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GEVREY REGULARITY FOR A CLASS OF SUMS OF SQUARES OF MONOMIAL VECTOR FIELDS

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ABSTRACT. Analytic or Gevrey hypoellipticity is proved for a class of sums of squares of vector fields having a symplectic characteristic manifold of dimension 2 and arbitrary (even) codimension. We note that this class contains examples for which the Treves stratification seems to work as well as examples for which the Treves stratification does not identify properly the non symplectic stratum.

1. INTRODUCTION AND STATEMENT OF THE RESULT

The purpose of the present paper is to prove a real analytic or Gevrey regularity result for a class of operators of the type "sum of squares". We consider a class of vector fields having monomial coefficients and such that the characteristic variety of the operator is a symplectic manifold of dimension 2 and codimension $2n$.

More precisely consider in $\mathbb{R}^n_x \times \mathbb{R}_y$ the second order differential operator

$$P(x, y, D_x, D_y) = \sum_{j=1}^{N} X_j(x, y, D_x, D_y)^2 + \sum_{k=1}^{n} \sum_{j=1}^{N'} \tilde{X}_{jk}(x, y, D_x)^2, \tag{1.1}$$

where the X_j, $\tilde{X}_{jk}$ are vector fields and moreover

$$X_j(x, y, D_x, D_y) = \begin{cases} D_j & \text{for } j = 1, \dots, n; \\ c_j y^{a_j} x^{\alpha_j} D_y & \text{for } j = n+1, \dots, N. \end{cases} \tag{1.2}$$

Here a_j denotes a nonnegative integer and α_j is a multiindex such that $|\alpha_j| > 0$. We use the notation $D_j = -i\partial_{x_j}$ and $D_y = -i\partial_y$.

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The vector fields $\tilde{X}_{jk}$ are defined as

$$\tilde{X}_{jk}(x, y, D_x) = \tilde{c}_{jk} y^{b_{jk}} x^{\beta_{jk}} D_k, \tag{1.3}$$

where $|\beta_{jk}| \geq 0$, $b_{jk} \geq 0$ and $k \in \{1, \dots, n\}$. The constants c_{jk}, $\tilde{c}_{jk}$ are real and non zero.

We make the following assumptions on the above vector fields:

(H1) The vector fields X_j, $j = 1, \dots, N$, satisfy Hörmander condition, i.e. the algebra generated by the X_j's and their iterated brackets has dimension $n + 1$.

(H2) The characteristic set of the operator P, $\mathrm{Char}(P)$, is the set $\Sigma = \{x = \xi = 0, \eta \neq 0\}$.

We point out that Assumption (H1) implies that P is C^∞ hypoelliptic ([8]), while (H2) ensures that its characteristic variety is the codimension 2 real analytic symplectic manifold $\{(0, y; 0, \eta) \mid \eta \neq 0\}$.

Remark 1.1. *As it will become quite evident from the proof below we might give up the assumption that the vector fields are monomials by allowing the coefficients c_j, $\tilde{c}_j$ to be real analytic functions which do not vanish at the origin. The Theorem proved below would be exactly the same.*

We chose to deal with constant c_j, $\tilde{c}_j$ in order to not burden both the notation and the proof below.

It is evident that a broad variety of model operators can be written in the form P is written. Just to cite the best known we mention here

$$\sum_{j=1}^{n} D_j^2 + \sum_{j=1}^{n} x_j^{2k_j} D_y^2, \tag{1.4}$$

where $k_j \in \mathbb{N}$, $j = 1, \dots, n$.

Other famous models are

$$P_1(x_1, y, D_1, D_y) = D_1^2 + x_1^{2(q-1)} D_y^2 + y^{2k} x_1^{2(\ell-1)} D_y^2, \tag{1.5}$$

when $n = 1$, $\ell, q \geq 2$, $\ell < q$ and $k \in \mathbb{N}$ (the generalized Métivier operator, see e.g. [13] for its basic form).

It is known that (1.4) is analytic hypoelliptic and we refer to [19] and [20] for a general statement including (1.4) when all the k_j are equal to 1. As for (1.5) it can be shown that it is Gevrey s hypoelliptic for

$$s \geq \frac{kq}{kq - q + \ell}. \tag{1.6}$$

There is no proof of the optimality of the above regularity to our knowledge.

Definition 1.1. *If $\Omega \subset \mathbb{R}^n$ is an open set we say that the function u belongs to the Gevrey class of order $s \geq 1$, $G^s(\Omega)$, if $u \in C^\infty(\Omega)$ and for every compact set $K \subset \Omega$ there exists a positive constant C_K such that*

$$\sup_K |\partial_x^\alpha u(x)| \leq C_K^{|\alpha|+1} \alpha!^s.$$

We point out that even though the operator in (1.5) has a symplectic real analytic characteristic manifold it is not analytic hypoelliptic. When $\ell = 1$, $q = 2$ and $k = 1$ this has been shown by Métivier in [13]; on the other hand it is also true that, in Métivier's case, the characteristic set is a real analytic manifold which is not symplectic. In a forthcoming paper we introduce a method leading among other things to the optimality proof of the value in (1.6).

Motivated by this type of model operators as well as by one model introduced by Oleĭnik and Radkevič (see [16], [17]), F. Treves in 1996, [22], introduced the idea that, in order to predict the analytic regularity of a sum of squares, one should examine a stratification of the characteristic variety into real analytic submanifolds. He conjectured that if each stratum is a symplectic manifold then the operator is analytic hypoelliptic and viceversa.

To this end he proposed a stratification obtained using the iterated Poisson brackets of the symbols of the vector fields. Unfortunately it has been shown in [1], [3] that the Poisson brackets do not identify the right strata in certain cases. For the operator in (1.5) however the Poisson bracket method gives the strata $\Sigma_1 = \{(0, y; 0, \eta) \mid \eta \neq 0, y \neq 0\}$ and $\Sigma_2 = \{(0, 0; 0, \eta) \mid \eta \neq 0\}$. Note that Σ_2 is not symplectic; actually it is a Hamilton leaf lying on the fiber of the cotangent bundle.

Actually the operator P in (1.1) generalizes this type of scenario, possibly admitting a non symplectic stratum of the type of Σ_2, even though P may be an operator for which the Poisson stratification fails to identify the non symplectic stratum. Since the characteristic manifold has dimension 2 the stratum can only be one-dimensional.

In order to state the result we need some notation.

Denote by $\mathscr{Q}$ the set of all vector fields X_j, $j \geq n+1$, for which $a_j = 0$. Hence if $X_j \in \mathscr{Q}$ we have that $X_j(x, D_y) = c_j x^{\alpha_j} D_y$, $j \geq n+1$.

In $\mathbb{Z}_+^n$ consider the set

$$B = \{\alpha_j \in \mathbb{Z}_+^n \mid X_j = c_j x^{\alpha_j} D_y \text{ for } X_j \in \mathscr{Q}\}. \tag{1.7}$$

The important object for determining the regularity of the solutions is the Newton polyhedron associated to B.

Definition 1.2. *Denote by* $\mathbf{NP}(B)$ *the Newton polyhedron containing the set* B *defined above:*

$$\mathbf{NP}(B) = \text{convex hull } \{\alpha + \mathbb{R}^n_+ \mid \alpha \in B\}, \tag{1.8}$$

where $\mathbb{R}^n_+ = \times_{j=1}^n \mathbb{R}^+$. *We denote by* $\partial\mathbf{NP}(B)$ *its boundary: by definition it is the union of all the compact faces of* $\mathbf{NP}(B)$.

$$\partial\mathbf{NP}(B) = \bigcup_\ell \mathbf{F}_\ell(B). \tag{1.9}$$

Let us now consider the set of all vector fields X_j whose monomial coefficient has a factor y

$$E' = \{X_j \mid X_j = c_j y^{a_j} x^{\alpha_j} D_y, \text{ with } a_j > 0\}.$$

We are interested only in those for which the x-exponent lies underneath the Newton polyhedron. Define

$$E = \{\alpha_j \mid X_j \in E', \alpha_j \notin \mathbf{NP}(B)\}. \tag{1.10}$$

Let $\gamma \in E$ and consider the half line through γ, $\{\lambda\gamma \mid \lambda > 0\}$. Because of Assumption (H2) this half line intersects $\partial\mathbf{NP}(B)$ in one point. We define $t(\gamma)$ so that

$$\{t(\gamma)\gamma\} = \{\lambda\gamma \mid \lambda > 0\} \cap \partial\mathbf{NP}(B)\,. \tag{1.11}$$

For instance if the half line meets $\partial\mathbf{NP}(B)$ in the face $\mathbf{F}_1(B)$ and $\mathbf{F}_1(B)$ is the convex hull of the vectors $\beta_1, \dots, \beta_m$, with integer components and $m \geq n$. Without loss of generality, we can assume that the first n vectors $\beta_1, \dots, \beta_n$ are linearly independent. Therefore, we have

$$t(\gamma) = \frac{\langle \wedge_{j=2}^n (\beta_j - \beta_1), \beta_1\rangle}{\langle \wedge_{j=2}^n (\beta_j - \beta_1), \gamma\rangle}.$$

Note that, by definition, $t(\gamma) > 1$ (see also Figure 1.)

Our result is

Theorem 1.1. *Using the above notation, for* $\gamma \in E$ *define*

$$s(\gamma) = \left(1 - \frac{1}{a(\gamma)} \frac{m}{m+1}\left(1 - \frac{1}{t(\gamma)}\right)\right)^{-1}, \tag{1.12}$$

where we denoted by $a(\gamma)$ *the* y*-exponent in the field containing* x^γ *and by* $m+1$ *the length of the minimal Poisson bracket generating the Lie algebra, according to Assumption (H1).*

Then the operator P *in* (1.1) *is Gevrey hypoelliptic of order* s, *with*

$$s = \begin{cases} \max_{\gamma\in E} s(\gamma) & \text{if } E \neq \emptyset, \\ 1 & \text{if } E = \emptyset. \end{cases} \tag{1.13}$$

We explicitly note that P *in* (1.1) *is analytic hypoelliptic if* $E = \emptyset$.

The first example is the operator in (1.5). It is Gevrey hypoelliptic of the order shown in (1.6). Here $n = 1$ and the Newton polygon is made by one point, $\alpha = q - 1$. $\gamma = \ell - 1$, the corresponding a is equal to k and finally $m + 1 = q$. Thus

$$s(\gamma) = \left(1 - \frac{1}{k}\frac{q-1}{q}\left(1 - \frac{\ell - 1}{q-1}\right)\right)^{-1} = \frac{kq}{kq - q + \ell},$$

which coincides with the value in (1.6).

We observe here that the Lie algebra is generated with the Poisson brackets of length q: $(\mathrm{ad}(X_1)^{q-1}X_2$. But computing a shorter bracket, $(\mathrm{ad}(X_1)^{\ell-1}X_3$, we find a symbol vanishing on $\{x_1 = \xi_1 = 0, y = 0\}$, which is non symplectic.

The second example is more complicated: it is in three variables. Let—$x \in \mathbb{R}^2$—

(1.14)

$$P_2(x, y, D_x, D_y) = D_1^2 + D_2^2 + x_2^{2(r-1)} D_y^2 + x_1^{2(q-1)} D_y^2 + x_1^{2(p-1)} y^{2a} D_y^2.$$

Here we assume that $1 < p < q < r$. As a consequence $m + 1 = q$. The characteristic manifold is $\{(0, 0, y; 0, 0, \eta) \mid \eta \neq 0\}$. The Newton polygon associated to (1.14) is shown in Figure 2.

The Treves stratification detects a non symplectic stratum in this case, given again by $\{(0, 0, 0; 0, 0, \eta) \mid \eta \neq 0\}$.

Finally we would like to mention the following example:

(1.15)

$$P_3(x, y, D_x, D_y) = D_1^2 + D_2^2 + x_2^{2(r-1)} D_y^2 + x_1^{2(q-1)} D_y^2 + x_1^{2(p-1)} y^{2a} D_y^2.$$

where $1 < r < p < q$. Its Newton polygon is shown in Figure 3.

P_3 and the operator in [1] consists in the fact that there is a non symplectic "stratum" whose Hamilton leaf lies on the fiber of the cotangent bundle. It is known that this situation is much more difficult to treat compared to leaves lying on the base of the cotangent bundle. At the moment we have no optimality proof for the Gevrey regularity (1.16) of (1.15). We also remark that the optimality of (1.16) would imply that the Treves conjecture does not hold in dimension 3.

Finally a last remark concerning P_1 in (1.5). Since the x-space dimension is 1 in this case, the Newton polygon is degenerate, i.e. it is just one point on the real positive line:

0

where $C_\varphi > 0$ *and independent of* p.

This kind of cutoff functions has been explicitly constructed in [9]. We moreover are going to need cutoff functions of the above type but only depending on the variable y. In fact if a cutoff function $\varphi = \varphi(x, y)$ is identically equal to 1 in a small neighborhood of the origin, each time it gets derived with respect to x it has a support bounded away from $\{x = 0\}$, i.e. it has support in the ellipticity region of the operator, where analyticity is well known. For this reason we assume, without loss of generality, that u is compactly supported with respect to x near $\{x = 0\}$.

Our purpose is to prove that if u is a smooth function such that $Pu \in G^{s_1}(\Omega)$, for $1 \leq s_1 \leq s$, s defined in (1.13), then we have an estimate of the form

$$\|X(D_y^k \varphi_p) D_y^\ell u\| \leq C^{p+1} p!^s, \quad \ell + k \leq p. \tag{2.4}$$

Here φ_p is a cutoff function of the type defined above, X denotes any of the vector fields involved in the definition of P and C is positive and independent of p.

Actually, for technical reasons, it is useful to prove a slightly more general estimate of the form

$$\begin{aligned} \|\varphi_p^{(h)} D_y^\ell u\|_{\frac{1}{m+1}} &+ \max_{j=1,\dots,N} \|X_j \varphi_p^{(h)} D_y^\ell u\| \\ &+ \max_{\substack{j=1,\dots,n \\ k=1,\dots,N'}} \|\tilde{X}_{jk} \varphi_p^{(h)} D_y^\ell u\| \leq C_1^{\ell+1} C_2^h p^{s(h+\ell)}, \end{aligned} \tag{2.5}$$

with h, $\ell(m+1) \in \mathbb{N}$, $h + \ell \leq p$, C_1 and C_2 positive constants independent of p. Here $\varphi_p^{(h)} = D_y^h \varphi_p$. Note that the above estimate is meaningful if ℓ is large; for bounded values of ℓ it is an immediate consequence of the C^∞-regularity of u (due to the Hörmander theorem) and of the estimates (2.3).

To prove (2.5) we proceed by induction on $h + \ell$: the estimate is trivially verified if $h + \ell = 0$. Assume that (2.5) is true if $h + \ell < p'$. We want to show that it holds when $h + \ell = p'$. To this end we again use induction on ℓ. If $0 < \ell < 1$ then (2.5) is easily deduced using the properties of the cutoff function φ_p. Assume that (2.5) is true for $h + \ell = p'$, but $\ell < p'' \leq p'$ and show that it holds when $\ell = p''$.

For the sake of simplicity we take $p'' = p' = p$. Thus we want to show that

$$\|\varphi_p D_y^p u\|_{\frac{1}{m+1}} + \max_{j=1,\dots,N} \|X_j \varphi_p D_y^p u\| + \max_{\substack{j=1,\dots,n \\ k=1,\dots,N'}} \|\tilde{X}_{jk} \varphi_p D_y^p u\| \leq C_1^{p+1} p^{sp}, \tag{2.6}$$

assuming that for $\ell < p$, $h + \ell \leq p$ we have

$$\|\varphi_p^{(h)} D_y^\ell u\|_{\frac{1}{m+1}} + \max_{j=1,\dots,N} \|X_j \varphi_p^{(h)} D_y^\ell u\| + \max_{\substack{j=1,\dots,n \\ k=1,\dots,N'}} \|\tilde{X}_{jk} \varphi_p^{(h)} D_y^\ell u\| \leq C_1^{\ell+1} C_2^h p^{s(h+\ell)}. \tag{2.7}$$

Replacing u with $\varphi_p D_y^p u$ in (2.1) we obtain

$$\|\varphi_p D_y^p u\|^2_{\frac{1}{m+1}} + \sum_{j=1}^N \|X_j \varphi_p D_y^p u\|^2 + \sum_{j=1}^n \sum_{k=1}^{N'} \|\tilde{X}_{jk} \varphi_p D_y^p u\|^2 \leq C \left(\langle P \varphi_p D_y^p u, \varphi_p D_y^p u \rangle + \|\varphi_p D_y^p u\|^2 \right), \tag{2.8}$$

where again $u \in C_0^\infty(\Omega)$.

First of all we show how to treat the error term in the right hand side of (2.8) (as well as any similar terms coming from the estimate of the scalar product):

Proposition 2.1 (([2])). *Using the above notation we have the inequality*

$$\|\varphi_p D_y^p u\| \leq C p^{-\frac{1}{m+1}} \|\varphi_p D_y^p u\|_{\frac{1}{m+1}} + C^{p+1} p^p, \tag{2.9}$$

where C is positive and independent of p.

Proof. Let χ be a smooth function such that $\chi(t) = 1$ if $|t| \geq 2$ and $\chi(t) = 0$ if $|t| \leq 1$. Consider the pseudodifferential operator $\chi(p^{-1} D_y)$. We have that $\chi(p^{-1} D_y) \in OPS^0_{0,0}$, the Hörmander (ρ, δ)–class of order 0 with $\rho = \delta = 0$. Then

$$\|\varphi_p D_y^p u\| \leq \|(1 - \chi(p^{-1} D_y)) \varphi_p D_y^p u\| + \|\chi(p^{-1} D_y) \varphi_p D_y^p u\|. \tag{2.10}$$

Consider the first summand on the r.h.s. above. Even though $1 - \chi$ has a compact support, we have no composition formula for χ. Observe that

$$\varphi_p D_y^p u = \sum_{s=0}^p (-1)^s \binom{p}{s} D_y^{p-s} \left(\varphi_p^{(s)} u \right), \tag{2.11}$$

so that
(2.12)
$$\|(1-\chi(p^{-1}D_y))\varphi_p D_y^p u\| \leq \sum_{s=0}^{p} \binom{p}{s} \left\|(1-\chi(p^{-1}D_y))D_y^{p-s}\left(\varphi_p^{(s)} u\right)\right\|.$$

We immediately verify that

$$\sigma\left((1-\chi(p^{-1}D_y))D_y^{p-s}\right) = (1-\chi(p^{-1}\eta))\eta^{p-s} \in S^0_{0,0},$$

because $p^{-1}|\eta| \leq 2$ in the support of $1-\chi$. Here $\sigma(A)$ denotes the symbol of the pseudodifferential operator A.

We also verify that

$$\max_{0\leq\alpha\leq\ell} \sup_{\eta} \left|\partial_\eta^\alpha \left(1-\chi(p^{-1}\eta)\eta^{p-s}\right)\right| \leq C^{p-s+1} p^{p-s}.$$

Here $C > 0$ is a suitable constant independent of p and ℓ is given in Theorem (A.1). This bounds the $S^0_{0,0}$–seminorms of $(1-\chi(p^{-1}D_y))D_y^{p-s}$ needed to apply the Calderón–Vaillancourt theorem (see Thm. (A.1) in the appendix) so that we obtain that

$$\|(1-\chi(p^{-1}D_y))D_y^{p-s}\|_{\mathscr{L}(L^2,L^2)} \leq C^{p-s+1}p^{p-s},$$

for a new positive constant C. Hence, using the definition of the cutoff function φ_p, we deduce

$$\begin{aligned}
\|(1-\chi(p^{-1}D_y))\varphi_p D_y^p u\| &\leq \sum_{s=0}^{p} \binom{p}{s} C^{p-s+1} p^{p-s} \|\varphi_p^{(s)} u\| \\
&\leq C^{p+1} C_\varphi p^p \|u\| \sum_{s=0}^{p} \binom{p}{s} \left(\frac{C_\varphi}{C}\right)^s \\
&\leq C_1^{p+1} p^p,
\end{aligned}$$

where C_1 is independent of p if we choose $C \geq C_\varphi$, which is always possible.

Consider now the second summand in (2.10). We have

$$\begin{aligned}
&\|\chi(p^{-1}D_y)\varphi_p D_y^p u\| \\
&\qquad = p^{-\frac{1}{m+1}} \|p^{\frac{1}{m+1}} \chi(p^{-1}D_y) D^{-\frac{1}{m+1}} \circ D^{\frac{1}{m+1}} \varphi_p D_y^p u\|,
\end{aligned}$$

where $D^s = \operatorname{Op}\left((1+|\xi|^2+\eta^2)^{s/2}\right)$, for any $s \in \mathbb{R}$. Again using the support of χ we see that

$$\begin{aligned}
&\sigma\left(p^{\frac{1}{m+1}} \chi(p^{-1}D_y) D^{-\frac{1}{m+1}}\right) \\
&\qquad = p^{\frac{1}{m+1}} \chi(p^{-1}\eta)(1+|\xi|^2+\eta^2)^{-\frac{1}{2(m+1)}} \in S^0_{0,0},
\end{aligned}$$

with the $S^0_{0,0}$–seminorms uniformly bounded w.r.t. p. Thus the Calderón–Vaillancourt theorem yields

$$\|p^{\frac{1}{m+1}}\chi(p^{-1}D_y)D^{-\frac{1}{m+1}}\|\|_{\mathscr{L}(L^2,L^2)} \leq C,$$

where C is a positive constant independent of p. So we have

$$\begin{aligned}\|\chi(p^{-1}D_y)\varphi_p D_y^p u\| &\leq Cp^{-\frac{1}{m+1}}\|D^{\frac{1}{m+1}}\varphi_p D_y^p u\| \\ &\leq Cp^{-\frac{1}{m+1}}\|\varphi_p D_y^p u\|_{\frac{1}{m+1}}.\end{aligned}$$

This completes the proof of the proposition. $\square$

In the next sections we are going to examine the terms in (2.8) originating from the scalar product on the r.h.s. and containing the X_j and those containing the $\tilde{X}_{jk}$ separately. As we shall see the main contribution comes from those X_j whose coefficients are represented as points below the Newton polyhedron.

3. Proof of the Theorem: Dealing with the Fields X_j

Let us consider the scalar product

$$\langle P\varphi_p D_y^p u, \varphi_p D_y^p u\rangle \tag{3.1}$$

in (2.8). We write

$$P\varphi_p D_y^p u = \varphi_p D_y^p P u + \left[P, \varphi_p D_y^p\right] u.$$

Since we are assuming that Pu is of Grevrey order s at least in a neighborhood of the point we are looking at, it is easily seen that the scalar product containing the first summand above yields an G^s growth rate when we bound it via Cauchy–Schwartz.

We have to treat the commutator term. First of all, by (1.1),

$$\left[P, \varphi_p D_y^p\right] = \sum_{j=1}^{N}\left[X_j^2, \varphi_p D_y^p\right] + \sum_{k=1}^{n}\sum_{j=1}^{N'}\left[\tilde{X}_{jk}^2, \varphi_p D_y^p\right]. \tag{3.2}$$

Furthermore we have, for a generic vector field X and a generic cutoff function φ,

$$\left[X^2, \varphi D^p\right] = 2X\left[X, \varphi D^p\right] - \left[X, \left[X, \varphi D^p\right]\right]. \tag{3.3}$$

First of all observe that our cutoff function φ_p depends on y only.

Now from (2.8) and (3.3) we obtain that

$$
\begin{aligned}
(3.4)\quad & \left|\langle P\varphi_p D_y^p u, \varphi_p D_y^p u\rangle\right| \\
&\le \left|\langle \varphi_p D_y^p P u, \varphi_p D_y^p u\rangle\right| + \left|\langle \left[P, \varphi_p D_y^p\right] u, \varphi_p D_y^p u\rangle\right| \\
&\le C\left(\|\varphi_p D_y^p P u\|^2 + \|\varphi_p D_y^p u\|^2\right) + \sum_{j=n+1}^{N} \left|\langle \left[X_j^2, \varphi_p D_y^p\right] u, \varphi_p D_y^p u\rangle\right| \\
&\qquad + \sum_{k=1}^{n}\sum_{j=n}^{N'} \left|\langle \left[\tilde{X}_{jk}^2, \varphi_p D_y^p\right] u, \varphi_p D_y^p u\rangle\right| .
\end{aligned}
$$

Moreover we observe that when $Pu \in G^s$ the first term in the next to last line above is easy to estimate:

$$\|\varphi_p D_y^p P u\| \le C^{p+1} p!^s$$

The second term in the next to last line of (3.4) is treated using Proposition 2.1 and this yields the term $p^{-\frac{1}{m+1}} \|\varphi_p D_y^p u\|_{\frac{1}{m+1}}$ that can be reabsorbed in the l.h.s. of (2.8) since $p >> 1$.

Let us estimate the terms containing the X_j^2. By (3.3) we have

$$
\begin{aligned}
(3.5)\quad & \langle \left[X_j^2, \varphi_p D_y^p\right] u, \varphi_p D_y^p u\rangle \\
&= 2\langle X_j \left[X_j, \varphi_p D_y^p\right] u, \varphi_p D_y^p u\rangle - \langle \left[X_j, \left[X_j, \varphi_p D_y^p\right]\right] u, \varphi_p D_y^p u\rangle
\end{aligned}
$$

In this section and in the sections 4 and 5 we estimate the single and double commutator terms containing the fields X_j, for $j = 1, \dots, N$. In the last section we include the estimate for the vector fields $\tilde{X}_{jk}$ as well as the final argument.

There are several cases:

i) The vector field X_j is one of the X_j when $j = 1, \dots, n$.
ii) We have $X_j \in \mathcal{Q}$. In particular for these X_j we have $a_j = 0$.
iii) The field X_j is such that $a_j > 0$.

3.1. **Case i).** For $j = 1, \dots, n$, $X_j = D_j$, so that $[X_j, \varphi_p D_y^p] = 0$ as well as $[X_j, [X_j, \varphi_p D_y^p]\,] = 0$. Thus these fields give no contribution.

One might object that this is due to the fact that we chose a simple case where the coefficients of the vector fields X_j, $j = 1, \dots, n$, are 1. No doubt this is true and simplifies the argument at this stage, but a more complicated model in which the sum $\sum_{j=1}^n D_j^2$ is replaced by an elliptic operator of the form $\sum_{r,s=1}^n a_{rs}(x,y) X_r X_s$ would have also led us to the same conclusion of Theorem 1.1, provided the coefficients are real valued, $a_{rs} \in C^\omega(\Omega)$ and the quadratic form is uniformly elliptic, i.e. $\sum_{r,s=1}^n a_{rs}(x,y)\xi_r\xi_s \ge c|\xi|^2$.

This takes care of the first case.

3.2. **Case ii).** Let us consider the case when $n < j \leq N$ and $a_j = 0$, i.e. $X_j = c_j x^{\alpha_j} D_y$, $\alpha_j \in \mathbb{Z}_+^n$, and of course $|\alpha_j| > 0$. Here $\mathbb{Z}_+$ denotes the non negative integers. Note that these fields correspond to points that are either on the Newton polyhedron or above it.

Let us consider the commutator

$$c_j \left[x^{\alpha_j} D_y, \varphi_p D_y^p \right] = c_j x^{\alpha_j} \varphi_p' D_y^p,$$

where we use the notation $\varphi_p^{(k)} = D_y^k \varphi_p$.

Using the formula

$$\varphi_p' D_y^p = \sum_{\ell=0}^{p-1} (-1)^\ell D_y \varphi_p^{(\ell+1)} D_y^{p-1-\ell} + (-1)^p \varphi_p^{(p+1)}, \tag{3.6}$$

we may write

$$\begin{aligned}(3.7)\quad & c_j \left[x^{\alpha_j} D_y, \varphi_p D_y^p \right] \\ &= \sum_{\ell=0}^{p-1} (-1)^\ell c_j x^{\alpha_j} D_y \varphi_p^{(\ell+1)} D_y^{p-1-\ell} + (-1)^p c_j x^{\alpha_j} \varphi_p^{(p+1)} \\ &= \sum_{\ell=0}^{p-1} (-1)^\ell X_j \varphi_p^{(\ell+1)} D_y^{p-1-\ell} + (-1)^p c_j x^{\alpha_j} \varphi_p^{(p+1)}.\end{aligned}$$

Let us estimate the first summand in the r.h.s of (3.5) for the X_j's considered in this section. Since the X_j are formally self adjoint we have, for a positive δ small enough

$$\begin{aligned}(3.8)\quad & 2 \left| \langle X_j \left[X_j, \varphi_p D_y^p \right] u, \varphi_p D_y^p u \rangle \right| \\ &= 2 \left| \langle \left[X_j, \varphi_p D_y^p \right] u, X_j \varphi_p D_y^p u \rangle \right| \\ &\leq \delta^2 \| X_j \varphi_p D_y^p u \|^2 + \frac{1}{\delta^2} \| \left[X_j, \varphi_p D_y^p \right] u \|^2.\end{aligned}$$

The first term can be absorbed on the left hand side of (2.8) provided δ is sufficiently small. As for the second term, we have, because of (3.7),

$$\frac{1}{\delta} \| \left[X_j, \varphi_p D_y^p \right] u \| \leq \frac{1}{\delta} \sum_{\ell=0}^{p-1} \| X_j \varphi_p^{(\ell+1)} D_y^{p-1-\ell} u \| + \frac{C}{\delta} \| \varphi_p^{(p+1)} u \|,$$

where $C > 0$ is a constant depending on the support of φ_p, on α_j, but independent of p.

Keeping into account the estimates (2.3) for φ_p and the inductive hypothesis (2.7), the above terms can be bounded as we claimed, i.e. by $C_1^{p+1} p^{sp}$.

The procedure followed above allows us to decrease the $y-$derivatives on u and correspondingly increase the $y-$derivative on the cutoff function φ_p. Since the latter has actually an analytic behaviour, by abuse of terminology we say that the terms yield an analytic growth rate.

Consider now the double commutator term in (3.5). We have

$$
\begin{aligned}
(3.9)\quad & \left|\langle\left[X_j,\left[X_j,\varphi_p D_y^p\right]\right]u,\varphi_p D_y^p u\rangle\right| \\
&= c_j^2\left|\langle x^{\alpha_j}\varphi_p'' D_y^p u, x^{\alpha_j}\varphi_p D_y^p u\rangle\right| \\
&\le C\left(\frac{1}{p^2}\|x^{\alpha_j}\varphi_p'' D_y^p u\|^2 + p^2\|x^{\alpha_j}\varphi_p D_y^p u\|^2\right) \\
&\le C_1\Bigg(\sum_{\ell=0}^{p-1}\frac{1}{p^2}\|X_j\varphi_p^{(\ell+2)} D_y^{p-(\ell+1)}u\|^2 + \frac{1}{p^2}\|\varphi_p^{(p+2)}u\|^2 \\
&\qquad + \sum_{\ell=0}^{p-1} p^2\|X_j\varphi_p^{(\ell)} D_y^{p-(\ell+1)}u\|^2 + p^2\|\varphi_p^{(p)}u\|^2\Bigg).
\end{aligned}
$$

Each term above, by the properties of φ_p and the inductive hypothesis give once more an analytic growth rate.

Hence we have bounded all terms originating from the vector fields considered in Case ii).

3.3. **Case iii).** Consider now vector fields of the form

$$X_j = c_j y^{a_j} x^{\alpha_j} D_y, \quad a_j > 0.$$

Depending on the value of the multiindex α_j, these fields may be represented with points below the Newton polyhedron or inside the Newton polyhedron. The latter are very easy to treat and we shall mention them in Section 5. Hence we are going to assume that

$$\alpha_j \notin \mathbf{NP}(B)\,. \tag{3.10}$$

Of course (3.10) implies that $\operatorname{dist}(\alpha_j, \partial\mathbf{NP}(B)) > 0$.

Remark 3.1. *We point out explicitly that if y is bounded away from zero there is nothing to prove, since these vector fields can be treated as we just did in the previous section, since at that point we may divide by y. This in particular gives that the operator P in* (1.1) *is analytic hypoelliptic away from $y = 0$.*

Let us start with the simple commutator in (3.5).

$$
\begin{aligned}
\left[X_j,\varphi_p D_y^p\right] &= c_j x^{\alpha_j}\left(y^{a_j}\left[D_y,\varphi_p D_y^p\right] + \left[y^{a_j},\varphi_p D_y^p\right]D_y\right) \\
&= c_j x^{\alpha_j} y^{a_j}\varphi_p' D_y^p + c_j x^{\alpha_j}\varphi_p\left[y^{a_j}, D_y^p\right]D_y.
\end{aligned}
$$

Since

$$\left[y^{a_j}, D_y^p\right] = -\sum_{k=1}^{a_j} \binom{p}{k} \frac{a_j!}{(a_j-k)!}(-i)^k y^{a_j-k} D_y^{p-k}, \tag{3.11}$$

we may write

$$\begin{aligned}\left[X_j, \varphi_p D_y^p\right] &= c_j x^{\alpha_j} y^{a_j} \varphi_p' D_y^p \\ &\quad -c_j x^{\alpha_j}\varphi_p \sum_{\ell=1}^{a_j} \binom{p}{\ell} \frac{a_j!}{(a_j-\ell)!}(-1)^\ell y^{a_j-\ell} D_y^{p-\ell+1}.\end{aligned} \tag{3.12}$$

Using (3.6) to reconstruct a vector field in the first term on the right of the above identity, we may conclude that

$$\begin{aligned}\left[X_j, \varphi_p D_y^p\right] &= \sum_{k=1}^{p}(-1)^{k-1} X_j \varphi_p^{(k)} D_y^{p-k} \\ &\quad + c_j y^{a_j} x^{\alpha_j} (-1)^p \varphi_p^{(p+1)} \\ &\quad -c_j x^{\alpha_j}\varphi_p \sum_{\ell=1}^{a_j} \binom{p}{\ell} \frac{a_j!}{(a_j-\ell)!}(-1)^\ell y^{a_j-\ell} D_y^{p-\ell+1}.\end{aligned} \tag{3.13}$$

Let us now examine the scalar product

$$\begin{aligned}&2\langle X_j \left[X_j, \varphi_p D_y^p\right] u, \varphi_p D_y^p u\rangle \\ &\qquad = 2\langle \left[X_j, \varphi_p D_y^p\right] u, X_j \varphi_p D_y^p u\rangle \\ &\qquad\qquad + 2\langle \left[X_j, \varphi_p D_y^p\right] u, \frac{a_j}{i} y^{a_j-1} x^{\alpha_j} \varphi_p D_y^p u\rangle,\end{aligned}$$

since the field X_j is not formally self adjoint. Here u is a function in $C_0^\infty(\Omega)$.

We can now estimate the modulus of the above scalar product; for a convenient positive small δ

$$\begin{aligned}&2\left|\langle X_j \left[X_j, \varphi_p D_y^p\right] u, \varphi_p D_y^p u\rangle\right| \\ &\quad \leq C\left[(1+\delta^{-2})\| \left[X_j, \varphi_p D_y^p\right] u\|^2 + \delta^2 \|X_j \varphi_p D_y^p u\|^2 + \|\varphi_p D_y^p u\|^2\right]\end{aligned} \tag{3.14}$$

The second term above is absorbable on the left hand side of (2.8) provided δ is chosen conveniently small. The last term is treated as indicated in Proposition 2.1. We are left with the norm containing the commutator, for which we have the expression (3.13). Thus

$$\| \left[X_j, \varphi_p D_y^p\right] u\| \leq \sum_{k=1}^{p} \|X_j \varphi_p^{(k)} D_y^{p-k} u\| + C_1 \|\varphi_p^{(p+1)} u\| + C_2 \sum_{\ell=1}^{a_j} \binom{p}{\ell} \|x^{\alpha_j} y^{a_j-\ell} \varphi_p D_y^{p-\ell+1} u\|, \tag{3.15}$$

where C_1, C_2 are positive constants independent of p but depending on the support of u.

By the inductive hypothesis (2.7) the norms in the first sum above give an analytic growth rate with respect to p, since they are exactly of the same type as our starting term, with less derivatives landing onto u.

The term with $\varphi_p^{(p+1)}$ can be bound using the estimate (2.3) of the cutoff function φ_p. We remark also that the constant C_1 can be chosen to be small if we shrink the neighborhood of the origin Ω where the support of u is contained. Hence this term gives an analytic growth rate too.

We are left with the terms in the second sum above.

First observe that

$$\binom{p}{\ell} = \frac{p(p-1)\cdots(p-\ell+1)}{\ell!} \leq p^\ell.$$

Hence we are going to estimate

$$C_2 p^\ell \|x^{\alpha_j} y^{a_j-\ell} \varphi_p D_y^{p-\ell+1} u\|, \qquad \ell = 1, \dots, a_j. \tag{3.16}$$

We are going to use the elementary estimate

$$xy \leq \frac{x^\lambda}{\lambda} + \frac{y^\mu}{\mu}, \qquad x, y \geq 0, \quad \frac{1}{\lambda} + \frac{1}{\mu} = 1. \tag{3.17}$$

Pick a $\theta \in]\frac{a_j-\ell}{a_j}, 1[$

$$p^\ell |y|^{a_j-\ell} = p^{(1-\theta)\ell} \cdot p^{\theta\ell} |y|^{a_j-\ell} \leq p^{(1-\theta)\ell\lambda} + p^{\theta\ell\mu} |y|^{(a_j-\ell)\mu} \leq p^{(1-\theta)a_j} + p^{\theta\frac{\ell a_j}{a_j-\ell}} |y|^{a_j},$$

if $\lambda^{-1} + \mu^{-1} = 1$ and we choose

$$\lambda = \frac{a_j}{\ell}, \quad \mu = \frac{a_j}{a_j - \ell} \quad .$$

If $\ell = a_j$ we simply skip this step.

Note that the choice of θ in the interval specified above is a sort of *Ansatz* that we shall verify in what follows.

Thus we get

$$p^{\ell}\|x^{\alpha_j}y^{a_j-\ell}\varphi_p D_y^{p-\ell+1}u\| \leq p^{(1-\theta)a_j}\|x^{\alpha_j}\varphi_p D_y^{p+1-\ell}u\| + p^{\theta\frac{\ell a_j}{a_j-\ell}}\|y^{a_j}x^{\alpha_j}\varphi_p D_y^{p+1-\ell}u\|. \tag{3.18}$$

Consider the second term in the r.h.s. above. Using the identity

$$\varphi_p D_y^{p'} = \sum_{j=0}^{p'-1}(-1)^j D_y \varphi_p^{(j)} D_y^{p'-1-j} + (-1)^{p'}\varphi_p^{(p')}, \qquad p' \leq p, \tag{3.19}$$

we have

$$p^{\theta\frac{\ell a_j}{a_j-\ell}}\|y^{a_j}x^{\alpha_j}\varphi_p D_y^{p+1-\ell}u\| \leq C\sum_{h=0}^{p-\ell} p^{\theta\frac{\ell a_j}{a_j-\ell}}\|X_j\varphi_p^{(h)}D_y^{p-\ell-h}u\| + Cp^{\theta\frac{\ell a_j}{a_j-\ell}}\|\varphi_p^{(p+1-\ell)}u\|. \tag{3.20}$$

Let us look at the terms in the sum above. First note that the decrease in the y-derivatives of u is strictly positive since $\ell + h \geq 1$.

Our second remark is that, since $|\varphi^{(h)}| \leq C_\varphi^{h+1}p^h$, a decrease of h units in the order of D_y contributes a factor p^h, while a decrease of ℓ units in the order of D_y contributes a factor

$$p^{\theta\frac{a_j}{a_j-\ell}}.$$

A similar argument holds for the second term in the right hand side of (3.20).

Hence an iteration of this procedure yields a growth rate of the form

$$p!^{\theta\frac{a_j}{a_j-\ell}}, \tag{3.21}$$

since $\theta\frac{a_j}{a_j-\ell} > 1$ by our *Ansatz* above.

Let us now focus on the first term in (3.18). First remark that

$$p^{(1-\theta)a_j}|x^{\alpha_j}| = p^{\rho}\cdot p^{(1-\theta)a_j-\rho}|x^{\alpha_j}|,$$

so that, by (3.17) we have

$$p^{(1-\theta)a_j}|x^{\alpha_j}| \leq p^{\rho\sigma_1} + p^{((1-\theta)a_j-\rho)\sigma_2}|x^{\sigma_2\alpha_j}|,$$

where

$$\frac{1}{\sigma_1} + \frac{1}{\sigma_2} = 1.$$

Choose now, for reasons that will become clear in a short while,

$$\sigma_2 = t(\alpha_j),$$

where the right hand side has been defined in (1.11). Note that this is fine since $t(\alpha_j) > 1$ by definition when α_j lies below the Newton polyhedron. The above choice gives

$$\sigma_1 = \frac{t(\alpha_j)}{t(\alpha_j) - 1}.$$

For the sake of simplicity we shall write just t instead of $t(\alpha_j)$. Replacing σ_j, $j = 1, 2$ with their values, we see that the first term in (3.18) is estimated as

$$p^{(1-\theta)a_j} \| x^{\alpha_j} \varphi_p D_y^{p+1-\ell} u \| \leq p^{\rho \frac{t}{t-1}} \| \varphi_p D_y^{p+1-\ell} u \| + p^{((1-\theta)a_j - \rho)t} \| x^{t\alpha_j} \varphi_p D_y^{p+1-\ell} u \|. \tag{3.22}$$

We note at this point that the choice of $\rho > 0$ can be done so that the exponents of p above are ≥ 1. We may state this as an *Ansatz* that will be confirmed later.

Consider the second term on the right hand side of (3.22). By definition $t\alpha_j \in \partial \mathbf{NP}(B)$. This implies that $t\alpha_j$ belongs to a face of $\partial \mathbf{NP}(B)$. Since we may always suppose that the faces of the Newton polyhedron are convex, we get that there are finitely many vertices of $\partial \mathbf{NP}(B)$, say $\beta_1, \dots, \beta_q$, $q \in \mathbb{N}$ depending on α_j only, and q numbers $t_1, \dots, t_q$, with $t_r \in]0, 1[$, $\sum_{r=1}^{q} t_r = 1$, such that

$$t\alpha_j = \sum_{r=1}^{q} t_r \beta_r,$$

whence we have

$$|x^{t\alpha_j}| \leq \sum_{r=1}^{q} t_r |x^{\beta_r}|.$$

Therefore it follows

$$\| x^{t\alpha_j} \varphi_p D_y^{p+1-\ell} u \| \leq \sum_{r=1}^{q} \| x^{\beta_r} \varphi_p D_y^{p+1-\ell} u \|.$$

This is bounded, using (3.19), as

$$\begin{aligned} &p^{((1-\theta)a_j - \rho)t} \| x^{t\alpha_j} \varphi_p D_y^{p+1-\ell} u \| \\ &\leq p^{((1-\theta)a_j - \rho)t} \sum_{r=1}^{q} \left[\sum_{\lambda=0}^{p-\ell} \| x^{\beta_r} D_y \varphi_p^{(\lambda)} D_y^{p-\ell-\lambda} u \| + \| x^{\beta_r} \varphi_p^{(p+1-\ell)} u \| \right] \\ &\leq C' p^{((1-\theta)a_j - \rho)t} \sum_{r=1}^{q} \left[\sum_{\lambda=0}^{p-\ell} \| X_{j_r} \varphi_p^{(\lambda)} D_y^{p-\ell-\lambda} u \| + \| \varphi_p^{(p+1-\ell)} u \| \right], \end{aligned} \tag{3.23}$$

X_{j_r} being the vector fields $c_{j_r} x^{\beta_r} D_y$, $r = 1, ..., q$.
An argument similar to that which led us to (3.21) allows us to conclude, using the inductive hypothesis, that the terms under the inner summation give a growth rate

$$p!^{((1-\theta)a_j - \rho)\frac{t}{\ell}}. \tag{3.24}$$

Finally the last term $\|\varphi_p^{(p+1-\ell)} u\|$ appearing in (3.23) is easy since it gives a growth rate of analytic type due to (2.3). We are left then with the first term on the right hand side of (3.22)

$$p^{\rho \frac{t}{t-1}} \|\varphi_p D_y^{p+1-\ell} u\| \tag{3.25}$$

for which is not possible to reconstruct a vector field. To treat this term we have to resort to the subelliptic part of estimate (2.8). In particular we would like to pull back a fraction of the y-derivative. The problem is that a fraction of a derivative is a pseudodifferential operator and we have to commute it with the cutoff function φ_p whose derivatives we can estimate up to order Rp—see Definition 2.1. To this end we are going to use Lemma B.1 and Corollary B.1 of [1], but before that we introduce a couple of auxiliary cutoff functions, χ_p, ω_p (see also [1] for the method used in the following.)

Let ω_p, χ_p denote Ehrenpreis cutoff function in $C^\infty(\mathbb{R})$, such that ω_p, χ_p are non negative and moreover $\omega_p(x) = 1$ for $|x| > 2$, $\omega_p(x) = 0$ for $|x| < 1$. We also assume that $\chi_p(x) = 0$ for $|x| < 3$ and $\chi_p(x) = 1$ for $|x| > 4$. Then $\omega_p \chi_p = \chi_p$.

Consider the norm in (3.25). We have

$$\begin{aligned} \|\varphi_p D_y^{p+1-\ell} u\| \leq \|\varphi_p (1 - \chi_p(p^{-1} D_y)) D_y^{p+1-\ell} u\| \\ + \|\varphi_p \chi_p(p^{-1} D_y) D_y^{p+1-\ell} u\|. \end{aligned} \tag{3.26}$$

Now the first term is easily bounded by arguing as done in Prop. 2.1 since on the support of $1 - \chi_p$, $|\eta| \leq 4p$. Hence

$$p^{\rho \frac{t}{t-1}} \|\varphi_p (1 - \chi_p(p^{-1} D_y)) D_y^{p+1-\ell} u\| \leq C_1^p p!$$

and we get

$$p^{\rho \frac{t}{t-1}} \|\varphi_p D_y^{p+1-\ell} u\| \ \leq \ p^{\rho \frac{t}{t-1}} \|\varphi_p \chi_p(p^{-1} D_y) D_y^{p+1-\ell} u\| \ + \ C_1^p p!. \tag{3.27}$$

Consider the other summand above. We have

$$
\begin{aligned}
(3.28)\quad & \|\varphi_p \chi_p(p^{-1}D_y) D_y^{p+1-\ell} u\| \\
&= \|\varphi_p \omega_p(p^{-1}D_y)\chi_p(p^{-1}D_y) D_y^{\frac{1}{m+1}} D_y^{p-\ell+\frac{m}{m+1}} u\| \\
&\leq \|\omega_p(p^{-1}D_y) D_y^{\frac{1}{m+1}} \varphi_p \chi_p(p^{-1}D_y) D_y^{p-\ell+\frac{m}{m+1}} u\| \\
&\quad + \|[\omega_p(p^{-1}D_y) D_y^{\frac{1}{m+1}}, \varphi_p]\ \chi_p(p^{-1}D_y) D_y^{p-\ell+\frac{m}{m+1}} u\|.
\end{aligned}
$$

In view of the Calderón-Vaillancourt theorem (see Theorem. A.1 in Appendix) $\omega_p(p^{-1}D_y)$ is L^2 bounded and we conclude that

$$
\begin{aligned}
(3.29)\quad \|\varphi_p \chi_p(p^{-1}D_y) D_y^{p+1-\ell} u\| &\leq \|\varphi_p \chi_p(p^{-1}D_y) D_y^{p-\ell+\frac{m}{m+1}} u\|_{\frac{1}{m+1}} \\
&\quad + \|[\omega_p(p^{-1}D_y) D_y^{\frac{1}{m+1}}, \varphi_p]\ \chi_p(p^{-1}D_y) D_y^{p-\ell+\frac{m}{m+1}} u\|.
\end{aligned}
$$

Summing up (3.27), (3.29) we obtain

$$
\begin{aligned}
(3.30)\quad p^{\rho\frac{t}{t-1}} \|\varphi_p D_y^{p+1-\ell} u\| &\leq p^{\rho\frac{t}{t-1}} \|\varphi_p \chi_p(p^{-1}D_y) D_y^{p-\ell+\frac{m}{m+1}} u\|_{\frac{1}{m+1}} \\
&\quad + p^{\rho\frac{t}{t-1}} \|[\omega_p(p^{-1}D_y) D_y^{\frac{1}{m+1}}, \varphi_p]\ \chi_p(p^{-1}D_y) D_y^{p-\ell+\frac{m}{m+1}} u\|
\end{aligned}
$$

where the harmless term $C_1^p p!$ is omitted.
The first term above allows us to apply the inductive hypothesis, as we shall see in the following. We only have to take care of the norm containing the commutator.

For clarity let us state the lemma we need

Lemma 3.1 ([1] Appendix B). *Let $0 < \theta < 1$. Then*

$$
\begin{aligned}
(3.31)\quad & [\omega_p(p^{-1}D_y) D_y^{\theta}, \varphi_p(y)] \chi_p(p^{-1}D_y) D_y^{p-\ell+1-\theta} \\
&= \sum_{k=1}^{p-\ell+1} a_{p,k}(y, D_y) \chi_p(p^{-1}D_y) D_y^{p-\ell+1},
\end{aligned}
$$

where $a_{p,k}$ is a pseudodifferential operator of order $-k$ such that
(3.32)
$$
|\partial_y^\beta \partial_\eta^\alpha a_{p,k}(y,\eta)| \leq C_a^{k+1+\alpha+\beta} p^{k+\alpha+\beta} \eta^{-k-\alpha}, \quad 1 \leq k \leq p-\ell+1, \alpha, \beta \leq p.
$$

Furthermore

Corollary 3.1 ([1] Appendix B). *For $1 \leq k \leq p-\ell$ in (3.31) we have*

$$
\begin{aligned}
(3.33)\quad & a_{p,k}(y, D_y) \chi_p(p^{-1}D_y) D_y^{p-\ell+1} \\
&= \frac{\theta(\theta-1)\cdots(\theta-k+1)}{k!} D_y^k \varphi_p(y) \chi_p(p^{-1}D_y) D_y^{p-\ell+1-k}.
\end{aligned}
$$

Using both Lemma 3.1 and Corollary 3.1 we deduce that

$$
\begin{aligned}
\|[\omega_p(p^{-1}D_y)D_y^{\frac{1}{m+1}}, \varphi_p]\ \chi_p(p^{-1}D_y)D_y^{p-\ell+\frac{m}{m+1}}u\| \\
&= \sum_{k=1}^{p-\ell+1} \|a_{p,k}(y,D_y)\chi_p(p^{-1}D_y)D_y^{p-\ell+1}u\| \\
&\leq \sum_{k=1}^{p-\ell} \|\varphi_p^{(k)}\chi_p(p^{-1}D_y)D_y^{p-\ell+1-k}u\| \\
&\qquad + \|a_{p,p-\ell+1}(y,D_y)\chi_p(p^{-1}D_y)D_y^{p-\ell+1}u\|.
\end{aligned}
$$

If we go back to (3.30) we find that

$$
\begin{aligned}
p^{\rho\frac{t}{t-1}}\|\varphi_p D_y^{p+1-\ell}u\| \\
&\leq p^{\rho\frac{t}{t-1}}\Big[\|\varphi_p\chi_p(p^{-1}D_y)D_y^{p-\ell+\frac{m}{m+1}}u\|_{\frac{1}{m+1}} \\
&\quad + \sum_{k=1}^{p-\ell}\|\varphi_p^{(k)}\chi_p(p^{-1}D_y)D_y^{p-\ell+1-k}u\| \\
&\qquad + \|a_{p,p-\ell+1}(y,D_y)\chi_p(p^{-1}D_y)D_y^{p-\ell+1}u\|\Big]
\end{aligned}
\tag{3.34}
$$

Let us start by discussing the first term on the right hand side of the above inequality:

$$
p^{\rho\frac{t}{t-1}}\|\varphi_p\chi_p(p^{-1}D_y)D_y^{p-\ell+\frac{m}{m+1}}u\|_{\frac{1}{m+1}}.
$$

We have by the Calderón-Vaillancourt theorem

$$
\begin{aligned}
p^{\rho\frac{t}{t-1}}\|\varphi_p\chi_p(p^{-1}D_y)D_y^{p-\ell+\frac{m}{m+1}}u\|_{\frac{1}{m+1}} \\
&\leq p^{\rho\frac{t}{t-1}}\|\varphi_p D_y^{p-\ell+\frac{m}{m+1}}u\|_{\frac{1}{m+1}} \\
&\quad + p^{\rho\frac{t}{t-1}}\|\varphi_p(1-\chi_p(p^{-1}D_y))D_y^{p-\ell+\frac{m}{m+1}}u\|_{\frac{1}{m+1}} \\
&\leq p^{\rho\frac{t}{t-1}}\|\varphi_p D_y^{p-\ell+\frac{m}{m+1}}u\|_{\frac{1}{m+1}} + C^{p+1}p!
\end{aligned}
$$

We may use the estimate (2.8) and induction, to obtain a bound of the type

$$
p^{\rho\frac{t}{t-1}}\|\varphi_p\chi_p(p^{-1}D_y)D_y^{p-\ell+\frac{m}{m+1}}u\|_{\frac{1}{m+1}} \leq C^{p+1}p!^{s_1(\rho)},
$$

where

$$
s_1(\rho) = \rho\frac{t}{t-1}\Big(\ell - \frac{m}{m+1}\Big)^{-1} = \rho\frac{t}{t-1}\cdot\frac{m+1}{\ell(m+1)-m}. \tag{3.35}
$$

On the other hand from (3.24) and (3.21) we found the exponents

$$s_2(\theta) = \theta \frac{a_j}{a_j - \ell}, \quad \text{and} \quad s_3(\theta, \rho) = ((1-\theta)a_j - \rho)\frac{t}{\ell}. \tag{3.36}$$

We choose θ, ρ so that

$$s_1(\rho) = s_2(\theta) = s_3(\theta, \rho). \tag{3.37}$$

This is the sole choice allowing us not to lose regularity in two out of three of the terms examined. We thus obtain

$$\theta = \frac{t(m+1)(a_j - \ell)}{a_j t(m+1) - m(t-1)}, \tag{3.38}$$

and
(3.39)

$$\rho = a_j \frac{\ell t(m+1) - (t-1)m}{a_j t(m+1) - (t-1)m} \cdot \frac{(t-1)(\ell(m+1) - m)}{(t-1)(\ell(m+1) - m) + \ell(m+1)}.$$

The Gevrey regularity we get—the common value of s_j for $j = 1, 2, 3$—is then

$$s_j = s_0 = \left(1 - \frac{1}{a_j}\left(1 - \frac{1}{t}\right)\frac{m}{m+1}\right)^{-1}, \tag{3.40}$$

which is the value we want to find if the index α_j is the only point underneath the Newton polyhedron (see (1.12)).

First of all we have to make sure that the *Ansatz* (or the *Ansätze*) we made are indeed verified.

Now $\theta > (a_j - \ell)/a_j$ is easily seen since $s_0 = \theta \frac{a_j}{a_j - \ell} > 1$. Let us verify that $1 - \theta > 0$.

$$1 - \theta = 1 - \frac{a_j - \ell}{a_j} s_0 > 0$$

is equivalent to

$$\ell > \left(1 - \frac{1}{t}\right)\frac{m}{m+1},$$

and this is obvious since both factors on the right are less than 1.

Next we verify that $\rho < (1-\theta)a_j$. We have

$$1 - \theta = \frac{\ell t(m+1) - (t-1)m}{a_j t(m+1) - (t-1)m},$$

so that

$$\rho = a_j(1-\theta)\frac{(t-1)(\ell(m+1) - m)}{(t-1)(\ell(m+1) - m) + \ell(m+1)},$$

and the fraction is less than 1.

Let us go back to (3.34) and discuss the other terms:

$$p^{\rho\frac{t}{t-1}}\|\varphi_p^{(k)}\chi_p(p^{-1}D_y)D_y^{p-\ell+1-k}u\|,$$

$k=1,\dots,p-\ell$, and

$$p^{\rho\frac{t}{t-1}}\|a_{p,p-\ell+1}(y,D_y)\chi_p(p^{-1}D_y)D_y^{p-\ell+1}u\|.$$

The last norm, by Lemma 3.1, (3.32), and by the Calderón Vaillancourt theorem, keeping into account the support of χ_p, is bounded by $C_3^{p+1}p^{\rho\frac{t}{t-1}}p^p \leq C_3'^{p+1}p!$, so that it gives an analytic growth rate.

Consider one of the norms involving derivatives of φ_p in (3.34), for $\ell=1,\dots,a_j$, $k=1,\dots,p-\ell$,

$$p^{\rho\frac{t}{t-1}}\|\varphi_p^{(k)}\chi_p(p^{-1}D_y)D_y^{p-\ell+1-k}u\|.$$

To evaluate this we are in the same position as before—see (3.28) and (3.29). An iteration of the above procedure and using the properties of the cutoff φ_p, we see that this type of terms have a bound of the form

$$C^{p+1}p!^{s(\ell,k)},$$

where

$$s(\ell,k)=\rho\frac{t}{t-1}\frac{1}{\ell+k-\frac{m}{m+1}}+\frac{k}{\ell+k-\frac{m}{m+1}}.$$

But

$$\begin{aligned}s(\ell,k)=\rho\frac{t}{t-1}\frac{1}{\ell-\frac{m}{m+1}}\frac{\ell-\frac{m}{m+1}}{\ell+k-\frac{m}{m+1}}+\frac{k}{\ell+k-\frac{m}{m+1}}&\\ =s_0\frac{\ell-\frac{m}{m+1}}{\ell+k-\frac{m}{m+1}}+\frac{k}{\ell+k-\frac{m}{m+1}}<s_0,&\end{aligned}$$

by (3.35) and (3.40).

This accomplishes the estimate of the single commutator terms in (3.5) for the vector fields where $a_j>0$. We must now estimate the double commutator terms.

4. Case iii). The double commutator terms

Let us go back to equation (3.5) to estimate the double commutator summand for the vector field $X_j = c_j y^{a_j} x^{\alpha_j} D_y$. From (3.12) we have

$$\begin{aligned}
&\big[X_j, \big[X_j, \varphi_p D_y^p\big]\big] \\
&\qquad\qquad = c_j^2 \left[y^{a_j} x^{\alpha_j} D_y \ , y^{a_j} x^{\alpha_j} \varphi_p' D_y^p\right] \\
&\quad - c_j^2 \left[y^{a_j} x^{\alpha_j} D_y \ , \sum_{\ell=1}^{a_j} \binom{p}{\ell} \frac{a_j!}{(a_j-\ell)!} (-i)^\ell x^{\alpha_j} \varphi_p y^{a_j-\ell} D_y^{p-\ell+1} \right] \\
&\quad = B_1 - B_2
\end{aligned}$$

B_1 is readily computed using the formula

$$\begin{aligned}
\left[a(x,y)D_y \ , \varphi(y) a(x,y) D_y^p\right] &= a(x,y)^2 \varphi'(y) D_y^p \\
&\quad - (p-1) a(x,y) \varphi(y) a_y'(x,y) D_y^p \\
&\quad - a(x,y)\varphi(y) \sum_{\ell=2}^{p} \binom{p}{\ell} a_y^{(\ell)}(x,y) D_y^{p+1-\ell},
\end{aligned} \tag{4.1}$$

where $\varphi'(y) = D_y \varphi(y)$ and $a_y^{(\ell)}(x,y) = D_y^\ell a(x,y)$. Hence replacing a with $y^{a_j} x^{\alpha_j}$ and φ with φ_p', we obtain

$$\begin{aligned}
B_1 = c_j^2 \Bigg(& x^{2\alpha_j} y^{2a_j} \varphi_p'' D_y^p - a_j(p-1)(-i) x^{2\alpha_j} y^{2a_j-1} \varphi_p' D_y^p \\
& - \sum_{\ell=2}^{a_j} \binom{p}{\ell} (-i)^\ell \frac{a_j!}{(a_j-\ell)!} x^{2\alpha_j} y^{2a_j-\ell} \varphi_p' D_y^{p+1-\ell} \Bigg).
\end{aligned} \tag{4.2}$$

Let us consider B_2. We have the formula

$$\begin{aligned}
\left[a(x,y)D_y \ , b_\ell(x,y) D_y^{p-\ell+1}\right] &= a(x,y) b_\ell'(x,y) D_y^{p+1-\ell} \\
&\quad - \sum_{k=1}^{p+1-\ell} \binom{p+1-\ell}{k} b_\ell(x,y) D_y^k a(x,y) D_y^{p+2-\ell-k},
\end{aligned} \tag{4.3}$$

where $b_\ell'(x,y) = D_y b_\ell(x,y)$.

Replacing a with $x^{\alpha_j}y^{a_j}$ and b_ℓ with $\binom{p}{\ell}\frac{a_j!}{(a_j-\ell)!}(-i)^\ell x^{\alpha_j}y^{a_j-\ell}\varphi_p$, we obtain that

$$
\begin{aligned}
(4.4)\quad B_2 = c_j^2 \sum_{\ell=1}^{a_j} \binom{p}{\ell} \frac{a_j!}{(a_j-\ell)!}(-i)^\ell \Bigg[& x^{2\alpha_j}y^{2a_j-\ell}\varphi_p' D_y^{p+1-\ell} \\
& + (-i)(a_j-\ell)x^{2\alpha_j}y^{2a_j-\ell-1}\varphi_p D_y^{p+1-\ell} \\
- \sum_{k=1}^{a_j} \binom{p+1-\ell}{k} & \frac{a_j!}{(a_j-k)!}(-i)^k x^{2\alpha_j}y^{2a_j-\ell-k}\varphi_p D_y^{p+2-\ell-k} \Bigg].
\end{aligned}
$$

Thus from (4.2) and (4.4) we finally get

$$
\begin{aligned}
(4.5)\quad & \big[X_j, \big[X_j, \varphi_p D_y^p\big]\big] \\
& = c_j^2 \Bigg[x^{2\alpha_j}y^{2a_j}\varphi_p'' D_y^p - a_j(p-1)(-i)x^{2\alpha_j}y^{2a_j-1}\varphi_p' D_y^p \\
& \quad - 2\sum_{\ell=2}^{a_j}\binom{p}{\ell}(-i)^\ell \frac{a_j!}{(a_j-\ell)!}x^{2\alpha_j}y^{2a_j-\ell}\varphi_p' D_y^{p+1-\ell} \\
& \qquad + ipa_j x^{2\alpha_j}y^{2a_j-1}\varphi_p' D_y^p \\
& \quad - \sum_{\ell=1}^{a_j}\binom{p}{\ell}(-i)^{\ell+1}\frac{a_j!}{(a_j-\ell-1)!}x^{2\alpha_j}y^{2a_j-\ell-1}\varphi_p D_y^{p+1-\ell} \\
& \quad + \sum_{\ell=1}^{a_j}\sum_{k=1}^{a_j}\binom{p}{\ell}\binom{p+1-\ell}{k}(-i)^{\ell+k}\frac{a_j!^2}{(a_j-\ell)!(a_j-k)!} \\
& \qquad \cdot x^{2\alpha_j}y^{2a_j-\ell-k}\varphi_p D_y^{p+2-\ell-k} \Bigg].
\end{aligned}
$$

Using this identity we compute the scalar product in (3.5) containing the double commutator:

$$\begin{aligned}
(4.6)\quad & \langle \left[X_j, \left[X_j, \varphi_p D_y^p\right]\right] u, \varphi_p D_y^p u\rangle \\
&= c_j^2 \Bigg[\langle x^{2\alpha_j} y^{2a_j} \varphi_p'' D_y^p u, \varphi_p D_y^p u\rangle \\
&\quad + i a_j (p-1) \langle x^{2\alpha_j} y^{2a_j-1} \varphi_p' D_y^p u, \varphi_p D_y^p u\rangle \\
&\quad - 2\sum_{\ell=2}^{a_j} \binom{p}{\ell} (-i)^{\ell} \frac{a_j!}{(a_j-\ell)!} \langle x^{2\alpha_j} y^{2a_j-\ell} \varphi_p' D_y^{p+1-\ell} u, \varphi_p D_y^p u\rangle \\
&\quad - \sum_{\ell=1}^{a_j-1} \binom{p}{\ell} (-i)^{\ell+1} \frac{a_j!}{(a_j-\ell-1)!} \langle x^{2\alpha_j} y^{2a_j-\ell-1} \varphi_p D_y^{p+1-\ell} u, \varphi_p D_y^p u\rangle \\
&\quad + \sum_{\ell=1}^{a_j} \sum_{k=1}^{a_j} \binom{p}{\ell} \binom{p+1-\ell}{k} \frac{a_j!^2}{(a_j-\ell)!(a_j-k)!} (-i)^{\ell+k} \\
&\quad \cdot \langle x^{2\alpha_j} y^{2a_j-\ell-k} \varphi_p D_y^{p+2-\ell-k} u, \varphi_p D_y^p u\rangle \Bigg] \\
&= \sum_{\rho=1}^{5} A_\rho .
\end{aligned}$$

Let us estimate A_ρ, $\rho = 1, \ldots, 5$.

4.1. A_1**.** Observe that

$$A_1 = c_j^2 \langle \frac{1}{p} x^{\alpha_j} y^{a_j} \varphi_p'' D_y^p u, p x^{\alpha_j} y^{a_j} \varphi_p D_y^p u\rangle .$$

We want to reconstruct, when possible, a vector field in each factor of the above scalar product. To this end we use (3.6) where φ_p' is replaced by the cutoff function appearing in the factors. Note that we consider the factor p as part of the cutoff since its role in A_1 is that of balancing the growth of the cutoff w.r.t. p.

Thus

$$\begin{aligned}
A_1 & \\
&= c_j^2 \langle x^{\alpha_j} y^{a_j} \Big[\sum_{\ell=0}^{p-1} (-1)^{\ell} D_y \frac{1}{p} \varphi_p^{(\ell+2)} D_y^{p-1-\ell} u + (-1)^p \frac{1}{p} \varphi_p^{(p+2)} \Big] u, \\
&\qquad x^{\alpha_j} y^{a_j} \Big[\sum_{\ell'=0}^{p-1} (-1)^{\ell'} D_y p \varphi_p^{(\ell')} D_y^{p-1-\ell'} + (-1)^p p \varphi_p^{(p)} \Big] u\rangle .
\end{aligned}$$

As a consequence

$$
\begin{aligned}
A_1 = \sum_{\ell,\ell'=1}^{p-1} &(-1)^{\ell+\ell'} \langle X_j \Big(\frac{\varphi_p^{(\ell+2)}}{p}\Big) D_y^{p-1-\ell} u, X_j (p\varphi_p^{(\ell')}) D_y^{p-1-\ell'} u\rangle \\
&+ \sum_{\ell=1}^{p-1} c_j (-1)^{\ell+p} \bigg[\langle X_j \Big(\frac{\varphi_p^{(\ell+2)}}{p}\Big) D_y^{p-1-\ell} u, x^{\alpha_j} y^{a_j} (p\varphi_p^{(p)}) u\rangle \\
&\qquad + \langle x^{\alpha_j} y^{a_j} \frac{1}{p} \varphi_p^{(p+2)} u, X_j (p\varphi_p^{(\ell)}) D_y^{p-1-\ell} u\rangle \bigg] \\
&\qquad + c_j^2 \langle x^{\alpha_j} y^{a_j} \frac{1}{p} \varphi_p^{(p+2)} u, x^{\alpha_j} y^{a_j} p \varphi_p^{(p)} u\rangle \\
&\qquad = \sum_{\ell,\ell'=1}^{p-1} A_{1,\ell,\ell'} + \sum_{\ell=1}^{p-1} (A_{1,\ell,+} + A_{1,\ell,-}) + A_{1,0}.
\end{aligned}
$$

The last term has a straightforward estimate since

$$|A_{1,0}| \leq C\delta C_\varphi^{2p+3} \|u\|^2 p^{2p+2},$$

where C_φ is the constant in Definition 2.1, C is a positive constant depending only on the operator and δ can be taken small since the neighborhood of the origin where u has its support can be chosen small.

Let us estimate next $A_{1,\ell,\ell'}$, since this motivates the insertion of the factor p on the left hand side of the scalar product to balance the higher derivative of φ_p.

We have

$$
\begin{aligned}
|A_{1,\ell,\ell'}| &= |\langle X_j \Big(\frac{\varphi_p^{(\ell+2)}}{p}\Big) D_y^{p-1-\ell} u, X_j (p\varphi_p^{(\ell')}) D_y^{p-1-\ell'} u\rangle| \\
&\leq \|X_j \Big(\frac{\varphi_p^{(\ell+2)}}{p}\Big) D_y^{p-1-\ell} u\| \; \|X_j (p\varphi_p^{(\ell')}) D_y^{p-1-\ell'} u\|.
\end{aligned}
$$

The above estimate allows us to apply the a priori estimate (2.8) with a slightly modified cutoff function $p^{-1}\varphi_p^{(\ell+2)}$ or $p\varphi_p^{(\ell')}$, respectively. Dividing the first function by C_φ and multiplying the second by the same positive constant we have that these two functions satisfy estimates of the form of those in Definition 2.1. Thus the term $A_{1,\ell.\ell'}$ yields only analytic growth.

Next we want to bound $A_{1,\ell,\pm}$. This is actually a blend of the two previous estimates:

$$|A_{1,\ell,+}| \leq C\delta C_\varphi^{p+2} \|u\| p^{p+1} \|X_j \Big(\frac{\varphi_p^{(\ell+2)}}{p}\Big) D_y^{p-1-\ell} u\|,$$

with an analogous meaning of the constants as well as $\delta > 0$. The norm above is treated as before. Finally the term $A_{1,\ell,-}$ is dealt with in a completely analogous way and we omit it.

4.2. A_2. We write

$$A_2 = i a_j c_j^2 \langle x^{\alpha_j} y^{a_j - 1} \varphi_p' D_y^p u, (p-1) x^{\alpha_j} y^{a_j} \varphi_p D_y^p u \rangle.$$

Hence, using formula (3.6) once more on the second factor of the scalar product,

$$\begin{aligned}(4.7)\quad |A_2| &\leq C \| x^{\alpha_j} y^{a_j-1} \varphi_p' D_y^p u \| \| c_j x^{\alpha_j} y^{a_j} p \varphi_p D_y^p u \| \\ &\leq C \| x^{\alpha_j} y^{a_j-1} \varphi_p' D_y^p u \| \left[\sum_{\ell=0}^{p-1} \| X_j p \varphi_p^{(\ell)} D_y^{p-1-\ell} u \| + \delta p \| \varphi_p^{(p)} u \| \right].\end{aligned}$$

Here C and δ have a meaning analogous to that of the previous subsection.

Now we already saw how to estimate the terms in brackets obtaining, at least for this branch of the induction tree, an analytic growth rate. On the other hand the first factor is a norm we already bounded in (3.16), obtaining a Gevrey-s_0 growth rate. Note that strictly speaking the difference between the factor in (4.7) above and the norm in (3.16) is that the norm (3.16) with $\ell = 1$ has no derivative of φ_p, but has an extra factor p in front. The two situations are analogous as far as the estimates are concerned.

4.3. A_3. We are going to estimate

$$A_{3,\ell} = \binom{p}{\ell} (-i)^\ell \frac{a_j!}{(a_j - \ell)!} \langle x^{2\alpha_j} y^{2a_j - \ell} \varphi_p' D_y^{p+1-\ell} u, \varphi_p D_y^p u \rangle,$$

the generic summand in (4.6) for $\ell = 2, ..., a_j$. Since the binomial factor is bound by p^ℓ, we have

$$\begin{aligned}(4.8)\quad |A_{3,\ell}| &\leq C |\langle p^{\ell-1} x^{\alpha_j} y^{a_j-\ell} \varphi_p' D_y^{p+1-\ell} u, x^{\alpha_j} y^{a_j} p \varphi_p D_y^p u \rangle| \\ &\leq C \left(p^\ell \| x^{\alpha_j} y^{a_j-\ell} \left(\frac{\varphi_p'}{p} \right) D_y^{p+1-\ell} u \| \right) \left(p \| x^{\alpha_j} y^{a_j-1} \varphi_p D_y^p u \| \right),\end{aligned}$$

where we used the fact that, near the origin, $|y^{a_j}| \leq |y^{a_j-1}|$.

Both norms above are of the form we bounded in (3.18). The first corresponds to the case when ℓ derivatives have been performed and we remark that the cutoff function $p^{-1}\varphi_p$ satisfies the same estimates as φ_p. The second norm corresponds to the case $\ell = 1$ in (3.18). As a consequence we get a Gevrey–s_0 growth rate from this term.

4.4. A_4. Using the same notation as in the preceding subsection, we set, for $\ell = 1, ..., a_j - 1$

$$A_{4,\ell} = \binom{p}{\ell}(-i)^{\ell+1}\frac{a_j!}{(a_j-\ell-1)!}\langle x^{2\alpha_j}y^{2a_j-\ell-1}\varphi_p D_y^{p+1-\ell}u, \varphi_p D_y^p u$$

and we observe that

$$\begin{aligned}(4.9)\quad |A_{4,\ell}| &\leq C|p^\ell\langle x^{\alpha_j}y^{a_j-\ell}\varphi_p D_y^{p+1-\ell}u, x^{\alpha_j}y^{a_j-1}\varphi_p D_y^p u\rangle| \\ &\leq C\left(p^\ell\|x^{\alpha_j}y^{a_j-\ell}\varphi_p D_y^{p+1-\ell}u\|\right)\left(p\|x^{\alpha_j}y^{a_j-1}\varphi_p D_y^p u\|\right).\end{aligned}$$

Note that the constant C has different meanings in different terms. Moreover here we just increased by one the number of p-factors in order to exactly match the expression already bounded in (3.18). As above we get a Gevrey-s_0 growth type.

4.5. A_5. Again we bound the summand

$$\begin{aligned}(4.10)\quad A_{5,\ell,k} = \binom{p}{\ell}\binom{p+1-\ell}{k}&\frac{a_j!^2}{(a_j-\ell)!(a_j-k)!}(-i)^{\ell+k} \\ &\cdot\langle x^{2\alpha_j}y^{2a_j-\ell-k}\varphi_p D_y^{p+2-\ell-k}u, \varphi_p D_y^p u\rangle\end{aligned}$$

in the last term of (4.6) with $\ell = 1, ..., a_j$, $k = 1, ..., a_j$, by noting that

$$\begin{aligned}(4.11)\quad |A_{5,\ell,k}| &\leq Cp^{\ell+k} \\ &\cdot|\langle x^{\alpha_j}y^{a_j-(\ell+k-1)}\varphi_p D_y^{p+1-(\ell+k-1)}u, x^{\alpha_j}y^{a_j-1}\varphi_p D_y^p u\rangle| \\ &\leq C\left(p^{\ell+k-1}\|x^{\alpha_j}y^{a_j-(\ell+k-1)}\varphi_p D_y^{p+1-(\ell+k-1)}u\|\right) \\ &\qquad\cdot\left(p\|x^{\alpha_j}y^{a_j-1}\varphi_p D_y^p u\|\right),\end{aligned}$$

and again both norms above match the norms bounded in (3.18) thus yielding a Gevrey-s_0 growth rate.

This completes the estimate of the double commutator terms for the vector fields X_j.

5. The vector fields X_j in the Newton polyhedron

In this section we study the vector fields $X_j = c_j y^{a_j}x^{\alpha_j}D_y, \quad a_j > 0$ that belong to the Newton polyhedron $\mathbf{NP}(B)$.
From (3.4) we have to estimate

$$\begin{aligned}(5.1)\quad &\langle\left[X_j^2, \varphi_p D_y^p\right]u, \varphi_p D_y^p u\rangle \\ &= 2\langle X_j\left[X_j, \varphi_p D_y^p\right]u, \varphi_p D_y^p u\rangle - \langle\left[X_j, \left[X_j, \varphi_p D_y^p\right]\right]u, \varphi_p D_y^p u\rangle.\end{aligned}$$

We proceed as done in Sect.3 Case *iii*) and we start by considering the first term with a single commutator in the r.h.s. above. Arguing as in (3.14) and in (3.15), we have

$$
\begin{aligned}
(5.2)\quad \| \left[X_j, \varphi_p D_y^p\right] u\| \leq & \sum_{k=1}^{p} \|X_j \varphi_p^{(k)} D_y^{p-k} u\| \\
& + C_1 \|\varphi_p^{(p+1)} u\| + C_2 \sum_{\ell=1}^{a_j} \binom{p}{\ell} \|x^{\alpha_j} y^{a_j-\ell} \varphi_p D_y^{p-\ell+1} u\|,
\end{aligned}
$$

By the inductive hypothesis (2.7) and from the regularity property (2.3) of the cutoff φ_p the first two summands in r.h.s. of (5.2) gives an analytic growth rate.

Therefore we are left with the last summand in the r.h.s. of (5.2). Since x^{α_j} belongs to the Newton polyhedron, there are finitely many vertices of $\partial\mathbf{NP}(B)$, say $\beta_1, \ldots, \beta_q$, $q \in \mathbb{N}$ depending on α_j only, such that

$$
|x^{\alpha_j}| \leq C \sum_{r=1}^{q} |x^{\beta_r}|.
$$

on the compact support of u. Thus we get

$$
\binom{p}{\ell} \|x^{\alpha_j} y^{a_j-\ell} \varphi_p D_y^{p-\ell+1} u\| \leq C p^\ell \sum_{r=1}^{q} \|x^{\beta_r} \varphi_p D_y^{p-\ell+1} u\|
$$

Using the formula

$$
(5.3)\qquad \varphi_p D_y^{p-\ell+1} = \sum_{k=0}^{p-\ell+1} (-1)^k D_y \varphi_p^{(k)} D_y^{p-\ell-k} + (-1)^p \varphi_p^{(p+1-\ell)},
$$

we can reconstruct the vector fields corresponding to the vertices $\beta_1, \ldots, \beta_q$ of $\partial\mathbf{NP}(B)$

$$
(5.4)\qquad X_{j_r} = c_{j_r} x^{\beta_r} D_y, \quad r = 1, ..., q.
$$

Thus we get

$$
\begin{aligned}
(5.5)\quad p^\ell \sum_{r=1}^{q} \|x^{\beta_r} \varphi_p D_y^{p-\ell+1} u\| \leq C p^\ell \sum_{r=1}^{q} \Bigg[& \sum_{k=0}^{p-\ell+1} \|X_{j_r} \varphi_p^{(k)} D_y^{p-\ell-k} u\| \\
& + \|\varphi_p^{(p+1-\ell)} u\| \Bigg].
\end{aligned}
$$

It is easy to see that the terms in the r.h.s. yield an analytic growth rate.

Finally we are left with the double commutator term in (5.1) and by (4.6) we have

$$\langle [X_j, [X_j, \varphi_p D_y^p]] \, u, \varphi_p D_y^p u\rangle = \sum_{\rho=1}^{5} A_\rho.$$

Arguing as done above, in each summand A_ρ we can reconstruct the vector fields (5.4) in both sides of the scalar product, yielding again terms with an analytic growth rate and this completes the analysis of the vector fields in the Newton polyhedron.

6. The vector fields $\tilde{X}_j$

In this section we examine the terms produced by the second summand on the right hand side of (3.2). We fix $j \in \{1, \dots, N'\}$ and $k \in \{1, \dots, n\}$ and want to evaluate (3.3) when X is replaced by $\tilde{X}_{jk} = \tilde{c}_{jk} y^{b_{jk}} x^{\beta_{jk}} D_k$. This is used in the right hand side of (3.4) in the same way as we did for the vector fields X_j. Thus we have

$$\begin{aligned} (6.1) \quad & \langle \left[\tilde{X}^2_{jk}, \varphi_p D_y^p\right] u, \varphi_p D_y^p u\rangle \\ &= 2\langle \tilde{X}_{jk} \left[\tilde{X}_{jk}, \varphi_p D_y^p\right] u, \varphi_p D_y^p u\rangle - \langle \left[\tilde{X}_{jk}, \left[\tilde{X}_{jk}, \varphi_p D_y^p\right]\right] u, \varphi_p D_y^p u\rangle \end{aligned}$$

First of all using the same argument we did for (3.14) we see that the only terms we have to treat are of two types:

α)

$$\| \left[\tilde{X}_{jk}, \varphi_p D_y^p\right] u\|;$$

β)

$$\langle \left[\tilde{X}_{jk}, \left[\tilde{X}_{jk}, \varphi_p D_y^p\right]\right] u, \varphi_p D_y^p u\rangle.$$

6.1. **Estimate of the norms in α).** First of all we compute

$$\begin{aligned} \left[\tilde{X}_{jk}, \varphi_p D_y^p\right] &= \varphi_p \left[\tilde{X}_{jk}, D_y^p\right] = \tilde{c}_{jk} \varphi_p \left[x^{\beta_{jk}} y^{b_{jk}} D_k, D_y^p\right] \\ &= \tilde{c}_{jk} \varphi_p \left[y^{b_{jk}}, D_y^p\right] x^{\beta_{jk}} D_k = -\tilde{c}_{jk} \varphi_p \left[D_y^p, y^{b_{jk}}\right] x^{\beta_{jk}} D_k \\ &= -\sum_{\sigma=1}^{p} \binom{p}{\sigma} \frac{b_{jk}!}{(b_{jk}-\sigma)!} (-i)^\sigma \tilde{c}_{jk} \varphi_p y^{b_{jk}-\sigma} x^{\beta_{jk}} D_y^{p-\sigma} D_k. \end{aligned}$$

It is of course understood that $\sigma \le b_{jk}$, or, in other words, that the involved factorials have a non negative argument.

Plugging this into the norm in the first item above we obtain

$$
\begin{aligned}
(6.2)\quad & \left\| \left[\tilde{X}_{jk}, \varphi_p D_y^p\right] u \right\| \\
&\leq \sum_{\sigma=1}^{p} \binom{p}{\sigma} \frac{b_{jk}!}{(b_{jk}-\sigma)!} |\tilde{c}_{jk}| \| x^{\beta_{jk}} y^{b_{jk}-\sigma} \varphi_p D_y^{p-\sigma} D_k u \| \\
&\qquad \leq C \sum_{\sigma=1}^{p} p^{\sigma} \| x^{\beta_{jk}} y^{b_{jk}-\sigma} \varphi_p D_y^{p-\sigma} D_k u \| \\
&\qquad\qquad \leq C\delta \sum_{\sigma=1}^{p} p^{\sigma} \| X_k \varphi_p D_y^{p-\sigma} u \|,
\end{aligned}
$$

where $\delta \geq \sup\{|x^{\beta_{jk}} y^{b_{jk}-\sigma}| \mid (x, y) \in \operatorname{supp} u\}$ and X_k has been defined in (1.2).

Each summand above is ready for an application of the inductive estimate (2.7).

We deduce thus that the Gevrey-s_0 regularity (see (3.40)) is again an upper bound for the growth rate of the terms in α).

6.2. **Estimate of the scalar products in β).** Using the expression above for the simple commutator we get that

$$
\begin{aligned}
& \left[\tilde{X}_{jk}, \left[\tilde{X}_{jk}, \varphi_p D_y^p\right]\right] \\
&= -\sum_{\sigma=1}^{p} \binom{p}{\sigma} \frac{b_{jk}!}{(b_{jk}-\sigma)!} (-i)^{\sigma} \tilde{c}_{jk}^2 \left[x^{\beta_{jk}} y^{b_{jk}} D_k, \varphi_p y^{b_{jk}-\sigma} x^{\beta_{jk}} D_y^{p-\sigma} D_k\right] \\
&= -\sum_{\sigma=1}^{p} \binom{p}{\sigma} \frac{b_{jk}!}{(b_{jk}-\sigma)!} (-i)^{\sigma} \tilde{c}_{jk}^2 \varphi_p y^{b_{jk}-\sigma} \left[x^{\beta_{jk}} y^{b_{jk}} D_k, x^{\beta_{jk}} D_y^{p-\sigma} D_k\right] \\
&= \sum_{\sigma=1}^{p} \binom{p}{\sigma} \frac{b_{jk}!}{(b_{jk}-\sigma)!} (-i)^{\sigma} \tilde{c}_{jk}^2 \varphi_p y^{b_{jk}-\sigma} (x^{\beta_{jk}} D_k)^2 \left[D_y^{p-\sigma}, y^{b_{jk}}\right] \\
&= \sum_{\sigma=1}^{p} \sum_{\sigma'=1}^{p-\sigma} \binom{p}{\sigma} \binom{p-\sigma}{\sigma'} \frac{b_{jk}!^2}{(b_{jk}-\sigma)!(b_{jk}-\sigma')!} (-1)^{\sigma+\sigma'} \tilde{c}_{jk}^2 \\
&\qquad\qquad \cdot \varphi_p y^{2b_{jk}-\sigma-\sigma'} (x^{\beta_{jk}} D_k)^2 D_y^{p-\sigma-\sigma'},
\end{aligned}
$$

with the usual convention about the factorials. We observe that this convention allows us to replace the p of the upper limit of the summations with b_{jk}, since p is definitely a large integer.

Note that

$$
(x^{\beta_{jk}} D_k)^2 = x^{2\beta_{jk}} D_k^2 + \frac{\langle \beta_{jk}, e_k \rangle}{i} x^{2\beta_{jk}-e_k} D_k,
$$

where e_k denotes the unit n-vector in the direction of the k-th axis. Then finally we have

$$
\begin{aligned}
(6.3)\quad & \left[\tilde{X}_{jk}, \left[\tilde{X}_{jk}, \varphi_p D_y^p\right]\right] \\
&= \sum_{\sigma=1}^{b_{jk}} \sum_{\sigma'=1}^{b_{jk}} \binom{p}{\sigma} \binom{p-\sigma}{\sigma'} \frac{b_{jk}!^2}{(b_{jk}-\sigma)!(b_{jk}-\sigma')!} (-1)^{\sigma+\sigma'} \tilde{c}_{jk}^2 \\
&\qquad \cdot \varphi_p y^{2b_{jk}-\sigma-\sigma'} x^{2\beta_{jk}} D_k^2 D_y^{p-\sigma-\sigma'} \\
&+ \sum_{\sigma=1}^{b_{jk}} \sum_{\sigma'=1}^{b_{jk}} \binom{p}{\sigma} \binom{p-\sigma}{\sigma'} \frac{b_{jk}!^2}{(b_{jk}-\sigma)!(b_{jk}-\sigma')!} (-1)^{\sigma+\sigma'} \tilde{c}_{jk}^2 \\
&\qquad \cdot \frac{\langle \beta_{jk}, e_k \rangle}{i} \varphi_p y^{2b_{jk}-\sigma-\sigma'} x^{2\beta_{jk}-e_k} D_k D_y^{p-\sigma-\sigma'} \\
&\qquad\qquad\qquad = E_2 + E_1.
\end{aligned}
$$

Denote by $E_{2,\sigma,\sigma'}$ and by $E_{1,\sigma,\sigma'}$ the summands for E_2 and for E_1 respectively. We start by estimating

$$\langle E_{2,\sigma,\sigma'} u, \varphi_p D_y^p u \rangle.$$

Because of (6.3), integrating by parts w.r.t. x_k and taking into account that $\operatorname{supp} u$ is a compact set, we have

$$
\begin{aligned}
& |\langle E_{2,\sigma,\sigma'} u, \varphi_p D_y^p u \rangle| \\
&\quad \leq C p^{\sigma+\sigma'} \Big[|\langle \varphi_p y^{b_{jk}-\sigma-\sigma'} x^{\beta_{jk}} X_k D_y^{p-\sigma-\sigma'} u, X_k \varphi_p D_y^p u \rangle| \\
&\qquad + |\langle \varphi_p y^{b_{jk}-\sigma-\sigma'} x^{\beta_{jk}-e_k} X_k D_y^{p-\sigma-\sigma'} u, \varphi_p D_y^p u \rangle| \Big] \\
&\quad \leq C\delta p^{\sigma+\sigma'} \Big[\| X_k \varphi_p D_y^{p-\sigma-\sigma'} u \| \;\; \| X_k \varphi_p D_y^p u \| \\
&\qquad\qquad + \| X_k \varphi_p D_y^{p-\sigma-\sigma'} u \| \;\; \| \varphi_p D_y^p u \| \Big],
\end{aligned}
$$

where $\delta \geq \sup\{ |y^{b_{jk}-\sigma-\sigma'} x^{\beta_{jk}-e_k}| \mid (x,y) \in \operatorname{supp} u \}$.

The second factor in each summand above can be absorbed on the right hand side of (2.8), so that, modulo a square, we are left with $p^{\sigma+\sigma'} \| X_k \varphi_p D_y^{p-\sigma-\sigma'} u \|$, which can be by the induction (see (2.7)). The same argument applies to the second factor of the second summand with just obvious changes.

Consider next $E_{1,\sigma,\sigma'}$. We have to estimate

$$\langle E_{1,\sigma,\sigma'} u, \varphi_p D_y^p u \rangle.$$

The above term is easier than the preceding one, since we cannot integrate by parts. The only remark is that the cutoff function φ_p can slide past the vector field X_k, since it does not depend on x, so that

$$|\langle E_{1,\sigma,\sigma'}u, \varphi_p D_y^p u\rangle| \leq C\delta p^{\sigma+\sigma'} \|X_k \varphi_p D_y^{p-\sigma-\sigma'} u\| \ \ \|\varphi_p D_y^p u\|.$$

The above quantity can be treated exactly as we did for the second term in $E_{2,\sigma,\sigma'}$.

This ends the proof of the theorem.

A. Appendix

For the sake of completeness we recall here some well-known facts used throughout the paper.

Definition A.1. *For any $m \in \mathbb{R}$, $\rho, \delta \in \mathbb{R}$ with $0 \leq \delta \leq \rho \leq 1, \delta < 1$, we denote by $S^m_{\rho,\delta}$ the set of all the functions $p(x,\xi) \in C^\infty(\mathbb{R}^{2n})$ such that for every multi-index α, β there exits a positive constant $C_{\alpha,\beta}$ for which*

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \leq C_{\alpha,\beta} \langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|},$$

where $\langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}$.

We denote by $OPS^m_{\rho,\delta}$ the class of the corresponding pseudodifferential operators $P = p(x,D)$.

It is trivial to see that the symbol class $S^m_{\rho,\delta}$ equipped with the semi-norms

$$|p|_\ell^{(m)} = \max_{|\alpha+\beta|\leq \ell} \sup_{(x,\xi)} \{|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \langle\xi\rangle^{-(m-\rho|\alpha|+\delta|\beta|)}\}, \quad \ell \in \mathbb{N}$$

is a Fréchet space.

The Calderón-Vaillancourt theorem shows the L^2-continuity properties of the pseudodifferential operators in the above classes (see [6] or, for a more general setting, [12] Chap. 7, Th.1.6). We state below a formulation of such a theorem for pseudodifferential operators of order zero.

Theorem A.1 (Calderón-Vaillancourt)**.** *Let $P = p(x,D) \in OPS^0_{\rho,\delta}$ with $\rho \leq \delta, \delta < 1$. Then there exist a positive integer ℓ and a positive constant M (depending only on n) such that*

$$\|Pu\| \leq M|p|_\ell^{(0)} \|u\|, \quad \textit{for every } u \in L^2(\mathbb{R}^n).$$

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