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Continuous dependence on boundary and Soret coefficients in double diffusive bidispersive convection

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Abstract

We develop a theory for double diffusive convection in a double porosity material which contains Brinkman terms. The Soret effect is included whereby a temperature gradient may directly influence salt concentration. The boundary conditions on the temperature and salt fields are of general type. Continuous dependence is established upon the Soret coefficient and upon the coefficients in the boundary conditions.

1 Introduction

Solutions to problems involving flow in double porosity, or bidisperse, materials are proving to be invaluable in modern life. Such theories have applications to many important areas such as chemical engineering, Enterría et al. [1], Huang et al. [2], Ly et al. [3]; in landslides, Borja et al. [4], Scotto di Santolo and Evangelista [5]; in self ignition of stockpiled coal, Hooman & Maas [6]; in land drainage and ensuring stormwater runoff does not pollute,

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Haws et al. [7], Jensen et al. [8]; in ensuring clean drinking water from an aquifer, Ghasemizadeh et al. [9], Fretwell et al. [10]; and there are many others.

A theory for flow in a bidisperse porous medium was proposed by Nield & Kuznetsov, see e.g. [12, 13], and the references therein, see also Nield[11]. In particular, Nield and Kuznetsov [12] produced a theory based on Brinkman porous media where temperature effects were paramount. Inclusion of temperature(s) is very important since thermal stresses can induce cracking and lead to the production of micro pores, see e.g. Gelet et al. [14], Kim and Hosseini [15], Rees et al. [16]. The theory of Nield and Kuznetsov has macro pores and smaller micro pores which lead to macro porosity, Φ, and micro porosity, ϵ. They have independent velocity, temperature and pressure fields in both the macro and micro pores, denoted by U_i^f, T^f and p^f, and by U_i^p, T^p and p^p, where f denotes macro pores while p represents the micro pores.

For many real life problems we believe a single temperature may suffice, while retaining independent velocity and pressure fields in the macro and micro pores. This approach has been applied successfully to various problems using Darcy porous media theory by Falsaperla et al. [17], Franchi et al. [18], Gentile and Straughan [19, 20], and by Straughan [22, 23, 24, 25, 26]. In this work we adopt a single temperature but we employ a Brinkman porous medium theory in both the macro and micro pores, in keeping with the original work of Nield and Kuznetsov [12]. It is worth pointing out that there are many situations where Brinkman theory is relevant, see e.g. Barletta et al. [27], Falsaperla et al. [28], Nield [29], and Rees [30].

We actually develop a Brinkman theory for the problem of thermosolutal flow in a bidisperse porous medium, where both salt and temperature field effects are present. In the single porosity case such effects are well known, see e.g. Barletta and Nield [31], Deepika [32], Nield and Kuznetsov [13]. To achieve our goal we employ a Boussinesq approximation in the buoyancy (body force) terms to allow us to include temperature and salt fields in a linear manner. The Boussinesq approximation is discussed at length in fluid mechanics and flows in single porosity media by Barletta [33], and by Nield and Barletta [34].

The analysis in this work is motivated primarily by a paper of Nield and Kuznetsov [13] who study double diffusive convection in a single porosity porous material with general boundary conditions and they discover that in a certain parameter range this problem can become singular. Our goal is to study continuous dependence upon boundary coefficients. Thus, our study is one of continuous dependence upon the model itself. Hirsch and Smale [35] pose the problem of what effect does changing the model have upon the solutions. In many ways continuous dependence of the solution in changes in the differential equations or boundary conditions is as important as continuous dependence upon the initial data, or stability. In fact, continuous dependence on the model occupies much recent research, see e.g. Celik and Hoang [36], Ciarletta et al.[37], Franchi et al. [18], Gentile and Straughan [21], Harfash [38], Hoang and Thinh [39], Kalantarova and Ugurlu [40], Li *et al.* [41], Liu [42], Liu and Xiao [43], Liu et al [44], Scott [45], Varsakelis and Papalexandris [46], Wang and Su [47].

Thus, we now present a model for double diffusive flow in a bidisperse porous medium where we allow for a Soret effect. The Soret effect is when a temperature gradient induces a change in solute concentration. This is manifest as a cross diffusion term which leads to complications in the analysis. Our goal is to demonstrate continuous dependence of the solution to the model in changes in the coefficients in the boundary conditions and also upon the Soret coefficient.

2 Governing equations

The basic equations governing the double diffusive flow in a double porosity material with Brinkman effects are now presented. Let $T(\mathbf{x}, \mathbf{t})$ be the temperature, let C be the concentration of solute, U_i^f and U_i^p are the velocities in the macro and micro pores, and p^f and p^p are the pressures in the macro and micro pores. The equations are

$$
\begin{aligned}
&\tilde{\mu}\Delta U_i^f - \frac{\mu}{K_f}U_i^f - \zeta(U_i^f - U_i^p) - p^f_{,i} + g_iT - h_iC = 0\,,\\
&U^f_{i,i} = 0\,,\\
&\tilde{\mu}\Delta U_i^p - \frac{\mu}{K_p}U_i^p - \zeta(U_i^p - U_i^f) - p^p_{,i} + g_iT - h_iC = 0\,,\\
&U^p_{i,i} = 0\,,\\
&(\rho c)_m T_{,t} + (\rho c)_f(U_i^f + U_i^p)T_{,i} = \kappa_m \Delta T\,,\\
&\epsilon_1 C_{,t} + (U_i^f + U_i^p)C_{,i} = \epsilon_2\Delta C + S\Delta T\,.
\end{aligned}
\tag{1}
$$

where f and p refer to macro and micro quantities, $\tilde{\mu}$ is the Brinkman viscosity coefficient, μ is the dynamic viscosity of the saturating fluid, K_f and K_p

are permeabilities, ζ is an interaction coefficient, ρc denotes the product of the density and specific heat at constant pressure with f denoting the fluid whereas m denotes a suitably averaged value in the porous medium. The term κ_m is an averaged value of the thermal conductivity, ϵ_1 and ϵ_2 arise from the equation for the solute, S is the Soret coefficient, and g_i and h_i are gravity terms which arise through use of a Boussinesq approximation. The Boussinesq approximation is discussed in detail in [33], Gouin and Ruggeri [**?**], Gouin et al. [**?**], Nield and Barletta [34] Rajagopal et al. [**?**, **?**].

Without loss of generality for the problem under consideration here we suppose that

$$|\mathbf{g}| \leq 1\,, \qquad |\mathbf{h}| \leq 1\,. \tag{2}$$

Throughout the article standard indicial notation is employed together with the Einstein summation convention, and Δ is the Laplace operator.

Equations $(1)_{1,2}$ are the balance of momentum and conservation of mass in the macro pores, while $(1)_{3,4}$ are the balance of momentum and conservation of mass in the micro pores, as derived by Nield and Kuznetsov [12]. Equation $(1)_5$ is the balance of energy for a single temperature, cf. Falsaperla et al. [28], Gentile and Straughan [19], and $(1)_6$ is the equation governing the solute concentration, see Straughan [23, 25], although we point out only Nield and Kuznetsov [12] employ Brinkman terms, the other writers restrict attention to Darcy theory.

It is convenient to replace U_i^f and U_i^p by u_i and v_i and to substitute μ/K_f and μ/K_p by μ and γ. We further divide the equation for T by κ_m and define $\alpha = (\rho c)_f/\kappa_m$, and rescale the time so that the term $T_{,t}$ has coefficient one. In this way, for the purpose of a continuous dependence analysis, the equations for double diffusive flow in a bidisperse porous medium may be written as

$$\begin{aligned}
&\tilde{\mu}\Delta u_i - \mu u_i - \zeta(u_i - v_i) - p_{,i} + g_i T - h_i C = 0\,,\\
&u_{i,i} = 0\,,\\
&\tilde{\mu}\Delta v_i - \gamma v_i - \zeta(v_i - u_i) - q_{,i} + g_i T - h_i C = 0\,,\\
&v_{i,i} = 0\,,\\
&T_{,t} + \alpha(u_i + v_i)T_{,i} = \Delta T\,,\\
&\epsilon_1 C_{,t} + (u_i + v_i)C_{,i} = \epsilon_2 \Delta C + S\Delta T\,.
\end{aligned} \tag{3}$$

Let $\mathcal{T}$ be an arbitrary positive number (fixed). Equations (3) hold on a bounded domain $\Omega \in \mathbb{R}^3$ for time $t \in (0, \mathcal{T}]$. On the boundary of Ω, Γ, we suppose that

$$u_i = 0\,, \qquad v_i = 0\,, \qquad \mathbf{x} \in \Gamma\,,\ t \in (0, \mathcal{T}]\,, \tag{4}$$

while the temperature and concentration satisfy the general boundary conditions

$$\frac{\partial T}{\partial n} = -L(T - T_a)\,, \qquad \mathbf{x} \in \Gamma\,,\ t \in (0, \mathcal{T}]\,, \tag{5}$$

and

$$\frac{\partial C}{\partial n} = -M(C - C_a)\,, \qquad \mathbf{x} \in \Gamma\,,\ t \in (0, \mathcal{T}]\,, \tag{6}$$

The coefficients L and M are given, and T_a and C_a are known ambient values. The derivative with respect to n is the unit outward normal derivative. Conditions (5) and (6) correspond to the boundary conditions employed by Nield and Kuznetsov [13] for a single porosity material.

The initial conditions are

$$T(\mathbf{x}, 0) = T_0(\mathbf{x})\,, \qquad C(\mathbf{x}, 0) = C_0(\mathbf{x})\,, \qquad \mathbf{x} \in \Omega. \tag{7}$$

Our goal is to derive continuous dependence results on the Soret coefficient S, together with dependence on the boundary coefficients L and M.

3 A priori estimates

In this section we derive a priori estimates on the temperature T, and on the salt concentration C, these estimates being essential for continuous dependence.

Let $\|\cdot\|$ and $(\cdot,\cdot)$ denote the norm and inner product on $L^2(\Omega)$ and let $\|\cdot\|_4$ denote the norm on $L^4(\Omega)$.

The first estimate proceeds by multiplying $(3)_5$ by T and integrating over Ω. After using the boundary conditions (4) and (5) and integrating by parts one may arrive at

$$\frac{d}{dt}\frac{1}{2}\|T\|^2 + \|\nabla T\|^2 + L\oint_\Gamma T(T - T_a)dA = 0\,.$$

Use the arithmetic-geometric mean inequality on the term involving $T\,T_a$ to find

$$\frac{d}{dt}\frac{1}{2}\|T\|^2 + \|\nabla T\|^2 + \frac{L}{2}\oint_\Gamma T^2\,dA \le \frac{L}{2}\oint_\Gamma T_a^2\,dA\,.$$

Upon integration of this inequality we obtain our first a priori estimate,

$$\|T\|^2 + 2\int_0^t \|\nabla T\|^2\,ds + L\int_0^t \oint_\Gamma T^2\,dA\,ds \le D_1(t)\,, \tag{8}$$

where $D_1(t)$ is the data term

$$D_1(t) = \|T_0\|^2 + L\int_0^t \oint_\Gamma T_a^2\,dA\,ds\,.$$

Next, we multiply $(3)_5$ by T^3 and integrate over Ω. After integration by parts and use of (4) and (5) one finds

$$\frac{d}{dt}\frac{1}{4}\|T\|_4^4 + 3\int_\Omega T^2 T_{,i}T_{,i}\,dx + L\oint_\Gamma T^4 dA = L\oint_\Gamma T^3 T_a\,dA\,.$$

Employ Young's inequality in the form $T^3T_a \le T^4/2 + 27\,T_a^4/32$ to obtain

$$\frac{d}{dt}\frac{1}{4}\|T\|_4^4 + 3\int_\Omega T^2 T_{,i}T_{,i}\,dx + \frac{L}{2}\oint_\Gamma T^4 dA = \frac{27}{32}L\oint_\Gamma T_a^4\,dA\,.$$

Upon integration this furnishes the a priori bound

$$\|T\|_4^4 + 12\int_0^t\int_\Omega T^2|\nabla T|^2\,dx\,ds + 2L\int_0^t\oint_\Gamma T^4\,dA\,ds \le D_2(t)\,, \tag{9}$$

where the data term $D_2(T)$ is given by

$$D_2(t) = \|T_0\|_4^4 + \frac{27L}{8}\int_0^t\oint_\Gamma T_a^4\,dA\,ds\,.$$

To derive the next a priori bound we multiply $(3)_6$ by C and integrate over Ω. After integration by parts and simultaneously employing (4)–(6) one may see that

$$\begin{aligned}&\frac{d}{dt}\frac{\epsilon_1}{2}\|C\|^2 + \epsilon_2\|\nabla C\|^2 + \epsilon_2 M\oint_\Gamma C^2 dA = \\ &= \epsilon_2 M\oint_\Gamma CC_a dA - S(\nabla C, \nabla T) - SL\oint_\Gamma CT\,dA + SL\oint_\Gamma CT_a da\,.\end{aligned}$$

We now employ the arithmetic-geometric mean inequality on the terms on the right to arrive at

$$\begin{aligned}&\frac{\epsilon_1}{2}\frac{d}{dt}\|C\|^2 + \frac{\epsilon_2}{2}\|\nabla C\|^2 + \frac{M\epsilon_2}{2}\oint_\Gamma C^2 dA \le \\ &\le M\epsilon_2\oint_\Gamma C_a^2 dA + \frac{2S^2L^2}{M\epsilon_2}\oint_\Gamma T_a^2 dA + \frac{S^2}{2\epsilon_2}\|\nabla T\|^2 + \frac{2S^2L}{M\epsilon_2}\oint_\Gamma T^2 dA\,.\end{aligned}$$

This inequality is now integrated in time and we add to this a suitable multiple of inequality (8), this multiple we call ω. In this way, for a constant ω given by

$$\omega = \max\left\{\frac{S^2}{2\epsilon_2}, \frac{4S^2L}{M\epsilon_2}\right\}$$

we may derive the third a priori estimate

$$\epsilon_1\|C\|^2 + \epsilon_2\int_0^t \|\nabla C\|^2\,ds + M\epsilon_2\int_0^t\oint_\Gamma C^2\,dA\,ds \le D_3(t)\,, \tag{10}$$

where the data term D_3 is given by

$$D_3(t) = 2M\epsilon_2\int_0^t\oint_\Gamma C_a^2\,dA\,ds + \epsilon_1\|C_0\|^2 + \frac{4S^2L^2}{M\epsilon_2}\int_0^t\oint_\Gamma T_a^2\,dA\,ds\,.$$

The constant ω has to be large enough that it ensures the terms in $\int_0^t\|\nabla T\|^2 ds$ and $\int_0^t\oint_\Gamma T^2 dA\,ds$ in (8) dominate the equivalent terms on the right of the inequality which arises.

4 Continuous dependence

Denote the boundary-initial problem consisting of equations (3) together with conditions (4)–(7) by $\mathcal{P}$. Let $\{u_i^1, v_i^1, p_1, q_1, T_1, C_1\}$ be a solution of $\mathcal{P}$ with initial data T_0, C_0 and values of the coefficients L, M and S being L_1, M_1, S_1. Let $\{u_i^2, v_i^2, p_2, q_2, T_2, C_2\}$ be another solution to $\mathcal{P}$ with the same initial data but now the values for the boundary and Soret coefficients L, M and S are L_2, M_2, S_2. Define the difference variables $\{\omega_i, r_i, \pi^f, \pi^p, \theta, \phi\}$ and l, m, s by

$$\begin{aligned} &w_i = u_i^1 - u_i^2\,, \quad r_i = v_i^1 - v_i^2\,, \quad \pi^f = p_1 - p_2\,, \\ &\pi^p = q_1 - q_2\,, \quad \theta = T_1 - T_2\,, \quad \phi = C_1 - C_2\,, \\ &l = L_1 - L_2\,, \quad m = M_1 - M_2\,, \quad s = S_1 - S_2\,. \end{aligned}$$

The boundary-initial value problem for the difference solution may be written as

$$\begin{aligned}
&0 = \tilde{\mu}\Delta w_i - \mu w_i - \zeta(w_i - r_i) - \pi^f_{,i} + g_i\theta - h_i\phi\,,\\
&w_{i,i} = 0\,,\\
&0 = \tilde{\mu}\Delta r_i - \gamma r_i - \zeta(r_i - w_i) - \pi^p_{,i} + g_i\theta - h_i\phi\,,\\
&r_{i,i} = 0\,,\\
&\theta_{,t} + \alpha(w_i + r_i)T_{1,i} + \alpha(u_i^2 + v_i^2)\theta_{,i} = \Delta\theta\,,\\
&\epsilon_1\phi_{,t} + (w_i + r_i)C_{1,i} + (u_i^2 + v_i^2)\phi_{,i} = \epsilon_2\Delta\phi + s\Delta T_1 + S_2\Delta\theta\,,
\end{aligned} \tag{11}$$

together with the boundary and initial conditions,

$$\begin{aligned}
&w_i = 0\,, \qquad r_i = 0\,, \qquad \mathbf{x} \in \Gamma\,,\\
&\frac{\partial\theta}{\partial n} = -L_1\theta - T_2 l + T_a l\,, \qquad \mathbf{x} \in \Gamma\,,\\
&\frac{\partial\phi}{\partial n} = -M_1\phi - C_2 m + C_a m\,, \qquad \mathbf{x} \in \Gamma\,,
\end{aligned} \tag{12}$$

and

$$\theta(\mathbf{x}, 0) = 0\,, \qquad \phi(\mathbf{x}, 0) = 0\,, \qquad \mathbf{x} \in \Omega. \tag{13}$$

To establish continuous dependence we multiply (11) $_1$by w_i and integrate over Ω, and we then multiply $(11)_3$ by r_i and integrate over Ω. After integration by parts and use of the boundary conditions we may add the results to obtain

$$\tilde{\mu}\|\nabla\mathbf{w}\|^2+\tilde{\mu}\|\nabla\mathbf{r}\|^2+\mu\|\mathbf{w}\|^2+\gamma\|\mathbf{r}\|^2+\zeta\|\mathbf{w}-\mathbf{r}\|^2 = (g_i\theta, w_i+r_i)-(h_i\phi, w_i+r_i)\,.$$

Next, let $k = \mu^{-1} + \gamma^{-1}$ and then use the arithmetic-geometric mean inequality on the right together with the bounds on g_i and h_i to find

$$\frac{1}{2}\left(\mu\|\mathbf{w}\|^2 + \gamma\|\mathbf{r}\|^2\right) + \zeta\|\mathbf{w} - \mathbf{r}\|^2 + \tilde{\mu}\left(\|\nabla\mathbf{w}\|^2 + \|\nabla\mathbf{r}\|^2\right) \le k\left(\|\theta\|^2 + \|\phi\|^2\right) \tag{14}$$

Now, multiply $(11)_5$ by θ and integrate over Ω. After integration by parts and use of the boundary conditions one may arrive at

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\|\theta\|^2 + \|\nabla\theta\|^2 + L_1\oint_\Gamma \theta^2 dA = \alpha\int_\Omega w_i T_1\theta_{,i}\,dx\\
&+ \alpha\int_\Omega r_i T_1\theta_{,i}\,dx - l\oint_\Gamma T_2\theta\,dA + l\oint_\Gamma T_a\theta\,dA\,.
\end{aligned} \tag{15}$$

We use the arithmetic-geometric mean inequality on the last two terms on the right. To handle the nonlinear terms we use the arithmetic-geometric mean inequality as follows

$$\int_\Omega w_i T_1 \theta_{,i}\, dx \leq \frac{1}{2\xi}\int_\Omega |\mathbf{w}|^2 |T_1|^2 dx + \frac{\xi}{2}\|\nabla\theta\|^2 \,, \tag{16}$$

for $\xi > 0$ at our disposal. Then use the Cauchy-Schwarz inequality followed by the Sobolev inequality $\|\mathbf{w}\|_4 \leq c_1\|\nabla\mathbf{w}\|$ to see that

$$\int_\Omega |\mathbf{w}|^2|T_1|^2 dx \leq \|\mathbf{w}\|_4^2\|T_1\|_4^2 \leq c_1^2\|\nabla\mathbf{w}\|^2\|T_1\|_4^2 \,.$$

Upon using this in (16) we obtain

$$\int_\Omega w_i T_1 \theta_{,i}\, dx \leq \frac{c_1^2}{2\xi}\|\nabla\mathbf{w}\|^2\|T_1\|_4^2 + \frac{\xi}{2}\|\nabla\theta\|^2 \,. \tag{17}$$

A similar inequality is derived for the r_i term, and then, setting $\xi = 1/2$, from (15) we easily obtain

$$\begin{aligned} &\frac{d}{dt}\|\theta\|^2 + \|\nabla\theta\|^2 + L_1\oint_\Gamma \theta^2 dA \leq 2\alpha^2 c_1^2\|\nabla\mathbf{w}\|^2\|T_1\|_4^2 \\ &+ 2\alpha^2 c_1^2\|\nabla\mathbf{r}\|^2\|T_1\|_4^2 + \frac{2}{L_1}l^2\oint_\Gamma T_a^2\, dA + \frac{2}{L_1}l^2\oint_\Gamma T_2^2\, dA \,. \end{aligned} \tag{18}$$

We now employ (14) in (18), integrate the result, and use (8) to arrive at

$$\begin{aligned} &\|\theta\|^2 + \int_0^t \|\nabla\theta\|^2 ds + L_1\int_0^t\oint_\Gamma \theta^2 dA\, ds \leq \\ &\leq \frac{2\alpha^2 c_1^2 k}{\tilde{\mu}}\int_0^t \|T_1\|_4^2\left(\|\theta\|^2 + \|\phi\|^2\right)\, ds \\ &+ \frac{2}{L_1}l^2\int_0^t\oint_\Gamma T_a^2\, dA\, ds + \frac{2}{L_1 L_2}l^2 D_1(t) \,. \end{aligned} \tag{19}$$

To proceed we now multiply $(11)_6$ by ϕ and integrate over Ω. After integration by parts and use of the boundary conditions we derive

$$
\begin{aligned}
&\frac{\epsilon_1}{2}\frac{d}{dt}\|\phi\|^2 + \epsilon_2\|\nabla\phi\|^2 + \epsilon_2 M_1 \oint_\Gamma \phi^2 dA = -\int_\Omega (w_i + r_i)\phi C_{1,i}\, dx \\
&- s(\nabla\phi, \nabla T_1) - m\epsilon_2 \oint_\Gamma C_2\phi\, dA + m\epsilon_2 \oint_\Gamma C_a\phi\, dA - L_1 s \oint_\Gamma T_1\phi\, dA \\
&+ L_1 s \oint_\Gamma T_a\phi\, dA - lS_2 \oint_\Gamma T_2\phi\, dA + lS_2 \oint_\Gamma T_a\phi\, dA \\
&- S_2 L_1 \oint_\Gamma \phi\theta\, dA - S_2(\nabla\phi, \nabla\theta)\,.
\end{aligned} \tag{20}
$$

The nonlinear terms are handled as follows. Firstly use the Cauchy-Schwarz inequality

$$
-\int_\Omega w_i\phi C_{1,i}\, dx \le \|\nabla C_1\| \left(\int_\Omega |\mathbf{w}|^2\phi^2 dx\right)^{1/2} \le \|\nabla C_1\|\|\mathbf{w}\|_3\|\phi\|_6\,, \tag{21}
$$

where $\|\cdot\|_p$ is the norm in $L^p(\Omega)$.

Now use the Cauchy-Schwarz inequality for $\mathbf{w}$, namely

$$
\int_\Omega |\mathbf{w}|^3 dx \le V^{1/2}\left(\int_\Omega |\mathbf{w}|^6 dx\right)^{1/2},
$$

where V is the Lebesgue measure of Ω. Hence, thanks to the Sobolev inequality

$$
\|\mathbf{w}\|_3 \le V^{1/6}\|\mathbf{w}\|_6 \le V^{1/6} c_1\|\nabla\mathbf{w}\|\,.
$$

Now the Sobolev inequality is applied to ϕ in the form

$$
\|\phi\|_6 \le \hat{c}_2\|\phi\|_{W^{1,2}} = \hat{c}_2\left(\int_\Omega |\nabla\phi|^2 dx + \int_\Omega \phi^2 dx\right)^{1/2}.
$$

The Poincaré inequality holds for a constant λ_1 in the form

$$
\lambda_1 \int_\Omega \phi^2 dx \le \int_\Omega |\nabla\phi|^2 dx + \oint_\Gamma \phi^2 dA\,.
$$

Therefore

$$
\|\phi\|_6 \le c_2\sqrt{\|\nabla\phi\|^2 + \oint_\Gamma \phi^2 dA} \tag{22}
$$

where $c_2 = \hat{c}_2 \left(1 + \lambda_1^{-1}\right)^{1/2}$.

Next, use (22) in (21) together with the estimate for $\|\mathbf{w}\|_3$ followed by the arithmetic-geometric mean inequality,

$$
\begin{aligned}
&-\int_\Omega w_i\,\phi\,C_{1,i}\,dx \le c_1 c_2 V^{1/6}\|\nabla C_1\|\|\nabla \mathbf{w}\|\sqrt{\|\nabla\phi\|^2 + \oint_\Gamma \phi^2 dA} \\
&\le \frac{c_1^2\,c_2^2\,V^{1/3}}{2\xi}\|\nabla C_1\|^2\|\nabla\mathbf{w}\|^2 + \frac{\xi}{2}\left(\|\nabla\phi\|^2 + \oint_\Gamma \phi^2 dA\right)
\end{aligned}
$$

where $\xi =$????. We now employ (14) to obtain

$$
\begin{aligned}
-\int_\Omega w_i\,\phi\,C_{1,i}\,dx \le & \frac{c_1^2\,c_2^2\,V^{1/3}k}{2\,\xi\tilde{\mu}}\left(\|\theta\|^2 + \|\phi\|^2\right)\|\nabla C_1\|^2 \\
& + \frac{\xi}{2}\left(\|\nabla\phi\|^2 + \oint_\Gamma \phi^2 dA\right).
\end{aligned} \tag{23}
$$

A similar inequality is derived for the r_i term. Then (23) and the analogous inequality for the r_i term are employed in (20). After use of the arithmetic-geometric inequality together with (8)–(10) we integrate (20) and obtain

$$
\begin{aligned}
&\|\phi\|^2 + \frac{\epsilon_2}{\epsilon_1}\int_0^t \|\nabla\phi\|^2 ds + M_1\frac{\epsilon_2}{\epsilon_1}\int_0^t \oint_\Gamma \phi^2 dA \le \\
&\quad \delta\int_0^t (\|\theta\|^2 + \|\phi\|^2)\|\nabla C_1\|^2 ds \\
&\quad + m^2 D_4(t) + l^2 D_5(t) + s^2 D_6(t) \\
&+ \quad \frac{S_2 L_1 \gamma_8}{\epsilon_1}\int_0^t \oint_\Gamma \theta^2 dA\,ds + \frac{S_2\gamma_7}{\epsilon_1}\int_0^t \|\nabla\theta\|^2 ds\,,
\end{aligned} \tag{24}
$$

where $\delta = 4c_1^2 c_2^2 V^{1/3}k/\epsilon_1\tilde{\mu}$, and $\gamma_7, \gamma_8 > 0$ are at our disposal, and D_4–D_6 are the data terms

$$
\begin{aligned}
D_4(t) &= \frac{D_3(t)}{M_2\epsilon_1\gamma_1} + \frac{\epsilon_2}{\epsilon_1\gamma_2}\int_0^t \oint_\Gamma C_a^2 dA\,ds\,, \\
D_5(t) &= \frac{S_2}{\epsilon_1\gamma_5 L_2}D_1(t) + \frac{S_2}{\epsilon_1\gamma_6}\int_0^t \oint_\Gamma T_a^2 dA\,ds\,, \\
D_6(t) &= \frac{D_1(t)}{\epsilon_1\gamma_3} + \frac{L_1}{\epsilon_1\gamma_4}\int_0^t \oint_\Gamma T_a^2 dA\,ds + \frac{D_1(t)}{2\gamma_9\epsilon_1}\,,
\end{aligned} \tag{25}
$$

for $\gamma_1, \gamma_2, \gamma_5, \gamma_6, \gamma_3, \gamma_4, \gamma_9 > 0$ to be selected opportunely.

For a constant $A > 0$ we now form $A(19) + (24)$. After a judicious choice of coefficients, namely

$$\begin{aligned} &\gamma_1 = \gamma_2 = \frac{\epsilon_1 M_1}{14\epsilon_2}\,, \quad \gamma_3 = \gamma_4 = \frac{\epsilon_1 M_1}{14 L_1}\,, \quad \gamma_5 = \gamma_6 = \frac{\epsilon_1 M_1}{14 S_2}\,, \\ &\gamma_7 = \frac{4S_2}{\epsilon_2}\,, \quad \gamma_8 = \frac{14 S_2 L_1}{\epsilon_1 M_1} \quad \text{and} \quad \gamma_9 = \frac{\epsilon_2}{2}\,, \end{aligned} \tag{26}$$

we derive the inequality

$$\begin{aligned} &\|\phi\|^2 + \|\theta\|^2 + \frac{\epsilon_2}{\epsilon_1}\int_0^t \|\nabla\phi\|^2 ds + M_1 \frac{\epsilon_2}{\epsilon_1}\int_0^t \oint_\Gamma \phi^2 dA\, ds \\ &+ \frac{1}{2}\int_0^t \|\nabla\theta\|^2 ds + \frac{L_1}{2}\int_0^t \oint_\Gamma \theta^2 dA\, ds \le D_4(t) m^2 \\ &+ D_7(t) l^2 + D_6(t) s^2 + \int_0^t \big(\|\theta\|^2 + \|\phi\|^2\big)\, \chi(s)\, ds\,, \end{aligned} \tag{27}$$

where $D_7(t)$ is a data term,

$$D_7(t) = D_5(t) + \frac{2}{L_1 L_2} D_1(t) + \frac{2}{L_1}\int_0^t \oint_\Gamma T_a^2 dA\, ds\,,$$

and where

$$\chi(s) = \delta \|\nabla C_1\|^2 + \frac{\alpha^2 c_1^2 k}{\tilde{\mu}} \|T_1\|_4^2\,.$$

Observe that the data terms $D_1, \ldots, D_7$, involve $\|T_0\|^2$, $\|C_0\|^2$, $\|T_0\|_4^4$, $\int_0^t \oint_\Gamma T_a^2 dAds$, $\int_0^t \oint_\Gamma T_a^4 dAds$, $\int_0^t \oint_\Gamma C_a^2 dAds$. Thus since $t \in [0, \mathcal{T})$ we may replace D_4, D_6 and D_7 by the constants $\bar{D}_4 = D_4(\mathcal{T})$, $\bar{D}_6 = D_6(\mathcal{T})$, $\bar{D}_7 = D_7(\mathcal{T})$. Then using Gronwall's inequality, see e.g. [48], we derive

$$\|\phi(t)\|^2 + \|\theta(t)\|^2 \le K(\bar{D}_4 m^2 + \bar{D}_7 l^2 + \bar{D}_6 s^2), \tag{28}$$

where the data constant K has form

$$K = 1 + \int_0^{\mathcal{T}} \chi(t) \exp\Big(\int_0^t \chi(s) ds\Big) dt.$$

We note that $\|T_1\|_4^4 \le D_2(t)$ and $\int_0^t \|\nabla C_1\|^2 ds \le D_3(t)$, and so K is a data term.

Inequality (28) demonstrates continuous dependence of a solution to $\mathcal{P}$ upon the boundary parameters L and M and upon the Soret coefficient S. Upon employingi eqref25 and (28) one may also obtain a continuous dependence estimate for $\int_0^t \|\nabla\phi\|^2 ds$, $\int_0^t \oint \phi^2 dAds$, $\int_0^t \|\nabla\theta\|^2 ds$ and $\int_0^t \oint \theta^2 dAds$.

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