b}, h_{b}, K_{b} as

$$ \varPsi_{\mathrm{b}}(\alpha) = \mathrm{e}^{\alpha J_{\mathrm{b}}}\, g_{\mathrm{b}}, \tag{3.6.23}$$

$$ \varUpsilon_{\mathrm{b}}(\alpha) = h_{\mathrm{b}} - \alpha K_{\mathrm{b}}, \quad \alpha \in \mathbb{R}[1]. \tag{3.6.24}$$

In the above relations, $g_{\mathrm{b}} \in \mathrm{M{\small AP}}(T[1]\pi^{-1}(U), \mathsf{G})$, $J_{\mathrm{b}} \in \mathrm{M{\small AP}}(T[1]\pi^{-1}(U), \mathfrak{e}[1])$, $h_{\mathrm{b}} \in \mathrm{M{\small AP}}(T[1]\pi^{-1}(U), \mathfrak{g}[1])$, $K_{\mathrm{b}} \in \mathrm{M{\small AP}}(T[1]\pi^{-1}(U), \mathfrak{e}[2])$.

Proposition 3.19. *g_{b}, J_{b}, h_{b}, K_{b} are related to g, J, h, K as*

$$ g_{\mathrm{b}} = \gamma g \gamma^{-1}, \tag{3.6.25}$$

$$J_{\rm b} = \mu\dot{}\,(\gamma, J) + \varGamma - \mu\dot{}\,(\gamma g\gamma^{-1}, \varGamma), \tag{3.6.26}$$

$$h_{\rm b} = \operatorname{Ad}\gamma(h - \sigma + \operatorname{Ad}g(\sigma)), \tag{3.6.27}$$

$$\begin{aligned} K_{\rm b} = \mu\dot{}\,(\gamma, K - \varSigma + \mu\dot{}\,(g, \varSigma) - \dot{}\mu\dot{}\,(\operatorname{Ad}g(\sigma), J) \\ - \dot{}\mu\dot{}\,(h - \sigma + \operatorname{Ad}g(\sigma), \mu\dot{}\,(\gamma^{-1}, \varGamma))). \end{aligned} \tag{3.6.28}$$

Restriction of g, J, h, K to $T[1]\pi^{-1}(U)$ is here also tacitly understood.

Proof. Substituting (3.3.11), (3.3.12), (3.5.8), (3.5.9) and (3.6.23), (3.6.24) into (3.6.17), (3.6.18) and using 3.5.2, 3.5.3 and 3.5.18 of I, one gets (3.6.25)–(3.6.28) by straightforward computations. □

Proposition 3.20. *$g_{\rm b}$, $J_{\rm b}$, $h_{\rm b}$, $K_{\rm b}$ obey*

$$d_{\pi^{-1}(U)}g_{\rm b}g_{\rm b}{}^{-1} = -h_{\rm b} - \dot\tau(J_{\rm b}), \tag{3.6.29}$$

$$d_{\pi^{-1}(U)}J_{\rm b} = -K_{\rm b} - \frac{1}{2}[J_{\rm b}, J_{\rm b}] - \dot{}\mu\dot{}\,(h_{\rm b}, J_{\rm b}), \tag{3.6.30}$$

$$d_{\pi^{-1}(U)}h_{\rm b} = -\frac{1}{2}[h_{\rm b}, h_{\rm b}] + \dot\tau(K_{\rm b}), \tag{3.6.31}$$

$$d_{\pi^{-1}(U)}K_{\rm b} = -\dot{}\mu\dot{}\,(h_{\rm b}, K_{\rm b}). \tag{3.6.32}$$

These relations are of the same form as (3.3.13)–(3.3.16). We have reobtained in this way in our basic formulation the familiar local description of 1–gauge transformations of strict higher gauge theory. More on this in subsect. 3.7.

Proof. (3.6.29)–(3.6.32) are shown by inserting (3.6.23), (3.6.24) into (3.6.19), (3.6.20) and using 3.5.13, 3.5.15, 3.5.22 and 3.5.24 of I. □

Concerning the 1–gauge transformed 2–connection ${}^{g,J,h,K}\omega$, ${}^{g,J,h,K}\varOmega$, ${}^{g,J,h,K}\theta$, ${}^{g,J,h,K}\varTheta$ we have the following result.

Proposition 3.21. *The basic components ${}^{g,J,h,K}\omega_{\rm b}$, ${}^{g,J,h,K}\varOmega_{\rm b}$, ${}^{g,J,h,K}\theta_{\rm b}$, ${}^{g,J,h,K}\varTheta_{\rm b}$ are given in terms of $\omega_{\rm b}$, $\varOmega_{\rm b}$, $\theta_{\rm b}$, $\varTheta_{\rm b}$ and $g_{\rm b}$, $J_{\rm b}$, $h_{\rm b}$, $K_{\rm b}$ by*

$${}^{g,J,h,K}\omega_{\rm b} = \operatorname{Ad}g_{\rm b}(\omega_{\rm b}) + h_{\rm b}, \tag{3.6.33}$$

$${}^{g,J,h,K}\varOmega_{\rm b} = \mu\dot{}\,(g_{\rm b}, \varOmega_{\rm b}) - \dot{}\mu\dot{}\,(\operatorname{Ad}g_{\rm b}(\omega_{\rm b}), J_{\rm b}) + K_{\rm b}, \tag{3.6.34}$$

$$ {}^{g,J,h,K}\theta_{\mathrm{b}} = \operatorname{Ad} g_{\mathrm{b}}(\theta_{\mathrm{b}}), \tag{3.6.35} $$

$$ {}^{g,J,h,K}\varTheta_{\mathrm{b}} = \mu\dot{}\,(g_{\mathrm{b}}, \varTheta_{\mathrm{b}}) - \dot{}\mu\dot{}\,(\operatorname{Ad} g_{\mathrm{b}}(\theta_{\mathrm{b}}), J_{\mathrm{b}}). \tag{3.6.36} $$

Proof. (3.6.33)–(3.6.36) follow from inserting (3.6.5), (3.6.6), (3.6.23), (3.6.24) into (3.6.21), (3.6.22) and using 3.5.18 of I. □

Following the remarks below eqs. (3.6.21), (3.6.22), we can regard the right hand sides of eqs. (3.6.33)–(3.6.36) as the expressions of the basic projected component gauge transforms ${}^{g_{\mathrm{b}},J_{\mathrm{b}},h_{\mathrm{b}},K_{\mathrm{b}}}\omega_{\mathrm{b}}$, ${}^{g_{\mathrm{b}},J_{\mathrm{b}},h_{\mathrm{b}},K_{\mathrm{b}}}\varOmega_{\mathrm{b}}$, ${}^{g_{\mathrm{b}},J_{\mathrm{b}},h_{\mathrm{b}},K_{\mathrm{b}}}\theta_{\mathrm{b}}$, ${}^{g_{\mathrm{b}},J_{\mathrm{b}},h_{\mathrm{b}},K_{\mathrm{b}}}\varTheta_{\mathrm{b}}$, respectively. Again, such expressions are formally identical to those holding for the ordinary projected components, viz (3.3.25)–(3.3.28) and so reproduce at the basic level the usual local description of 2–connection 1–gauge transformation of strict higher gauge theory. This matter will be reconsidered in subsect. 3.7.

The basic components of the 1–gauge transformation behave as expected when the 1–gauge transformation is special.

Proposition **3.22.** *If the* 1*–gauge transformation and the adapted coordinates are both special (cf. defs. 3.5, 3.9), then*

$$ {I_U}^* J_{\mathrm{b}} = 0, \tag{3.6.37} $$

$$ {I_U}^* K_{\mathrm{b}} = 0, \tag{3.6.38} $$

where $I_U : {\pi_0}^{-1}(U) \to \pi^{-1}(U)$ *is the inclusion map.*

Proof. This follows from (3.6.26), (3.6.28) upon substituting (3.3.33), (3.3.34) and (3.5.26), (3.5.27) □

Finally, we consider a 2–gauge transformation of P of modification and variation components E, C relative to the reference 2–connection ω, $\varOmega$, θ, $\varTheta$ (cf. subsect. 3.4, def. 3.6).

Definition **3.12.** *The basic modification and variation components of the* 2*–gauge transformation are the Lie group and algebra valued internal functions* $E_{\mathrm{b}} \in \mathrm{M{\small AP}}(T[1]\pi^{-1}(U), \mathsf{E})$ *and* $C_{\mathrm{b}} \in \mathrm{M{\small AP}}(T[1]\pi^{-1}(U), \mathfrak{e}[1])$ *given by*

$$E_{\rm b} = \mu(\gamma, E), \tag{3.6.39}$$

$$C_{\rm b} = \dot\mu\dot{}(\gamma, C) + \varGamma - \operatorname{Ad}\mu(\gamma, E)(\varGamma). \tag{3.6.40}$$

Again, restriction of E, C to $T[1]\pi^{-1}(U)$ is here also tacitly understood. Further, the names given to $E_{\rm b}$, $C_{\rm b}$ reflect their basicness.

Proposition 3.23. *$E_{\rm b}$, $C_{\rm b}$ are basic elements of the operation* $\operatorname{O{\scriptstyle P}} S_{\pi^{-1}(U)}$.

Proof. To show that $E_{\rm b}$, $C_{\rm b}$ are annihilated by all derivations $j_{\pi^{-1}(U)Z}$ and $l_{\pi^{-1}(U)Z}$ with $Z \in \mathrm{D}\mathfrak{m}$ we use relations (3.4.3)–(3.4.6) and (3.5.4)–(3.5.7). □

Proposition 3.24. *$E_{\rm b}$, $C_{\rm b}$ satisfy obey the relations*

$$d_{\pi^{-1}(U)}E_{\rm b}{E_{\rm b}}^{-1} = -C_{\rm b} - \dot{}\mu(\omega_{\rm b}, E_{\rm b}), \tag{3.6.41}$$

$$d_{\pi^{-1}(U)}C_{\rm b} = -\frac{1}{2}[C_{\rm b}, C_{\rm b}] - \dot{}\mu\dot{}(\omega_{\rm b}, C_{\rm b}) - \dot{}\mu(\theta_{\rm b}, E_{\rm b}) - \varOmega_{\rm b} + \operatorname{Ad}E_{\rm b}(\varOmega_{\rm b}). \tag{3.6.42}$$

As expected by now, these are analogous in form to (3.4.1), (3.4.2).

Proof. Combining (3.4.1), (3.4.2) and (3.5.2), (3.5.3) and carrying out a simple computation, (3.6.41), (3.6.42) are readily obtained. □

As to the 2–gauge transform ${}^{E,C}g$, ${}^{E,C}J$, ${}^{E,C}h$, ${}^{E,C}K$ of the 1–gauge transformation g, J, h, K (cf. subsect. 3.4, def. 3.7), the following result holds.

Proposition 3.25. *${}^{E,C}g_{\rm b}$, ${}^{E,C}J_{\rm b}$, ${}^{E,C}h_{\rm b}$, ${}^{E,C}K_{\rm b}$ are given in terms of $g_{\rm b}$, $J_{\rm b}$, $h_{\rm b}$, $K_{\rm b}$ and $E_{\rm b}$, $C_{\rm b}$ by the expressions*

$${}^{E,C}g_{\rm b} = \tau(E_{\rm b})g_{\rm b}, \tag{3.6.43}$$

$${}^{E,C}J_{\rm b} = \operatorname{Ad}E_{\rm b}(J_{\rm b}) + \dot{}\mu(\omega_{\rm b} - \operatorname{Ad}g_{\rm b}(\omega_{\rm b}) - h_{\rm b}, E_{\rm b}) + C_{\rm b}, \tag{3.6.44}$$

$${}^{E,C}h_{\rm b} = \operatorname{Ad}\tau(E_{\rm b})(h_{\rm b}) + \dot\tau(\dot{}\mu(\operatorname{Ad}g_{\rm b}(\omega_{\rm b}) + h_{\rm b}, E_{\rm b})), \tag{3.6.45}$$

$$\begin{aligned}{}^{E,C}K_{\rm b} = \operatorname{Ad}E_{\rm b}(K_{\rm b}) &+ \dot{}\mu\dot{}(\operatorname{Ad}(\tau(E_{\rm b})g_{\rm b})(\omega_{\rm b}), \dot{}\mu(\omega_{\rm b} - \operatorname{Ad}g_{\rm b}(\omega_{\rm b}) - h_{\rm b}, E_{\rm b}) \\ &+ C_{\rm b}) + \dot{}\mu(\operatorname{Ad}g_{\rm b}(\theta_{\rm b}) + \dot\tau(\dot{}\mu\dot{}(g_{\rm b}, \varOmega_{\rm b}) - \dot{}\mu\dot{}(\operatorname{Ad}g_{\rm b}(\omega_{\rm b}), J_{\rm b}) + K_{\rm b}), E_{\rm b}).\end{aligned} \tag{3.6.46}$$

Proof. Relations (3.6.43)–(3.6.46) are obtained by combining (3.6.25)–(3.6.28) and (3.6.39), (3.6.40) with (3.4.11)–(3.4.14) through simple computations. □

Above, we can regard the right hand sides of eqs. (3.6.43)–(3.6.46) as the expressions of the basic projected component gauge transform ${}^{E_{\rm b},C_{\rm b}}g_{\rm b}$, ${}^{E_{\rm b},C_{\rm b}}J_{\rm b}$, ${}^{E_{\rm b},C_{\rm b}}h$, ${}^{E_{\rm b},C_{\rm b}}K$, respectively. Such expressions are formally identical to those holding for the ordinary projected components, viz (3.4.11)–(3.4.14). Moreover, they provide a local basic description of 1–gauge transformation 2–gauge transformation of strict higher gauge theory.

Remark **3.1.** *The basic components $\omega_{\rm b}$, $\varOmega_{\rm b}$, $\theta_{\rm b}$, $\varTheta_{\rm b}$, $g_{\rm b}$, $J_{\rm b}$, $h_{\rm b}$, $K_{\rm b}$ and $E_{\rm b}$, $C_{\rm b}$ satisfy relations formally identical to (3.4.15)–(3.4.18).*

Proof. Indeed, the basic projected components formally obey the same relations as the ordinary projected ones. Moreover, the basic projected component 2-connection 1–gauge transformation and 1–gauge transformation 2–gauge transformation are formally given by the same expressions as their ordinary projected counterparts. ☐

For a given trivializing neighborhood $U \subset M$, the basic components of 2–connections and 1– and 2 gauge transformations are Lie valued internal functions on $T[1]\pi^{-1}(U)$ so that they are only locally defined. The problem arises of matching the local data pertaining to distinct but overlapping trivializing neighborhoods U, $U' \subset M$. Below, we denote by $\varLambda, \varDelta$ and $\varLambda', \varDelta'$ the adapted coordinates modelled on $\mathrm{D}\mathsf{M}$ of $\pi^{-1}(U)$ $\pi^{-1}(U')$, respectively.

The inclusion map $N^U_{U\cap U'} : \pi^{-1}(U \cap U') \to \pi^{-1}(U)$ induces a morphisms $Q^U_{U\cap U'} : S_{\pi^{-1}(U\cap U')} \to S_{\pi^{-1}(U)}$ of the morphism spaces of $\pi^{-1}(U \cap U')$ and $\pi^{-1}(U)$ and via this a morphism $\mathrm{O{\scriptstyle P}}\,Q^U_{U\cap U'} : \mathrm{O{\scriptstyle P}}\,S_{\pi^{-1}(U)} \to \mathrm{O{\scriptstyle P}}\,S_{\pi^{-1}(U\cap(U')}$ of the associated operations. Thus, if a function $F \in \mathrm{F{\scriptstyle UN}}(T[1]\pi^{-1}(U))$ obeys certain relations under the action of the derivations $j_{\pi^{-1}(U)Z}$, $l_{\pi^{-1}(U)Z}$ of $\mathrm{O{\scriptstyle P}}\,S_{\pi^{-1}(U)}$, its restriction $F|_{T[1]\pi^{-1}(U\cap U')} := N^U_{U\cap U'}{}^*F \in \mathrm{F{\scriptstyle UN}}(T[1]\pi^{-1}(U \cap U'))$ obeys formally identical relations under the action of the derivations $j_{\pi^{-1}(U\cap U')Z}$, $l_{\pi^{-1}(U\cap U')Z}$ of $\mathrm{O{\scriptstyle P}}\,S_{\pi^{-1}(U\cap U')}$. Similar remarks hold with U replaced by U'.

Definition **3.13.** *The local basic matching transformation and shift components are the Lie group and algebra valued internal functions $G_{\rm b} \in \mathrm{M{\scriptstyle AP}}(T[1]\pi^{-1}(U \cap U'),$*

DM) *and* $D_\mathrm{b} \in \mathrm{MAP}(T[1]\pi^{-1}(U \cap U'), \mathrm{D}\mathfrak{m}[1])$ *defined by*

$$G_\mathrm{b} = \varLambda'\varLambda^{-1}, \tag{3.6.47}$$

$$D_\mathrm{b} = \mathrm{Ad}\,\varLambda(\varDelta' - \varDelta). \tag{3.6.48}$$

Above, restriction of $\varLambda$, $\varDelta$ and $\varLambda'$, $\varDelta'$ to $T[1]\pi^{-1}(U \cap U')$ is understood for simplicity. The names given to G_b, D_b are justified by the following proposition.

Proposition 3.26. *G_b, D_b are basic elements of the operation* $\mathrm{OP}\,S_{\pi^{-1}(U\cap U')}$.

Proof. Using relations (3.5.4)–(3.5.7) and their primed counterpart, one easily verifies that G_b, D_b are annihilated by all derivations $j_{\pi^{-1}(U\cap U')Z}$ and $l_{\pi^{-1}(U\cap U')Z}$ with $Z \in \mathrm{D}\mathfrak{m}$. ☐

Proposition 3.27. *The following relation holds,*

$$D_\mathrm{b} = {G_\mathrm{b}}^{-1} d_{\pi^{-1}(U\cap U')} G_\mathrm{b} + {G_\mathrm{b}}^{-1} d_{\dot{\tau}} G_\mathrm{b}. \tag{3.6.49}$$

Proof. Identity (3.6.49) is easily verified substituting relation (3.5.2) and its primed counterpart into eq. (3.6.48). ☐

By virtue of 3.5.1, 3.5.12 of I, we can expand the local basic matching components G_b, D_b as

$$G_\mathrm{b}(\alpha) = \mathrm{e}^{\alpha F_\mathrm{b}} f_\mathrm{b}, \tag{3.6.50}$$

$$D_\mathrm{b}(\alpha) = s_\mathrm{b} - \alpha S_\mathrm{b}, \qquad \alpha \in \mathbb{R}[1], \tag{3.6.51}$$

where $f_\mathrm{b} \in \mathrm{MAP}(T[1]\pi^{-1}(U \cap U'), \mathsf{G})$, $F_\mathrm{b} \in \mathrm{MAP}(T[1]\pi^{-1}(U \cap U'), \mathfrak{e}[1])$ and $s_\mathrm{b} \in \mathrm{MAP}(T[1]\pi^{-1}(U \cap U'), \mathfrak{g}[1])$, $S_\mathrm{b} \in \mathrm{MAP}(T[1]\pi^{-1}(U \cap U'), \mathfrak{e}[2])$ are suitable projected local basic matching components.

Let γ, $\varGamma$, σ, $\varSigma$ and γ', $\varGamma'$, σ', $\varSigma'$ be the projected components of $\varLambda$, $\varDelta$ and $\varLambda'$, $\varDelta'$, respectively (cf. eqs. (3.5.8)–(3.5.9)).

Proposition 3.28. *The basic projected components f_b, F_b, s_b, S_b can be expressed in terms of the projected components γ, $\varGamma$, σ, $\varSigma$ and γ', $\varGamma'$, σ', $\varSigma'$ as*

$$f_\mathrm{b} = \gamma'\gamma^{-1}, \tag{3.6.52}$$

$$F_{\rm b} = \mathit{\Gamma}' - \dot\mu(\gamma'\gamma^{-1}, \mathit{\Gamma}), \tag{3.6.53}$$

$$s_{\rm b} = \operatorname{Ad}\gamma(\sigma' - \sigma), \tag{3.6.54}$$

$$S_{\rm b} = \dot\mu(\gamma, \mathit{\Sigma}' - \mathit{\Sigma}) - \dot{\dot\mu}(\operatorname{Ad}\gamma(\sigma' - \sigma), \mathit{\Gamma}). \tag{3.6.55}$$

Here, γ, $\mathit{\Gamma}$, σ, $\mathit{\Sigma}$ and γ', $\mathit{\Gamma}'$, σ', $\mathit{\Sigma}'$ are restricted to $T[1]\pi^{-1}(U \cap U')$.

Proof. Relations (3.6.52)–(3.6.55) follow straightforwardly from inserting (3.5.8)–(3.5.9), its primed counterpart and (3.6.50), (3.6.51) into (3.6.47), (3.6.48) and applying 3.5.2, 3.5.3 and 3.5.18 of I. $\square$

Proposition 3.29. *The basic projected components $f_{\rm b}$, $F_{\rm b}$, $s_{\rm b}$, $S_{\rm b}$ are related as*

$$s_{\rm b} = f_{\rm b}{}^{-1} d_{\pi^{-1}(U \cap U')} f_{\rm b} + \dot\tau(\dot\mu(f_{\rm b}{}^{-1}, F_{\rm b})), \tag{3.6.56}$$

$$S_{\rm b} = \dot\mu(f_{\rm b}{}^{-1}, d_{\pi^{-1}(U \cap U')} F_{\rm b} + [F_{\rm b}, F_{\rm b}]/2). \tag{3.6.57}$$

Proof. Substituting (3.6.50) into (3.6.49) and applying 3.5.23 and 3.5.25 of I, one readily obtains relations (3.6.56)–(3.6.57). $\square$

Consider next a 2–connection of $\hat{\mathcal{P}}$ of components A, B.

Proposition 3.30. *The basic components $A_{\rm b}$, $B_{\rm b}$ and ${A'}_{\rm b}$, ${B'}_{\rm b}$ of the 2–connection are related on $T[1]\pi^{-1}(U \cap U')$ as*

$${A'}_{\rm b} = \operatorname{Ad} G_{\rm b}(A_{\rm b} - D_{\rm b}), \tag{3.6.58}$$

$${B'}_{\rm b} = \operatorname{Ad} G_{\rm b}(B_{\rm b}). \tag{3.6.59}$$

Proof. Exploiting relations (3.6.1), (3.6.2), we can express A, B in terms of $A_{\rm b}$, $B_{\rm b}$, $\mathit{\Lambda}$, $\mathit{\Delta}$. Inserting these identities into the primed counterparts of (3.6.1), (3.6.2), we obtain expressions of ${A'}_{\rm b}$, ${B'}_{\rm b}$ in terms of $A_{\rm b}$, $B_{\rm b}$, $\mathit{\Lambda}$, $\mathit{\Delta}$, $\mathit{\Lambda}'$, $\mathit{\Delta}'$. These latter can be cast in the form (3.6.58), (3.6.59) employing (3.6.47), (3.6.48). $\square$

The matching relation (3.6.58), (3.6.59) can be written in terms of projected components.

Proposition 3.31. *The projected basic components $\omega_{\rm b}$, $\mathit{\Omega}_{\rm b}$, $\theta_{\rm b}$, $\mathit{\Theta}_{\rm b}$ and ${\omega'}_{\rm b}$, ${\mathit{\Omega}'}_{\rm b}$, ${\theta'}_{\rm b}$, ${\mathit{\Theta}'}_{\rm b}$ of the 2–connection are related on $T[1]\pi^{-1}(U \cap U')$ as*

$$\omega'_{\mathrm{b}} = \mathrm{Ad}\, f_{\mathrm{b}}(\omega_{\mathrm{b}} - s_{\mathrm{b}}), \tag{3.6.60}$$

$$\varOmega'_{\mathrm{b}} = \mu\dot{}(f_{\mathrm{b}}, \varOmega_{\mathrm{b}} - S_{\mathrm{b}}) - \dot{}\mu\dot{}(\mathrm{Ad}\, f_{\mathrm{b}}(\omega_{\mathrm{b}} - s_{\mathrm{b}}), F_{\mathrm{b}}), \tag{3.6.61}$$

$$\theta'_{\mathrm{b}} = \mathrm{Ad}\, f_{\mathrm{b}}(\theta_{\mathrm{b}}), \tag{3.6.62}$$

$$\varTheta'_{\mathrm{b}} = \mu\dot{}(f_{\mathrm{b}}, \varTheta_{\mathrm{b}}) - \dot{}\mu\dot{}(\mathrm{Ad}\, f_{\mathrm{b}}(\theta_{\mathrm{b}}), F_{\mathrm{b}}). \tag{3.6.63}$$

Relations (3.6.56), (3.6.57) entail that, at the basic level, eqs. (3.6.60)–(3.6.63) are of the same form as the matching relations of the projected components of a 2–connection in strict higher gauge theory. See subsect. 3.7 for more in this.

Proof. Inserting (3.6.5), (3.6.6), their primed counterparts and (3.6.50), (3.6.51), into (3.6.58), (3.6.59) and using 3.5.18 of I, we obtain the (3.6.60)–(3.6.63) by simple calculations. □

Next, consider a 1–gauge transformation of $\hat{\mathcal{P}}$ of components $\varPsi$, $\varUpsilon$.

Proposition 3.32. *The basic components $\varPsi_{\mathrm{b}}$, $\varUpsilon_{\mathrm{b}}$ and $\varPsi'_{\mathrm{b}}$, $\varUpsilon'_{\mathrm{b}}$ of the 1–gauge transformation are related on $T[1]\pi^{-1}(U \cap U')$ as*

$$\varPsi'_{\mathrm{b}} = G_{\mathrm{b}} \varPsi_{\mathrm{b}} G_{\mathrm{b}}^{-1}, \tag{3.6.64}$$

$$\varUpsilon'_{\mathrm{b}} = \mathrm{Ad}\, G_{\mathrm{b}}(\varUpsilon_{\mathrm{b}} - D_{\mathrm{b}} + \mathrm{Ad}\, \varPsi_{\mathrm{b}}(D_{\mathrm{b}})). \tag{3.6.65}$$

Proof. Exploiting relations (3.6.17), (3.6.18), we can express $\varPsi$, $\varUpsilon$ in terms of $\varPsi_{\mathrm{b}}$, $\varUpsilon_{\mathrm{b}}$, $\varLambda$, $\varDelta$. Inserting these identities into the primed counterparts of (3.6.17), (3.6.18), we obtain expressions of $\varPsi'_{\mathrm{b}}$, $\varUpsilon'_{\mathrm{b}}$ in terms of $\varPsi_{\mathrm{b}}$, $\varUpsilon_{\mathrm{b}}$, $\varLambda$, $\varDelta$, $\varLambda'$, $\varDelta'$. These latter can be cast in the form (3.6.64), (3.6.65) employing (3.6.47), (3.6.48). □

The matching relation (3.6.64), (3.6.65) can be written in terms of projected components.

Proposition 3.33. *The projected basic components g_{b}, J_{b}, h_{b}, K_{b} and g'_{b}, J'_{b}, h'_{b}, K'_{b} of the 1–gauge transformation are related on $T[1]\pi^{-1}(U \cap U')$ as*

$$g'_{\mathrm{b}} = f_{\mathrm{b}} g_{\mathrm{b}} f_{\mathrm{b}}{}^{-1}, \tag{3.6.66}$$

$$J'_{\mathrm{b}} = \mu\dot{}(f_{\mathrm{b}}, J_{\mathrm{b}}) + F_{\mathrm{b}} - \mu\dot{}(f_{\mathrm{b}} g_{\mathrm{b}} f_{\mathrm{b}}{}^{-1}, F_{\mathrm{b}}), \tag{3.6.67}$$

$$h'_{\mathrm{b}} = \operatorname{Ad} f_{\mathrm{b}}(h_{\mathrm{b}} - s_{\mathrm{b}} + \operatorname{Ad} g_{\mathrm{b}}(s_{\mathrm{b}})), \tag{3.6.68}$$

$$K'_{\mathrm{b}} = \mu\dot{}(f_{\mathrm{b}}, K_{\mathrm{b}} - S_{\mathrm{b}} + \mu\dot{}(g_{\mathrm{b}}, S_{\mathrm{b}}) - \dot{}\mu\dot{}(\operatorname{Ad} g_{\mathrm{b}}(s_{\mathrm{b}}), J_{\mathrm{b}}) \tag{3.6.69}$$
$$- \dot{}\mu\dot{}(h_{\mathrm{b}} - s_{\mathrm{b}} + \operatorname{Ad} g_{\mathrm{b}}(s_{\mathrm{b}}), \mu\dot{}(f_{\mathrm{b}}{}^{-1}, F_{\mathrm{b}}))).$$

In view of eqs. (3.6.56), (3.6.57), the (3.6.66)–(3.6.69) reproduce at the basic level the matching relations of the projected components of a 1–gauge transformation of strict higher gauge theory. We will come back to this in subsect. 3.7.

Proof. Inserting (3.6.23), (3.6.24), their primed counterparts and (3.6.50), (3.6.51), into (3.6.64), (3.6.65) and using 3.5.2, 3.5.3 and 3.5.18 of I, we obtain the (3.6.66)–(3.6.69) through straightforward computations. □

Finally consider a 2–gauge transformation of $\hat{\mathcal{P}}$ of components E, C.

Proposition 3.34. *The projected basic components E_{b}, C_{b} and E'_{b}, C'_{b} of the 2–gauge transformation are related on $T[1]\pi^{-1}(U \cap U')$ as*

$$E'_{\mathrm{b}} = \mu(f_{\mathrm{b}}, E_{\mathrm{b}}), \tag{3.6.70}$$

$$C'_{\mathrm{b}} = \mu\dot{}\,(f_{\mathrm{b}}, C_{\mathrm{b}}) + F_{\mathrm{b}} - \operatorname{Ad}\mu(f_{\mathrm{b}}, E_{\mathrm{b}})(F_{\mathrm{b}}). \tag{3.6.71}$$

Proof. By relations (3.6.39), (3.6.40), we can express E, C in terms of E_{b}, C_{b}, γ, $\varGamma$. Inserting these identities into the primed counterparts of (3.6.39), (3.6.40), we obtain expressions of E'_{b}, C'_{b} in terms of E_{b}, C_{b}, γ, $\varGamma$, γ', $\varGamma'$. These latter can be rewritten in the form (3.6.70), (3.6.71) employing (3.6.52), (3.6.53). □

It is noteworthy that the matching relations do not involve the underlying reference 2–connection.

The basic matching components behave as expected when the adapted coordinates used are special.

Proposition 3.35. *If the two sets of adapted coordinates involved are both special (cf. defs. 3.9), then one has*

$$I_{U\cap U'}{}^{*}F_{\mathrm{b}} = 0, \tag{3.6.72}$$

$$I_{U\cap U'}{}^{*}S_{\mathrm{b}} = 0, \tag{3.6.73}$$

where $I_{U\cap U'}:{\pi_0}^{-1}(U\cap U')\to\pi^{-1}(U\cap U')$ *is the inclusion map.*

Proof. This follows from (3.6.53), (3.6.55) upon substituting (3.5.26), (3.5.27) and its primed counterpart. □

Note that this property renders the matching relations (3.6.61), (3.6.63), respectively (3.6.67), (3.6.69), compatible with (3.6.15), (3.6.16), respectively (3.6.37), (3.6.38), in case the relevant 2–connection, respectively 1–gauge transformation, is special.

3.7 Relation to non Abelian differential cocycles

In this subsection, we shall explore whether 2–connections and 1–gauge transformations as defined in the synthetic theory of subsects. 3.2, 3.3 can be related to non Abelian differential cocycles and their equivalences [13, 29]. We consider again a synthetic principal $\hat{\mathscr{K}}$–2–bundle $\hat{\mathcal{P}}$ and its associated synthetic setup.

In subsect. 3.5, we have seen that we can describe the portion $\pi^{-1}(U)$ of P lying above a trivializing neighborhood U of M by means of adapted coordinates γ, $\varGamma$, σ, $\varSigma$. Since σ and $\varSigma$ are expressible in terms of γ, $\varGamma$ through relations (3.5.10), (3.5.11), only these latter are truly independent. So, we shall limit ourselves to their sole consideration.

In subsect. 3.6, using adapted coordinates we have constructed via (3.6.7)–(3.6.10) the local basic data $\omega_{\rm b}$, $\varOmega_{\rm b}$, $\theta_{\rm b}$, $\varTheta_{\rm b}$ associated with a 2–connection on $\pi^{-1}(U)$. Of these, $\theta_{\rm b}$, $\varTheta_{\rm b}$. can be expressed in terms of $\omega_{\rm b}$, $\varOmega_{\rm b}$ by eqs. (3.6.11), (3.6.12) and so can be disregarded in the following. Similarly, through adapted coordinates we have constructed via (3.6.25)–(3.6.28) also the local basic data $g_{\rm b}$, $J_{\rm b}$, $h_{\rm b}$, $K_{\rm b}$ associated with a 1–gauge transformation on $\pi^{-1}(U)$. Again, of these $h_{\rm b}$ and $K_{\rm b}$ can be given in term of $g_{\rm b}$, $J_{\rm b}$ by eqs. (3.6.29), (3.6.30) and so can be once more disregarded. In this way, the basic data ${}^{g,J}\omega_{\rm b}$, ${}^{g,J}\varOmega_{\rm b}$ of the 1–gauge transformed 2–connection can be expressed in terms of $\omega_{\rm b}$, $\varOmega_{\rm b}$ and $g_{\rm b}$, $J_{\rm b}$ only by the familiar higher gauge theoretic relations, as we found out by inserting (3.6.29), (3.6.30) into (3.6.33), (3.6.34).

In subsect. 3.6, further, we have seen that the matching of local basic 2–

connection and 1–gauge transformation data relative to overlapping neighborhoods U, U' of M is governed by local basic transition data $f_{\rm b}$, $F_{\rm b}$, $s_{\rm b}$, $S_{\rm b}$ given by eqs. (3.6.52)–(3.6.55) of which the latter two are expressible in terms of the former two by eqs. (3.6.56), (3.6.57) and so can be also safely left aside in the following.

We now choose an trivializing covering $\{U_i\}$ of M and for each set U_i adapted coordinates γ_i, $\varGamma_i$ and consider the associated local 2–connection, 1–gauge transformation and transition data. To relate the present framework to non Abelian differential cocycle theory, we shall restrict ourselves fake flat 2–connections as appropriate.

For a 2–connection, there are then defined for every set U_i of the covering local basic data $\omega_{{\rm b}i} \in \mathrm{M{\scriptsize AP}}(T[1]\pi^{-1}(U_i), \mathfrak{g}[1])$, $\varOmega_{{\rm b}i} \in \mathrm{M{\scriptsize AP}}(T[1]\pi^{-1}(U_i), \mathfrak{e}[2])$ via (3.6.7), (3.6.8). By the assumed fake flatness, these satisfy

$$d_{\pi^{-1}(U_i)}\omega_{{\rm b}i} + \frac{1}{2}[\omega_{{\rm b}i}, \omega_{{\rm b}i}] - \dot{\tau}(\varOmega_{{\rm b}i}) = 0. \tag{3.7.1}$$

For a 1–gauge transformation, local basic data $g_{{\rm b}i} \in \mathrm{M{\scriptsize AP}}(T[1]\pi^{-1}(U_i), \mathsf{G})$, $J_{{\rm b}i} \in \mathrm{M{\scriptsize AP}}(T[1]\pi^{-1}(U_i), \mathfrak{e}[1])$ can be similarly defined on each U_i via (3.6.25), (3.6.26).

For every couple of intersecting sets U_i, U_j of the covering, transition data $f_{{\rm b}ij} \in \mathrm{M{\scriptsize AP}}(T[1]\pi^{-1}(U_i \cap U_j), \mathsf{G})$, $F_{{\rm b}ij} \in \mathrm{M{\scriptsize AP}}(T[1]\pi^{-1}(U_i \cap U_j), \mathfrak{e}[1])$ are likewise built through (3.6.52), (3.6.53). The local 2–connection data $\omega_{{\rm b}i}$, $\varOmega_{{\rm b}i}$, match as

$$\omega_{{\rm b}i} = \mathrm{Ad}\, f_{{\rm b}ij}(\omega_{{\rm b}j}) - d_{\pi^{-1}(U_i \cap U_j)} f_{{\rm b}ij} {f_{{\rm b}ij}}^{-1} - \dot{\tau}(F_{{\rm b}ij}), \tag{3.7.2}$$

$$\varOmega_{{\rm b}i} = \mu\dot{}\,(f_{{\rm b}ij}, \varOmega_{{\rm b}j}) - d_{\pi^{-1}(U_i \cap U_j)} F_{{\rm b}ij} - \frac{1}{2}[F_{{\rm b}ij}, F_{{\rm b}ij}] - \dot{\mu}\dot{}\,(\omega_{{\rm b}i}, F_{{\rm b}ij}) \tag{3.7.3}$$

on $U_i \cap U_j$, as follows readily from eqs. (3.6.60), (3.6.61) using the (3.6.56), (3.6.57). Similarly, the the local 1–gauge transformation data $g_{\rm b}$, $J_{\rm b}$ match as

$$g_{{\rm b}i} = f_{{\rm b}ij} g_{{\rm b}j} {f_{{\rm b}ij}}^{-1}, \tag{3.7.4}$$

$$J_{{\rm b}i} = \mu\dot{}(f_{{\rm b}ij}, J_{{\rm b}j}) + F_{{\rm b}ij} - \mu\dot{}(g_{{\rm b}i}, F_{{\rm b}ij}) \tag{3.7.5}$$

by eqs. (3.6.66), (3.6.67).

By virtue of relations (3.6.52), (3.6.53), the data $f_{{\rm b}ij}$. $F_{{\rm b}ij}$ form a $\mathrm{D}\mathsf{M}$–valued

1–cocycle on P, as on every non empty triple intersection $U_i \cap U_j \cap U_k$

$$f_{\mathrm{b}ik} = f_{\mathrm{b}ij} f_{\mathrm{b}jk}, \tag{3.7.6}$$

$$F_{\mathrm{b}ik} = F_{\mathrm{b}ij} + \dot{\mu}\,(f_{\mathrm{b}ij}, F_{\mathrm{b}jk}). \tag{3.7.7}$$

By the way it is constructed, this cocycle is trivial.

Combining (3.6.29), (3.6.30) into (3.6.33), (3.6.34), the local basic data ${}^{g,J}\omega_{\mathrm{b}i}$, ${}^{g,J}\varOmega_{\mathrm{b}i}$ of the 1–gauge transformed 2–connection are found to be given by

$${}^{g,J}\omega_{\mathrm{b}i} = \mathrm{Ad}\, g_{\mathrm{b}i}(\omega_{\mathrm{b}i}) - d_{\pi^{-1}(U_i)} g_{\mathrm{b}i} {g_{\mathrm{b}i}}^{-1} - \dot{\tau}(J_{\mathrm{b}i}), \tag{3.7.8}$$

$${}^{g,J}\varOmega_{\mathrm{b}i} = \dot{\mu}\,(g_{\mathrm{b}i}, \varOmega_{\mathrm{b}i}) - d_{\pi^{-1}(U_i)} J_{\mathrm{b}i} - \frac{1}{2}[J_{\mathrm{b}i}, J_{\mathrm{b}i}] - \dot{\mu}\dot{}\,({}^{g,J}\omega_{\mathrm{b}i}, J_{\mathrm{b}i}) \tag{3.7.9}$$

for any covering set U_i.

Our aim next is ascertaining whether the above setup can be naturally related to (some internal variant of) non Abelian differential cocycle theory. We are going to submit a proposal in this sense. Before proceeding further, however, the following remark is in order. In an ordinary principal G–bundle P, basic forms of P are pull–backs via the bundle's projection map π of ordinary forms of the base M. The proof of this important property requires crucially that the right G–action of P is transitive on the fibers. In a principal $\hat{\mathscr{K}}$–2–bundle $\hat{\mathcal{P}}$, transitiveness holds only up to isomorphism. For this reason, basic forms of P do not necessarily arise as pull–backs via the bundle's projection map π of ordinary forms of the base M, though they may do. Our reformulation of differential cocycle theory hinges on this property.

We have found the following notion useful.

Definition 3.14. *A quasi trivializer consists in an assignment of a basic Lie group valued internal function $T_{\mathrm{b}ij} \in \mathrm{M{\scriptsize AP}}(T[1]\pi^{-1}(U_i \cap U_j), \mathsf{E})$ for each pair of intersecting covering sets U_i, U_j.*

We stress that the basicness of the $T_{\mathrm{b}ij}$ is crucial.

Definition 3.15. *A differential paracocycle is a pair of a fake flat 2–connection $\{\omega_{\mathrm{b}i}, \varOmega_{\mathrm{b}i}\}$ and a quasi trivializer $\{T_{\mathrm{b}ij}\}$ enjoying the following properties.*

1. *For any set U_i, Lie algebra valued internal functions $\bar{\omega}_i \in \mathrm{Map}(T[1]U_i, \mathfrak{g}[1])$, $\bar{\Omega}_i \in \mathrm{Map}(T[1]U_i, \mathfrak{e}[2])$ exist with the property that*

$$\omega_{\mathrm{b}i} = \pi^* \bar{\omega}_i, \tag{3.7.10}$$

$$\Omega_{\mathrm{b}i} = \pi^* \bar{\Omega}_i. \tag{3.7.11}$$

2. *For any two intersecting sets U_i, U_j, Lie group and algebra valued internal functions $\bar{f}_{ij} \in \mathrm{Map}(T[1](U_i \cap U_j), \mathsf{G})$, $\bar{F}_{ij} \in \mathrm{Map}(T[1](U_i \cap U_j), \mathfrak{e}[1])$ exist such that on $U_i \cap U_j$*

$$f_{\mathrm{b}ij} = \tau(T_{\mathrm{b}ij})\pi^* \bar{f}_{ij}, \tag{3.7.12}$$

$$F_{\mathrm{b}ij} = \mathrm{Ad}\, T_{\mathrm{b}ij}(\pi^* \bar{F}_{ij}) - \dot{}\mu(\pi^* \bar{\omega}_i, T_{\mathrm{b}ij}) - d_{\pi^{-1}(U_i \cap U_j)} T_{\mathrm{b}ij} {T_{\mathrm{b}ij}}^{-1}. \tag{3.7.13}$$

3. *For any three intersecting sets U_i, U_j, U_k, there is a Lie group valued internal function $\bar{T}_{ijk} \in \mathrm{Map}(T[1](U_i \cap U_j \cap U_k), \mathsf{E})$ such that on $U_i \cap U_j \cap U_k$*

$${T_{\mathrm{b}ik}}^{-1} \mu(f_{\mathrm{b}ij}, T_{\mathrm{b}jk}) T_{\mathrm{b}ij} = \pi^* \bar{T}_{ijk}. \tag{3.7.14}$$

The content of the above definition is motivated by the following result which it leads to.

Proposition 3.36. *The local 2–connection and transition data $\{\bar{\omega}_i, \bar{\Omega}_i, \bar{f}_{ij}, \bar{F}_{ij}, \bar{T}_{ijk}\}$ of a differential paracocycle $\{\omega_{\mathrm{b}i}, \Omega_{\mathrm{b}i}, T_{\mathrm{b}ij}\}$ constitute a differential cocycle. Indeed, the 2–connection data $\bar{\omega}_i$, $\bar{\Omega}_i$ satisfy the fake flatness condition*

$$d_{U_i} \bar{\omega}_i + \frac{1}{2}[\bar{\omega}_i, \bar{\omega}_i] - \dot{\tau}(\bar{\Omega}_i) = 0 \tag{3.7.15}$$

on every set U_i and the matching conditions

$$\bar{\omega}_i = \mathrm{Ad}\, \bar{f}_{ij}(\bar{\omega}_j) - d_{U_i \cap U_j} \bar{f}_{ij} {\bar{f}_{ij}}^{-1} - \dot{\tau}(\bar{F}_{ij}), \tag{3.7.16}$$

$$\bar{\Omega}_i = \mu\dot{}\,(\bar{f}_{ij}, \bar{\Omega}_j) - d_{U_i \cap U_j} \bar{F}_{ij} - \frac{1}{2}[\bar{F}_{ij}, \bar{F}_{ij}] - \dot{}\mu\dot{}\,(\bar{\omega}_i, \bar{F}_{ij}) \tag{3.7.17}$$

on every non empty intersection $U_i \cap U_j$. Moreover, the transition data $\bar{f}_{ij}$, $\bar{F}_{ij}$, $\bar{T}_{ijk}$ satisfy the consistency conditions

$$\bar{f}_{ik} = \tau(\bar{T}_{ijk}) \bar{f}_{ij} \bar{f}_{jk}, \tag{3.7.18}$$

$$\bar{F}_{ik} = \operatorname{Ad} \bar{T}_{ijk}(\bar{F}_{ij} + \dot{\mu}(\bar{f}_{ij}, \bar{F}_{jk})) - \dot{\mu}(\bar{\omega}_i, \bar{T}_{ijk}) - d_{U_i \cap U_j \cap U_k} \bar{T}_{ijk} \bar{T}_{ijk}{}^{-1} \tag{3.7.19}$$

on every non empty intersection $U_i \cap U_j \cap U_k$*. Finally,*

$$\bar{T}_{ikl} \bar{T}_{ijk} = \bar{T}_{ijl} \mu(\bar{f}_{ij}, \bar{T}_{jkl}) \tag{3.7.20}$$

on every non empty intersection $U_i \cap U_j \cap U_k \cap U_l$*.*

Proof. Relations (3.7.15)–(3.7.19) follow from substituting expressions (3.7.10)–(3.7.13) into relations (3.7.1)–(3.7.3), (3.7.6), (3.7.7) and using (3.7.14). The proof involves combined use of the identities of app. B of I. The property of π being a surjective submersion (cf. prop. 3.2 of I) is used to deduce that $\bar{\tau} = 0$ from any identity of the form $\pi^* \bar{\tau} = 0$ with $\bar{\tau}$ some local internal function on M. (3.7.20) follows directly from (3.7.14) through a simple calculation. □

The above result can be intuitively understood as follows. The local basic data $\{\omega_{\mathrm{b}i}, \Omega_{\mathrm{b}i}, f_{\mathrm{b}ij}, F_{\mathrm{b}ij}, 1_{\mathsf{E}}\}$ can be viewed as something like a trivial differential cocycle on P. By (3.7.10)–(3.7.14), the local basic data $\{\pi^* \bar{\omega}_i, \pi^* \bar{\Omega}_i, \pi^* \bar{f}_{ij}, \pi^* \bar{F}_{ij}, \pi^* \bar{T}_{ijk}\}$ form a trivial differential cocycle on P equivalent to the former. The fundamental cocycle relations obeyed by the data $\{\pi^* \bar{\omega}_i, \pi^* \bar{\Omega}_i, \pi^* \bar{f}_{ij}, \pi^* \bar{F}_{ij}, \pi^* \bar{T}_{ijk}\}$ are then satisfied also by the data $\{\bar{\omega}_i, \bar{\Omega}_i, \bar{f}_{ij}, \bar{F}_{ij}, \bar{T}_{ijk}\}$, since π is a surjective submersion. The local data $\{\bar{\omega}_i, \bar{\Omega}_i, \bar{f}_{ij}, \bar{F}_{ij}, \bar{T}_{ijk}\}$ constitute therefore a differential cocycle on M. Unlike its counterpart in P, this cocycle is generally non trivial since in eq. (3.7.14) $T_{\mathrm{b}ij}$ is not necessarily of the form $T_{\mathrm{b}ij} = \pi^* \bar{T}_{ij}$ for some internal function $\bar{T}_{ij} \in \mathrm{M{\scriptsize AP}}(T[1](U_i \cap U_j), \mathsf{E})$.

Definition 3.16. *Two differential paracocycles* $\{\omega_{\mathrm{b}i}, \Omega_{\mathrm{b}i}, T_{\mathrm{b}ij}\}$*,* $\{\tilde{\omega}_{\mathrm{b}i}, \tilde{\Omega}_{\mathrm{b}i}, \tilde{T}_{\mathrm{b}ij}\}$ *are said to be equivalent if for every set* U_i

$$\tilde{\omega}_{\mathrm{b}i} = \omega_{\mathrm{b}i}, \tag{3.7.21}$$

$$\tilde{\Omega}_{\mathrm{b}i} = \Omega_{\mathrm{b}i} \tag{3.7.22}$$

and for every intersecting set pair U_i*,* U_j *there is a Lie group valued internal function* $\bar{T}_{ij} \in \mathrm{M{\scriptsize AP}}(T[1](U_i \cap U_j), \mathsf{E})$ *such that*

$$\tilde{T}_{\mathrm{b}ij} = T_{\mathrm{b}ij} \pi^* \bar{T}_{ij}{}^{-1}. \tag{3.7.23}$$

Differential paracocycle equivalence is manifestly an equivalence relation as suggested by its name. Further, it implies the equivalence of the underlying differential cocycles.

Proposition 3.37. *If* $\{\omega_{\mathrm{b}i}, \Omega_{\mathrm{b}i}, T_{\mathrm{b}ij}\}$, $\{\tilde{\omega}_{\mathrm{b}i}, \tilde{\Omega}_{\mathrm{b}i}, \tilde{T}_{\mathrm{b}ij}\}$ *are two equivalent differential paracocycles, then their associated differential cocycles* $\{\bar{\omega}_i, \bar{\Omega}_i, \bar{f}_{ij}, \bar{F}_{ij}, \bar{T}_{ijk}\}$ $\{\bar{\tilde{\omega}}_i, \bar{\tilde{\Omega}}_i, \bar{\tilde{f}}_{ij}, \bar{\tilde{F}}_{ij}, \bar{\tilde{T}}_{ijk}\}$ *are equivalent. Indeed,*

$$\bar{\tilde{\omega}}_i = \bar{\omega}_i, \tag{3.7.24}$$

$$\bar{\tilde{\Omega}}_i = \bar{\Omega}_i \tag{3.7.25}$$

on each set U_i*,*

$$\bar{\tilde{f}}_{ij} = \tau(\bar{T}_{ij})\bar{f}_{ij}, \tag{3.7.26}$$

$$\bar{\tilde{F}}_{ij} = \mathrm{Ad}\,\bar{T}_{ij}(\bar{F}_{ij}) - \dot{}\mu(\bar{\tilde{\omega}}_i, \bar{T}_{ij}) - d_{U_i\cap U_j}\bar{T}_{ij}\bar{T}_{ij}{}^{-1} \tag{3.7.27}$$

on every non empty intersection $U_i \cap U_j$ *and*

$$\bar{\tilde{T}}_{ijk} = \bar{T}_{ik}\bar{T}_{ijk}\mu(\bar{f}_{ij}, \bar{T}_{jk}{}^{-1})\bar{T}_{ij}{}^{-1} \tag{3.7.28}$$

on every non empty intersection $U_i \cap U_j \cap U_k$

Proof. Relations (3.7.24), (3.7.25) are an immediate consequence of (3.7.21), (3.7.22) and (3.7.10), (3.7.11) and their tilded analogues. Relations (3.7.26), (3.7.27) follow from equating the tilded and untilded versions of expressions (3.7.12), (3.7.13) and use the resulting equations together with (3.7.23) to express $\bar{\tilde{f}}_{ij}$, $\bar{\tilde{F}}_{ij}$ in terms of $\bar{f}_{ij}$, $\bar{F}_{ij}$. The proof involves combined use of the identities of app. B of I. Finally, (3.7.28) follows from the tilded version of (3.7.14) upon using (3.7.23) and the untilded form of (3.7.14). □

Intuitively, the above result can be understood as follows. In P, the differential cocycles $\{\pi^*\bar{\omega}_i, \pi^*\bar{\Omega}_i, \pi^*\bar{f}_{ij}, \pi^*\bar{F}_{ij}, \pi^*\bar{T}_{ijk}\}$, $\{\pi^*\bar{\tilde{\omega}}_i, \pi^*\bar{\tilde{\Omega}}_i, \pi^*\bar{\tilde{f}}_{ij}, \pi^*\bar{\tilde{F}}_{ij}, \pi^*\bar{\tilde{T}}_{ijk}\}$ are equivalent to the cocycles $\{\omega_{\mathrm{b}i}, \Omega_{\mathrm{b}i}, f_{\mathrm{b}ij}, F_{\mathrm{b}ij}, 1_{\mathsf{E}}\}$, $\{\tilde{\omega}_{\mathrm{b}i}, \tilde{\Omega}_{\mathrm{b}i}, f_{\mathrm{b}ij}, F_{\mathrm{b}ij}, 1_{\mathsf{E}}\}$, respectively. Since the latter two coincide by (3.7.21), (3.7.22), the former two are equivalent. Thanks to (3.7.23), this property entails the equivalence of the cocycles $\{\bar{\omega}_i, \bar{\Omega}_i, \bar{f}_{ij}, \bar{F}_{ij}, \bar{T}_{ijk}\}$ $\{\bar{\bar{\omega}}_i, \bar{\bar{\Omega}}_i, \bar{\bar{f}}_{ij}, \bar{\bar{F}}_{ij}, \bar{\bar{T}}_{ijk}\}$. Note that this equivalence is not of the most general form, as it does not involve 1–gauge transformation.

The above analysis shows that the local basic data $\{\omega_{\mathrm{b}i}, \Omega_{\mathrm{b}i}\}$ of a fake flat 2–connection together with the data $\{T_{\mathrm{b}ij}\}$ of a quasi trivializer can fit into a differential paracocycle. This in turn is directly related to a genuine differential cocycle. The natural question arises about whether the local basic data $\{g_{\mathrm{b}i}, J_{\mathrm{b}i}\}$ of a 1–gauge transformation can fit into some object with somewhat analogous properties capable of relating in a meaningful way to an assigned differential paracocycle.

Definition **3.17.** *A gauge paraequivalence subordinated to a differential paracocycle $\{\omega_{\mathrm{b}i}, \Omega_{\mathrm{b}i}, T_{\mathrm{b}ij}\}$ consists of a 1–gauge transformation $\{g_{\mathrm{b}i}, J_{\mathrm{b}i}\}$ enjoying the following properties.*

1. *For any set U_i, there exist Lie group and algebra valued internal functions $\bar{g}_i \in \mathrm{MAP}(T[1]U_i, \mathsf{G})$, $\bar{J}_i \in \mathrm{MAP}(T[1]U_i, \mathfrak{e}[1])$ such that*

$$g_{\mathrm{b}i} = \pi^* \bar{g}_i, \tag{3.7.29}$$

$$J_{\mathrm{b}i} = \pi^* \bar{J}_i. \tag{3.7.30}$$

2. *For any two intersecting sets U_i, U_j, there exists a Lie group valued internal function $\bar{A}_{ij} \in \mathrm{MAP}(T[1](U_i \cap U_j), \mathsf{E})$ such that*

$$\mu(g_{\mathrm{b}i}, {T_{\mathrm{b}ij}}^{-1}) T_{\mathrm{b}ij} = \pi^* \bar{A}_{ij}. \tag{3.7.31}$$

The following proposition shows the naturality of the above definition.

Proposition **3.38.** *Let $\{\omega_{\mathrm{b}i}, \Omega_{\mathrm{b}i}, T_{\mathrm{b}ij}\}$ be a differential paracocycle and let $\{g_{\mathrm{b}i}, J_{\mathrm{b}i}\}$ be gauge paraequivalence subordinated to it. Then, $\{{}^{g,J}\omega_{\mathrm{b}i}r, {}^{g,J}\Omega_{\mathrm{b}i}, T_{\mathrm{b}ij}\}$ is a differential paracocycle as well. In terms of the cocycle and equivalence data of $\{\omega_{\mathrm{b}i}, \Omega_{\mathrm{b}i}, T_{\mathrm{b}ij}\}$ and $\{g_{\mathrm{b}i}, J_{\mathrm{b}i}\}$ the cocycle data of $\{{}^{g,J}\omega_{\mathrm{b}i}r, {}^{g,J}\Omega_{\mathrm{b}i}, T_{\mathrm{b}ij}\}$ read as*

$${}^{\bar{g},\bar{J}}\bar{\omega}_i = \mathrm{Ad}\,\bar{g}_i(\bar{\omega}_i) - d_{U_i}\bar{g}_i{\bar{g}_i}^{-1} - \dot{\tau}(\bar{J}_i), \tag{3.7.32}$$

$${}^{\bar{g},\bar{J}}\bar{\Omega}_i = \mu\dot{}(\bar{g}_i, \bar{\Omega}_i) - d_{U_i}\bar{J}_i - \frac{1}{2}[\bar{J}_i, \bar{J}_i] - \dot{\mu}\dot{}({}^{\bar{g},\bar{J}}\bar{\omega}_i, \bar{J}_i), \tag{3.7.33}$$

$$ {}^{\bar g,\bar J}\bar f_{ij} = \bar f_{ij}, \tag{3.7.34} $$

$$ {}^{\bar g,\bar J}\bar F_{ij} = \mathrm{Ad}\,\bar A_{ij}{}^{-1}(\bar J_i + \dot\mu(\bar g_i, \bar F_{ij})) - \dot\mu(\bar f_{ij}, \bar J_j) - \dot{}\mu({}^{\bar g,\bar J}\bar\omega_i, \bar A_{ij}{}^{-1}) - d_{U_i\cap U_j}\bar A_{ij}{}^{-1}\bar A_{ij}, \tag{3.7.35} $$

$$ {}^{\bar g,\bar J}\bar T_{ijk} = \bar T_{ijk}. \tag{3.7.36} $$

Proof. Inserting (3.7.10), (3.7.11) and (3.7.29), (3.7.30) into (3.7.8), (3.7.9), one readily finds that ${}^{g,J}\omega_{\mathrm{b}i} = \pi^*{}^{\bar g,\bar J}\bar\omega_i$, ${}^{g,J}\Omega_{\mathrm{b}i} = \pi^*{}^{\bar g,\bar J}\bar\Omega_i$ with ${}^{\bar g,\bar J}\bar\omega_i$, ${}^{\bar g,\bar J}\bar\Omega_i$ given by (3.7.32), (3.7.33), respectively. (3.7.34) is evident by relation (3.7.12) expressing $f_{\mathrm{b}ij}$. To verify (3.7.35), one has to show that $F_{\mathrm{b}ij}$ can be expressed as in (3.7.13) with $\bar\omega_i$, $\bar F_{ij}$ replaced by ${}^{\bar g,\bar J}\bar\omega_i$, ${}^{\bar g,\bar J}\bar F_{ij}$ as given by (3.7.32), (3.7.35), respectively. This is straightforward using (3.7.5) together with (3.7.12), (3.7.13) and (3.7.31) and the identities of app. B of I. (3.7.36) is evident from relation (3.7.14). □

The following proposition describes the global matching of the local data of a gauge paraequivalence.

Proposition 3.39. *Let $\{g_{\mathrm{b}i}, J_{\mathrm{b}i}\}$ be a gauge paraequivalence subordinated to the differential paracocycle $\{\omega_{\mathrm{b}i}, \Omega_{\mathrm{b}i}, T_{\mathrm{b}ij}\}$. Then,*

$$ \bar g_i = \tau(\bar A_{ij})\bar f_{ij}\bar g_j\bar f_{ij}{}^{-1}, \tag{3.7.37} $$

$$ \bar J_i = \mathrm{Ad}\,\bar A_{ij}(\dot\mu(\bar f_{ij}, \bar J_j) + {}^{\bar g,\bar J}\bar F_{ij}) - \dot{}\mu({}^{\bar g,\bar J}\bar\omega_i, \bar A_{ij}) - d_{U_i\cap U_j}\bar A_{ij}\bar A_{ij}{}^{-1} - \dot\mu(\bar g_i, \bar F_{ij}) \tag{3.7.38} $$

on every non empty intersection $U_i \cap U_j$. Moreover,

$$ \bar A_{ik} = \mu(\bar g_i, \bar T_{ijk})\bar A_{ij}\mu(\bar f_{ij}, \bar A_{jk})\bar T_{ijk}{}^{-1} \tag{3.7.39} $$

on every non empty intersection $U_i \cap U_j \cap U_k$.

Proof. Inserting (3.7.12) and (3.7.29) into (3.7.4) and rearranging the resulting factors in the right hand side using also (3.7.31), relation (3.7.37) is obtained. To show (3.7.38), one substitutes (3.7.12), (3.7.13) and (3.7.29), (3.7.30) into (3.7.5). In the first insertion of (3.7.13), one expresses $F_{\mathrm{b}ij}$ in terms of ${}^{\bar g,\bar J}\bar\omega_i$, ${}^{\bar g,\bar J}\bar F_{ij}$; in the second, one writes $F_{\mathrm{b}ij}$ through $\bar\omega_i$, $\bar F_{ij}$. Use of the identities of app. B of I leads to (3.7.38) straightforwardly. (3.7.39) follows from combining (3.7.4), (3.7.12), (3.7.14), (3.7.29), (3.7.31) trough a simple algebraic computation. □

We note that eq. (3.7.38) is an equivalent rewriting of eq. (3.7.35). However, we deduced (3.7.38) from (3.7.5) by suitably expressing the latter relation in terms of barred objects. So, eq. (3.7.38) does not constitute anything new, but it merely shows the consistency of eqs. (3.7.5) and (3.7.35).

Definition **3.18.** *Two pairs of differential paracocycles and subordinated gauge paraequivalences* $\{\omega_{\mathrm{b}i}, \Omega_{\mathrm{b}i}, T_{\mathrm{b}ij}\}$, $\{g_{\mathrm{b}i}, J_{\mathrm{b}i}\}$, $\{\tilde{\omega}_{\mathrm{b}i}, \tilde{\Omega}_{\mathrm{b}i}, \tilde{T}_{\mathrm{b}ij}\}$, $\{\tilde{g}_{\mathrm{b}i}, \tilde{J}_{\mathrm{b}i}\}$ *are equivalent if* $\{\omega_{\mathrm{b}i}, \Omega_{\mathrm{b}i}, T_{\mathrm{b}ij}\}$, $\{\tilde{\omega}_{\mathrm{b}i}, \tilde{\Omega}_{\mathrm{b}i}, \tilde{T}_{\mathrm{b}ij}\}$ *are equivalent differential paracocycles and furthermore for every set* U_i

$$\tilde{g}_{\mathrm{b}i} = g_{\mathrm{b}i}, \tag{3.7.40}$$

$$\tilde{J}_{\mathrm{b}i} = J_{\mathrm{b}i}. \tag{3.7.41}$$

Equivalence of differential paracocycle and subordinated gauge paraequivalence pairs is manifestly an equivalence relation as suggested by its name.

Proposition **3.40.** *If* $\{\omega_{\mathrm{b}i}, \Omega_{\mathrm{b}i}, T_{\mathrm{b}ij}\}$, $\{g_{\mathrm{b}i}, J_{\mathrm{b}i}\}$, $\{\tilde{\omega}_{\mathrm{b}i}, \tilde{\Omega}_{\mathrm{b}i}, \tilde{T}_{\mathrm{b}ij}\}$, $\{\tilde{g}_{\mathrm{b}i}, \tilde{J}_{\mathrm{b}i}\}$ *are equivalent pairs of differential paracocycles and subordinated gauge paraequivalences, then identities* (3.7.24)–(3.7.28) *hold and moreover*

$$\bar{\tilde{g}}_i = \bar{g}_i, \tag{3.7.42}$$

$$\bar{\tilde{J}}_i = \bar{J}_i \tag{3.7.43}$$

on each set U_i *and*

$$\bar{\tilde{A}}_{ij} = \mu(\bar{g}_i, \bar{T}_{ij})\bar{A}_{ij}\bar{T}_{ij}{}^{-1} \tag{3.7.44}$$

on every non empty intersection $U_i \cap U_j$*.*

Proof. Since $\{\omega_{\mathrm{b}i}, \Omega_{\mathrm{b}i}, T_{\mathrm{b}ij}\}$, $\{\tilde{\omega}_{\mathrm{b}i}, \tilde{\Omega}_{\mathrm{b}i}, \tilde{T}_{\mathrm{b}ij}\}$ are equivalent differential cocycles according to def. 3.18, eqs. (3.7.24)–(3.7.28) hold by virtue of prop. 3.37. Relations (3.7.42), (3.7.43) are an immediate consequence of (3.7.40), (3.7.41) and (3.7.29), (3.7.30) and their tilded analogues. (3.7.44) follows form (3.7.31) and its tilded form and (3.7.23). □

Gauge paraequivalences subordinated to the same differential paracocycle form a group.

Proposition **3.41.** *The gauge paraequivalences* $\{g_{\mathrm{b}i}, J_{\mathrm{b}i}\}$ *subordinated to a fixed differential paracocycle* $\{\omega_{\mathrm{b}i}, \Omega_{\mathrm{b}i}, T_{\mathrm{b}ij}\}$ *constitute a subgroup of the* 1*–gauge group.*

Proof. Suppose that $\{g_{1\mathrm{b}i}, J_{1\mathrm{b}i}\}$, $\{g_{2\mathrm{b}i}, J_{2\mathrm{b}i}\}$ are gauge paraequivalences subordinated to $\{\omega_{\mathrm{b}i}, \Omega_{\mathrm{b}i}, T_{\mathrm{b}ij}\}$ and that $\{g_{3\mathrm{b}i}, J_{3\mathrm{b}i}\}$ is their product as 1–gauge transformations, so that $g_{3\mathrm{b}i} = g_{2\mathrm{b}i} g_{1\mathrm{b}i}$, $J_{3\mathrm{b}i} = J_{2\mathrm{b}i} + \dot\mu(g_{2\mathrm{b}i}, J_{1\mathrm{b}i})$. Then, $g_{3\mathrm{b}i}$, $J_{3\mathrm{b}i}$ satisfy (3.7.29)–(3.7.31) too with $\bar g_{3i} = \bar g_{2i}\bar g_{1i}$, $\bar J_{3i} = \bar J_{2i} + \dot\mu(\bar g_{2i}, \bar J_{1i})$ and $\bar A_{3ij} = \mu(\bar g_{2i}, \bar A_{1ij})\bar A_{2ij}$. Similarly, suppose that $\{g_{1\mathrm{b}i}, J_{1\mathrm{b}i}\}$ is a gauge paraequivalence subordinated to $\{\omega_{\mathrm{b}i}, \Omega_{\mathrm{b}i}, T_{\mathrm{b}ij}\}$ and that $\{g_{2\mathrm{b}i}, J_{2\mathrm{b}i}\}$ is its inverse as a 1–gauge transformation, so that $g_{2\mathrm{b}i} = {g_{1\mathrm{b}i}}^{-1}$, $J_{2\mathrm{b}i} = -\dot\mu(g_{1\mathrm{b}i}, J_{1\mathrm{b}i})$. Then, $g_{2\mathrm{b}i}$, $J_{2\mathrm{b}i}$ satisfies (3.7.29)–(3.7.31) too with $\bar g_{2i} = {\bar g_{1i}}^{-1}$, $\bar J_{2i} = -\dot\mu({\bar g_{1i}}^{-1}, \bar J_{1i})$ and $\bar A_{2ij} = \mu({\bar g_{1i}}^{-1}, {\bar A_{1ij}}^{-1})$. This is enough to show the proposition. □

We assume now that for each set U_i of the covering the adapted coordinates γ_i, Γ_i can be chosen to be special (cf. def. 3.9). Then, by (3.5.26)

$$I_i{}^*\Gamma_i = 0, \tag{3.7.45}$$

where $I_i : {\pi_0}^{-1}(U_i) \to \pi^{-1}(U_i)$ is the injection map.

Proposition **3.42.** *The basic matching data* $F_{\mathrm{b}ij}$ *satisfy*

$$I_{ij}{}^*F_{\mathrm{b}ij} = 0 \tag{3.7.46}$$

for each non empty intersection $U_i \cap U_j$

Above, $I_{ij} : {\pi_0}^{-1}(U_i \cap U_j) \to \pi^{-1}(U_i \cap U_j)$ is the injection map.

Proof. Eq. (3.7.46) follows immediately from (3.6.53) and (3.7.45). □

Proposition **3.43.** *If* $\{\omega_{\mathrm{b}i}, \Omega_{\mathrm{b}i}, T_{\mathrm{b}ij}\}$ *is a differential paracocycle, then for each non empty intersection* $U_i \cap U_j$

$$\begin{aligned}\mathrm{Ad}\, I_{ij}{}^*T_{\mathrm{b}ij}({\pi_0}^*\bar F_{ij}) - \dot\mu({\pi_0}^*\bar\omega_i, I_{ij}{}^*T_{\mathrm{b}ij}) \\ - d_{{\pi_0}^{-1}(U_i\cap U_j)} I_{ij}{}^*T_{\mathrm{b}ij} {I_{ij}{}^*T_{\mathrm{b}ij}}^{-1} = 0.\end{aligned} \tag{3.7.47}$$

Proof. Eq. (3.7.47) is a direct consequence of (3.7.13) and (3.7.46). □

Definition 3.19. *A differential paracocycle* $\{\omega_{\mathrm{b}i}, \Omega_{\mathrm{b}i}, T_{\mathrm{b}ij}\}$ *is said to be special if the underlying* 2*–connection is special (cf. def. 3.2).*

By (3.6.16), then, in each set U_i

$$I_i{}^*\Omega_{\mathrm{b}i} = 0. \tag{3.7.48}$$

We note that by (3.7.3) the condition of specialty is globally consistent if (3.7.46) holds.

Proposition 3.44. *If the differential paracocycle* $\{\omega_{\mathrm{b}i}, \Omega_{\mathrm{b}i}, T_{\mathrm{b}ij}\}$ *is special, then*

$$\bar{\Omega}_i = 0 \tag{3.7.49}$$

in each set U_i.

Proof. By virtue of (3.7.11) and the relation $\pi \circ I_i = \pi_0|_{\pi_0{}^{-1}(U_i)}$, (3.7.48) implies that $0 = I_i{}^*\pi^*\bar{\Omega}_i = \pi_0{}^*\bar{\Omega}_i$. Since π_0 is a surjective submersion (cf. prop. 3.2 of I), (3.7.49) holds. □

Proposition 3.45. *If the differential paracocycle* $\{\omega_{\mathrm{b}i}, \Omega_{\mathrm{b}i}, T_{\mathrm{b}ij}\}$ *is special, so is any other paracocycle* $\{\tilde{\omega}_{\mathrm{b}i}, \tilde{\Omega}_{\mathrm{b}i}, \tilde{T}_{\mathrm{b}ij}\}$ *equivalent to it.*

Proof. By (3.7.21), (3.7.22) and (3.6.7), (3.6.8) and their tilded counterparts, the 2–connections underlying two equivalent paracocycles are equal. So, if the first paracocycle is special, so is the second by virtue of def. 3.19. □

Definition 3.20. *A gauge paraequivalence* $\{g_{\mathrm{b}i}, J_{\mathrm{b}i}\}$ *subordinated to a differential paracocycle* $\{\omega_{\mathrm{b}i}, \Omega_{\mathrm{b}i}, T_{\mathrm{b}ij}\}$ *is said to be special if the underlying* 1*–gauge transformation is special.*

By (3.6.37), then, in each set U_i

$$I_i{}^*J_{\mathrm{b}i} = 0. \tag{3.7.50}$$

We note that by (3.7.5) the condition of specialty is globally consistent if (3.7.46) holds.

Proposition **3.46.** *It the gauge paraequivalence* $\{g_{\mathrm{b}i}, J_{\mathrm{b}i}\}$ *subordinated to a differential paracocycle* $\{\omega_{\mathrm{b}i}, \Omega_{\mathrm{b}i}, T_{\mathrm{b}ij}\}$ *is special, then in each set* U_i*.*

$$\bar{J}_i = 0. \tag{3.7.51}$$

Proof. This follows from (3.7.30) through a reasoning similar to that leading to (3.7.49). □

Proposition **3.47.** *If* $\{\omega_{\mathrm{b}i}, \Omega_{\mathrm{b}i}, T_{\mathrm{b}ij}\}$*,* $\{g_{\mathrm{b}i}, J_{\mathrm{b}i}\}$ *is a pair of a differential paracocycle and a subordinated gauge paraequivalence with* $\{g_{\mathrm{b}i}, J_{\mathrm{b}i}\}$ *special and* $\{\tilde{\omega}_{\mathrm{b}i}, \tilde{\Omega}_{\mathrm{b}i}, \tilde{T}_{\mathrm{b}ij}\}$*,* $\{\tilde{g}_{\mathrm{b}i}, \tilde{J}_{\mathrm{b}i}\}$ *is a pair of a differential paracocycle and a subordinated gauge paraequivalence equivalent to the former, then* $\{\tilde{g}_{\mathrm{b}i}, \tilde{J}_{\mathrm{b}i}\}$ *is special.*

Proof. By (3.7.40), (3.7.41) and (3.6.25), (3.6.26) and their tilded counterparts, the 1–gauge transformations underlying two equivalent differential paracocycle and subordinated gauge paraequivalence pairs are equal. So, if the first paraequivalence is special, so is the second by virtue of def. 3.20. □

The reader certainly noticed that we did not include 2–gauge symmetry in our discussion. The reason for this is that, apparently, there is no way of making it fitting into the framework described in this subsection. An analysis of the global matching of the local basic data $E_{\mathrm{b}i}$, $C_{\mathrm{b}i}$ of a 2–gauge transformation would unavoidably be based on relations (3.6.70), (3.6.71). Forcing on $E_{\mathrm{b}i}$, $C_{\mathrm{b}i}$ relations analogous to (3.7.10), (3.7.11) and (3.7.29), (3.7.30) does not seem to yield any reasonable relation on M. This is an open problem requiring further investigation including possibly a revision of the synthetic theory of subsect. 3.4.

4 Appraisal of the results obtained

It is important to critically assess the strengths and weaknesses of the operational synthetic formulation of the total space theory of principal 2–bundles and 2–connections and 1– and 2–gauge transformations thereof developed in this paper. A number of points can be raised concerning its viability and its eventual relationship with other approaches. We are going address some of these issues in this section.

4.1 Some open problems

The geometry of a principal $\hat{\mathscr{K}}$–2–bundle $\hat{\mathcal{P}}$ is characterized not only by the right $\hat{\mathsf{K}}$–action but also by the morphism composition of $\hat{P}$. Our operational formulation relies heavily of the former while it leaves the latter in the background (cf. subsect. 3.8 of I). However, the second is a constitutive element of the principal 2–bundle structure as basic as the first.

Since morphisms belonging to different fibers of a principal 2–bundle can never be composed, morphism composition is essentially a local operation. For a chosen local neighborhood U of M, through a pair of reciprocally weakly inverse trivializing functors $\hat{\varPhi}_U : \hat{\pi}^{-1}(U) \to U \times \hat{\mathsf{K}}$ and $\tilde{\hat{\varPhi}}_U : U \times \hat{\mathsf{K}} \to \hat{\pi}^{-1}(U)$ composition of morphisms of $\hat{\pi}^{-1}(U)$ is turned into composition of corresponding morphisms of $\hat{\mathsf{K}}$ and viceversa. It is known [30] that the groupoid structure of a strict 2–group such as $\hat{\mathsf{K}}$ can be reduced to the group one as follows. With any morphism $A \in \hat{\mathsf{K}}$ there is associated a morphism $\hat{\varrho}(A) \in \hat{\mathsf{K}}$ given by

$$\hat{\varrho}(A) = \hat{t}(A)^{-1}A, \tag{4.1.1}$$

such that for composable morphisms $A,\ B \in \hat{\mathsf{K}}$

$$\hat{\varrho}(B \circ A) = \hat{\varrho}(B)\hat{\varrho}(A). \tag{4.1.2}$$

So, as $\hat{t}(B \circ A) = \hat{t}(B)$, right composition of B by A is equivalent to right multiplication of B by $\hat{\varrho}(A)$. It follows that, for any two composable morphisms $X,\ Y \in \hat{P}$, right composition of Y by X can be reduced, in the appropriate categorical sense, to the right action of some element of $A_X \in \hat{\mathsf{K}}$ depending on X on Y.

Shifting to the synthetic setup of $\hat{\mathcal{P}}$, it is in the way explained above that the operation $\mathrm{O\scriptstyle P}\, S_P$ indirectly includes the morphism composition structure of $\hat{P}$ in spite of the fact that its synthetic counterpart P has no groupoid structure (cf. subsect. 3.2 of I) A more explicit incorporation of this latter in our formulation would be desirable.

To construct the basic theory, we proposed a notion of coordinates adapted to the local product structure $U \times \mathsf{K}$ of a principal $\hat{\mathscr{K}}$–2–bundle $\hat{\mathcal{P}}$ in subsect. 3.5. These coordinates are Lie valued internal functions on $T[1]\pi^{-1}(U)$ behaving in a certain way under the action of the derivations of the operation $\mathrm{O\scriptstyle P}\, S_{\pi^{-1}(U)}$. The definition provided is essentially algebraic. It leads to a seemingly viable basic formulation of principal 2–bundle 2–connection and 1–gauge transformation theory. However, the eventual relation of adapted coordinates to trivialization functors remains blurred at best and calls for further investigation.

To make contact with other widely studied formulations of 2–connections and 1–gauge transformations of principal 2–bundles, we introduced the notions of differential paracocycle and gauge paraequivalence in subsect. 3.7. The definitions of these entities we gave are admittedly somewhat *ad hoc*. The cocycle data $\{\bar{\omega}_i, \bar{\Omega}_i, \bar{f}_{ij}, \bar{F}_{ij}, \bar{T}_{ijk}\}$ associated with a differential paracocycle $\{\omega_{\mathrm{b}i}, \Omega_{\mathrm{b}i}, T_{\mathrm{b}ij}\}$ are simply assumed to exist as part of the definition of this latter. Similarly, the equivalence data $\{\bar{g}_i, \bar{J}_i, \bar{A}_{ij}\}$ associated with a gauge paraequivalence $\{g_{\mathrm{b}i}, J_{\mathrm{b}i}\}$ are again assumed to exist. It would be desirable instead to have a formulation where the cocycle and equivalence data can be constructively shown to exist in analogy to the ordinary theory.

The viability of the formulation furnished here remains to be tested in concrete examples. This left for future work.

4.2 Toward a more geometric interpretation

In this paper, we worked out an operational synthetic total space theory of 2–connections and 1– and 2–gauge transformations for strict principal 2–bundles adopting a graded differential geometric approach and mimicking to a large extent the corresponding formulation of connection and gauge transformation theory for ordinary principal bundles. In the ordinary case, however, these notions have also a more conventional intuitive geometric interpretation in terms of the overall geometry of the principal bundle and its fibered structure. The natural question arises whether a similar interpretation exists also in the higher theory.

As already observed in subsect. 3.1 of I, at the moment no definition of 2–connection on a strict principal 2–bundle akin to that of the ordinary theory formulated in terms of a horizontal invariant distribution in the tangent bundle of the bundle is available. There exits however a definition of 1–gauge transformation analogous to that of the ordinary theory as an equivariant fiber preserving bundle automorphism formulated by Wockel in ref. [7]. The interpretation of 2–connections as defined in subsect. 3.2 along the lines just indicated remains an open problem. It is conversely possible to attempt a comparison of the notion of 1–gauge transformation of subsect. 3.3 and Wockel's categorical one.

For a given strict principal $\hat{\mathscr{K}}$–2–bundle $\hat{\mathcal{P}}$, the synthetic counterpart of the gauge 2–group $\mathrm{Fun}^{\hat{\mathsf{K}}}(\hat{P}, \hat{\mathsf{K}}_{\mathrm{Ad}})$ (cf. subsect. 3.1 of I) is the group $\mathrm{Fun}^{\mathsf{K}}(P, \mathsf{K}_{\mathrm{Ad}})$ of K–equivariant maps of $\mathrm{Map}(P, \mathsf{K})$ restricting to K_0–equivariant maps of $\mathrm{Map}(P_0, \mathsf{K}_0)$. $\mathrm{Fun}^{\mathsf{K}}(P, \mathsf{K}_{\mathrm{Ad}})$ is formally analogous to $\mathrm{Fun}^{\hat{\mathsf{K}}}(\hat{P}, \hat{\mathsf{K}}_{\mathrm{Ad}})$ in several respects, but by the lack of a groupoid structure of K (cf. subsect. 3.2 of I) it has no morphisms and is thus a mere mapping group. $\mathrm{Fun}^{\mathsf{K}}(P, \mathsf{K}_{\mathrm{Ad}})$ cannot be directly equated with the 1–gauge group as defined earlier in subsect. 3.3. Rather, $\mathrm{Fun}^{\mathsf{K}}(P, \mathsf{K}_{\mathrm{Ad}})$ can be identified as a distinguished subgroup of the special subgroup of the 1–gauge transformation group, as we show next.

Recalling that $\mathsf{K} = \mathrm{D}\mathsf{M}$, an element of $\mathrm{Fun}^{\mathsf{K}}(P, \mathsf{K}_{\mathrm{Ad}})$ is an instance of an internal function $\varPsi \in \mathrm{M{\scriptsize AP}}(T[1]P, \mathrm{D}\mathsf{M})$ that is $\mathrm{D}\mathsf{M}$–horizontal and $\mathrm{D}\mathsf{M}$–equivariant and restricts to an internal function $\varPsi_0 \in \mathrm{M{\scriptsize AP}}(T[1]P_0, \mathrm{D}\mathsf{M}_0)$ that is $\mathrm{D}\mathsf{M}_0$–horizontal and $\mathrm{D}\mathsf{M}_0$–equivariant, the action of $\mathrm{D}\mathsf{M}$, respectively $\mathrm{D}\mathsf{M}_0$, on itself being the right conjugation one. $\mathrm{D}\mathsf{M}$–horizontality translates directly into relation (3.3.3). $\mathrm{D}\mathsf{M}$–equivariance is equivalent to the condition that

$$R_F{}^*\varPsi = F^{-1}\varPsi F \tag{4.2.1}$$

for $F \in \mathrm{D}\mathsf{M}$, where in the right hand side F is identified with its image under the isomorphism $z_{\mathsf{M}} : \mathrm{D}\mathsf{M} \to \mathrm{D}\mathsf{M}^{+}$ defined in eq. 3.6.1 of I. In infinitesimal form, expressing F as $1+tZ$, where $Z \in \mathrm{D}\mathfrak{m}$ and t is a formal parameter such that $t^2 = 0$, this relation takes the form (3.3.5). Thus Ψ is the transformation component of a 1–gauge transformation. Since Ψ restricts on $T[1]P_0$ to a $\mathrm{D}\mathsf{M}_0$–valued $\mathrm{D}\mathsf{M}_0$–horizontal and $\mathrm{D}\mathsf{M}_0$–equivariant internal map, this 1–gauge transformation is special.

In the present formulation of the theory, 2–gauge transformations of $\hat{\mathcal{P}}$ cannot be obviously related to morphisms of the gauge 2–group $\mathrm{Fun}^{\hat{\mathsf{K}}}(\hat{P}, \hat{\mathsf{K}}_{\mathrm{Ad}})$, because the synthetic form $\mathrm{Fun}^{\mathsf{K}}(P, \mathsf{K}_{\mathrm{Ad}})$ of this latter does not have any. Moreover, 2–gauge transformations are supposed to act on 1–gauge transformations (cf. subsect. 3.4) and do so in a proper way depending on an assigned 2–connection (cf. subsect. 3.2). As long we do not have a purely geometric total space theory of 2–connections, any attempt to relate 2–gauge transformations to morphisms of the gauge 2–group is premature at best.

4.3 Comparison with other formulations

An interesting total space formulation of 2–connections theory has been worked out by Waldorf in refs. [17, 18]. We anticipate that Waldorf's theory is not obviously equivalent to ours and most likely it is not. We outline it briefly below referring the interested reader to the cited papers for a full exposition.

Waldorf's approach is based on a special differential geometric framework. For a given principal $\hat{\mathscr{K}}$–2–bundle $\hat{\mathcal{P}}$, its main ingredients are the morphism and object manifolds $\hat{P}$ and $\hat{P}_0$ of $\hat{\mathcal{P}}$ and the Lie group crossed module $\mathsf{M} = (\mathsf{E}, \mathsf{G}, \tau, \mu)$ associated with the structure Lie 2–group $\hat{\mathscr{K}}$. He defines a vector space $A^{\bullet}(\hat{P}, \hat{\mathfrak{k}})$ of $\hat{\mathfrak{k}}$–valued differential forms of $\hat{P}$ and endows it with a structure of differential graded Lie algebra with Lie bracket $[-,-]$ and differential D. $A^p(\hat{P}, \hat{\mathfrak{k}})$ is a certain subspace of the vector space $\Omega^p(\hat{P}_0, \mathfrak{g}) \oplus \Omega^p(\hat{P}, \mathfrak{e}) \oplus \Omega^{p+1}(\hat{P}_0, \mathfrak{e})$ defined by algebraic constraints expressed in terms of the face maps of the nerve of the groupoid $\hat{P}$, the simplicial complex $\hat{P}_{\bullet} = \cdots \substack{\longrightarrow\\[-1em]\longrightarrow\\[-1em]\longrightarrow\\[-1em]\longrightarrow} \hat{P}_2 \substack{\longrightarrow\\[-1em]\longrightarrow\\[-1em]\longrightarrow} \hat{P}_1 \rightrightarrows \hat{P}_0$ of composable sequences of $\hat{P}$. Each p–form possesses therefore three components. The Lie bracket combines the form wedge product and the Lie bracket of the Lie algebra $\mathfrak{e} \rtimes_{\dot{\mu}} \mathfrak{g}$. The differential D is constructed assembling the de Rahm differentials $d_{\hat{P}}$, $d_{\hat{P}_0}$, the face maps of $\hat{P}_\bullet$ and target map $\dot{\tau}$. An adjoint action Ad of functors $\hat{P} \to \hat{\mathsf{K}}$ on $A^\bullet(\hat{P}, \hat{\mathfrak{k}})$ preserving degree is also defined.

A 2–connection is defined again in terms of its behaviour under the right $\hat{\mathsf{K}}$ action of $\hat{P}$. A $\hat{\mathsf{K}}$–valued variable Q is considered and a Maurer–Cartan 1–form $\mathit{\Gamma} \in A^1(\hat{\mathsf{K}}, \hat{\mathfrak{k}})$ obeying the Maurer–Cartan equation $D\mathit{\Gamma} + [\mathit{\Gamma}, \mathit{\Gamma}]/2 = 0$ is defined. A 2–connection of $\hat{P}$ is a 1–form $A \in A^1(\hat{P}, \hat{\mathfrak{k}})$ such that

$$R_Q{}^*A = \operatorname{Ad} Q^{-1}(A) + \mathit{\Gamma}, \tag{4.3.1}$$

which must be viewed as a 1–form of $A^1(\hat{P} \times \hat{\mathsf{K}}, \hat{\mathfrak{k}})$. The curvature of A is a 2–form $B \in A^2(\hat{P}, \hat{\mathfrak{k}})$ defined by

$$B = DA + \frac{1}{2}[A, A]. \tag{4.3.2}$$

By (4.3.1), it obeys

$$R_Q{}^*B = \operatorname{Ad} Q^{-1}(B). \tag{4.3.3}$$

More explicitly, denoting by g and (H, h) the G and $\mathsf{E} \rtimes_\mu \mathsf{G}$ variables underlying Q above, a 2–connection A consists of a triplet of forms $\omega \in \Omega^1(\hat{P}_0, \mathfrak{g})$, $\mathit{\Omega}' \in \Omega^1(\hat{P}, \mathfrak{e})$, $\mathit{\Omega} \in \Omega^2(\hat{P}_0, \mathfrak{e})$ satisfying the simplicial constraints and such that

$$R_g{}^*\omega = \operatorname{Ad} g^{-1}(\omega) + g^{-1}dg, \tag{4.3.4}$$

$$R_{(H,h)}{}^*\mathit{\Omega}' = \mu\dot{}\,(h^{-1}, \operatorname{Ad} H(\mathit{\Omega}' + \dot{}\mu(\hat{s}^*\omega, H)) + H^{-1}dH), \tag{4.3.5}$$

$$R_g{}^*\mathit{\Omega} = \mu\dot{}\,(g^{-1}, \mathit{\Omega}), \tag{4.3.6}$$

where $\hat{s}$, $\hat{t}t$ are the source and target maps of $\hat{P}$. The curvature of the 2–connection is the triplet of forms $\theta \in \Omega^2(\hat{P}_0, \mathfrak{g})$, $\mathit{\Theta}' \in \Omega^2(\hat{P}, \mathfrak{e})$, $\mathit{\Theta} \in \Omega^3(\hat{P}_0, \mathfrak{e})$ given by

$$\theta = d_{\hat{P}_0}\omega + \frac{1}{2}[\omega, \omega] - \dot{\tau}(\mathit{\Omega}), \tag{4.3.7}$$

$$\mathit{\Theta}' = (\hat{t}^* - \hat{s}^*)\mathit{\Omega} + d_{\hat{P}}\mathit{\Omega}' + \frac{1}{2}[\mathit{\Omega}', \mathit{\Omega}'] + \dot{}\mu\dot{}\,(\hat{s}^*\omega, \mathit{\Omega}'), \tag{4.3.8}$$

$$\mathit{\Theta} = d_{\hat{P}_0}\mathit{\Omega} + \dot{}\mu\dot{}\,(\omega, \mathit{\Omega}) \tag{4.3.9}$$

satisfying certain simplicial constraints and such that

$$R_g{}^*\theta = \mathrm{Ad}\, g^{-1}(\theta), \tag{4.3.10}$$

$$R_{(H,h)}{}^*\varTheta' = \dot\mu\,(h^{-1}, \mathrm{Ad}\, H(\varTheta' + \dot\mu(\hat{s}^*\theta, H))), \tag{4.3.11}$$

$$R_g{}^*\varTheta = \dot\mu\,(g^{-1}, \varTheta). \tag{4.3.12}$$

The following differences between Waldorf's formulation, henceforth marked as W, and the formulation presented in this paper emerge, marked as O, emerge even leaving aside the non synthetic nature of W and the synthetic one of O.

In W, a 2–connection has three components ω, $\varOmega'$, $\varOmega$ whereas, in O, it has only two components ω, $\varOmega$. In W, ω, $\varOmega$ are forms on $\hat{P}_0$, while, in O, ω, $\varOmega$ are forms on P. It is not possible to forget $\varOmega'$ in W because it enters into the simplicial constraints together with ω, nor it is possible to set $\varOmega' = 0$ because this would be inconsistent with (4.3.5). Apparently, the components ω, $\varOmega$ of W correspond to the pull-back components $\omega_0 = I^*\omega$, $\varOmega_0 = I^*\varOmega$ of O (cf. subsect. 3.2). Relations (4.3.4), (4.3.6) of W in infinitesimal form are compatible with relations (3.2.17), (3.2.18) of O under the operation morphism $\mathrm{O{\scriptsize P}}\,L$. Similar remarks apply when comparing the three curvature components θ, $\varTheta'$, $\varTheta$ of W and the two components θ, $\varTheta$ of O. From these remarks, it appears that W is not obviously equivalent to O and most likely it is not. Yet, the two formulations may yield at the end equivalent descriptions of 2–connections on the bundle's base manifold. This remains an issue deserving further investigation.

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