ARCHIVIO ISTITUZIONALE
DELLA RICERCA

Alma Mater Studiorum Università di Bologna
Archivio istituzionale della ricerca

Operational total space theory of principal 2-bundles I: operational geometric framework

This is the final peer-reviewed author's accepted manuscript (postprint) of the following publication:

Published Version:
Roberto Zucchini (2020). Operational total space theory of principal 2-bundles I: operational geometric framework. JOURNAL OF GEOMETRY AND PHYSICS, 156, 1-28 [10.1016/j.geomphys.2020.103826].

Availability:
This version is available at: https://hdl.handle.net/11585/771857 since: 2020-09-16

Published:

DOI: http://doi.org/10.1016/j.geomphys.2020.103826

Terms of use:

Some rights reserved. The terms and conditions for the reuse of this version of the manuscript are specified in the publishing policy. For all terms of use and more information see the publisher's website.

This item was downloaded from IRIS Università di Bologna (https://cris.unibo.it/).
When citing, please refer to the published version.

(Article begins on next page)

07 September 2026

This is the final peer-reviewed accepted manuscript of:

Roberto Zucchini, *Operational total space theory of principal 2-bundles I: Operational geometric framework*, Journal of Geometry and Physics, Volume 156, 2020, 103826.

The final published version is available online at: https://doi.org/10.1016/j.geomphys.2020.103826

Rights / License:

The terms and conditions for the reuse of this version of the manuscript are specified in the publishing policy. For all terms of use and more information see the publisher's website.

This item was downloaded from IRIS Università di Bologna (https://cris.unibo.it/)

When citing, please refer to the published version.

arXiv:1905.10057v2 [math-ph] 29 Jun 2019

Operational total space theory of principal 2–bundles I: operational geometric framework

Roberto Zucchini

Dipartimento di Fisica ed Astronomia,
Università di Bologna,
I.N.F.N., sezione di Bologna,
viale Berti Pichat, 6/2
Bologna, Italy
Email: roberto.zucchini@unibo.it, zucchinir@bo.infn.it

Abstract: It is a classic result that the geometry of the total space of a principal bundle with reference to the action of the bundle's structure group is codified in the bundle's operation, a collection of derivations comprising the de Rham differential and the contraction and Lie derivatives of all vertical vector fields and obeying the six Cartan relations. In particular, connections and gauge transformations can be defined through the way they are acted upon by the operation's derivations. In this paper, the first of a series of two extending the ordinary theory, we construct an operational total space theory of strict principal 2–bundles with regard to the action of the structure strict 2–group. Expressing this latter via a crossed module (E,G), the operation is based on the derived Lie group $\mathfrak{e}[1]\rtimes\mathsf{G}$. In the second paper, an original formulation of the theory of 2–connections and 1– and 2–gauge transformations based on the operational framework worked out here will be provided.

MSC: 81T13 81T45 58A50 58E40 55R65

Contents

1 Introduction **3**

1.1 The operational theory and its scope 4

1.2 Main features of the operational theory of principal 2–bundles 6

2 Operations and principal bundles **9**

2.1 Operations and Lie group spaces: our geometric paradigm 9

2.2 Total space operation of a principal bundle 13

3 Operational total space theory of strict principal 2–bundles **15**

3.1 Total space theory of principal 2–bundles 15

3.2 Synthetic formulation of principal 2–bundle theory 22

3.3 Lie group and algebra crossed modules 28

3.4 Derived Lie groups and algebras 31

3.5 Derived Lie group and algebra valued internal functions 35

3.6 Derived Lie group and algebra cross modality 43

3.7 Lie group crossed module spaces 45

3.8 Total space operations of a principal 2–bundle 47

A Notation and conventions **51**

B Basic identities **54**

1 Introduction

Principal 2–bundles with strict structure 2–group are most often described through appropriate sets of transitional data with respect to an open cover of the base manifold, or cocycles, a framework originally put forward in the foundational works by Schreiber [1] and Baez and Schrieber [2, 3]. This approach points directly to Girauds non Abelian cohomology [4] which provides the associated classifying theory. The first formulation of the total space theory of principal 2–bundles as Lie groupoids was given by Bartels [5] and Baez and Schreiber [2,3], making systematic use of the ideas and techniques of categorification. The total space perspective was further developed by Wockel [6], who also obtained a classification of principal 2–bundles up to Morita equivalence by non Abelian cohomology and provided a categorical characterization of the the gauge 2–group. Schommer-Pries [7] extended Wockel's theory by constructing a bicategory of principal 2–bundles.

Special incarnations of principal 2–bundles are provided by the non Abelian bundle gerbes of Aschieri, Cantini and Jurco [8] and the G–gerbes of Laurent–Gengoux, Stiénon and Xu [9] and Ginot and Stiénon [10]. According to Nikolaus and Waldorf [11], there are indeed three more distinct but equivalent ways of presenting by principal 2–bundles with strict structure 2–group beside the one already recalled: *i*) by smooth non Abelian Čech cocycles of the 2–group; *ii*) by classifying maps to classifying space of the 2–group; *iii*) by 2–group bundle gerbes.

The theory of 2–connections and 1– and 2–gauge transformations on principal 2–bundles has been the object of intense inquiry because of its potential applications in higher gauge theory and string theory. It takes different forms depending on the description of 2–bundles used. We mention in particular the work of Breen and Messing [12], Schreiber [1,13], Baez and Schreiber [2,3] and Jurco, Saemann and Wolf [14, 15] characterizing in various ways connections and gauge transformations in terms of transition data, Aschieri, Cantini and Jurco [8] treating connections and gauge transformations in the framework of bundle gerbes and Laurent–Gengoux, Stiénon and Xu [9] considering connections of G–gerbes. A

theory of 2–connections working in Wockels total space principal 2–bundle theory [6] has been worked out only recently by Waldorf [16, 17].

1.1 The operational theory and its scope

The present endeavours puts forward a proposal for an operational total space theory of principal 2–bundles. We begin by reviewing the operational approach to ordinary principal bundle theory on which the higher extension is modelled.

Let P be a principal bundle P over M with structure group G. P is so a manifold with a free fiberwise transitive right G–action. With any Lie algebra element $x \in \mathfrak{g}$ there is then associated a vertical vector field C_{Px}. It is a classic result [18] that the underlying total space geometry of P is codified in its *operation* $\mathrm{Op}\, S_P$. This is the geometrical structure consisting of the graded algebra $\Omega^\bullet(P)$ of differential forms of P and the collection of graded derivations of $\Omega^\bullet(P)$ comprising the de Rham differential d_P and the contraction and Lie derivatives j_{Px}, l_{Px} of the vector fields C_{Px}. The derivations obey the six Cartan relations,

$$[d_P, d_P] = 0, \tag{1.1.1}$$

$$[d_P, j_{Px}] = l_{Px}, \tag{1.1.2}$$

$$[d_P, l_{Px}] = 0, \tag{1.1.3}$$

$$[j_{Px}, j_{Py}] = 0, \tag{1.1.4}$$

$$[l_{Px}, j_{Py}] = j_{[x,y]}, \tag{1.1.5}$$

$$[l_{Px}, l_{Py}] = l_{[x,y]}. \tag{1.1.6}$$

The differential forms of P annihilated by all derivations j_{Px}, l_{Px} with $x \in \mathfrak{g}$ are called *basic* and form a subalgebra $\Omega^\bullet{}_{\mathrm{b}}(P)$ of $\Omega^\bullet(P)$. $\Omega^\bullet{}_{\mathrm{b}}(P)$ is in one–to–one correspondence and thus can be identified with the differential form algebra $\Omega^\bullet(M)$ on M.

A connections ω can be characterized as a $\mathfrak{g}$–valued 1–form of P acted upon in a prescribed way by the operation's derivations. Similarly, a gauge transformation g can be defined as a G–valued map behaving in a certain way under the action

of the derivations. Gauge transformations form an infinite dimensional Lie group with a left action on the space of connections.

On any sufficiently small neighborhood U of M, on which the G–bundle P is isomorphic to the trivial G–bundle $U \times \mathsf{G}$, a connection ω is completely characterized by a basic $\mathfrak{g}$–valued 1–form ω_{b} on U. Similarly, a gauge transformation g is characterized by a basic G–valued map g_{b} on U. Local connection and gauge transformation data relative to distinct overlapping trivializing neighborhoods of M match in a prescribed way codified in a G–valued Čech 1–cocycle. All these properties are described by the *basic* formulation of principal bundle geometry, which is the one implicitly used in most physical literature.

The operational approach to principal bundle theory admits an elegant reformulation in the language of *graded differential geometry*. For a principal G–bundle P, the graded differential form algebra $\Omega^\bullet(P)$ is described as the graded function algebra $\mathrm{Fun}(T[1]P)$ of the shifted tangent bundle $T[1]P$ of P and the operation derivations d_P, j_{Px}, l_{Px}, $x \in \mathfrak{g}$, as graded vector fields on $T[1]P$. Connection and gauge transformations can then be defined as earlier as degree 1 $\mathfrak{g}$– and degree 0 G–valued functions on $T[1]P$ behaving in a prescribed way under the action of d_P, j_{Px}, l_{Px}. Gauge transformation of connections can be implemented in the familiar way. The basic formulation goes through essentially with no changes.

Our investigation has shown that an analogous graded geometric operational total space formulation of principal 2–bundle theory can be worked out. The results of our research are expounded in a series of two papers. In the first paper, referred to as I, we set the foundations of the operational total space theory of principal 2–bundles. In the second paper, referred to as II [19], based on the operational setup worked out in I, we provide an original formulation of the theory of 2–connections and 1– and 2–gauge transformations of strict principal 2–bundles.

To make the nature of our work more easily appreciable by the reader, we now outline briefly the intuitive ideas underlying the higher extension. The ordinary theory is the model against which the higher one is built. In a nutshell, the higher theory can be outlined as follows. With the 2–bundle's structure 2–group,

there is associated a derived graded group. The structure 2-group's action on the 2–bundle is then shown to induce a derived group's action on a synthetic form of the 2–bundle. With the latter, there is associated an operation, that suitably describes the 2–bundle's geometry and in particular 2–connections and 1– and 2–gauge transformations.

The passage from the ordinary to the higher theory is not free of subtleties. The graded nature of the derived group makes it necessary to further reshape the ordinary theory before constructing the higher one by analogy. For a principal bundle P, the ordinary graded function algebra $\mathrm{Fun}(T[1]P)$ must be replaced by the *internal* graded function algebra $\mathrm{Fun}(T[1]P)$. While $\mathrm{Fun}(T[1]P)$ has a single grading, the shifted tangent bundle grading of $T[1]P$, $\mathrm{Fun}(T[1]P)$ has two, the tangent bundle grading and a further internal grading. $\mathrm{Fun}(T[1]P)$ can therefore be identified with the subalgebra of $\mathrm{Fun}(T[1]P)$ of zero internal grading.

Extending the function algebra from $\mathrm{Fun}(T[1]P)$ to $\mathrm{Fun}(T[1]P)$ as indicated introduces in the original operational framework internal multiplicities. In the familiar formulation of gauge theory, this would correspond to add the ghost degree to the form one. It endows in this way connections and gauge transformations with ghostlike partners making the whole geometrical framework akin to that employed in the AKSZ formulation of BV theory [20] (see also [21]). In ordinary principal bundle geometry doing so is discretionary. In higher principal bundle theory, as it turns out, it is unavoidable.

1.2 Main features of the operational theory of principal 2–bundles

In this paper, I of the series, we build an operation describing the fibered geometry of a given principal 2–bundle. The attendant geometric framework will constitute the backdrop for the 2–connection and 1– and 2–gauge transformation theory of the companion paper, II of the series. The formulation proposed, which is non trivial in many respects, is outlined in this subsection.

The operational framework (cf. subsect. 2.1) is the paradigm on which the whole architecture of the present work rests. It is by design the most appropriate approach for the study of the total space differential geometry of ordinary prin-

cipal bundles (cf. subsect. 2.2) and, once suitably adapted, has the potential of shedding light into that of principal 2–bundles as argued below.

A principal 2–bundle consists of a morphism manifold $\hat{P}$ with an object submanifold $\hat{P}_0$ constituting a groupoid, a base manifold M, compatible projection maps $\hat{\pi} : \hat{P} \to M$ and $\hat{\pi}_0 : \hat{P}_0 \to M$ forming a functor, a morphism group $\hat{\mathsf{K}}$ with an object subgroup $\hat{\mathsf{K}}_0$ organized as a strict Lie 2-group and compatible right actions $\hat{R} : \hat{P} \times \hat{\mathsf{K}} \to \hat{P}$ and $\hat{R}_0 : \hat{P}_0 \times \hat{\mathsf{K}}_0 \to \hat{P}_0$ building a functor and respecting $\hat{\pi}$ and $\hat{\pi}_0$. The 2–bundle enjoys further the property of locally trivializability, that is on any small enough neighborhood U of M the groupoid $(\hat{P}|_U, \hat{P}_0|_U)$ is equivariantly projection preservingly equivalent to the groupoid $(U \times \hat{\mathsf{K}}, U \times \hat{\mathsf{K}}_0)$ with the obvious projection and right action structures.

By virtue of the right $\hat{\mathsf{K}}$– and $\hat{\mathsf{K}}_0$–actions on the manifolds $\hat{P}$ and $\hat{P}_0$, there are two operations associated with any given principal 2–bundle, which we could include in an operational framework. However, proceeding in this naive way would not allow us to make contact with the standard formulation of strict higher gauge theory. To achieve this end, we have to make a further step: shifting to what we call the synthetic formulation for its formal affinity to smooth infinitesimal analysis of synthetic differential geometry.

In the synthetic approach (cf. subsects. 3.2–3.8), one adjoins to the given principal 2–bundle a synthetic structure consisting of morphism and object manifolds P and P_0, the base manifold M, projection maps π and π_0, morphism and object groups K and K_0 and right K– and K_0– actions R and R_0 on P and P_0. The synthetic setup is obtained from the original non synthetic one through the following formal construction. Describe the strict Lie 2–group $(\hat{\mathsf{K}}, \hat{\mathsf{K}}_0)$ by its associated Lie group crossed module (E, G) so that $\hat{\mathsf{K}} = \mathsf{E} \rtimes \mathsf{G}$ and $\hat{\mathsf{K}}_0 = \mathsf{G}$. Then, $\mathsf{K} = \mathfrak{e}[1] \rtimes \mathsf{G}$ and $\mathsf{K}_0 = \mathsf{G}$. Next, formally extend the $\hat{\mathsf{K}}$–action $\hat{R}$ to a K–action. Then, P is the K–action image of $\hat{P}_0$ and $P_0 = \hat{P}_0$, R is the restriction of $\hat{R}$ to P and $R_0 = \hat{R}$. Here, K and P must be regarded as spaces of functions from $\mathbb{R}[-1]$ to $\mathsf{E} \rtimes \mathsf{G}$ and P, respectively. We remark that, although the synthetic structure shares many of the properties of the underlying principal 2–bundle, it is not one since neither pairs $(\mathsf{K}, \mathsf{K}_0)$ and (P, P_0) have a groupoid structure.

There are two operations $\mathrm{Op}\, S_P$ and $\mathrm{Op}\, S_{P0}$ which codify the K– and K_0–actions R and R_0 of P and P_0. By their synthetic nature, as it will be shown in II, they have the right properties for making the desired contact with strict higher gauge theory. We describe them in greater detail next.

The Lie group $\mathrm{D}\mathsf{M} = \mathfrak{e}[1] \rtimes \mathsf{G}$ and its subgroup $\mathrm{D}\mathsf{M}_0 = \mathsf{G}$ encountered above as well as the Lie algebras $\mathrm{D}\mathfrak{m} = \mathfrak{e}[1] \rtimes \mathfrak{g}$ and its subalgebra $\mathrm{D}\mathfrak{m}_0 = \mathfrak{g}$ are called derived and play a fundamental role in the architecture of our construction. As partly anticipated, $\mathrm{D}\mathsf{M}$ may be thought of as the Lie group of internal functions $\mathrm{e}^{\bar{\alpha} L}\, a$, $\bar{\alpha} \in \mathbb{R}[-1]$, with $a \in \mathsf{G}$ and $L \in \mathfrak{e}[1]$. Similarly, $\mathrm{D}\mathfrak{m}$ may be thought of as the Lie algebra of internal functions $x + \bar{\alpha} X$, $\bar{\alpha} \in \mathbb{R}[-1]$, with $x \in \mathfrak{g}$ and $X \in \mathfrak{e}[1]$. Note however that, in the spirit of the superfield formulation of supersymmetric field theories, here $\bar{\alpha}$ is simply a book–keeping parameter and not an extra variable injected in the theory and allowed to vary in its full range.

The synthetic morphism operation $\mathrm{Op}\, S_P$ consists of the internal function algebra $\mathrm{Fun}(T[1]P)$ acted upon the de Rham vector field d_P and the contraction and Lie vector fields j_{PZ}, l_{PZ}, $Z \in \mathrm{D}\mathfrak{m}$. In this case, it is necessary and not just merely optional to use the internal function algebra rather than the ordinary one, because the latter would not be preserved by the j_{PZ}, l_{PZ} due to the non trivial gradation of $\mathrm{D}\mathfrak{m}$.

The synthetic object operation $\mathrm{Op}\, S_{P0}$ has a similar constitution. The underlying function algebra is the internal algebra $\mathrm{Fun}(T[1]P_0)$ and the operation vector fields are the de Rham vector field d_{P_0} and the contraction and Lie vector fields $j_{P_0 Z_0}$, $l_{P_0 Z_0}$, $Z_0 \in \mathrm{D}\mathfrak{m}_0$. In this case, due to the trivial gradation of $\mathrm{D}\mathfrak{m}_0$, we could have restricted ourselves to the ordinary algebra, but we decided to opt for the internal one to allow for a simple relation to the operation $\mathrm{Op}\, S_P$. We remark that $\mathrm{Op}\, S_{P0}$ plays a subordinate role as compared to $\mathrm{Op}\, S_P$ and serves mainly the purpose of elucidating how the ordinary theory is extended by the higher one in our operational setup.

With our principal 2–bundle operational setup in place, the ground is set for the operational theory of 2–connections and 1– and 2–gauge transformations presented in II.

2 Operations and principal bundles

The operational framework is the paradigm on which the whole architecture of the present endeavour rests. It is indeed a most adequate approach for the study of the total space differential geometry of ordinary principal bundles and for this reason, once suitably adapted, holds the potential for providing new insights into that of principal 2–bundles. In this section, we review it and its application to ordinary principal bundle theory to set notation and terminology and to make the more technically involved applications of it to principal 2–bundle theory more easily accessible.

In what follows, all algebras considered will be tacitly assumed to be graded commutative, unital, associative and real unless otherwise stated.

2.1 Operations and Lie group spaces: our geometric paradigm

In this subsection, we review the main definitions and results of operation and Lie group space theory. For a comprehensive treatment, we refer the reader to [18]. We begin by introducing operations.

Definition **2.1.** *An operation O consists of a graded commutative algebra A and a Lie algebra $\mathfrak{g}$ together with a derivation d of A of degree 1 and for each element $X \in \mathfrak{g}$ two derivations j_X, l_X of A of degree -1, 0, respectively, satisfying the six Cartan relations*

$$[d, d] = 0, \tag{2.1.1}$$

$$[d, j_X] = l_X, \tag{2.1.2}$$

$$[d, l_X] = 0, \tag{2.1.3}$$

$$[j_X, j_Y] = 0, \tag{2.1.4}$$

$$[l_X, j_Y] = j_{[X,Y]}, \tag{2.1.5}$$

$$[l_X, l_Y] = l_{[X,Y]} \tag{2.1.6}$$

for $X, Y \in \mathfrak{g}$, where all commutators are graded.

We shall denote the operation O by the list $(A, \mathfrak{g}, d, j, l)$ of its constituent data, or simply $(A, \mathfrak{g})$ when no confusion is possible, and say that O is an operation of $\mathfrak{g}$ on A.

Definition **2.2.** *A morphism $\chi : O \to O'$ of operations consists of a graded commutative algebra morphism $\phi : A \to A'$ and a Lie algebra morphism $h : \mathfrak{g}' \to \mathfrak{g}$ such that the relations*

$$d'\phi = \phi\, d, \tag{2.1.7}$$

$$j'_X\phi = \phi\, j_{h(X)}, \tag{2.1.8}$$

$$l'_X\phi = \phi\, l_{h(X)} \tag{2.1.9}$$

are obeyed for all $X \in \mathfrak{g}'$.

We shall denote an operation morphisms $\chi : O \to O'$ by the list $(\phi : A \to A', h : \mathfrak{g}' \to \mathfrak{g})$ of its constituent data and shall omit the specification of sources and targets if no confusion can arise.

Operations and operation morphisms thereof constitute a category **Op**.

Assume that an operation morphism $\chi : O \to O'$ given by the pair $(\phi : A \to A', h : \mathfrak{g}' \to \mathfrak{g})$ is such that $A = A'$ and $\phi = \mathrm{id}_A$. We then have

$$d' = d, \tag{2.1.10}$$

$$j'_X = j_{h(X)}, \tag{2.1.11}$$

$$l'_X = l_{h(X)} \tag{2.1.12}$$

for $X \in \mathfrak{g}'$ by (2.1.7)–(2.1.9). In this case, the morphism χ is fully specified by the underlying Lie algebra morphism h. We then denote O' by h^*O and call it the pull-back of O by h.

Let $O = (A, \mathfrak{g}, d, j, l)$ be an operation. Since by (2.1.1) $d^2 = 0$, (A, d) is a cochain complex.

Definition **2.3.** *The cohomology of O is cohomology of (A, d).*

(A, d) contains a distinguished subcomplex (A_{basic}, d), where A_{basic}, the basic sub-

algebra of A, consists of all horizontal and invariant elements $a \in A$,

$$j_X a = 0, \tag{2.1.13}$$

$$l_X a = 0 \tag{2.1.14}$$

for $X \in \mathfrak{g}$.

Definition **2.4.** *The basic cohomology of O is the cohomology of (A_{basic}, d).*

Next, we introduce Lie group spaces.

Definition **2.5.** *A Lie group space S is a graded manifold P carrying a right action $R : P \times \mathsf{G} \to P$ of a Lie group G.*

We shall denote the space S through the list (P, G, R) of its defining data.

Definition **2.6.** *A morphism $T : S' \to S$ of Lie group spaces consists of a map $F : P' \to P$ of graded manifolds and a Lie group morphism $\eta : \mathsf{G}' \to \mathsf{G}$ such that for $a \in \mathsf{G}'$ one has*

$$R_{\eta(a)} \circ F = F \circ R'_a. \tag{2.1.15}$$

We shall denote a Lie group space morphism $T : S' \to S$ employing the list $(F : P' \to P,\ \eta : \mathsf{G}' \to \mathsf{G})$ of its constituent data and shall omit the specification of sources and targets if no confusion is possible.

Lie group spaces and their morphisms form a category **Lsp**.

Consider a Lie group space S. For each $X \in \mathfrak{g}$, the Lie algebra of G, the vertical vector field V_X of the action R associated with X is defined. The internal function algebra $\textsc{Fun}(T[1]P)$ (cf. app. A) of the shifted tangent bundle $T[1]P$ of P is acted upon by the de Rham derivation d_P of degree 1 and for $X \in \mathfrak{g}$ by the contraction and Lie derivations j_{PX}, l_{PX} of V_X of degree -1, 0, all realized as graded vector fields on $T[1]P$.

Proposition **2.1.** *With any Lie group space $S = (P, \mathsf{G}, R)$ there is associated the operation $\textsc{Op}\, S = (\textsc{Fun}(T[1]P), \mathfrak{g}, d_P, j_P, l_P)$.*

The reason why the internal function algebra $\textsc{Fun}(T[1]P)$ is used instead than

the ordinary algebra $\mathrm{Fun}(T[1]P)$ is that for the Lie groups spaces considered in this paper $\mathrm{Fun}(T[1]P)$ is not closed under all operation derivations j_{PX}, l_{PX} while $\textsc{Fun}(T[1]P)$ is. For the standard Lie group spaces encountered in geometry, $\mathrm{Fun}(T[1]P)$ is closed and thus it is possible and indeed customary to use it to construct the operation $\mathrm{Op}S = (\mathrm{Fun}(T[1]P), \mathfrak{g}, d_P, j_P, l_P)$. In what follows, we shall consider mostly the operation $\textsc{Op}\,S$.

Similarly, morphisms of Lie group spaces induce morphisms of the associated operations by the equivariance condition (2.1.15).

Proposition **2.2.** *Let $T : S' \to S$ be a morphism of Lie group spaces specified by the pair $(F : P' \to P, \eta : \mathsf{G}' \to \mathsf{G})$. Then, T induces a morphism $\textsc{Op}\,T : \textsc{Op}\,S \to \textsc{Op}\,S'$ of operations specified by the pair $(F^* : \textsc{Fun}(T[1]P) \to \textsc{Fun}(T[1]P'), h : \mathfrak{g}' \to \mathfrak{g})$, where F^* is the pull-back of function spaces by F and h is the Lie algebra morphism associated with η by Lie differentiation.*

The map $\textsc{Op}$ that associates to each space its operation and to each morphism of spaces its morphism of operations is a functor from the space category $\mathbf{Lsp}^{\mathrm{op}}$ into the operation category $\mathbf{Op}$.

Consider the special case of a morphism $T : S' \to S$ of Lie group spaces specified by a pair $(F : P' \to P, \eta : \mathsf{G}' \to \mathsf{G})$, where $P = P'$ as graded manifolds and $F = \mathrm{id}_P$ as a graded manifold map. Then, by (2.1.15)

$$R'_a = R_{\eta(a)} \tag{2.1.16}$$

with $a \in \mathsf{G}'$. We then denote S' as $\eta^* S$ and call it the pull-back of S by η. Furthermore, as the graded algebra morphism $F^* = \mathrm{id}_{\mathrm{Fun}(T[1]P)}$, the operation $\textsc{Op}\,S'$ is just the pull–back $h^*\,\textsc{Op}\,S$ of the operation $\textsc{Op}\,S$ by the Lie algebra morphism $h : \mathfrak{g}' \to \mathfrak{g}$ associated with the Lie group morphism η by Lie differentiation. Note that by construction

$$\textsc{Op}\,\eta^* S = h^*\,\textsc{Op}\,S. \tag{2.1.17}$$

For a Lie group space $S = (P, \mathsf{G}, R)$, the cochain complex $(\mathrm{Fun}(T[1]P), d_P)$ is as well–known isomorphic to the de Rham complex $(\Omega^\bullet(P), d_{dR\,P})$. When

the action R is free, and so the quotient P/G is a manifold, the basic complex $(\mathrm{Fun}(T[1]P)_{\mathrm{basic}}, d_P)$ is isomorphic to the de Rham complex $(\Omega^\bullet(P/\mathsf{G}), d_{dR\,P/\mathsf{G}})$. Note however that $(\mathrm{Fun}(T[1]P)_{\mathrm{basic}}, d_P)$ is defined even when R is not free and this identification strictly speaking fails to hold making it possible by way of generalization to meaningfully interpret it in a weaker sense as the complex $(\Omega^\bullet(P/\mathsf{G}), d_{dR\,P/\mathsf{G}})$.

2.2 Total space operation of a principal bundle

The total space theory of principal bundles, surveyed in this subsection, is concerned with the more geometrically intuitive features of principal bundles, in particular its fibered structure and structure group action. Our expositioin, which follows mainly [18], is organized in such a way to render the naturalness of the operational formulation apparent.

Let G be a Lie group.

Definition **2.7.** *A principal G–bundle P consists of a manifold P, a further manifold M, a surjective submersion $\pi : P \to M$ and a right G–action R on P acting freely and transitively on the fibers of π.*

P, G, M and π are called the total space, the structure group, the base and the projection of the bundle, respectively.

In a principal G–bundle P, each fiber of P is diffeomorphic to G, the orbits of the action R are the fibers themselves and the orbit space P/G is diffeomorphic to the base manifold M. Further, the vertical subbundle $V_P = \ker T\pi$ of the tangent bundle TP of P, where $T\pi : TP \to TM$ is the tangent map of the bundle's projection π, is trivial.

A connection of P is a G–invariant distribution $H \subset TP$ that is pointwise transverse to V_P. The connection is said to be flat, if the distribution H is integrable. Connections of P form an affine space $\mathcal{A}_P$.

A gauge transformation of P is a G–equivariant fiber preserving diffeomorphism Φ of P. Gauge transformations form a group $\mathrm{Aut}^G_\pi(P)$ under composition, called the gauge group. $\mathrm{Aut}^G_\pi(P)$ turns out to be isomorphic to the group

$\mathrm{Fun}^G(P, \mathsf{G}_{\mathrm{Ad}})$ of G–equivariant maps of P into G_{Ad}, where G_{Ad} is G with the conjugation right action.

If H is a connection and Φ a gauge transformation of P, then ${\Phi^{-1}}_* H$ is also a connection of P since the equivariance of Φ renders ${\Phi^{-1}}_* H$ a G–invariant distribution in TP. In this way, a left action $(\Phi, H) \to {}^{\Phi}H = {\Phi^{-1}}_* H$ of the gauge group $\mathrm{Aut}^G_\pi(P)$ on the connection space $\mathcal{A}_P$ is defined.

With a principal G–bundle P there is associated a Lie group space $S_P = (P, \mathsf{G}, R)$ and through this an operation $\mathrm{Op}\, S_P = (\mathrm{Fun}(T[1]P), \mathfrak{g}, d_P, j_P, l_P)$. $\mathrm{Op}\, S_P$ furnishes a very elegant and natural differential geometric framework for the study of connections and gauge transformations of P, since these are essential defined by the way they behave under the G–action of P. We shall review the resulting theory in greater detail in sect. 2 of II. Here, in I, we shall use the operational theory of principal bundles just outlined as a prototype for the corresponding theory of strict principal 2–bundles introduced and studied in next section.

3 Operational total space theory of strict principal 2–bundles

In this section, using as a model the operational total space theory of ordinary principal bundles reviewed in sect. 2, we present an operation based formulation of the total space theory of strict principal 2–bundles. The construction expounded below relies in an essential way on the description of the bundle's structure 2–group as a Lie group crossed module and involves a synthetic recasting of the categorical formulation of the theory that leads one into the realm of graded differential geometry. The end result is a compact, elegant formulation closely related to the superfield formalism employed in many areas of quantum field theory.

3.1 Total space theory of principal 2–bundles

Since the goal we set ourselves is generalizing the operation based theory of ordinary principal bundles to strict principal 2–bundles, a suitable total space theory of such 2–bundles is required. This has been worked out by Bartels [5] and Wockel [6] (see also [11]) systematically categorifying the standard notion of principal bundle. In this subsection, we preset a review of this topic based mainly on [6]. Here, we claim neither originality nor completeness and mathematical rigour and shall restrict ourselves to provide those basic notions which are required to justify the constructions presented in this paper.

Definition **3.1.** *A strict Lie 2–group $\hat{\mathscr{K}}$ is a group object in the category* **DiffCat** *of smooth categories. $\hat{\mathscr{K}}$ consists thus of the following data.*

1. *A Lie groupoid $\hat{\mathsf{K}}$.*
2. *A smooth multiplication functor $\hat{\varkappa} : \hat{\mathsf{K}} \times \hat{\mathsf{K}} \to \hat{\mathsf{K}}$.*
3. *A smooth inversion functor $\hat{\imath} : \hat{\mathsf{K}} \to \hat{\mathsf{K}}$.*
4. *A distinguished object $\hat{1}$.*

$\hat{\varkappa}$, $\hat{\imath}$ and $\hat{1}$ obey further the usual axioms of group theory at the functor level.

Under mild assumptions, a strict Lie 2–group $\hat{\mathscr{K}}$ can be described equivalently by the following set of data.

1. A Lie group $\hat{\mathsf{K}}$.
2. A Lie subgroup $\hat{\mathsf{K}}_0$ of $\hat{\mathsf{K}}$.
3. A Lie groupoid structure $\hat{\mathsf{K}} \rightrightarrows \hat{\mathsf{K}}_0$ all of whose structure maps are Lie group morphisms with the identity assigning map $\hat{\mathsf{K}}_0 \longrightarrow \hat{\mathsf{K}}$ being the inclusion map $\hat{\mathsf{K}}_0 \subset \hat{\mathsf{K}}$.

Here, we describe the groupoid $\hat{\mathsf{K}}$ underlying $\hat{\mathscr{K}}$ through its morphism and object manifolds $\hat{\mathsf{K}}$ and $\hat{\mathsf{K}}_0$. With a convenient and harmless abuse of notation, we denote in the same way both the groupoid and its morphism manifold, since this latter obviously by itself supports the whole 2–group structure. For this reason, we shall often identify $\hat{\mathscr{K}}$ through the group $\hat{\mathsf{K}}$ or more explicitly the group pair $\hat{\mathsf{K}}$, $\hat{\mathsf{K}}_0$. Whenever necessary, we shall mention the relevant structure maps.

Definition **3.2.** *A strict principal $\hat{\mathscr{K}}$–2–bundle $\mathcal{P}$ is a principal bundle object in the category* **DiffCat** *of smooth categories. $\mathcal{P}$ consists thus of the following data.*

1. *A Lie groupoid $\hat{P}$.*
2. *A discrete smooth category M.*
3. *A surjective submersion smooth projection functor $\hat{\pi} : \hat{P} \to M$.*
4. *A strict Lie 2–group $\hat{\mathsf{K}}$.*
5. *A smooth right $\hat{\mathsf{K}}$–action functor $\hat{R} : \hat{P} \times \hat{\mathsf{K}} \to \hat{P}$ such that*

$$\hat{\pi} \circ \hat{R} = \hat{\pi} \circ \mathrm{pr}_1 \tag{3.1.1}$$

   strictly on the nose.

6. *For each small enough open neighborhood $U \subset M$, two reciprocally weakly inverse $\hat{\mathsf{K}}$–equivariant trivializing smooth functors $\hat{\Phi}_U : \hat{\pi}^{-1}(U) \to U \times \hat{\mathsf{K}}$ and $\tilde{\hat{\Phi}}_U : U \times \hat{\mathsf{K}} \to \hat{\pi}^{-1}(U)$ such that*

$$\mathrm{pr}_U \circ \hat{\Phi}_U = \hat{\pi}\big|_{\hat{\pi}^{-1}(U)}, \tag{3.1.2}$$

$$\hat{\pi} \circ \tilde{\hat{\Phi}}_U = \mathrm{pr}_U \tag{3.1.3}$$

strictly on the nose, where $U \times \hat{\mathsf{K}}$ *is endowed with the right* $\hat{\mathsf{K}}$*–action functor given by right* $\hat{\mathsf{K}}$*–multiplication on the factor* $\hat{\mathsf{K}}$*.*

The adjective 'semistrict' is used instead of 'strict' in [6].

We can unpack the above definition and analyze a strict principal $\hat{\mathscr{K}}$–2–bundle $\hat{\mathcal{P}}$ in more explicit terms under mild assumptions. Though this is a straightforward task, we decided to review it in some detail to make the reading of the rest of the paper easier.

The groupoid structure of $\hat{\mathcal{P}}$ is described by the following elements.

1. A smooth manifold $\hat{P}$.

2. A submanifold $\hat{P}_0$ of $\hat{P}$.

3. A Lie groupoid structure $\hat{P} \rightrightarrows \hat{P}_0$ whose identity assigning map $\hat{P}_0 \longrightarrow \hat{P}$ is the inclusion map $\hat{P}_0 \subset \hat{P}$.

Here, analogously to the way we did for 2–groups, we characterize the groupoid $\hat{P}$ underlying $\hat{\mathcal{P}}$ through its morphism and object manifolds $\hat{P}$ and $\hat{P}_0$, identifying further the groupoid the former. The base structure of $\hat{\mathcal{P}}$ is given by an ordinary manifold, adding a further element.

4. A manifold M.

We reduce so the discrete smooth category M underlying $\hat{\mathcal{P}}$, which has only identity morphisms, to its object manifold, which we also denote as M. The projection structure of $\hat{\mathcal{P}}$ is consequently given by an ordinary map.

5. A smooth surjective submersion projection map $\hat{\pi} : \hat{P} \to M$.

In line with the conventions we are outlining, here we describe the projection functor $\hat{\pi}$ of $\hat{\mathcal{P}}$ through its action on the morphism manifold $\hat{P}$ also denoted as $\hat{\pi}$. $\hat{\pi}$ restricts to a smooth surjective submersion projection map $\hat{\pi}_0 : \hat{P}_0 \to M$

giving the functor's action on the object manifold $\hat{P}_0$. By functoriality, $\hat{\pi}$ and $\hat{\pi}_0$ satisfy certain relations. We mention only that

$$\hat{\pi} = \hat{\pi}_0 \circ \hat{s} = \hat{\pi}_0 \circ \hat{t}, \tag{3.1.4}$$

where $\hat{s}$, $\hat{t}$ are the source and target map of the groupoid $\hat{P}$. The structure 2–group of $\hat{\mathcal{P}}$ is analyzed next as follows.

6. A Lie group $\hat{\mathsf{K}}$.

7. A Lie subgroup $\hat{\mathsf{K}}_0$ of $\hat{\mathsf{K}}$.

8. A Lie 2–group structure $\hat{\mathsf{K}} \rightrightarrows \hat{\mathsf{K}}_0$

   whose identity assigning map $\hat{\mathsf{K}}_0 \longrightarrow \hat{\mathsf{K}}$ is the inclusion map $\hat{\mathsf{K}}_0 \subset \hat{\mathsf{K}}$.

We explained the reason for this earlier. The right $\hat{\mathsf{K}}$–action structure of $\hat{\mathcal{P}}$ is given in terms of a further element.

9. A smooth right $\hat{\mathsf{K}}$–action map $\hat{R} : \hat{P} \times \hat{\mathsf{K}} \to \hat{P}$ such that

$$\hat{\pi} \circ \hat{R} = \hat{\pi} \circ \mathrm{pr}_1 \tag{3.1.5}$$

(cf. eq. (3.1.1)). Again, we describe the $\hat{\mathsf{K}}$–action functor $\hat{R}$ of $\hat{\mathcal{P}}$ through its action on the morphism manifold $\hat{P}$, which we also denote as $\hat{R}$. $\hat{R}$ restricts to a smooth right $\hat{\mathsf{K}}_0$–action map $\hat{R}_0 : \hat{P}_0 \times \hat{\mathsf{K}}_0 \to \hat{P}_0$ giving the functor's action on the object manifold $\hat{P}_0$. $\hat{R}$ and $\hat{R}_0$ obey certain relations by virtue of functoriality. Finally, by the property of local trivializability, a full collection of local trivializing maps is available.

10. For any small open neighborhood U of M, two trivializing $\hat{\mathsf{K}}$–equivariant smooth maps $\hat{\Phi}_U : \hat{\pi}^{-1}(U) \to U \times \hat{\mathsf{K}}$ and $\tilde{\hat{\Phi}}_U : U \times \hat{\mathsf{K}} \to \hat{\pi}^{-1}(U)$ such that

$$\mathrm{pr}_U \circ \hat{\Phi}_U = \hat{\pi}\big|_{\hat{\pi}^{-1}(U)}, \tag{3.1.6}$$

$$\hat{\pi} \circ \tilde{\hat{\Phi}}_U = \mathrm{pr}_U \tag{3.1.7}$$

(cf. eqs. (3.1.2), (3.1.3)). Analogously to before, we describe the trivializing functors $\hat{\Phi}_U$ and $\tilde{\hat{\Phi}}_U$ of $\hat{\mathcal{P}}$ on U through their actions on the morphism manifold

$\hat{P}$ denoted as $\hat{\Phi}_U$ and $\tilde{\hat{\Phi}}_U$ too. $\hat{\Phi}_U$ and $\tilde{\hat{\Phi}}_U$ restrict to trivializing $\hat{\mathsf{K}}_0$–equivariant smooth maps $\hat{\Phi}_{U0} : {\hat{\pi}_0}^{-1}(U) \to U \times \hat{\mathsf{K}}_0$ and $\tilde{\hat{\Phi}}_{U0} : U \times \hat{\mathsf{K}}_0 \to {\hat{\pi}_0}^{-1}(U)$ giving the functors' actions on the object manifolds ${\hat{\pi}_0}^{-1}(U)$ and $U \times \hat{\mathsf{K}}_0$. $\hat{\Phi}_U$ and $\tilde{\hat{\Phi}}_U$ and $\hat{\Phi}_{U0}$ and $\tilde{\hat{\Phi}}_{U0}$ obey a large set of relations stemming from functoriality and reciprocal weak invertibility. Further, there are a $\hat{\mathsf{K}}$–equivariant map $\hat{T}_U : \hat{\pi}^{-1}(U) \to \hat{\mathsf{K}}$, where $\hat{\mathsf{K}}$ is endowed with the right multiplication action, such that

$$\hat{\Phi}_U(X) = (\hat{\pi}(X), \hat{T}_U(X)) \tag{3.1.8}$$

for $X \in \hat{\pi}^{-1}(U)$ and for each $m \in U$ a $\hat{\mathsf{K}}$–equivariant map $\hat{X}_{Um} : \hat{\mathsf{K}} \to \hat{\pi}^{-1}(U)$ with the property that

$$\tilde{\hat{\Phi}}_U(m, A) = \hat{X}_{Um}(A) \tag{3.1.9}$$

for $A \in \hat{\mathsf{K}}$ obeying many further relations deriving from $\hat{\Phi}_U$ and $\tilde{\hat{\Phi}}_U$ being weakly inverse functors. The associated $\hat{\mathsf{K}}_0$–equivariant maps $\hat{T}_{U0} : {\hat{\pi}_0}^{-1}(U) \to \hat{\mathsf{K}}_0$ and $\hat{X}_{U0m} : \hat{\mathsf{K}}_0 \to {\hat{\pi}_0}^{-1}(U)$ are again restrictions.

The fact that the local trivializing functors $\hat{\Phi}_U$ and $\tilde{\hat{\Phi}}_U$ of a strict principal 2–bundle $\hat{\mathcal{P}}$ are only reciprocally weakly inverse makes $\hat{\mathcal{P}}$ unlike an ordinary principal bundle. In general, $\hat{\Phi}_U$ is not injective and one can use it to coordinatize $\hat{\pi}^{-1}(U)$ only in a weaker sense than in the ordinary theory. Likewise, $\tilde{\hat{\Phi}}_U$ is generally non surjective and one can use it to parametrize $\hat{\pi}^{-1}(U)$ only up to suitable isomorphism.

Because of the above properties, the $\hat{\mathsf{K}}$–action $\hat{R}$ on $\hat{P}$ is fiberwise free but fiberwise transitive only up to appropriate isomorphism. Furthermore, a $\hat{\mathsf{K}}$–invariant map $f : \hat{P} \to \mathbb{R}$ cannot be identified with a map $\bar{f} : M \to \mathbb{R}$ in general. However, as locally on a neighborhood U $\hat{\pi}^{-1}(U)$ is $\hat{\mathsf{K}}$–equivariantly equivalent to $U \times \hat{\mathsf{K}}$, which is an ordinary principal $\hat{\mathsf{K}}$–bundle, a $\hat{\mathsf{K}}$–invariant map $f : \hat{\pi}^{-1}(U) \to \mathbb{R}$ can be described as a map $\bar{f}_U : U \to \mathbb{R}$. However, for fixed U, $\bar{f}_U$ is built through and generally depends on the underlying trivializing functor $\tilde{\hat{\Phi}}_U$ and is not therefore uniquely associated to f. With the appropriate notion of horizontality, similar statements hold also for differential forms.

To the best of our knowledge, there does not exist at the moment any defini-

tion of connection on a strict principal 2–bundle akin to that of ordinary principal bundle theory formulated in terms of a horizontal invariant distribution in the tangent bundle of the bundle. There exits however a definition of gauge transformation analogous to that of the ordinary theory as an equivariant fiber preserving bundle automorphism.

A weak 2–group $\mathscr{W}$ is a group object in the category **Cat** of categories, that is a groupoid W endowed with a multiplication functor $\varkappa : \mathsf{W} \times \mathsf{W} \to \mathsf{W}$, an inversion functor $\iota : \mathsf{W} \to \mathsf{W}$ and a distinguished object 1 obeying the usual axioms of group theory at the functor level up to natural isomorphisms satisfying suitable coherence conditions. Strict 2–groups are those weak 2–groups for which all these natural isomorphisms are identities. Replacing the category **Cat** with **DiffCat**, one defines similarly the notion of weak Lie 2–group and strict Lie 2–group, the latter of which was introduced earlier.

A given category C is characterized by the weak 2–group $\mathrm{Aut}(C)$ whose objects are the weakly invertible functors $F : C \to C$, whose morphisms are the natural isomorphisms $\beta : F \Rightarrow G$ and whose 2–group structure is as follows. The group structure of $\mathrm{Aut}(C)_0$ is given by functor composition and weak inversion, the group structure of $\mathrm{Aut}(C)_1$ is given by the so called Godement composition and inversion of natural isomorphisms, the groupoid structure of $\mathrm{Aut}(C)_1$ consists in the usual composition and inversion of natural isomorphisms.

There is a notion of gauge 2–group of a principal 2–bundle extending the familiar notion of gauge group of the ordinary theory. The objects and morphisms of the gauge 2–group are closely related to the 1– and 2–gauge transformations in higher gauge theory and therefore are of considerable interest for us. There are a few categorical equivalent ways of presenting the gauge 2–group, which we review next. Below, we let $\hat{\mathcal{P}}$ be a strict principal $\hat{\mathscr{K}}$–2–bundle with base M.

The gauge 2–group of $\hat{\mathcal{P}}$ can be characterized as its automorphism 2–group.

Definition **3.3.** *The automorphism 2–group of $\hat{\mathcal{P}}$, $\mathrm{Aut}^{\hat{\mathsf{K}}}_{\hat{\pi}}(\hat{P})$, consists in the weak sub–2–group of $\mathrm{Aut}(\hat{P})$ of $\hat{\mathsf{K}}$–equivariant weakly invertible functors $F : \hat{P} \to \hat{P}$ preserving the bundle projection $\hat{\pi}$,*

$$\hat{\pi} \circ F = \hat{\pi} \tag{3.1.10}$$

strictly, and $\hat{\mathsf{K}}$*–equivariant natural isomorphisms* $\beta : F \Rightarrow G$*.*

The gauge 2–group of $\hat{\mathcal{P}}$ can be characterized also as its equivariant structure 2–group valued morphism 2–group.

Definition 3.4. *The equivariant structure* 2*–group valued morphism* 2*–group of* $\hat{\mathcal{P}}$*,* $\mathrm{Fun}^{\hat{\mathsf{K}}}(\hat{P}, \hat{\mathsf{K}}_{\mathrm{Ad}})$*, is the strict* 2*–group whose objects and morphisms are respectively the* $\hat{\mathsf{K}}$*–equivariant functors* $F : \hat{P} \to \hat{\mathsf{K}}$ *and* $\hat{\mathsf{K}}$*–equivariant natural isomorphisms* $\beta : F \Rightarrow G$ *and whose* 2*–group operations are those induced pointwise by those of* $\hat{\mathsf{K}}$*, where* $\hat{\mathsf{K}}_{\mathrm{Ad}}$ *denotes* $\hat{\mathsf{K}}$ *with the conjugation right action.*

The following theorem, shown in ref. [6], extends a basic property of the gauge groups of ordinary principal bundles to principal 2–bundles.

Theorem 3.1. *The automorphism* 2*–group* $\mathrm{Aut}^{\hat{\mathsf{K}}}_{\hat{\pi}}(\hat{P})$ *and the equivariant structure* 2*–group valued morphism* 2*–group* $\mathrm{Fun}^{\hat{\mathsf{K}}}(\hat{P}, \hat{\mathsf{K}}_{\mathrm{Ad}})$ *are equivalent as weak 2-groups.*

In this way, in the appropriate categorical sense, we can identify $\mathrm{Aut}^{\hat{\mathsf{K}}}_{\hat{\pi}}(\hat{P})$ and $\mathrm{Fun}^{\hat{\mathsf{K}}}(\hat{P}, \hat{\mathsf{K}}_{\mathrm{Ad}})$. This second realization of the gauge 2–group of $\hat{\mathcal{P}}$ is more convenient and we shall refer to it in the following. Under the assumptions that M is compact, $\hat{\mathsf{K}}$ is locally exponentiable and the $\hat{\mathsf{K}}$–action is principal on $\hat{P}$ and $\hat{P}_0$, one can prove that $\mathrm{Fun}^{\hat{\mathsf{K}}}(\hat{P}, \hat{\mathsf{K}}_{\mathrm{Ad}})$ is a strict Lie 2–group. Moreover, $\mathrm{Fun}^{\hat{\mathsf{K}}}(\hat{P}, \hat{\mathsf{K}}_{\mathrm{Ad}})$ enjoys a simple concrete description.

1. The objects of $\mathrm{Fun}^{\hat{\mathsf{K}}}(\hat{P}, \hat{\mathsf{K}}_{\mathrm{Ad}})$ are $\hat{\mathsf{K}}$–equivariant functors $F : \hat{P} \to \hat{\mathsf{K}}_{\mathrm{Ad}}$.

2. The morphisms of $\mathrm{Fun}^{\hat{\mathsf{K}}}(\hat{P}, \hat{\mathsf{K}}_{\mathrm{Ad}})$ from an object F to another G are the $\hat{\mathsf{K}}_0$–equivariant maps $\beta : \hat{P}_0 \to \hat{\mathsf{K}}_{\mathrm{Ad}}$ such that $\beta(x) : F_0(x) \to G_0(x)$ for $x \in \hat{P}_0$.

The $\hat{\mathsf{K}}_0$–equivariant map $F_0 : \hat{P}_0 \to \hat{\mathsf{K}}_{0\mathrm{Ad}}$ associated with the object F is again a restriction.

A useful local description of the gauge 2–group of $\hat{\mathcal{P}}$ is also available. On a trivializing neighborhood $U \subset M$, where $\hat{\pi}^{-1}(U)$ is $\hat{\mathsf{K}}$–equivariantly equivalent to $U \times \hat{\mathsf{K}}$, the 2–groups $\mathrm{Fun}^{\hat{\mathsf{K}}}(\hat{\pi}^{-1}(U), \hat{\mathsf{K}}_{\mathrm{Ad}})$ and $\mathrm{Fun}^{\hat{\mathsf{K}}}(U \times \hat{\mathsf{K}}, \hat{\mathsf{K}}_{\mathrm{Ad}})$ are equivalent as weak 2–groups. The latter 2–group has in turn the following description.

Definition **3.5.** *For a neighborhood $U \subset M$, we let* $\mathrm{Fun}(U, \hat{\mathsf{K}})^{\mathrm{hop}}$ *be the strict* 2*–group whose objects are functors $V : U \to \hat{\mathsf{K}}$, whose morphisms are natural isomorphisms $\sigma : V \Rightarrow W$ and whose* 2*–group operations are those induced pointwise by those of $\hat{\mathsf{K}}^{\mathrm{hop}}$, the* 2 *group $\hat{\mathsf{K}}$ with the opposite multiplication functor.*

Proposition **3.1.** *The* 2*–group* $\mathrm{Fun}^{\hat{\mathsf{K}}}(U \times \hat{\mathsf{K}}, \hat{\mathsf{K}}_{\mathrm{Ad}})$ *is isomorphic as a strict* 2*–group to the* 2*–group* $\mathrm{Fun}(U, \hat{\mathsf{K}})^{\mathrm{hop}}$

The 2–group $\mathrm{Fun}(U, \hat{\mathsf{K}})^{\mathrm{hop}}$ has a very simple structure strongly reminiscent that of the gauge group of ordinary principal bundles.

1. An object V of $\mathrm{Fun}(U, \hat{\mathsf{K}})$ is specified by a map $V : U \to \hat{\mathsf{K}}_0$.

2. A morphism $\sigma : V \Rightarrow W$ of $\mathrm{Fun}(U, \hat{\mathsf{K}})$ is similarly specified by a map $\sigma : M \to \hat{\mathsf{K}}$ such that $\sigma(m) : V(m) \to W(m)$ for $m \in U$.

Under the strict 2–group isomorphism $\mathrm{Fun}(U, \hat{\mathsf{K}})^{\mathrm{hop}} \simeq \mathrm{Fun}^{\hat{\mathsf{K}}}(U \times \hat{\mathsf{K}}, \hat{\mathsf{K}}_{\mathrm{Ad}})$, the object H_V of $\mathrm{Fun}^{\hat{\mathsf{K}}}(U \times \hat{\mathsf{K}}, \hat{\mathsf{K}}_{\mathrm{Ad}})$ corresponding to an object V of $\mathrm{Fun}(U, \hat{\mathsf{K}})$ is given by the expression

$$H_V(m, A) = A^{-1} V(m) A \tag{3.1.11}$$

with $m \in U$ and $A \in \hat{\mathsf{K}}$. The morphism $\theta_\sigma : H_V \to H_W$ of $\mathrm{Fun}^{\hat{\mathsf{K}}}(U \times \hat{\mathsf{K}}, \hat{\mathsf{K}}_{\mathrm{Ad}})$ corresponding to a morphism $\sigma : V \Rightarrow W$ of $\mathrm{Fun}(U, \hat{\mathsf{K}})$ is similarly specified by

$$\theta_\sigma(m, a) = a^{-1} \sigma(m) a \tag{3.1.12}$$

with $m \in U$ and $a \in \hat{\mathsf{K}}_0$.

In the next subsection, we shall introduce the synthetic description of principal 2–bundles, which brings the categorical formulation expounded above closer to higher gauge theory.

3.2 Synthetic formulation of principal 2–bundle theory

The total space theory of principal 2–bundles expounded in subsect. 3.1 is elegant and geometrically intuitive, but it has certain shortcomings. In such framework,

no viable definition of 2–connection and 1–gauge transformation is available that is satisfactorily close to the corresponding notions of higher gauge theory, which is our main concern. (See however refs. [16, 17] for an interesting attempt in this direction.) As we shall show in II, these problems can be solved by switching from the categorical framework to a distinct but related description, which we shall call synthetic, for lack of a better term and with an abuse of language, because the notions it is based on are akin in spirit to those of synthetic differential geometry. The synthetic approach turns out to be closely related to the standard formulation of strict higher gauge theory, as we require.

It is difficult to justify the change of perspective we are espousing in simple intuitive terms. The synthetic formulation is essentially validated *a posteriori* by the viability of the results it leads to.

A fundamental property of strict 2–groups is their equivalence to group crossed modules described in greater detail in subsect. 3.3.

Theorem 3.2. *For a strict Lie 2–group $\hat{\mathscr{K}}$ with source and target maps $\hat{s}$, $\hat{t}$, there exists a Lie group isomorphism*

$$\hat{\mathsf{K}} \simeq \ker \hat{s} \rtimes_{\hat{\lambda}} \hat{\mathsf{K}}_0, \tag{3.2.1}$$

where the semidirect product group structure is defined with respect to the left action $\hat{\lambda} : \hat{\mathsf{K}}_0 \times \ker \hat{s} \to \ker \hat{s}$ given by

$$\hat{\lambda}(a, Z) = aZa^{-1} \tag{3.2.2}$$

with $a \in \hat{\mathsf{K}}_0$, $Z \in \ker \hat{s}$. Furthermore, under the isomorphism (3.2.1), the Lie groupoid structure of $\hat{\mathsf{K}}$ can be expressed completely in terms of the group structures of $\hat{\mathsf{K}}_0$ and $\ker \hat{s}$, the action map $\hat{\lambda}$ and the target map

$$\hat{\epsilon} = \hat{t}\big|_{\ker \hat{s}}. \tag{3.2.3}$$

Viewing the structure 2–group $\hat{\mathscr{K}}$ of the relevant principal 2–bundle $\hat{\mathcal{P}}$ as a Lie group crossed module $(\hat{\mathsf{K}}_0, \ker \hat{s})$ leads to a geometric framework of the higher theory closer in form and in spirit to that of the ordinary one. It is also a

necessary starting point for the synthetic formulation, which essentially amounts to trading $\ker \hat{s}$ with $\mathrm{Lie}\ker\hat{s}[1]$.

Definition 3.6. *The synthetic form* K *of the morphism group* $\hat{\mathsf{K}}$ *is the graded Lie group of the internal functions* $Q \in \textsc{Map}(\mathbb{R}[-1], \hat{\mathsf{K}})$ *of the form*

$$Q(\bar{\alpha}) = \mathrm{e}^{\bar{\alpha}L}\, a, \qquad \bar{\alpha} \in \mathbb{R}[-1] \tag{3.2.4}$$

with $a \in \hat{\mathsf{K}}_0$, $L \in \mathrm{Lie}\ker\hat{s}[1]$. *The synthetic form* K_0 *of the object group* $\hat{\mathsf{K}}_0$ *is the graded Lie subgroup of* K *constituted by the elements of the special form* $q(\bar{\alpha}) = a$.

The group operations are pointwise multiplication and inversion. K_0 can be identified canonically with $\hat{\mathsf{K}}_0$.

The synthetic groups K, K_0 stem from the morphism and object groups $\hat{\mathsf{K}}$, $\hat{\mathsf{K}}_0$ of the strict Lie 2–group $\hat{\mathscr{K}}$. Furthermore, K_0 is a subgroup of K. However, K, K_0 are not the morphism and object groups of any synthetic strict Lie 2–group $\mathscr{K}$, as not all the groupoid structure maps of $\hat{\mathsf{K}}$ can be directly extended to K. A graded Lie group morphism $s : \mathsf{K} \to \mathsf{K}_0$ extending the source morphism $\hat{s}$ of $\hat{\mathsf{K}}$ does in fact exist.

Definition 3.7. *The synthetic form of* $\hat{s}$ *is the element* $s \in \mathrm{Hom}(\mathsf{K}, \mathsf{K}_0)$ *given by*

$$s(Q) = \hat{s} \circ Q \tag{3.2.5}$$

with $Q \in \mathsf{K}$.

s is well–defined, since, writing Q as in (3.2.4), s is given by $s(Q)(\bar{\alpha}) = \hat{s}(Q(\bar{\alpha})) = a$ and so has a range lying in K_0 as required. A partner morphism $t \in \mathrm{Hom}(\mathsf{K}, \mathsf{K}_0)$ extending the target morphism $\hat{t}$ of $\hat{\mathsf{K}}$ instead does not. The would–be target map t would read as $t(Q)(\bar{\alpha}) = \hat{t}(Q(\bar{\alpha})) = \mathrm{e}^{\bar{\alpha}\dot{\hat{t}}(L)}\, a$, where $\dot{\hat{t}}$ is the Lie differential of $\hat{t}$, and thus have a range generally lying outside K_0. Consequently, composition also cannot be defined in K and K has no groupoid structure.

Definition 3.8. *The synthetic form* P *of the morphism manifold* $\hat{P}$ *is the graded manifold of the internal functions* $V \in \textsc{Map}(\mathbb{R}[-1], \hat{P})$ *of the form*

$$V(\bar{\alpha}) = \hat{R}(x, Q(\bar{\alpha})), \qquad \bar{\alpha} \in \mathbb{R}[-1] \tag{3.2.6}$$

with $x \in \hat{P}_0 \subset \hat{P}$, $Q \in \mathsf{K}$. *The synthetic form* P_0 *of the object manifold* $\hat{P}_0$ *is the graded submanifold of* P *constituted by of the points of the special form* $v(\bar{\alpha}) = x$.

Since $\textsc{Map}(\mathbb{R}[-1], \hat{P}) \simeq T[1]\hat{P}$, the shifted tangent bundle of $\hat{P}$, P is a vector subbundle of $T[1]\hat{P}$. Indeed, by the way it is defined and in analogy to the ordinary notion, P can be described as $V[1]\hat{P}|_{\hat{P}_0}$, the shifted vertical subbundle of $\hat{P}$ restricted to $\hat{P}_0$. P_0 can be identified canonically with $\hat{P}_0$, which in turn can be regarded as the zero section of $V[1]\hat{P}|_{\hat{P}_0}$.

The synthetic manifolds P, P_0 answer to the morphism and object manifolds $\hat{P}$, $\hat{P}_0$ of the 2–bundle $\hat{\mathcal{P}}$. P_0 is further a submanifold of P. However, P, P_0 are not the morphism and object manifolds of any synthetic principal $\mathscr{K}$–2–bundle $\mathcal{P}$. To begin with, the synthetic groups K, K_0 cannot constitute the 2–bundle's structure 2–group $\mathscr{K}$ for reasons explained earlier. Moreover, not all the groupoid structure maps of $\hat{P}$ can be consistently extended to P. A map $s : P \to P_0$ extending the source map $\hat{s}$ of $\hat{P}$ is in fact available.

Definition 3.9. *The synthetic form of* $\hat{s}$ *is the map* $s \in \mathrm{Map}(P, P_0)$ *given by*

$$s(V) = \hat{s} \circ V \tag{3.2.7}$$

with $V \in P$.

s is well–defined, as, for $V \in P$ of the form (3.2.6) with $Q \in \mathsf{K}$ as in (3.2.4), s reads as $s(V)(\bar{\alpha}) = \hat{s}(V(\bar{\alpha})) = \hat{s}(\hat{R}(x, Q(\bar{\alpha}))) = \hat{R}_0(\hat{s}(x), \hat{s}(Q(\bar{\alpha}))) = \hat{R}_0(x, a)$ and so has a range lying in P_0 as wished. A mate target map $t \in \mathrm{Map}(P, P_0)$ extending the target map $\hat{t}$ of $\hat{P}$ instead is not. The supposedly target map t would be given by $t(V)(\bar{\alpha}) = \hat{t}(V(\bar{\alpha})) = \hat{t}(\hat{R}(x, Q(\bar{\alpha}))) = \hat{R}_0(\hat{t}(x), \hat{t}(Q(\bar{\alpha}))) = \hat{R}_0(x, \mathrm{e}^{\bar{\alpha}\dot{\hat{t}}(L)}\, a)$ and thus have a range lying outside P_0. As a consequence, composition also cannot be defined in P and P has no groupoid structure. In spite of this findings, many of the properties of $\hat{P}$ and $\hat{P}_0$ as part of the 2–bundle $\hat{\mathcal{P}}$ do extend in an appropriate form to P and P_0. We study these in some detail next.

Projection maps π and π_0 of P induced by $\hat{\pi}$ and $\hat{\pi}_0$ respectively exist and enjoy the expected properties.

Definition **3.10.** *The synthetic form of $\hat{\pi}$ is the map $\pi \in \mathrm{Map}(P, M)$ given by*

$$\pi(V) = \hat{\pi} \circ V, \tag{3.2.8}$$

with $V \in P$. The synthetic form of $\hat{\pi}_0$ is the map $\pi_0 \in \mathrm{Map}(P_0, M)$ resulting from restricting π to P_0.

π extends $\hat{\pi}$ because, writing $V \in P$ in the form (3.2.6) above, π is given by $\pi(V)(\bar{\alpha}) = \hat{\pi}(V(\bar{\alpha})) = \hat{\pi}(\hat{R}(x, Q(\bar{\alpha}))) = \hat{\pi}_0(x)$ and thus has a range lying in M.

Proposition **3.2.** *π and π_0 are surjective submersions. Further, $\pi = \pi_0 \circ s$.*

Compare the above relation with (3.1.4).

Proof. Letting $\rho : P \to \hat{P}_0$ be the projection map of $P \simeq V[1]\hat{P}|_{\hat{P}_0}$, one has $\pi = \hat{\pi} \circ \rho$. As $\hat{\pi}$, ρ are both surjective submersions, so is π. Likewise, letting $\rho_0 : P_0 \to \hat{P}_0$ be the canonical identification of P_0 and $\hat{P}_0$, one has $\pi_0 = \hat{\pi}_0 \circ \rho_0$. Since $\hat{\pi}_0$, ρ_0 are both surjective submersions, so is π_0. The relation $\pi_0 \circ s = \pi$ follows from the identity $\hat{\pi}_0 \circ \hat{s} = \hat{\pi}$, (3.2.5) and (3.2.7). □

The right actions $\hat{R}$ and $\hat{R}_0$ of $\hat{\mathsf{K}}$ and $\hat{\mathsf{K}}_0$ on $\hat{P}$ and $\hat{P}_0$ similarly give rise to right actions R and R_0 of K and K_0 on P and P_0 with the expected properties.

Definition **3.11.** *The synthetic form of $\hat{R}$ is the map $R \in \mathrm{Map}(P \times \mathsf{K}, P)$ given by*

$$R(V, A) = \hat{R} \circ (V \times A) \tag{3.2.9}$$

for $V \in P$, $A \in \mathsf{K}$. The synthetic form of $\hat{R}_0$ is the map $R_0 \in \mathrm{Map}(P_0 \times \mathsf{K}_0, P_0)$ yielded by restriction of R to $P_0 \times \mathsf{K}_0$.

R extends $\hat{R}$ because, writing $V \in P$ in the form (3.2.6) above, R is given by $R(V, A)(\bar{\alpha}) = \hat{R}(\hat{R}(x, Q(\bar{\alpha})), A(\bar{\alpha})) = \hat{R}(x, Q(\bar{\alpha})A(\bar{\alpha}))$ and thus has a range lying in P. This calculation also shows that R_0, as the restriction of R to $P_0 \times \mathsf{K}_0$, has a range lying in P_0.

Proposition **3.3.** *R and R_0 are right actions of K on P and K_0 on P_0, respectively. Further, $\pi \circ R = \pi \circ \mathrm{pr}_1$ and $\pi_0 \circ R_0 = \pi_0 \circ \mathrm{pr}_1$.*

The above relations answer to (3.1.5).

Proof. These properties follow trivially from the corresponding properties of $\hat{R}$ and $\hat{R}_0$ and $\hat{\pi}$ and $\hat{\pi}_0$. □

For any trivializing neighborhood $U \subset M$ and trivializing functor $\hat{\Phi}_U$, there exist trivializing maps Φ_U and Φ_{U0} of P and P_0 induced by $\hat{\Phi}_U$ and $\hat{\Phi}_{U0}$.

Definition 3.12. *The synthetic form of $\hat{\Phi}_U$ is the map $\Phi_U \in \mathrm{Map}(\pi^{-1}(U), U \times \mathsf{K})$ given for $V \in \pi^{-1}(U) \subset P$ by*

$$\Phi_U(V) = \hat{\Phi}_U \circ V. \tag{3.2.10}$$

The synthetic form of $\hat{\Phi}_{U0}$ is the map $\Phi_{U0} \in \mathrm{Map}({\pi_0}^{-1}(U), U \times \mathsf{K}_0)$ yielded by restricting Φ_U to ${\pi_0}^{-1}(U)$.

Φ_U and Φ_{U0} having the ranges indicated follows from the following proposition.

Proposition 3.4. *There is a K–equivariant map $T_U \in \mathrm{Map}(\pi^{-1}(U), \mathsf{K})$ such that*

$$\Phi_U(V) = (\pi(V), T_U(V)), \qquad T_U(V) = \hat{T}_U \circ V, \tag{3.2.11}$$

where the map $\hat{T}_U$ is defined through relation (3.1.8). Further, T_U restricts to a K_0–equivariant map $T_{U0} \in \mathrm{Map}({\pi_0}^{-1}(U), \mathsf{K}_0)$.

(3.2.11) extends (3.1.8) and shows that Φ_U, as $\hat{\Phi}_U$, is projection preserving.

Proof. T_U has range in K, as, writing $V \in \pi^{-1}(U)$ in the form (3.2.6), T_U is given by $T_U(V)(\bar{\alpha}) = \hat{T}_U(V(\bar{\alpha})) = \hat{T}_U(\hat{R}(x, Q(\bar{\alpha}))) = \hat{T}_{U0}(x)Q(\bar{\alpha})$ with $\hat{T}_{U0}(x) \in \hat{\mathsf{K}}_0$. (3.2.11) follows immediately from (3.1.8) and (3.2.8). The K–equivariance of T_U follows from the $\hat{\mathsf{K}}$–equivariance of $\hat{T}_U$. The calculation above shows also that, for $v \in {\pi_0}^{-1}(U)$, T_{U0} is given by $T_{U0}(v)(\bar{\alpha}) = \hat{T}_{U0}(x)$ so that T_{U0} has range in K_0. The K_0–equivariance of T_{U0} follows from the $\hat{\mathsf{K}}_0$–equivariance of $\hat{T}_{U0}$. □

It is possible to define the synthetic analog $\tilde{\Phi}_U$ of a weak inverse $\tilde{\hat{\Phi}}_U$ of $\hat{\Phi}_U$ and its object restriction. We shall not consider them here, since we shall not need them in the following.

In the rest of this section, of a mostly technical nature, we shall work out a graded differential geometric framework for strict principal 2–bundle theory based on the operational apparatus of subsect. 2.1 and the synthetic description expounded above. The operational synthetic formulation allows for an original theory of 2–connections and 1– and 2–gauge transformations, developed in depth in the companion paper II, very close in form and spirit to the principal bundle theoretic framework reviewed in subsect. 1.1.

3.3 Lie group and algebra crossed modules

In subsect. 3.2, we have seen that thanks to the isomorphism (3.2.1) a strict Lie 2–group is fully encoded in a pair of Lie groups equipped with two structure maps with certain properties. These data constitute a Lie group crossed module, a notion which we review in this subsection mainly to set our notation and terminology. We refer the reader to the papers [22, 23] for a comprehensive treatment of this subject.

Definition **3.13.** *A Lie group crossed module* M *consists of two Lie groups* E *and* G *together with a Lie group morphism* $\tau : \mathsf{E} \to \mathsf{G}$ *and an action* $\mu : \mathsf{G} \times \mathsf{E} \to \mathsf{E}$ *with the following properties.*

1. *For* $a \in \mathsf{G}$, $\mu(a, \cdot) \in \mathrm{Aut}(\mathsf{E})$.

2. *The map* $a \in \mathsf{G} \to \mu(a, \cdot) \in \mathrm{Aut}(\mathsf{E})$ *is a Lie group morphism.*

3. *Equivariance: for* $a \in \mathsf{G}$, $A \in \mathsf{E}$

$$\tau(\mu(a, A)) = a\tau(A)a^{-1}. \tag{3.3.1}$$

4. *Peiffer identity: for* $A, B \in \mathsf{E}$

$$\mu(\tau(A), B) = ABA^{-1}. \tag{3.3.2}$$

In what follows, we shall often denote a Lie group crossed module M by the list of its constituent data $(\mathsf{E}, \mathsf{G}, \tau, \mu)$, or simply (E, G) when no confusion can occur.

Definition **3.14.** *A morphism $\beta : \mathsf{M}' \to \mathsf{M}$ of Lie group crossed modules consists of two Lie group morphisms $\phi : \mathsf{G}' \to \mathsf{G}$ and $\Phi : \mathsf{E}' \to \mathsf{E}$ with the following properties.*

1. *For $A \in \mathsf{E}'$*

$$\tau(\Phi(A)) = \phi(\tau'(A)). \tag{3.3.3}$$

2. *For $a \in \mathsf{G}'$, $A \in \mathsf{E}'$*

$$\Phi(\mu'(a, A)) = \mu(\phi(a), \Phi(A)). \tag{3.3.4}$$

We shall concisely denote a Lie group crossed module morphism $\beta : \mathsf{M}' \to \mathsf{M}$ by the list $(\Phi : \mathsf{E}' \to \mathsf{E}, \phi : \mathsf{G}' \to \mathsf{G})$ of its defining data and shall omit the specification of sources and targets when no confusion is possible.

Lie group crossed modules and morphisms thereof constitute a category denoted by **Lgcm**.

Definition **3.15.** *A Lie algebra crossed module $\mathfrak{m}$ consists of two Lie algebras $\mathfrak{e}$ and $\mathfrak{g}$ together with a Lie algebra morphism $t : \mathfrak{e} \to \mathfrak{g}$ and an action $m : \mathfrak{g} \times \mathfrak{e} \to \mathfrak{e}$ with the following properties.*

1. *For $u \in \mathfrak{g}$, $m(u, \cdot) \in \mathrm{Der}(\mathfrak{e})$.*

2. *The map $u \in \mathfrak{g} \to m(u, \cdot) \in \mathrm{Der}(\mathfrak{e})$ is a Lie algebra morphism.*

3. *Equivariance: for $u \in \mathfrak{g}$, $U \in \mathfrak{e}$*

$$t(m(u, U)) = [u, t(U)]. \tag{3.3.5}$$

4. *Peifer identity: for $U, V \in \mathfrak{e}$*

$$m(t(U), V) = [U, V]. \tag{3.3.6}$$

As for a Lie group crossed module, we shall denote a Lie algebra crossed module $\mathfrak{m}$ by the list $(\mathfrak{e}, \mathfrak{g}, t, m)$ of its defining data or simply by $(\mathfrak{e}, \mathfrak{g})$ when no confusion can arise.

Definition **3.16.** *A morphism $p : \mathfrak{m}' \to \mathfrak{m}$ of Lie algebra crossed modules consists of two Lie algebra morphisms $h : \mathfrak{g}' \to \mathfrak{g}$ and $H : \mathfrak{e}' \to \mathfrak{e}$ with the following properties.*

1. *For $U \in \mathfrak{e}'$*

$$t(H(U)) = h(t'(U)). \tag{3.3.7}$$

2. *For $u \in \mathfrak{g}'$, $U \in \mathfrak{e}'$*

$$H(m'(u, U)) = m(h(u), H(U)). \tag{3.3.8}$$

We shall concisely denote a Lie algebra crossed module morphism $p : \mathfrak{m}' \to \mathfrak{m}$ by the list $(H : \mathfrak{e}' \to \mathfrak{e}, h : \mathfrak{g}' \to \mathfrak{g})$ of its defining data and shall omit the specification of sources and targets when no confusion is possible.

Lie algebra crossed modules and morphisms thereof constitute a category denoted by **Lacm**.

Proposition **3.5.** *With a Lie group crossed module $\mathsf{M} = (\mathsf{E}, \mathsf{G}, \tau, \mu)$, there is associated a Lie algebra crossed module $\mathfrak{m} = (\mathfrak{e}, \mathfrak{g}, t, m)$, where $\mathfrak{e} = \operatorname{Lie}\mathsf{E}$, $\mathfrak{g} = \operatorname{Lie}\mathsf{G}$,*

$$t = \dot{\tau}, \tag{3.3.9}$$

$$m = \dot{\mu}. \tag{3.3.10}$$

Above, $\dot{}$ denotes Lie differentiation with respect to the relevant Lie group argument. In the second relation above, Lie differentiation is carried out with respect to both arguments. In the following, we shall often encounter the Lie differential $\dot{\mu} : \mathsf{G} \times \mathfrak{e} \to \mathfrak{e}$. For $a \in \mathsf{G}$, $\dot{\mu}(a, \cdot) \in \operatorname{Aut}(\mathfrak{e})$ and the map $a \in \mathsf{G} \to \dot{\mu}(a, \cdot) \in \operatorname{Aut}(\mathfrak{e})$ is a Lie group morphism. The Lie differential $\dot{\mu} : \mathfrak{g} \times \mathsf{E} \to \mathfrak{e}$ shall also be considered. See app. B for details about the precise definition and main properties of these objects.

Proposition **3.6.** *With every Lie group crossed module morphism $\beta : \mathsf{M}' \to \mathsf{M} = (\Phi : \mathsf{E}' \to \mathsf{E}, \phi : \mathsf{G}' \to \mathsf{G})$ there is associated by Lie differentiation a Lie algebra crossed module morphism $p : \mathfrak{m}' \to \mathfrak{m} = (H : \mathfrak{e}' \to \mathfrak{e}, h : \mathfrak{g}' \to \mathfrak{g})$, where*

$$h = \dot{\phi}, \tag{3.3.11}$$

$$H = \dot{\Phi}. \tag{3.3.12}$$

The morphism p shall be denoted by $\dot{\beta}$ in the following.

The map that associates with each Lie group crossed module M its Lie algebra crossed module $\mathfrak{m}$ and with each Lie group crossed module morphism $\beta : \mathsf{M}' \to \mathsf{M}$ its Lie algebra crossed module morphism $\dot{\beta} : \mathfrak{m}' \to \mathfrak{m}$ is a functor of the category **Lgcm** into the category **Lacm**.

3.4 Derived Lie groups and algebras

The derived Lie group and algebra of a Lie group and algebra crossed module, respectively, constitute the fundamental algebraic structures of the formulation of 2–connection and 1– and 2–gauge transformation theory studied in depth in part II. In this subsection, we formulate these notions and study in detail their properties.

Let L be an ordinary Lie group with Lie algebra $\mathfrak{l}$. The internal function space $\textsc{Map}(\mathbb{R}[-1], \mathsf{L})$ with the Lie group structure inherited form that of L is a graded Lie group. Every element $f \in \textsc{Map}(\mathbb{R}[-1], \mathsf{L})$ can be written uniquely as $f(\bar{\alpha}) = \mathrm{e}^{\bar{\alpha}x}\, l$, $\bar{\alpha} \in \mathbb{R}[-1]$, with $l \in \mathsf{L}$ and $x \in \mathfrak{l}[1]$.

Analogous remarks hold for an ordinary Lie algebra $\mathfrak{l}$. The internal function space $\textsc{Map}(\mathbb{R}[-1], \mathfrak{l})$ with the Lie algebra structure inherited form that of $\mathfrak{l}$ is a graded Lie algebra. Any element $z \in \textsc{Map}(\mathbb{R}[-1], \mathfrak{l})$ can so be written uniquely as $z(\bar{\alpha}) = x + \bar{\alpha} y$, $\bar{\alpha} \in \mathbb{R}[-1]$, with $x \in \mathfrak{l}$ and $y \in \mathfrak{l}[1]$.

For a Lie group crossed module $\mathsf{M} = (\mathsf{E}, \mathsf{G}, \tau, \mu)$, consider the graded Lie group $\textsc{Map}(\mathbb{R}[-1], \mathsf{E} \rtimes_\mu \mathsf{G})$, where $\mathsf{E} \rtimes_\mu \mathsf{G}$ denotes the semidirect product of the groups E and G with respect to the G–action μ. $\textsc{Map}(\mathbb{R}[-1], \mathsf{E} \rtimes_\mu \mathsf{G})$ contains among others elements of the special form

$$P(\bar{\alpha}) = \mathrm{e}^{\bar{\alpha}L}\, a, \quad \bar{\alpha} \in \mathbb{R}[-1], \tag{3.4.1}$$

with $a \in \mathsf{G}$, $L \in \mathfrak{e}[1]$ forming form a distinguished subset $\mathrm{D}\mathsf{M}$.

For any P, $Q \in \mathrm{D}\mathsf{M}$ with $P(\bar{\alpha}) = \mathrm{e}^{\bar{\alpha}L}\, a$, $Q(\bar{\alpha}) = \mathrm{e}^{\bar{\alpha}N}\, b$, one has

$$PQ(\bar{\alpha}) = \mathrm{e}^{\bar{\alpha}(L+\dot{\mu}(a,N))}\, ab, \tag{3.4.2}$$

$$P^{-1}(\bar{\alpha}) = \mathrm{e}^{-\bar{\alpha}\dot{\mu}(a^{-1},L)}\, a^{-1}. \tag{3.4.3}$$

Therefore, DM is a graded Lie subgroup of $\textsc{Map}(\mathbb{R}[-1], \mathsf{E} \rtimes_{\mu} \mathsf{G})$.

Proposition 3.7. DM *is a graded Lie group. The graded Lie group isomorphism*

$$\mathsf{DM} \simeq \mathfrak{e}[1] \rtimes_{\dot{\mu}} \mathsf{G} \tag{3.4.4}$$

holds, where $\mathfrak{e}$ *is regarded as an Abelian Lie group.*

Above, $\mathfrak{e}[1] \rtimes_{\dot{\mu}} \mathsf{G}$ denotes the semidirect product of the Lie groups $\mathfrak{e}[1]$ and G with respect to the G–action $\dot{\mu}$. The graded Lie group DM is called the derived Lie group of M.

Proof. The statement follows immediately from (3.4.2), (3.4.3). □

Proposition 3.8. *With every Lie group crossed module morphism* $\beta : \mathsf{M}' \to \mathsf{M} = (\Phi : \mathsf{E}' \to \mathsf{E}, \phi : \mathsf{G}' \to \mathsf{G})$ *there is associated a graded Lie group morphism* $\mathrm{D}\beta : \mathsf{DM}' \to \mathsf{DM}$. *With respect to the factorization* (3.4.4), $\mathrm{D}\beta$ *reads*

$$\mathrm{D}\beta = \dot{\Phi} \times \phi. \tag{3.4.5}$$

$\mathrm{D}\beta$ is called the derived Lie group morphism of β.

Proof. The statement follows readily from (3.4.2), (3.4.3) and ϕ, Φ being group morphisms with Φ satisfying (3.3.4). □

The map D associating with each Lie group crossed module its derived Lie group and with each Lie group module morphism its derived Lie group morphism constitutes a functor from the Lie group crossed module category **Lgcm** into the graded Lie group category **gLg**.

Next, for a Lie algebra crossed module $\mathfrak{m} = (\mathfrak{e}, \mathfrak{g}, t, m)$, consider the internal function space $\textsc{Map}(\mathbb{R}[-1], \mathfrak{e} \rtimes_m \mathfrak{g})$, where $\mathfrak{e} \rtimes_m \mathfrak{g}$ denotes the semidirect product of the Lie algebras $\mathfrak{e}$ and $\mathfrak{g}$ with respect to the $\mathfrak{g}$–action m. $\textsc{Map}(\mathbb{R}[-1], \mathfrak{e} \rtimes_m \mathfrak{g})$ contains in particular the elements of the form

$$Y(\bar{\alpha}) = u + \bar{\alpha}U, \quad \bar{\alpha} \in \mathbb{R}[-1], \tag{3.4.6}$$

with $u \in \mathfrak{g}$, $U \in \mathfrak{e}[1]$ spanning a special subspace $\mathrm{D}\mathfrak{m}$.

For any $Y, W \in \mathrm{D}\mathfrak{m}$ such that $Y(\bar{\alpha}) = u + \bar{\alpha}U$, $W(\bar{\alpha}) = v + \bar{\alpha}V$, one has

$$[Y, W](\bar{\alpha}) = [u, v] + \bar{\alpha}(m(u, V) - m(v, U)). \tag{3.4.7}$$

Thus, $\mathrm{D}\mathfrak{m}$ is a graded Lie subalgebra of $\textsc{Map}(\mathbb{R}[-1], \mathfrak{e} \rtimes_m \mathfrak{g})$.

Proposition 3.9. $\mathrm{D}\mathfrak{m}$ *is a graded Lie algebra. The graded Lie algebra isomorphism*

$$\mathrm{D}\mathfrak{m} \simeq \mathfrak{e}[1] \rtimes_m \mathfrak{g} \tag{3.4.8}$$

holds, where $\mathfrak{e}$ *is regarded as an Abelian Lie algebra.*

Above, $\mathfrak{e}[1] \rtimes_m \mathfrak{g}$ denotes the semidirect product of the Lie algebras $\mathfrak{e}[1]$ and $\mathfrak{g}$ with respect to the $\mathfrak{g}$–action m. The graded Lie algebra $\mathrm{D}\mathfrak{m}$ is called the derived Lie algebra of $\mathfrak{m}$.

Proof. The claim is an immediate consequence of (3.4.7). □

Proposition 3.10. *With every Lie algebra crossed module morphism* $p : \mathfrak{m}' \to \mathfrak{m} = (H : \mathfrak{e}' \to \mathfrak{e}, h : \mathfrak{g}' \to \mathfrak{g})$ *there is associated a graded Lie algebra morphism* $\mathrm{D}p : \mathrm{D}\mathfrak{m}' \to \mathrm{D}\mathfrak{m}$. *With respect to the factorization* (3.4.8), $\mathrm{D}p$ *reads*

$$\mathrm{D}p = H \times h. \tag{3.4.9}$$

$\mathrm{D}p$ is called the derived Lie algebra morphism of p.

Proof. The claim follows from (3.4.7) and H, h being Lie algebra morphisms with H satisfying (3.3.8). □

The map D associating with each Lie algebra crossed module its derived Lie algebra and with each Lie algebra crossed module morphism its derived Lie algebra morphism is a functor from the Lie algebra crossed module category **Lacm** into the graded Lie algebra category **gLa**.

Let $\mathsf{M} = (\mathsf{E}, \mathsf{G}, \tau, \mu)$ be a Lie group crossed module and let $\mathfrak{m} = (\mathfrak{e}, \mathfrak{g}, \dot{\tau}, \dot{\mu})$ be its associated Lie algebra crossed module.

Proposition 3.11. $\mathrm{D}\mathfrak{m}$ *is the Lie algebra of* $\mathrm{D}\mathsf{M}$.

Proof. A curve in DM is a smooth map $\Gamma : \mathbb{R} \to \mathsf{DM}$ such that $\Gamma(0) = 1_{\mathsf{E} \rtimes_\mu \mathsf{G}}$. By (3.4.1), Γ is of the form

$$\Gamma(t)(\bar{\alpha}) = \mathrm{e}^{\bar{\alpha}\Xi(t)}\, \gamma(t) \tag{3.4.10}$$

for smooth maps $\gamma : \mathbb{R} \to \mathsf{G}$ with $\gamma(0) = 1_{\mathsf{G}}$ and $\Xi : \mathbb{R} \to \mathfrak{e}[1]$ with $\Xi(0) = 0$. Differentiating Γ at $t = 0$, one finds

$$Y(\bar{\alpha}) := \Gamma(t)^{-1} \frac{d\Gamma(t)}{dt}(\bar{\alpha})\bigg|_{t=0} = \gamma(t)^{-1} \frac{d\gamma(t)}{dt}\bigg|_{t=0} + \bar{\alpha} \frac{d\Xi(t)}{dt}\bigg|_{t=0}. \tag{3.4.11}$$

So, $Y \in \mathrm{D}\mathfrak{m}$, as it is of the form (3.4.6). Moreover, every element $Y \in \mathrm{D}\mathfrak{m}$ can be obtained in this way by a suitable choice of the maps γ and Ξ. □

An explicit expression of the adjoint action of DM on $\mathrm{D}\mathfrak{m}$ is available.

Proposition 3.12. *For $P \in \mathsf{DM}$ and $Y \in \mathrm{D}\mathfrak{m}$ respectively of the form (3.4.1) and (3.4.6), one has*

$$\mathrm{Ad}\, P(Y)(\bar{\alpha}) = \mathrm{Ad}\, a(u) + \bar{\alpha}(\dot{\mu}(a, U) - \dot{\mu}(\mathrm{Ad}\, a(u), L)), \tag{3.4.12}$$

$$\mathrm{Ad}\, P^{-1}(Y)(\bar{\alpha}) = \mathrm{Ad}\, a^{-1}(u) + \bar{\alpha}\dot{\mu}(a^{-1}, U + \dot{\mu}(u, L)). \tag{3.4.13}$$

Proof. Let $D \in \mathsf{DM}$ be given by

$$D(\bar{\alpha}) = \mathrm{e}^{\bar{\alpha}U}\, \mathrm{e}^{u}. \tag{3.4.14}$$

Using the identities (3.4.2), (3.4.3), one finds

$$PDP^{-1}(\bar{\alpha}) = \mathrm{e}^{\bar{\alpha}(\dot{\mu}(a,U) + L - \dot{\mu}(\mathrm{e}^{\mathrm{Ad}\, a(u)}, L))}\, \mathrm{e}^{\mathrm{Ad}\, a(u)}, \tag{3.4.15}$$

$$P^{-1}DP(\bar{\alpha}) = \mathrm{e}^{\bar{\alpha}\dot{\mu}(a^{-1}, U - L + \dot{\mu}(\mathrm{e}^{u}, L))}\, \mathrm{e}^{\mathrm{Ad}\, a^{-1}(u)}. \tag{3.4.16}$$

Linearizing these relations with respect to u, U, one obtains (3.4.12), (3.4.13) immediately. □

Let M, M' be Lie group crossed modules and $\mathfrak{m}$, $\mathfrak{m}'$ be their associated Lie algebra crossed modules.

Proposition 3.13. *If $\beta : \mathsf{M}' \to \mathsf{M}$ is a Lie group crossed module morphism and*

$\dot\beta : \mathfrak{m}' \to \mathfrak{m}$ *is its associated Lie algebra crossed module morphism, then the Lie differential of the derived Lie group morphism* $\mathrm{D}\beta$ *is the derived Lie algebra morphism* $\mathrm{D}\dot\beta$*, that is*

$$\dot{\mathrm{D}}\beta = \mathrm{D}\dot\beta. \tag{3.4.17}$$

Proof. Let $\beta = (\Phi, \phi)$. From (3.4.5), it follows that $\dot{\mathrm{D}}\beta = \dot\Phi \times \dot\phi$. From (3.4.9) with $p = \dot\beta$ and (3.3.11), (3.3.12), we have $\mathrm{D}\dot\beta = \dot\Phi \times \dot\phi$ too. Identity (3.4.17) so holds true. □

For a Lie group crossed module $\mathsf{M} = (\mathsf{E}, \mathsf{G}, \tau, \mu)$, it is possible to construct appropriate Maurer–Cartan elements of the derived Lie group $\mathrm{D}\mathsf{M}$ valued in the derived Lie algebra $\mathrm{D}\mathfrak{m}$. Let M denote a $\mathrm{D}\mathsf{M}$ variable.

Definition **3.17.** *The Maurer-Cartan forms of* $\mathrm{D}\mathsf{M}$ *are the* 1*–forms* dMM^{-1}, $M^{-1}dM \in \Omega^1(\mathrm{D}\mathsf{M}) \otimes \mathrm{D}\mathfrak{m}$*, where* d *is the de Rham differential of* $\mathrm{D}\mathsf{M}$*.*

Explicit formulae of the Maurer-Cartan forms can be gotten by expressing M as

$$M(\bar\alpha) = \mathrm{e}^{\bar\alpha E}\, g, \quad \bar\alpha \in \mathbb{R}[-1], \tag{3.4.18}$$

where g and E are a G and an $\mathfrak{e}[1]$ variable, respectively, in conformity with (3.4.1),

Proposition **3.14.** *The Maurer-Cartan forms of* $\mathrm{D}\mathsf{M}$ *are given by*

$$dMM^{-1}(\bar\alpha) = dgg^{-1} - \bar\alpha(dE - \dot\mu(dgg^{-1}, E)), \tag{3.4.19}$$

$$M^{-1}dM(\bar\alpha) = g^{-1}dg - \bar\alpha\dot\mu(g^{-1}, dE). \tag{3.4.20}$$

Proof. (3.4.19), (3.4.20) follow straightforwardly from the well–known variational identities $\delta\,\mathrm{e}^{\bar\alpha E}\,\mathrm{e}^{-\bar\alpha E} = \frac{\exp(\bar\alpha\,\mathrm{ad}\,E)-1}{\bar\alpha\,\mathrm{ad}\,E}\delta(\bar\alpha E)$, $\mathrm{e}^{-\bar\alpha E}\,\delta\,\mathrm{e}^{\bar\alpha E} = \frac{1-\exp(-\bar\alpha\,\mathrm{ad}\,E)}{\bar\alpha\,\mathrm{ad}\,E}\delta(\bar\alpha E)$ with $\delta = d$ upon taking the nilpotence of $\bar\alpha$ into account. □

3.5 Derived Lie group and algebra valued internal functions

In a graded geometric framework such as ours, 2–connections and 1–gauge transformations of a given principal 2–bundle, which we shall study in part II, are

instances of derived Lie group and algebra valued internal functions. In this subsection, we study in detail the spaces of such type of functions.

We begin with a few basic remarks. There exist a few definitions of infinite dimensional Lie groups and algebras and group and algebra morphisms and Lie differentiation, e. g. Fréchet, diffeological etc. In a typical infinite dimensional Lie theoretic setting, such notions coincide in their algebraic content but differ in their differential geometric one (unless the dimension is actually finite). For this reason, as long as only the algebraic structure is relevant and the smooth structure plays no role, we are free to leave the definition adopted unspecified. In such cases, we shall speak of *virtual* infinite dimensional Lie groups and algebras and Lie group and algebra morphisms and Lie differentiation and remain safely in the realm of pure algebra and formal analysis. Any statement of virtual infinite dimensional Lie theory turns into a statement of a specific kind of it provided the underlying geometric data satisfy suitable conditions.

We illustrate the above line of thought by reviewing very briefly the prototypical examples occurring in ordinary infinite dimensional Lie theory. Let M be a manifold. The space of functions from M to a Lie group L, $\mathrm{Map}(M, \mathsf{L})$, with the multiplication and inversion induced pointwise by those of L is a virtual infinite dimensional Lie group. Furthermore, any Lie group morphism $\lambda : \mathsf{L}' \to \mathsf{L}$ induces by left composition a virtual infinite dimensional Lie group morphism $\mathrm{Map}(M, \lambda) : \mathrm{Map}(M, \mathsf{L}') \to \mathrm{Map}(M, \mathsf{L})$. Similarly, the space of functions from M to a Lie algebra $\mathfrak{l}$, $\mathrm{Map}(M, \mathfrak{l})$, with the bracket induced pointwise by that of $\mathfrak{l}$ is a virtual infinite dimensional Lie algebra. Further, any Lie algebra morphism $l : \mathfrak{l}' \to \mathfrak{l}$ induces by left composition a virtual infinite dimensional Lie algebra morphism $\mathrm{Map}(M, l) : \mathrm{Map}(M, \mathfrak{l}') \to \mathrm{Map}(M, \mathfrak{l})$. When $\mathfrak{l}$ is the Lie algebra of L, moreover, $\mathrm{Map}(M, \mathfrak{l})$ is the virtual Lie algebra of $\mathrm{Map}(M, \mathsf{L})$. The above virtual assertions become actual ones once a particular infinite dimensional Lie theoretic framework is adopted, provided M meets suitable restrictions. For instance, in the Fréchet framework, those properties are verified if M is compact.

To the best of our knowledge, a suitable extension of the above framework to a graded differential geometric setting such as the one studied in this subsection has

not been worked out yet. Fortunately, we shall not need it. The purely algebraic notions of virtual infinite dimensional graded Lie group and algebra and group and algebra morphism, which are the obvious extensions to the graded setting of the corresponding notions of the ungraded one described above, will suffice.

Below, we consider a fixed graded manifold N. The space of internal functions from N to a graded Lie group P, $\textsc{Map}(N, \mathsf{P})$, with the multiplication and inversion induced pointwise by those of P is a virtual infinite dimensional graded Lie group. Furthermore, a virtual infinite dimensional graded Lie group morphism $\textsc{Map}(N, \kappa) : \textsc{Map}(N, \mathsf{P}') \to \textsc{Map}(N, \mathsf{P})$ is determined by any graded Lie group morphism $\kappa : \mathsf{P}' \to \mathsf{P}$. Similarly, the space of internal functions from N to a graded Lie algebra $\mathfrak{p}$, $\textsc{Map}(N, \mathfrak{p})$, with the bracket induced pointwise by that of $\mathfrak{p}$ is a virtual infinite dimensional graded Lie algebra. A virtual infinite dimensional graded Lie algebra morphism $\textsc{Map}(N, k) : \textsc{Map}(N, \mathfrak{p}') \to \textsc{Map}(N, \mathfrak{p})$ is furthermore determined by any graded Lie algebra morphisms $k : \mathfrak{p}' \to \mathfrak{p}$. Similar statements hold also under full degree extension of the graded Lie algebras and algebra morphisms involved (cf. app. A). When $\mathfrak{p}$ is the Lie algebra of P, moreover, $\textsc{Map}(N, \mathfrak{p})$ is the virtual Lie algebra of $\textsc{Map}(N, \mathsf{P})$.

Let $\mathsf{M} = (\mathsf{E}, \mathsf{G}, \tau, \mu)$ be a Lie group crossed module and let $\mathrm{D}\mathsf{M}$ be its derived Lie group (cf. subsect. 3.4). The internal function space $\textsc{Map}(N, \mathrm{D}\mathsf{M})$ is then a virtual infinite dimensional graded Lie group. $\textsc{Map}(N, \mathrm{D}\mathsf{M})$ is most appropriately described by noticing that it is isomorphic to a subgroup of the virtual Lie group $\textsc{Map}(\mathbb{R}[1] \times N, \mathsf{E} \rtimes_\mu \mathsf{G})$. Indeed, by (3.4.1), an element $F \in \textsc{Map}(N, \mathrm{D}\mathsf{M})$ reduces to a pair of internal functions $m \in \textsc{Map}(N, G)$, $Q \in \textsc{Map}(N, \mathfrak{e}[1])$ defining the internal function $F \in \textsc{Map}(\mathbb{R}[1] \times N, \mathsf{E} \rtimes_\mu \mathsf{G})$, denoted here by the same symbol for simplicity, given by

$$F(\alpha) = \mathrm{e}^{\alpha Q}\, m, \quad \alpha \in \mathbb{R}[1]. \tag{3.5.1}$$

Moreover, on account of (3.4.2), (3.4.3), for any $F, G \in \textsc{Map}(N, \mathrm{D}\mathsf{M})$ such that $F(\alpha) = \mathrm{e}^{\alpha Q}\, m$, $G(\alpha) = \mathrm{e}^{\alpha R}\, n$, we have

$$FG(\alpha) = \mathrm{e}^{\alpha(Q + \dot{\mu}(m,R))}\, mn, \tag{3.5.2}$$

$$F^{-1}(\alpha) = \mathrm{e}^{-\alpha\dot{\mu}(m^{-1},Q)}\, m^{-1}. \tag{3.5.3}$$

The structure of the elements as well as the group multiplication and inversion of $\textsc{Map}(N, \mathsf{DM})$ is hence formally the same as that of DM given in eqs. (3.4.1) and (3.4.2), (3.4.3) except for an inversion of degrees.

Proposition 3.15. *The virtual infinite dimensional graded Lie group isomorphism*

$$\textsc{Map}(N, \mathsf{DM}) \simeq \textsc{Map}(N, \mathfrak{e}[1]) \rtimes_{\textsc{Map}(N,\dot{\mu})} \textsc{Map}(N, G) \tag{3.5.4}$$

holds true.

(3.5.4) is the function space analog of (3.4.4).

Proof. (3.5.4) follows from (3.4.4) by acting with the functor $\textsc{Map}(N, -)$. □

Proposition 3.16. *With every Lie group crossed module morphism* $\beta : \mathsf{M}' \to \mathsf{M} = (\Phi : \mathsf{E}' \to \mathsf{E}, \phi : \mathsf{G}' \to \mathsf{G})$ *there is associated a virtual infinite dimensional graded Lie group morphism* $\textsc{Map}(N, \mathrm{D}\beta) : \textsc{Map}(N, \mathsf{DM}') \to \textsc{Map}(N, \mathsf{DM})$. *With respect to the factorization* (3.5.4), $\textsc{Map}(N, \mathrm{D}\beta)$ *reads as*

$$\textsc{Map}(N, \mathrm{D}\beta) = \textsc{Map}(N, \dot{\Phi}) \times \textsc{Map}(N, \phi). \tag{3.5.5}$$

This is the function space analog of (3.4.5).

Proof. (3.5.5) follows immediately from (3.4.5) through the action of the functor $\textsc{Map}(N, -)$ again. □

Let $\mathfrak{m} = (\mathfrak{e}, \mathfrak{g}, t, m)$ be a Lie algebra crossed module and let $\mathrm{D}\mathfrak{m}$ be its derived Lie algebra (cf. subsect. 3.4). The internal function space $\textsc{Map}(N, \mathrm{D}\mathfrak{m})$ is then a virtual infinite dimensional graded Lie algebra. $\textsc{Map}(N, \mathrm{D}\mathfrak{m})$ is best described by observing that it is isomorphic to a subalgebra of the virtual Lie algebra $\textsc{Map}(\mathbb{R}[1] \times N, \mathfrak{e} \rtimes_m \mathfrak{g})$. Indeed, by (3.4.6), an element $S \in \textsc{Map}(N, \mathrm{D}\mathfrak{m})$ can be decomposed in a pair of internal functions $j \in \textsc{Map}(N, \mathfrak{g})$, $J \in \textsc{Map}(N, \mathfrak{e}[1])$, which together in turn define the internal function $S \in \textsc{Map}(\mathbb{R}[1] \times N, \mathfrak{e} \rtimes_m \mathfrak{g})$, denoted by the same symbol, given by

$$S(\alpha) = j + \alpha J, \quad \alpha \in \mathbb{R}[1]. \tag{3.5.6}$$

Further, on account of (3.4.7), for any $S, T \in \mathrm{MAP}(N, \mathrm{D}\mathfrak{m})$ with $S(\alpha) = j + \alpha J$, $T(\alpha) = k + \alpha K$, we have

$$[S, T](\alpha) = [j, k] + \alpha(m(j, K) - m(k, J)). \tag{3.5.7}$$

Again, so, the structure of the elements and the Lie bracket of $\mathrm{MAP}(N, \mathrm{D}\mathfrak{m})$ is formally the same as that of $\mathrm{D}\mathfrak{m}$ given in eqs. (3.4.6) and (3.4.7) except for an inversion of degrees.

Proposition **3.17.** *The virtual infinite dimensional graded Lie algebra isomorphism*

$$\mathrm{MAP}(N, \mathrm{D}\mathfrak{m}) \simeq \mathrm{MAP}(N, \mathfrak{e}[1]) \rtimes_{\mathrm{MAP}(N,m)} \mathrm{MAP}(N, \mathfrak{g}) \tag{3.5.8}$$

holds true.

(3.5.8) is the function space analog of (3.4.8).

Proof. (3.5.8) follows from (3.4.8) by acting with the functor $\mathrm{MAP}(N, -)$. □

Proposition **3.18.** *With every Lie algebra crossed module morphism* $p : \mathfrak{m}' \to \mathfrak{m} = (H : \mathfrak{e}' \to \mathfrak{e}, h : \mathfrak{g}' \to \mathfrak{g})$ *there is associated a virtual infinite dimensional graded Lie algebra morphism* $\mathrm{MAP}(N, \mathrm{D}p) : \mathrm{MAP}(N, \mathrm{D}\mathfrak{m}') \to \mathrm{MAP}(N, \mathrm{D}\mathfrak{m})$. *With respect to the factorization* (3.5.8), $\mathrm{MAP}(N, \mathrm{D}p)$ *reads explicitly as*

$$\mathrm{MAP}(N, \mathrm{D}p) = \mathrm{MAP}(N, H) \times \mathrm{MAP}(N, h). \tag{3.5.9}$$

This is the function space analog of (3.4.9).

Proof. Relation (3.5.9) follows readily from (3.4.9) through the action of the functor $\mathrm{MAP}(N, -)$ again. □

The derived Lie algebra $\mathrm{D}\mathfrak{m}$ has a full degree prolongation

$$\mathrm{ZD}\mathfrak{m} = \mathrm{Z}\mathfrak{g} \oplus \mathrm{Z}\mathfrak{e} = \textstyle\bigoplus_{p=-\infty}^{\infty} \mathrm{D}\mathfrak{m}[p], \tag{3.5.10}$$

where $\mathrm{D}\mathfrak{m}[p] = \mathfrak{e}[p+1] \oplus \mathfrak{g}[p]$ with $\mathrm{D}\mathfrak{m}[0] = \mathrm{D}\mathfrak{m}$ as a graded vector space

(cf. eq. (A.0.9)). As $\mathrm{D}\mathfrak{m}$ is a graded Lie algebra, the internal function space $\mathrm{MAP}(N, \mathrm{ZD}\mathfrak{m})$ is a virtual infinite dimensional graded Lie algebra containing $\mathrm{MAP}(N, \mathrm{D}\mathfrak{m})$ as a Lie subalgebra. $\mathrm{MAP}(N, \mathrm{ZD}\mathfrak{m})$ is moreover differential, as we now describe. To begin with, we note that $\mathrm{MAP}(N, \mathrm{ZD}\mathfrak{m})$ can be identified with a subalgebra of the virtual Lie algebra $\mathrm{MAP}(\mathbb{R}[1] \times N, \mathrm{Z}(\mathfrak{e} \rtimes_m \mathfrak{g}))$. Indeed, by (3.5.10), $\mathrm{MAP}(N, \mathrm{ZD}\mathfrak{m})$ decomposes as the direct sum

$$\mathrm{MAP}(N, \mathrm{ZD}\mathfrak{m}) = \bigoplus_{p=-\infty}^{\infty} \mathrm{MAP}(N, \mathrm{D}\mathfrak{m}[p]). \tag{3.5.11}$$

From (3.4.6), reasoning as we did earlier, an element $S \in \mathrm{MAP}(N, \mathrm{D}\mathfrak{m}[p])$ comprises internal functions $j \in \mathrm{MAP}(N, \mathfrak{g}[p])$, $J \in \mathrm{MAP}(N, \mathfrak{e}[p+1])$ combining in an internal function $S \in \mathrm{MAP}(\mathbb{R}[1] \times N, (\mathfrak{e} \rtimes_m \mathfrak{g})[p])$ given by

$$S(\alpha) = j + (-1)^p \alpha J, \quad \alpha \in \mathbb{R}[1]. \tag{3.5.12}$$

Further, on account of (3.4.7), one finds that the Lie bracket of a couple of elements $S \in \mathrm{MAP}(N, \mathrm{D}\mathfrak{m}[p])$, $T \in \mathrm{MAP}(N, \mathrm{D}\mathfrak{m}[q])$ such that $S(\alpha) = j + (-1)^p \alpha J$, $T(\alpha) = k + (-1)^q \alpha K$ is the element $[S, T] \in \mathrm{MAP}(N, \mathrm{D}\mathfrak{m}[p+q])$ given by

$$[S, T](\alpha) = [j, k] + (-1)^{p+q} \alpha (m(j, K) - (-1)^{pq} m(k, J)). \tag{3.5.13}$$

Identifying $\mathrm{D}\mathfrak{m}[0]$ with $\mathrm{D}\mathfrak{m}$ as usual, it is apparent here that $\mathrm{MAP}(N, \mathrm{ZD}\mathfrak{m})$ contains $\mathrm{MAP}(N, \mathrm{D}\mathfrak{m})$ as a virtual graded Lie subalgebra. In addition to a Lie algebra structure, $\mathrm{MAP}(N, \mathrm{ZD}\mathfrak{m})$ is also endowed with a cochain complex structure,

$$\begin{aligned} \cdots \xrightarrow{d_t} \mathrm{MAP}(N, \mathrm{D}\mathfrak{m}[p-1]) \xrightarrow{d_t} \mathrm{MAP}(N, \mathrm{D}\mathfrak{m}[p]) \\ \xrightarrow{d_t} \mathrm{MAP}(N, \mathrm{D}\mathfrak{m}[p+1]) \xrightarrow{d_t} \cdots, \end{aligned} \tag{3.5.14}$$

where the coboundary d_t acts as

$$d_t S(\alpha) = t\left(\frac{d}{d\alpha} S(\alpha)\right) = (-1)^p t(J) \tag{3.5.15}$$

with $S \in \mathrm{MAP}(N, \mathrm{D}\mathfrak{m}[p])$ of the form (3.5.12).

Under prolongation, the isomorphism in eq. (3.5.8) generalizes as stated in the next proposition.

Proposition **3.19.** *There is a virtual infinite dimensional differential graded Lie algebra isomorphism*

$$\mathrm{MAP}(N, \mathrm{ZD}\mathfrak{m}) \simeq \mathrm{MAP}(N, \mathrm{Z}\mathfrak{e}) \rtimes_{\mathrm{MAP}(N, s^{-1}t), \mathrm{MAP}(N,m)} \mathrm{MAP}(N, \mathrm{Z}\mathfrak{g}). \tag{3.5.16}$$

Above, $\mathrm{MAP}(N, \mathrm{Z}\mathfrak{e}) \rtimes_{\mathrm{MAP}(N,t), \mathrm{MAP}(N,m)} \mathrm{MAP}(N, \mathrm{Z}\mathfrak{g})$ denotes the differential semidirect product of the graded Lie algebras $\mathrm{MAP}(N, \mathrm{Z}\mathfrak{e})$ and $\mathrm{MAP}(N, \mathrm{Z}\mathfrak{g})$, that is the ordinary Lie algebra semidirect product $\mathrm{MAP}(N, \mathrm{Z}\mathfrak{e}) \rtimes_{\mathrm{MAP}(N,m)} \mathrm{MAP}(N, \mathrm{Z}\mathfrak{g})$ with the cochain complex structure $\mathrm{MAP}(N, s^{-1}t) : \mathrm{MAP}(N, \mathrm{Z}\mathfrak{e}) \to \mathrm{MAP}(N, \mathrm{Z}\mathfrak{g})$, where s^{-1} denotes the desuspension isomorphism lowering degree by one unit.

Proof. (3.5.16) follows by inspection of the structure of the Lie bracket $[-,-]$ shown in eq. (3.5.13) and the form of the coboundary d_t shown in eq. (3.5.15). and the fact that d_t differentiates the Lie bracket (3.5.13) as it can be checked by a straightforward calculation. □

The d_t–cohomology of the complex is $\mathrm{MAP}(N, \mathrm{ZD}\mathfrak{h})$, where $\mathfrak{h}$ is the Lie algebra crossed module $(\ker t, \mathfrak{g}/\operatorname{ran} t)$ whose target morphism vanishes and whose $\mathfrak{g}/\operatorname{ran} t$–action is that induced by m. We shall not need this result however.

Proposition **3.20.** *Every Lie algebra crossed module morphism* $p : \mathfrak{m}' \to \mathfrak{m} = (H : \mathfrak{e}' \to \mathfrak{e},\ h : \mathfrak{g}' \to \mathfrak{g})$ *induces a virtual infinite dimensional differential graded Lie algebra morphism* $\mathrm{MAP}(N, \mathrm{ZD}p) : \mathrm{MAP}(N, \mathrm{ZD}\mathfrak{m}') \to \mathrm{MAP}(N, \mathrm{ZD}\mathfrak{m})$. $\mathrm{MAP}(N, \mathrm{ZD}p)$ *reads explicitly as*

$$\mathrm{MAP}(N, \mathrm{ZD}p) = \mathrm{MAP}(N, \mathrm{Z}H) \times \mathrm{MAP}(N, \mathrm{Z}h), \tag{3.5.17}$$

where $\mathrm{Z}H$, $\mathrm{Z}h$ are the prolongations of H, h, respectively.

Proof. This follows from the factorization (3.4.9) under full degree extension for the semidirect product structure and and from relation (3.3.7) for the differential structure. □

If $\mathfrak{m}$ is the Lie algebra crossed module of a Lie group crossed module M, $\mathrm{MAP}(N, \mathrm{D}\mathfrak{m})$ is the virtual Lie algebra of $\mathrm{MAP}(N, \mathrm{D}\mathsf{M})$. The adjoint action of $\mathrm{D}\mathsf{M}$ on $\mathrm{D}\mathfrak{m}$ induces an adjoint action of $\mathrm{MAP}(N, \mathrm{D}\mathsf{M})$ on $\mathrm{MAP}(N, \mathrm{ZD}\mathfrak{m})$.

Proposition 3.21. *For $F \in \mathrm{MAP}(N, \mathrm{D}\mathsf{M})$ and $S \in \mathrm{MAP}(N, \mathrm{D}\mathfrak{m}[p])$ respectively of the form (3.5.1) and (3.5.12), one has*

$$\mathrm{Ad}\, F(S)(\alpha) = \mathrm{Ad}\, m(j) + (-1)^p \alpha(\dot{\mu}(m, J) - \dot{\mu}(\mathrm{Ad}\, m(j), Q)), \tag{3.5.18}$$

$$\mathrm{Ad}\, F^{-1}(S)(\alpha) = \mathrm{Ad}\, m^{-1}(j) + (-1)^p \alpha \dot{\mu}(m^{-1}, J + \dot{\mu}(j, Q)). \tag{3.5.19}$$

Proof. Relations (3.5.18), (3.5.19) are an immediate consequence if identities (3.4.12), (3.4.13). □

Proposition 3.22. *For a Lie group crossed module morphism $\beta : \mathsf{M}' \to \mathsf{M}$ with associated Lie algebra crossed module morphism $\dot{\beta} : \mathfrak{m}' \to \mathfrak{m}$, the virtual Lie algebra morphism $\mathrm{MAP}(N, \mathrm{D}\dot{\beta})$ is the virtual Lie differential of the Lie group morphism $\mathrm{MAP}(N, \mathrm{D}\beta)$,*

$$\dot{\mathrm{MAP}}(N, \mathrm{D}\beta) = \mathrm{MAP}(N, \mathrm{D}\dot{\beta}). \tag{3.5.20}$$

This is the function space counterpart of relation (3.4.17).

Proof. There really is no proof to be given here. In the spirit of the virtual Lie theory, (3.5.20) is essentially the definition of $\dot{\mathrm{MAP}}(N, \mathrm{D}\beta)$. □

Suppose that the manifold N is equipped with a degree p derivation D. Given an internal function $C \in \mathrm{MAP}(N, \mathrm{D}\mathsf{M})$, one can then construct elements DCC^{-1}, $C^{-1}DC \in \mathrm{MAP}(N, \mathrm{D}\mathfrak{m}[p])$ by pull–back C of the Maurer–Cartan forms dMM^{-1}, $M^{-1}dM$ of $\mathrm{D}\mathsf{M}$ (cf. def. 3.17) followed by contraction with D seen as a vector field on N. Explicit formulae can be obtained by expressing C as

$$C(\alpha) = \mathrm{e}^{\alpha O}\, r, \quad \alpha \in \mathbb{R}[1], \tag{3.5.21}$$

with $r \in \mathrm{MAP}(N, \mathsf{G})$, $O \in \mathrm{MAP}(N, \mathfrak{e}[1])$, in conformity with (3.5.1),

Proposition 3.23. *DCC^{-1}, $C^{-1}DC$ are given explicitly by*

$$DCC(\alpha)^{-1} = Drr^{-1} + (-1)^p \alpha(DO - \dot{\mu}(Drr^{-1}, O)), \tag{3.5.22}$$

$$C^{-1}DC(\alpha) = r^{-1}Dr + (-1)^p \alpha \dot{\mu}(r^{-1}, DO). \tag{3.5.23}$$

Proof. Relations (3.5.22), (3.5.23) are obtained proceeding as indicated above

using the expressions of dMM^{-1}, $M^{-1}dM$ given in eqs. (3.4.19), (3.4.20). □

The coboundary $d_{\dot\tau}$ of $\mathrm{MAP}(N, \mathrm{ZD}\mathfrak{m})$ as a cochain complex allows for the construction of two special elements $d_{\dot\tau}CC^{-1}$, $C^{-1}d_{\dot\tau}C \in \mathrm{MAP}(N, \mathrm{D}\mathfrak{m}[1])$. Explicit formulae can be obtained by expressing again C as in (3.5.21).

Proposition **3.24.** *$d_{\dot\tau}CC^{-1}$, $C^{-1}d_{\dot\tau}C$ are given by the expressions*

$$d_{\dot\tau}CC(\alpha)^{-1} = \dot\tau(O) + \frac{1}{2}\alpha[O,O], \tag{3.5.24}$$

$$C^{-1}d_{\dot\tau}C(\alpha) = \mathrm{Ad}\, r^{-1}(\dot\tau(O)) - \frac{1}{2}\alpha\dot\mu\,(r^{-1},[O,O]). \tag{3.5.25}$$

Proof. (3.5.24), (3.5.25) follow straightforwardly from the well–known variational identities $\delta\, \mathrm{e}^{\alpha O}\, \mathrm{e}^{-\alpha O} = \frac{\exp(\alpha\,\mathrm{ad}\, O)-1}{\alpha\,\mathrm{ad}\, O}\delta(\alpha O)$, $\mathrm{e}^{-\alpha O}\, \delta\, \mathrm{e}^{\alpha O} = \frac{1-\exp(-\alpha\,\mathrm{ad}\, O)}{\alpha\,\mathrm{ad}\, O}\delta(\alpha O)$ with $\delta = d_{\dot\tau} = \dot\tau d/d\alpha$ and taking the nilpotence of α into account. □

3.6 Derived Lie group and algebra cross modality

In the setup of subsect. 3.5, one may consider the degenerate finite dimensional case where the graded manifold N is the singleton manifold $*$. It is then natural to express everything in terms of the cross functor $(-)^+ = \mathrm{MAP}(*, -)$ introduced in app. A. This allows us to obtain a "cross modality" of the derived Lie group and algebras introduced in subsect. 3.4.

In subsect. 3.4, it has been shown that with a given Lie group crossed module $\mathsf{M} = (\mathsf{E}, \mathsf{G}, \tau, \mu)$ there is associated a derived Lie group $\mathrm{D}\mathsf{M}$. Through the cross functor $(-)^+$, the crossed modality derived Lie group $\mathrm{D}\mathsf{M}^+$ can be also constructed. $\mathrm{D}\mathsf{M}^+$ enjoys properties analogous to those of $\mathrm{D}\mathsf{M}$. The elements $F \in \mathrm{D}\mathsf{M}^+$ are of the form (3.5.1) with $m \in \mathsf{G}^+ \simeq \mathsf{G}$, $Q \in \mathfrak{e}[1]^+ \simeq \mathbb{R}[-1] \otimes \mathfrak{e}$ and the group operations read as in (3.5.2), (3.5.3), analogously to (3.4.1) and (3.4.2), (3.4.3). By (3.5.4), we have further that $\mathrm{D}\mathsf{M}^+ \simeq \mathfrak{e}[1]^+ \rtimes_{\dot\mu^+} \mathsf{G}^+$ as graded Lie groups, similarly to (3.4.4). Finally, by virtue of (3.5.5), for any Lie group crossed module morphism $\beta : \mathsf{M}' \to \mathsf{M} = (\Phi : \mathsf{E}' \to \mathsf{E}, \phi : \mathsf{G}' \to \mathsf{G})$ we have a graded Lie group morphism $\mathrm{D}\beta^+ : \mathrm{D}\mathsf{M}'^+ \to \mathrm{D}\mathsf{M}^+$ with $\mathrm{D}\beta^+ = \dot\Phi^+ \times \phi^+$, analogously to (3.4.5). The derived Lie group $\mathrm{D}\mathsf{M}$ and its cross modality $\mathrm{D}\mathsf{M}^+$ are

however related in a deeper way as now we explain.

The graded structure of DM stems from its being the subgroup of the internal function Lie group $\textsc{Map}(\mathbb{R}[-1], \mathsf{E} \rtimes_\mu \mathsf{G})$ formed by the elements of the form (3.4.1). The degree 1 carried by the variable $\bar{\alpha} \in \mathbb{R}[-1]$ is excluded in the degree counting. Upon including it instead, DM becomes a graded Lie group concentrated in degree 0, that can be treated as an ordinary Lie group. Completely analogous considerations apply for the cross modality derived Lie group DM^+. In the following, we shall regard these Lie groups in the way just described.

Definition **3.18.** *Let* $z_{\mathsf{M}} : \mathsf{DM} \to \mathsf{DM}^+$ *be the map defined by*

$$z_{\mathsf{M}} P(\alpha) = \mathrm{e}^{\alpha\, \zeta_{\mathfrak{e},1}(L)}\, a, \quad \alpha \in \mathbb{R}[1], \tag{3.6.1}$$

for $P \in \mathsf{DM}$ *of the form* (3.4.1).

Above, $\zeta_{\mathfrak{e},1} : \mathfrak{e}[1] \xrightarrow{\simeq} \mathfrak{e}[1]^+$ is the suspension isomorphism defined in app. A.

Proposition **3.25.** z_{M} *is a Lie group isomorphism.*

Proof. Using (3.4.2), the linearity of $\zeta_{\mathfrak{e},1}$ and (3.5.2), one readily checks that z_{M} is a Lie group morphism. The invertibility of z_{M} follows from that of $\zeta_{\mathfrak{e},1}$. □

In subsect. 3.4, it has been also shown that with a given Lie algebra crossed module $\mathfrak{m} = (\mathfrak{e}, \mathfrak{g}, t, m)$ there is associated a derived Lie algebra $\mathrm{D}\mathfrak{m}$. Through the cross functor $(-)^+$, the crossed modality derived Lie algebra $\mathrm{D}\mathfrak{m}^+$ can be also constructed. $\mathrm{D}\mathfrak{m}^+$ enjoys properties analogous to those of $\mathrm{D}\mathfrak{m}$. The elements $S \in \mathrm{D}\mathfrak{m}^+$ are of the form (3.5.6) with $j \in \mathfrak{g}^+ \simeq \mathfrak{g}$, $J \in \mathfrak{e}[1]^+ \simeq \mathbb{R}[-1] \otimes \mathfrak{e}$ and the Lie algebra operations read as in (3.5.7), similarly to (3.4.6) and (3.4.7). By (3.5.8), we have also that $\mathrm{D}\mathfrak{m}^+ \simeq \mathfrak{e}[1]^+ \rtimes_{m^+} \mathfrak{g}^+$ as graded Lie algebras, analogously to (3.4.8). Finally, by virtue of (3.5.9), with every Lie algebra crossed module morphism $p : \mathfrak{m}' \to \mathfrak{m} = (H : \mathfrak{e}' \to \mathfrak{e}, h : \mathfrak{g}' \to \mathfrak{g})$ there is associated a graded Lie algebra morphism $\mathrm{D}p^+ : \mathrm{D}\mathfrak{m}'^+ \to \mathrm{D}\mathfrak{m}^+$ with $\mathrm{D}p^+ = H^+ \times h^+$, similarly to (3.4.9) Just as in the group case, the derived Lie algebra $\mathrm{D}\mathfrak{m}$ and its cross modality $\mathrm{D}\mathfrak{m}^+$ are related also in another deeper way.

The graded structure of $\mathrm{D}\mathfrak{m}$ stems from its being the subalgebra of the internal

function Lie algebra $\textsc{Map}(\mathbb{R}[-1], \mathfrak{e} \rtimes_m \mathfrak{g})$ formed by the elements of the form (3.4.6). Including the degree 1 of the variable $\bar{\alpha} \in \mathbb{R}[-1]$ in the degree counting, $\mathrm{D}\mathfrak{m}$ becomes a graded Lie algebra concentrated in degree 0 and as such can be treated as an ordinary Lie algebra. Completely analogous considerations apply for the cross mode derived Lie algebra $\mathrm{D}\mathfrak{m}^+$. In the following, as in the group case, we shall regard these Lie algebras in the way just described.

Definition **3.19.** *Let* $\zeta_\mathfrak{m} : \mathrm{D}\mathfrak{m} \to \mathrm{D}\mathfrak{m}^+$ *be the map defined by*

$$\zeta_\mathfrak{m} Y(\alpha) = u + \alpha\, \zeta_{\mathfrak{e},1}(U), \quad \alpha \in \mathbb{R}[1], \tag{3.6.2}$$

for $Y \in \mathrm{D}\mathfrak{m}$ *of the form* (3.4.6)*.*

Proposition **3.26.** $\zeta_\mathfrak{m}$ *is a Lie algebra isomorphism.*

Proof. Using (3.4.7), the linearity of $\zeta_{\mathfrak{e},1}$ and (3.5.7), one readily checks that $\zeta_\mathfrak{m}$ is a Lie algebra morphism. The invertibility of $\zeta_\mathfrak{m}$ follows from that of $\zeta_{\mathfrak{e},1}$. □

When $\mathfrak{m}$ is the Lie algebra crossed module of a Lie group crossed module M, $\mathrm{D}\mathfrak{m}^+$ is the Lie algebra of the Lie group $\mathrm{D}\mathsf{M}^+$. Further, for a Lie group crossed module morphism $\beta : \mathsf{M}' \to \mathsf{M}$ with associated Lie algebra crossed module morphism $\dot{\beta} : \mathfrak{m}' \to \mathfrak{m}$, the Lie algebra morphism $\mathrm{D}\dot{\beta}^+$ is the Lie differential $\dot{\mathrm{D}}\beta^+$ of the Lie group morphism $\mathrm{D}\beta^+$.

Proposition **3.27.** $\zeta_\mathfrak{m}$ *is the Lie differential of* z_M*.*

$$\zeta_\mathfrak{m} = \dot{z}_\mathsf{M}. \tag{3.6.3}$$

Proof. This is apparent from inspecting (3.6.1), (3.6.2). □

3.7 Lie group crossed module spaces

The morphism and object manifolds of a principal 2–bundle are equipped with the right action of the morphism and object groups of the structure Lie 2–group, respectively. They are so special cases of Lie group spaces (cf. def. 2.5). In the synthetic crossed module theoretic version of the theory, Lie group spaces of this kind is instances of Lie group crossed module spaces, a notion we introduce next.

Definition **3.20.** *A Lie group crossed module space S is a Lie group space of the special form (P, DM, R) for some graded manifold P, Lie group crossed module M and right action R of DM on P.*

Above, the derived Lie group DM (cf. subsect. 3.4) is regarded as an ordinary Lie group in the sense explained in subsect. 3.6. We shall denote the space S through the list (P, M, R) of its defining data.

A notion of Lie group crossed module space morphism can be formulated as a specialization of that of Lie group space morphism. (cf. def. 2.6).

Definition **3.21.** *A morphism $T : S' \to S$ of Lie group crossed module spaces consists in a morphism of the Lie group spaces (P', DM', R'), (P, DM, R) underlying S', S of the special form $(F : P' \to P,\ \mathrm{D}\beta : \mathsf{DM}' \to \mathsf{DM})$ for some graded manifold morphism $F : P' \to P$ and Lie group crossed module morphism $\beta : \mathsf{M}' \to \mathsf{M}$.*

Above, $\mathrm{D}\beta$ is the derived Lie group morphism of β (cf. subsect. 3.4). We shall denote the morphism $T : S' \to S$ by the list $(F : P' \to P, \beta : \mathsf{M}' \to \mathsf{M})$ of its defining data, omitting indicating sources and targets when possible.

In this way, Lie group crossed module spaces and their morphisms form a category **Lcmsp** that can be identified with a subcategory of the category **Lsp** of Lie group spaces defined in subsect. 2.1.

With any Lie group crossed module space there is associated an operation on general grounds (cf. subsect. 2.1, prop. 2.1).

Definition **3.22.** *For any Lie group crossed module space $S = (P, \mathsf{M}, R)$, we let $\mathrm{Op}S$ be the operation $(\mathrm{Fun}(T[1]P), \mathrm{D}\mathfrak{m})$ of the Lie group space (P, DM, R) underlying S.*

Above, $\mathfrak{m}$ is the Lie algebra crossed module associated with M (cf. subsect. 3.4). Further, $\mathrm{D}\mathfrak{m}$ is regarded as an ordinary Lie algebra in the sense explained in subsect. 3.6. In keeping with our notational conventions, we shall denote $\mathrm{Op}S$ concisely as $(\mathrm{Fun}(T[1]P), \mathfrak{m})$. We shall also denote the operation derivations by d_P, j_P, l_P. In similar fashion, any morphism of Lie group crossed module spaces

determines a morphism of the associated operations again on general grounds (cf. subsect. 2.1, prop. 2.2).

Definition **3.23.** *For any Lie group crossed module space morphism* $T : S' \to S = (F : P' \to P, \beta : \mathsf{M}' \to \mathsf{M})$*, we denote by* $\mathrm{Op}T : \mathrm{Op}S \to \mathrm{Op}S'$ *the morphism* $(F^* : \mathrm{Fun}(T[1]P) \to \mathrm{Fun}(T[1]P'), \mathrm{D}\dot{\beta} : \mathrm{D}\mathfrak{m}' \to \mathrm{D}\mathfrak{m})$ *of the operations* $(\mathrm{Fun}(T[1]P), \mathrm{D}\mathfrak{m})$*,* $(\mathrm{Fun}(T[1]P'), \mathrm{D}\mathfrak{m}')$ *underlying* $\mathrm{Op}S$*,* $\mathrm{Op}S'$*.*

Above, $\dot{\beta}$ is the Lie algebra crossed module morphism yielded by the Lie group crossed module morphism β by Lie differentiation and $\mathrm{D}\dot{\beta}$ is the derived Lie algebra morphism of $\dot{\beta}$ (cf. subsects. 3.3, 3.4).

Consider the special case of a morphism $T : S' \to S$ of Lie group crossed module spaces specified by a pair $(F : P' \to P, \beta : \mathsf{M}' \to \mathsf{M})$, where $P = P'$ as graded manifolds and $F = \mathrm{id}_P$ as a graded manifold map. We then denote S' by β^*S and call it the pull-back of S by β and the operation $\mathrm{Op}S'$ by $\dot{\beta}^* \mathrm{Op}S$ and call it the pull-back by $\dot{\beta}$ of $\mathrm{Op}S$, so that $\mathrm{Op}\beta^*S = \dot{\beta}^* \mathrm{Op}S$. In fact, as Lie group spaces, S' is just the pull–back $\mathrm{D}\beta^*S$ of S by the Lie group morphism $\mathrm{D}\beta$. Furthermore, as operations of Lie group spaces, $\mathrm{Op}S'$ is just the pull–back $\mathrm{D}\dot{\beta}^* \mathrm{Op}S$ of the operation $\mathrm{Op}S$ by the Lie algebra morphism $\mathrm{D}\dot{\beta}$. See again subsect. 2.1.

In the spirit of subsect. 2.1 above, we can think of a Lie group crossed module space $S = (P, \mathsf{M}, R)$ as a generalized principal $\mathrm{D}\mathsf{M}$–bundle over $P/\mathrm{D}\mathsf{M}$. In this way, we can identify the complexes $(\mathrm{Fun}(T[1]P), d_P)$ and $(\mathrm{Fun}(T[1]P)_{\mathrm{basic}}, d_P)$ of the associated operation $\mathrm{Op}S$ with the de Rham complexes $(\Omega^\bullet(P), d_{dR\,P})$ and $(\Omega^\bullet(P/\mathrm{D}\mathsf{M}), d_{dR\,P/\mathrm{D}\mathsf{M}})$, respectively.

3.8 Total space operations of a principal 2–bundle

In this final subsection, we introduce and study the morphism and object space of a principal 2–bundle and their associated operations. In this way, we have set the foundations for the operational total space theory of principal 2–bundles, in particular of the 2–connection and 1– and 2– gauge transformation theory of II, which is the goal of the present endeavour.

Theor. 3.2 states that any strict Lie 2–group $\hat{\mathscr{K}}$ is fully described by a Lie group crossed module.

Proposition 3.28. *The Lie group crossed module codifying* $\hat{\mathscr{K}}$ *is* $\mathsf{M} = (\mathsf{E}, \mathsf{G}, \tau, \mu)$, *where* $\mathsf{E} = \ker \hat{s}$, $\mathsf{G} = \hat{\mathsf{K}}_0$, $\tau = \hat{\epsilon}$ *and* $\mu = \hat{\lambda}$.

Here, $\hat{s}$ is the source map of $\hat{\mathsf{K}}$ and $\hat{\lambda}$ and $\hat{\epsilon}$ are the action and target maps defined in eqs. (3.2.2) and (3.2.3), respectively.

Proof. This is just a restatement of theor. 3.2 in the notation of subsect. 3.3. □

Proposition 3.29. *The Lie group crossed module* M *contains the submodule* $\mathsf{M}_0 = (1_{\mathsf{E}}, \mathsf{G}, \tau_0, \mu_0)$, *where* τ_0 *and* μ_0 *are the restrictions of* τ *and* μ *to* $1_{\mathsf{E}} \subset \mathsf{E}$ *and* $\mathsf{G} \times 1_{\mathsf{E}} \subset \mathsf{G} \times \mathsf{E}$, *respectively.*

Note that τ_0 and μ_0 are necessarily trivial.

Proof. The statement follows from τ_0 and μ_0 being restrictions of τ and μ. □

In fact, M_0 is just the discrete crossed module of the group G. M_0 provides a crossed module theoretic coding of G, which will turn out to be quite useful in the following.

With the Lie Lie 2–group $\hat{\mathscr{K}}$, there are associated the synthetic Lie groups K, K_0 (cf. def. 3.6). Recalling the derived construction expounded in subsect. 3.4, it is apparent that K is nothing but the derived Lie group $\mathrm{D}\mathsf{M}$ of M

$$\mathsf{K} = \mathrm{D}\mathsf{M}. \tag{3.8.1}$$

Analogously, K_0 can be described as the derived group $\mathrm{D}\mathsf{M}_0$,

$$\mathsf{K}_0 = \mathrm{D}\mathsf{M}_0. \tag{3.8.2}$$

In the synthetic formulation, with a principal $\hat{\mathscr{K}}$–2–bundle $\hat{\mathcal{P}}$ there are associated synthetic forms of the morphism and object manifolds $\hat{P}$, $\hat{P}_0$, viz P, P_0 (cf. def. 3.8). Importantly, P, P_0 are equipped with right K–, K_0–actions R, R_0, respectively (cf. def. 3.11 and prop. 3.3).

By (3.8.1), the K–action R of P can be expressed as one of $\mathrm{D}\mathsf{M}$. Relying on

framework constructed in subsect. 3.7, we can then state the following definition.

Definition **3.24.** *The morphism space of the principal* 2*–bundle* P *is the Lie group crossed module space* $S_P = (P, \mathsf{M}, R)$.

Similarly, by (3.8.2), the K_0 –action R_0 of P_0 can be expressed as one of $\mathrm{D}\mathsf{M}_0$. In the same framework, we can state a further definition.

Definition **3.25.** *The object space of the principal* 2*–bundle* P *is the Lie group crossed module space* $S_{P0} = (P_0, \mathsf{M}_0, R_0)$.

As detailed in subsect. 3.7, with the spaces S_P and S_{P0} there are associated operations $\mathrm{Op}\, S_P = (\mathrm{Fun}(T[1]P), \mathfrak{m})$ and $\mathrm{Op}\, S_{P0} = (\mathrm{Fun}(T[1]P_0), \mathfrak{m}_0)$.

S_{P0} and $\mathrm{Op}\, S_{P0}$ should be compatible with S_P and $\mathrm{Op}\, S_P$ in the appropriate sense, as P_0 is a submanifold of P, M_0 is a submodule of M and R_0 is a restriction of R. Such congruity is codified by an appropriate crossed module space morphism and its appended operation morphism along the lines of subsect. 3.7.

Definition **3.26.** *Let* $L : S_{P0} \to S_P$ *be the Lie group crossed module space morphism* $L = (I, \varsigma)$*, where* $I : P_0 \to P$

and $\varsigma : \mathsf{M}_0 \to \mathsf{M}$ *are the inclusion maps of* P_0 *and* M_0 *into* P *and* M*, respectively.*

Associated with L there is a morphism $\mathrm{Op}\, L : \mathrm{Op}\, S_P \to \mathrm{Op}\, S_{P0}$ of operations, namely $\mathrm{Op}\, L = (I^*, \dot{\varsigma})$, $I^* : \mathrm{Fun}(P) \to \mathrm{Fun}(P_0)$ being the restriction pull-back and $\dot{\varsigma} : \mathfrak{m}_0 \to \mathfrak{m}$ the Lie algebra crossed module morphism induced by the ς by Lie differentiation.

Our operational synthetic theory so closely parallels that of the graded geometric version of the ordinary theory reviewed in subsect. 1.1 with a few important differences. To begin with, unlike the ordinary theory, it is not directly based on the relevant principal $\hat{\mathscr{K}}$–2–bundle $\hat{\mathcal{P}}$ but on its attached synthetic setup. This hinges on the synthetic morphism and object groups K, K_0 and manifolds P, P_0, which, as discussed at length in subsect. 3.2, do not constitute a true synthetic $\mathscr{K}$–2–bundle $\mathcal{P}$ in spite of having many properties of one. Moreover, again unlike the ordinary theory, two operations, viz $\mathrm{Op}\, S_P$, $\mathrm{Op}\, S_{P0}$, rather

that just one appear and are potentially relevant. More technically, furthermore, the internal function algebras $\textsc{Fun}(T[1]P)$, $\textsc{Fun}(T[1]P_0)$ instead of the ordinary algebras $\mathrm{Fun}(T[1]P)$, $\mathrm{Fun}(T[1]P_0)$ are used here, a feature that the end is bound to make a difference at the end.

The above dissimilarities notwithstanding, the ordinary theory provides a simple model, reference to which considerably aids intuition. In particular, it is useful to think of the whole synthetic setup of K, K_0 and P, P_0 as if it were some kind of synthetic $\mathscr{K}$–2–bundle $\mathcal{P}$, though, as we have recalled, strictly speaking it is not lacking as it does a compositional structure.

A Notation and conventions

We recall below some of the basic notions and conventions of graded algebra and geometry we use throughout the present paper.

For a pair M, N of graded manifolds, we denote by $\textsc{Map}(M,N)$ the set of internal functions of M into N. Thus, when expressed in terms of local body and soul coordinates t^a and y^r of M, the components of one such function with respect to local body and soul coordinates u^i and z^h of N are polynomials in the y^r with coefficients which are smooth functions of the t^a of possibly non zero degree. When N is a graded vector space, group, Lie algebra etc., so is $\textsc{Map}(M,N)$ with the pointwise operations induced by those of N.

In what follows, E stands for an ungraded finite dimensional real vector space. Much of what we shall say can be formulated also for a graded vector space, but we shall not need to do so.

For any integer p, we denote by $E[p]$ the degree p shift of E, a copy of E placed in degree $-p$. If we conventionally think of E as a graded vector space supported in degree 0, as it is customarily done, we may identify $E[0]$ with E itself.

A linear coordinate of $E[p]$ is just a non zero element of $E[p]^\vee = E^\vee[-p]$ and has hence degree p, where $^\vee$ denotes duality. Given a set of vectors $e_i \in E$ constituting a basis, there exists a unique set of linear coordinates ${x_p}^i$ of $E[p]$ dual to the basis, that is such that ${x_p}^i(e_j) = s^p{\delta^i}_j$, where s^p denotes p–fold suspension raising degree by p units.

If we equip $E[p]$ with a set of linear coordinates ${x_p}^i$ as above, $E[p]$ becomes a graded manifold with singleton body. As such, $E[p]$ is concentrated in degree p because the ${x_p}^i$ have degree p. For a graded manifold M, the set $\textsc{Map}(M,E[p])$ of $E[p]$–valued internal functions of M has so a natural structure of graded vector space supported in degree p.

The particular case where $E=\mathbb{R}$ deserves a special mention for its relevance. For each p, we have the graded vector space

$$\textsc{Fun}_p(M) = \textsc{Map}(M,\mathbb{R}[p]) \tag{A.0.1}$$

of $\mathbb{R}[p]$–valued internal functions. The spaces $\textsc{Fun}_p(M)$ for varying p span together the graded vector space

$$\textsc{Fun}(M) = \bigoplus_{p=-\infty}^{\infty} \textsc{Fun}_p(M). \tag{A.0.2}$$

Thanks to the existence of graded commutative products $\textsc{Fun}_p(M) \times \textsc{Fun}_q(M) \to \textsc{Fun}_{p+q}(M)$, $\textsc{Fun}(M)$ is a graded commutative algebra. $\textsc{Fun}(M)$ is just the algebra of internal functions of M and for each p $\textsc{Fun}_p(M)$ is the subspace of $\textsc{Fun}(M)$ of degree p internal functions.

There exists a canonical graded vector space isomorphism

$$\textsc{Map}(M, E[p]) \simeq \textsc{Fun}_p(M) \otimes E. \tag{A.0.3}$$

Indeed, upon choosing basis vectors e_i of E, a given function $f \in \textsc{Map}(M, E[p])$ is fully specified by a set of degree p internal functions $f^i \in \textsc{Fun}_p(M)$, the components of f with respect to the linear coordinates ${x_p}^i$ dual to the e_i. The f^i in turn define an element $\hat{f} \in \textsc{Fun}_p(M) \otimes E$ given by

$$\hat{f} = f^i \otimes e_i. \tag{A.0.4}$$

$\hat{f}$ is by construction independent from the choice of the basis e_i. The correspondence $f \to \hat{f}$ yields the isomorphism (A.0.3). By (A.0.3), the functions of $\textsc{Map}(M, E[p])$ can be regarded as specially structured collections of functions of $\textsc{Fun}_p(M)$. In this way, further, any linear operation on $\textsc{Fun}(M)$, e. g. a derivation, immediately induces a corresponding operation on $\textsc{Map}(M, E[p])$.

The function space $\textsc{Map}(E[p], E[p])$ contains a tautological element x_p,

$$x_p = {x_p}^i \otimes e_i, \tag{A.0.5}$$

of degree p. x_p corresponds to the identity function $\mathrm{id}_{E[p]}$ of $E[p]$ regarded as a graded manifold.

The internal function spaces $\textsc{Map}(*, N)$ with $*$ the singleton manifold play an important role in the analysis of this paper. $\textsc{Map}(*, -)$ is a functor from the category of graded manifolds to the category of sets. We call it the cross functor and denote it with the simplified notation $(-)^+$.

When restricted to graded vector spaces regarded as graded manifolds, the cross functor can be described rather explicitly. For $\mathbb{R}$, by (A.0.1), we have

$$\mathbb{R}[p]^+ = \textsc{Fun}_p(*) \simeq \mathbb{R}[-p]. \tag{A.0.6}$$

By (A.0.3), for the vector space E, we have similarly

$$E[p]^+ \simeq \textsc{Fun}_p(*) \otimes E \simeq \mathbb{R}[-p] \otimes E. \tag{A.0.7}$$

Thus, by (A.0.4), chosen a basis e_i of E an element $v \in E[p]^+$ expands as

$$v = v^i \otimes e_i, \tag{A.0.8}$$

where $v^i \in \mathbb{R}[-p]$. Notice that $\mathbb{R}[p]^+$ differs from $\mathbb{R}[p]$ for $p \neq 0$, being in fact $\mathbb{R}[p]^+ = s^{2p}\mathbb{R}[p]$. Similarly, $E[p]^+$ differs from $E[p]$ but it is related to $E[p]$ be means of the $2p$–fold suspension isomorphism $\zeta_{E,p} : E[p] \xrightarrow{\simeq} E[p]^+$ defined by the sequence $E[p] \simeq \mathbb{R} \otimes E[p] \xrightarrow{s^p \otimes s^p} s^p\mathbb{R} \otimes s^p E[p] \simeq \mathbb{R}[-p] \otimes E \simeq E[p]^+$.

It is sometimes useful to assemble the degree shifts $E[p]$ of E for all p's into the full degree extension $\mathrm{Z}E$ of E, the infinite dimensional graded vector space

$$\mathrm{Z}E = \bigoplus_{p=-\infty}^{\infty} E[p]. \tag{A.0.9}$$

Similarly, we can assemble the $E[p]$–valued internal functions for all p's into the graded vector space

$$\textsc{Map}(M, \mathrm{Z}E) = \bigoplus_{p=-\infty}^{\infty} \textsc{Map}(M, E[p]). \tag{A.0.10}$$

When E is endowed with additional structures such as those of graded commutative algebra, graded Lie algebra, etc, these are inherited by $\textsc{Map}(M, \mathrm{Z}E)$.

An important instance is the full degree extension $\mathrm{Z}\mathbb{R}$ of $\mathbb{R}$. $\textsc{Map}(M, \mathrm{Z}\mathbb{R})$ is a graded commutative algebra thanks to the ordinary multiplicative structure of $\mathbb{R}$. Indeed, $\textsc{Map}(M, \mathrm{Z}\mathbb{R})$ is nothing but the internal function algebra $\textsc{Fun}(M)$ described earlier.

In this paper, we are mostly concerned with internal function spaces of graded manifolds such as $\textsc{Map}(M, N)$. The above setup can also be formulated for ordinary function spaces $\mathrm{Map}(M, N)$ with similar results but also a few noticeable exceptions. Such spaces are more restrictive than the internal ones, since the

coefficient functions of soul coordinates polynomials are required to be degree 0 functions of the body coordinates. For the singleton manifold $M = *$, one has for instance $\mathrm{Map}(*, \mathbb{R}[p]) = \delta_{p,0}\mathbb{R}$ and $\mathrm{Map}(*, E[p]) = \delta_{p,0}E$.

B Basic identities

Let $\mathsf{M} = (\mathsf{E}, \mathsf{G}, \tau, \mu)$ be a Lie group crossed module. We collect below a number of structure relations which are used throughout the main text of the paper.

The relevant differentiated structure maps are $\dot{\tau} : \mathfrak{e} \to \mathfrak{g}$, $\dot{\mu} : \mathsf{G} \times \mathfrak{e} \to \mathfrak{e}$, $\dot{\mu} : \mathfrak{g} \times \mathsf{E} \to \mathfrak{e}$ and $\dot{\dot{\mu}} : \mathfrak{g} \times \mathfrak{e} \to \mathfrak{e}$. They are defined as

$$\dot{\tau}(X) = \frac{d\tau(C(v))}{dv}\Big|_{v=0}, \tag{B.0.1}$$

$$\dot{\mu}(a, X) = \frac{d}{dv}\mu(a, C(v))\Big|_{v=0}, \tag{B.0.2}$$

$$\dot{\mu}(x, A) = \frac{d}{du}\mu(c(u), A)A^{-1}\Big|_{u=0}, \tag{B.0.3}$$

$$\dot{\dot{\mu}}(x, X) = \frac{\partial}{\partial u}\Big(\frac{\partial\mu(c(u), C(v))}{\partial v}\Big|_{v=0}\Big)\Big|_{u=0} \tag{B.0.4}$$

for $a \in \mathsf{G}$, $A \in \mathsf{E}$, $x \in \mathfrak{g}$, $X \in \mathfrak{e}$, where $c(u)$ and $C(v)$ are curves in G and E with $c(u)\big|_{u=0} = 1_{\mathsf{G}}$ and $C(v)\big|_{v=0} = 1_{\mathsf{E}}$ and $dc(u)/du\big|_{u=0} = x$ and $dC(v)/dv\big|_{v=0} = X$, respectively, whose choice is immaterial.

The following algebraic identities hold:

$$\dot{\tau}(\dot{\mu}(x, A)) = x - \mathrm{Ad}\,\tau(A)(x), \tag{B.0.5}$$

$$\dot{\mu}(\dot{\tau}(X), A) = X - \mathrm{Ad}\,A(X), \tag{B.0.6}$$

$$\dot{\mu}([x, y], A) = \dot{\dot{\mu}}(x, \dot{\mu}(y, A)) - \dot{\dot{\mu}}(y, \dot{\mu}(x, A)) - [\,\dot{\mu}(x, A), \dot{\mu}(y, A)], \tag{B.0.7}$$

$$\dot{\mu}(x, AB) = \dot{\mu}(x, A) + \mathrm{Ad}\,A(\dot{\mu}(x, B)), \tag{B.0.8}$$

$$\dot{\mu}(\mathrm{Ad}\,a(x), \mu(a, A)) = \dot{\mu}(a, \dot{\mu}(x, A)), \tag{B.0.9}$$

$$\mathrm{Ad}\,A(\dot{\dot{\mu}}(x, X)) = \dot{\dot{\mu}}(x, \mathrm{Ad}\,A(X)) - [\,\dot{\mu}(x, A), \mathrm{Ad}\,A(X)], \tag{B.0.10}$$

where $a \in \mathsf{G}$, $A, B \in \mathsf{E}$, $x, y \in \mathfrak{g}$, $X \in \mathfrak{e}$.

The following variational identities hold:

$$\delta\mu(a,A)\mu(a,A)^{-1} = \dot{\mu}(a, \dot{\mu}(a^{-1}\delta a, A) + \delta A A^{-1}), \tag{B.0.11}$$

$$\delta\dot{\mu}(a,X) = \dot{\mu}(a, \dot{\mu}(a^{-1}\delta a, X) + \delta X), \tag{B.0.12}$$

$$\delta\,\dot{\mu}(x,A) = \dot{\mu}(\delta x, A) + \dot{\mu}(x, \delta A A^{-1}) - [\,\dot{\mu}(x,A), \delta A A^{-1}], \tag{B.0.13}$$

where $a \in \mathsf{G}$, $A \in \mathsf{E}$, $x \in \mathfrak{g}$, $X \in \mathfrak{e}$.

Acknowledgements. The author thanks R. Picken, J. Huerta and C. Saemann for useful discussions. He acknowledges financial support from INFN Research Agency under the provisions of the agreement between University of Bologna and INFN. He also thanks the organizer of the 2018 EPSRC Durham Symposium on "Higher Structures in M-Theory" during which part of this work was done.

References

[1] U. Schreiber, *From loop space mechanics to nonAbelian strings*, Ph. D. thesis, Universität Duisburg–Essen (2005)
arXiv:hep-th/0509163.

[2] J. C. Baez and U. Schreiber, *Higher gauge theory: 2-connections on 2-bundles*,
arXiv:hep-th/0412325.

[3] J. C. Baez and U. Schreiber, *Higher gauge theory*, in *Categories in algebra, geometry and mathematical physics*, eds. A. Davydov et al.,
Contemp. Math. **431** AMS (2007) 7 [arXiv:math/0511710].

[4] J. Giraud, *Cohomologie non-abèlienne*,
Grundl. Math. Wiss. **197**, Springer Verlag (1971).

[5] T. K. Bartels, *Higher gauge theory: 2-bundles*, Ph.D. thesis, University of California, Riverside (2006)
arXiv:math/0410328 [math.CT].

[6] C. Wockel, *Principal 2-bundles and their gauge 2-groups*,
Forum Math. **23** (3) (2011) 565, [arXiv:0803.3692 [math.DG]].

[7] C. Schommer-Pries, *Central extensions of smooth 2-groups and a finite-dimensional string 2-group*,
Geom. Topol. **15** (2011) [arXiv:0911.2483 [math.AT]].

[8] P. Aschieri, L. Cantini and B. Jurco, *Non Abelian bundle gerbes, their differential geometry and gauge theory*,
Commun. Math. Phys. **254** (2005) 367 [arXiv:hep-th/0312154].

[9] C. Laurent–Gengoux, M. Stienon and P. Xu, *Non Abelian differentiable gerbes*,
Adv. Math. **220** (2009) 1357 [arXiv:math/0511696 [math.DG]].

[10] G. Ginot and M. Stiénon, *G–gerbes, principal 2–group bundles and characteristic classes*,
J. Symplectic Geom. **13** 4 (2015) 1001 [arXiv:0801.1238 [math.AT]].

[11] T. Nikolaus and K. Waldorf, *Four equivalent versions of non-Abelian gerbes*,
Pacific J. Math., **264** 2 (2013) 355 [arXiv:1103.4815 [math.DG]].

[12] L. Breen and W. Messing, *Differential geometry of gerbes*,
Adv. Math. **198** (2005) 732 [arXiv:math/0106083 [math.AG]].

[13] U. Schreiber, *Differential cohomology in a cohesive infinity-topos*,
arXiv:1310.7930 [math-ph].

[14] B. Jurco, C. Saemann and M. Wolf, *Semistrict higher gauge theory*,
JHEP **1504** (2015) 087 [arXiv:1403.7185 [hep-th]].

[15] B. Jurco, C. Saemann and M. Wolf, *Higher groupoid bundles, higher spaces, and self–dual tensor field equations*,
Fortsch. Phys. **64** (2016) 674 [arXiv:1604.01639 [hep-th]].

[16] K. Waldorf, *A global perspective to connections on principal 2-bundles*,
Forum Math. **30** 4 (2017) 809 [arXiv:1608.00401 [math.DG]].

[17] K. Waldorf, *Parallel transport in principal 2-bundles*,
arXiv:1704.08542 [math.DG].

[18] W. Greub, S. Halperin and R. Vanstone, *Connections, Curvature and Cohomology*,
vol. 3, Pure and Applied Mathematics **47**, Academic Press (1976).

[19] R. Zucchini, *Operational total space theory of principal 2–bundles II: 2–connections and 1– and 2–gauge transformations.*

[20] M. Alexandrov, M. Kontsevich, A. Schwartz, and O. Zaboronsky, *The geometry of the master equation and topological quantum field theory*,
Int. J. Mod. Phys. A **12** (1997) 1405 [arXiv:hep-th/9502010].

[21] R. Zucchini, *Algebraic formulation of higher gauge theory*, J. Math. Phys. **58** (2017) no.6, 062301 [arXiv:1702.01545 [hep-th]].

[22] J. Baez and A. Lauda, *Higher dimensional algebra V: 2-groups*, Theor. Appl. Categor. **12** (2004) 423 [arXiv:math.0307200].

[23] J. C. Baez and A. S. Crans, *Higher dimensional algebra VI: Lie* 2*–algebras*, Theor. Appl. Categor. **12** (2004) 492 [arXiv:math/0307263].