The Hammersley–Chapman–Robbins inequality for repeatedly monitored quantum system

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Abstract

We derive the Hammersley-Chapman-Robbins inequality for discrete quantum parameter models in the presence of time dependent measurements. The extension determines a discrete counterpart of the classical Fisher information. We provide an illustration concerning a quantum optics problem.

Keywords: Parametric quantum models, Fisher information, Time-dependent measurements, Discrete parameter space

1. Introduction

Statistical inference when the parameter space is reduced to a lattice was first considered by Hammersley (1950) who was mainly concerned with the problem of estimating the molecular weight of insulin with a Normal distribution with known variance and unknown integer mean. Here, chemical theory restricts the unknown molecular weight to be a positive integer. In Khan (1973, 1978, 2000, 2003) the general admissibility conditions for the mean estimator proposed by Hammersley (1950) are investigated. Cox and Hinkley (1974) discuss the construction of confidence intervals, LaMotte (2008) treats sufficiency and minimal sufficiency for models in which the parameter space and the sample space are finite. Teunissen (2007) extends the theory of minimum mean squared prediction error by introducing new classes of predictors based on the principle of equivariance. Recently, Choirat and Seri (2012) discuss consistency, asymptotic theory, information inequalities and their relations with efficiency and superefficiency for a general class of M-estimators in a discrete parameter setting.

On the side of applications, Baram (1978) and Baram and Sandell Jr (1978a,b) highlight

the relevance of discrete parameter models in signal processing by deriving the conditions for consistent selection among a finite set of stationary Gaussian models. Moreover, from an information theory perspective, Poor and Verdu (1995) establish a lower bound on the probability of error in multi-hypothesis testing and Kanaya (1995) studies the asymptotic relation between the posterior entropy and the maximum a posteriori error probability.

This paper is motivated by the physical problem of estimating the state of a quantum system. The relevance of quantum state estimation is connected with the possibility to track and control the dynamics of a quantum system, with possible applications to quantum feedback control, high-precision measurement and quantum computing, among others (Ramakrishna and Rabitz, 1996; Wiseman and Milburn, 2009; Dong and Petersen, 2010; Barreiro et al., 2011; Dunjko and Briegel, 2018). In general, the state of the system is assumed to be a smooth function of an unknown parameter so that classical inference based on maximum likelihood theory can be applied. In this framework, tractable lower bounds of the variance of an unbiased estimator of the parameter of interest have been extensively studied (see Helstrom 1976; Holevo 1982, and more recently Barndorff-Nielsen et al. 2003; Luati 2004, 2011; Yang et al. 2019). However, there are some practically important examples in which the smoothness assumptions do not hold, e.g., the photon-number states, the direction the magnetic field points to, or an atom whose possible oscillation frequency belongs to a finite set of values. So far, little has been done in order to study such cases, especially in a straightforward and tractable way. Tsuda and Matsumoto (2005) introduced a general framework for quantum state estimation for non differentiable models based on a quantum analogue of classical Fisher information and in a static setting.

The purpose of this paper is to derive the Hammersley–Chapman–Robbins (HCR) bound (Hammersley, 1950; Chapman and Robbins, 1951) for discrete parameter models in repeatedly monitored quantum systems. The bound determines a discrete counterpart of the classical Fisher information for a time-dependent sequence of quantum measurements. An illustration concerning a quantum optics problem is also provided.

The paper is organized as follows. The next Section introduces the probabilistic framework

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of time dependent quantum measurements. The main result, the HCR lower bound, is derived in Section 2, following the approach of Hammersley (1950). Section 3 presents an illustration and Section 4 concludes the paper.

1.1. Preliminaries

The state of a quantum system can be represented by a density matrix $\rho(\theta)$, that is, a nonnegative, self-adjoint and trace-one linear operator acting on a n-dimensional complex Hilbert space $\mathcal{H}_n$ (Petz, 2007; Nielsen and Chuang, 2010). Note that $\rho(\theta)$ depends on an unknown parameter $\theta \in \Theta \subseteq \mathbb{R}^k$ which is the object of our inferential problem. The probability distribution of a random variable $X : (\Omega, \mathcal{F}, P) \to (\mathbb{G}, \mathcal{G}, P_X)$, describing the outcome of an experiment performed on the system in the state $\rho(\theta)$, is given by the trace rule for probability

$$P_X(G; \theta) = \mathrm{tr}\{\rho(\theta) M(G)\}, \forall G \in \mathcal{G},$$

where M is the measurement applied to the system. The latter is represented by a set of nonnegative Hermitian matrices defined on the measure space $(\mathbb{G}, \mathcal{G})$ and taking values in $\mathcal{H}_n$, satisfying $M(\mathbb{G}) = \mathbf{I}$, the identity matrix, $M(\varnothing) = \mathbf{O}$, the null matrix, and $M\left(\bigcup\limits_{h=1}^{\infty} G_h\right) = \sum_{h=1}^{\infty} M(G_h)$ if $G = \bigcup\limits_{h=1}^{\infty} G_h, G_h \cap G_k = \varnothing, \forall h, k = 1, \ldots, \infty, h \neq k$. If M is dominated by a σ-finite measure ν on $(\mathbb{G}, \mathcal{G})$ such that $M(G) = \int_G m(x) \nu(dx), \forall G \in \mathcal{G}$, where $m(x)$ is a non-negative Hermitian matrix-valued function, then the density of X is

$$p_X(x; \theta) = \mathrm{tr}\{\rho(\theta) m(x)\}, \ \forall x \in \mathbb{G}.$$

By the nature of quantum mechanics, the measurement of a quantum system causes a change in its state. Therefore, the complete description of a quantum experiment requires both the probability distribution of X and the specification of the state of the system after the measurement, and can be obtained through a collection of effect operators $O(x)$ satisfying $\sum_x O(x)^* O(x) = \mathbf{I}$. Thus, conditionally on the outcome x, the state of a quantum system is

$$\rho(\theta; x) = \frac{O(x)\rho(\theta)O(x)^*}{\text{tr}\{O(x)\rho(\theta)O(x)^*\}},$$

where the denominator gives the probability of observing the outcome x (Barndorff-Nielsen et al., 2003).

Let $\rho_0(\theta)$ be the initial state of the system at time $t = 0$ and consider the experiment described by X. Let also $x_1, \ldots, x_T$ be the sequence of outcomes indexed by time, with $T \subseteq \mathbb{Z}$, corresponding to repeated measurements on the system. After the first experiment, with outcome x_1, the normalized state of the system is,

$$\rho_1(\theta; x_1) = \frac{O(x_1)\rho_0(\theta)O(x_1)^*}{\text{tr}\{O(x_1)\rho_0(\theta)O(x_1)^*\}},$$

while, after the second measurement, one has

$$\rho_2(\theta; x_2 | x_1) = \frac{O(x_2)\rho_1(\theta)O(x_2)^*}{\text{tr}\{O(x_2)\rho_1(\theta)O(x_2)^*\}}$$

where the denominator represents the probability of observing the outcome x_2 conditionally on the outcome x_1, that is

$$
\begin{aligned}
p_{X_2|X_1}(x_2; x_1, \theta) &= \text{tr}\{O(x_2)\rho_1(\theta; x_1)O(x_2)^*\} \\
&= \text{tr}\left\{O(x_2)\frac{O(x_1)\rho_0(\theta)O(x_1)^*}{\text{tr}\{O(x_1)\rho_0(\theta)O(x_1)^*\}}O(x_2)^*\right\}.
\end{aligned}
$$

The joint probability of observing x_1 and x_2 is given by

$$
\begin{aligned}
p(x_1, x_2; \theta) &= p_{X_2|X_1}(x_2; x_1, \theta)p(x_1; \theta) \\
&= \text{tr}\left\{O(x_2)\frac{O(x_1)\rho_0(\theta)O(x_1)^*}{\text{tr}\{O(x_1)\rho_0(\theta)O(x_1)^*\}}O(x_2)^*\right\}\text{tr}\{O(x_1)\rho_0(\theta)O(x_1)^*\} \\
&= \text{tr}\{O(x_2)O(x_1)\rho_0(\theta)O(x_1)^*O(x_2)^*\}.
\end{aligned}
$$

Following the same reasoning, after the t-th measurement, the state of the system is

$$\rho_t(\theta; x_t | x_1, \ldots, x_{t-1}) = \frac{O(x_t)\rho_{t-1}(\theta; x_{t-1} | x_1, \ldots, x_{t-2})O(x_t)^*}{\mathrm{tr}\{O(x_t)\rho_{t-1}(\theta; x_{t-1} | x_1, \ldots, x_{t-2})O(x_t)^*\}} \tag{1}$$

where $\rho_{t-1}(\theta; x_{t-1} | x_1, \ldots, x_{t-2})$ represents the state of the system after the $(t-1)$-th measurement, with outcome x_{t-1} and conditional on the outcome sequence $x_1, \ldots, x_{t-2}$. The joint probability of the sequence until time t is given by

$$p(x_1, \ldots, x_t; \theta) = p_{X_t | X_{t-1}}(x_t; x_1, \ldots, x_{t-1}, \theta)p(x_1, \ldots, x_{t-1}; \theta)$$

and, iterating recursively, this expression can be rewritten as follows

$$p(x_1, \ldots, x_t; \theta) = \prod_{r=1}^{t} p(x_r; x_1, \ldots, x_{r-1}, \theta) \tag{2}$$

$$= \mathrm{tr}\{O(x_t) \ldots O(x_1)\rho_0(\theta)O(x_1)^* \ldots O(x_t)^*\}.$$

Note that, by using equation (2) it is possible to express $\rho_t(\theta; x_t | x_1, \ldots, x_{t-1})$ in (1) as

$$\rho_t(\theta; x_t | x_1, \ldots, x_{t-1}) = \frac{O(x_t) \ldots O(x_1)\rho_0(\theta)O(x_1)^* \ldots O(x_t)^*}{\mathrm{tr}\{O(x_t) \ldots O(x_1)\rho_0(\theta)O(x_1)^* \ldots O(x_t)^*\}}$$

$$= \frac{\tilde{\rho}_t(\theta)}{\mathrm{tr}\{\tilde{\rho}_t(\theta)\}}$$

where $\tilde{\rho}_t(\theta) = O(x_t) \ldots O(x_1)\rho_0(\theta)O(x_1)^* \ldots O(x_t)^*$ represents the un-normalized state, whose trace, $\mathrm{tr}\{\tilde{\rho}_t(\theta)\}$, can be thought of as a generalization of the likelihood function of θ given the outcome sequence $x_1, \ldots, x_t$ (Gammelmark and Mølmer, 2013),

$$L(\theta; x_1, \ldots, x_t) = \mathrm{tr}\{O(x_t) \ldots O(x_1)\rho_0(\theta)O(x_1)^* \ldots O(x_t)^*\} = \mathrm{tr}\{\tilde{\rho}_t(\theta)\}. \tag{3}$$

Clearly, the above formulation can be easily extended to continuous-time measurements, by taking the limit as $T \to \infty$ for a fixed time span (Gammelmark and Mølmer, 2013).

2. HCR bound for time-dependent sequences of quantum measurements

In the current setting, the parameter space is an enumerable set of values $\Theta = \{\theta_1, \theta_2, \ldots, \theta_p\}$ which makes differentiation of the likelihood function in (3) inadmissible. For this reason, instead of derivatives we shall consider differences of the likelihood function evaluated at different parameter values. In the following, the derivation of the lower bound for the scalar parameter θ is presented. The result for the multidimensional vector of parameters can be obtained in the same fashion.

Theorem 1: *Let $L(\theta; \mathbf{x}) = \text{tr}\{\tilde{\rho}_T(\theta)\}$ represent the likelihood function of θ given the observation sequence $\mathbf{x} = (x_1, \ldots, x_T)$, where $X_t : (\Omega, \mathcal{F}, P) \to (\mathbb{G}, \mathcal{G}, P_X) \, \forall t = 1, \ldots, T$. Then, for $\theta \in \Theta = \{\theta_1, \theta_2, \ldots, \theta_p\} \subset \mathbb{R}$, $\theta \neq \theta_0$, where θ_0 is the true unknown parameter value, the lower bound of the variance of an unbiased estimator $\hat{\theta}$ is*

$$\mathbb{V}[\hat{\theta}] \geq \sup_{\theta \neq \theta_0} \frac{(\theta - \theta_0)^2}{\displaystyle\int_{\mathbb{G}^T} \frac{\text{tr}^2\{\tilde{\rho}_T(\theta)\}}{\text{tr}\{\tilde{\rho}_T(\theta_0)\}} \, \nu(\mathbf{dx}) - 1} \tag{4}$$

where $\mathbb{V}[\hat{\theta}]$ is the variance of $\hat{\theta}$, and

$$i_T(\theta) = \int_{\mathbb{G}^T} \frac{\text{tr}^2\{\tilde{\rho}_T(\theta)\}}{\text{tr}\{\tilde{\rho}_T(\theta_0)\}} \, \nu(\mathbf{dx}) - 1$$

represents the Fisher information quantity for T repeated measurements, $\forall \theta \in \Theta$.

Proof. The expected value of the estimator $\hat{\theta}$ is

$$\mathbb{E}[\hat{\theta}] = a = \theta + b(\theta) = \int_{\mathbb{G}^T} \hat{\theta} \, \text{tr}\{\tilde{\rho}_T(\theta)\} \, \nu(\mathbf{dx})$$

where $b(\theta) = \mathbb{E}[\hat{\theta}] - \theta$ indicates the bias of $\hat{\theta}$, and the variance is

$$\mathbb{V}[\hat{\theta}] = \int_{\mathbb{G}^T} (\hat{\theta} - \theta_0)^2 \, \text{tr}\{\tilde{\rho}_T(\theta)\} \, \nu(\mathbf{dx}).$$

In this framework, differentiating the likelihood function with respect to the parameter θ is

in general not admissible since the value $\theta + d\theta$ may not belong to the restricted parameter space Θ. As a possible solution, we could consider differences of the likelihood function for different parameter values, i.e. for $\theta_1, \theta_2 \in \Theta$:

$$\int_{\mathbb{G}^T} [\mathrm{tr}\{\tilde{\rho}_T(\theta_1)\} - \mathrm{tr}\{\tilde{\rho}_T(\theta_2)\}] \, \nu(\mathbf{dx}) = 0. \tag{5}$$

Multiplying both sides of equation (5) by $\hat{\theta}$ one has

$$\int_{\mathbb{G}^T} \hat{\theta} \, [\mathrm{tr}\{\tilde{\rho}_T(\theta_1)\} - \mathrm{tr}\{\tilde{\rho}_T(\theta_2)\}] \, \nu(\mathbf{dx}) = (a_1 - a_2), \tag{6}$$

and subtracting equation (5) multiplied by a_2 from equation (6) gives

$$(a_1 - a_2) = \int_{\mathbb{G}^T} (\hat{\theta} - a_2) \, [\mathrm{tr}\{\tilde{\rho}_T(\theta_1)\} - \mathrm{tr}\{\tilde{\rho}_T(\theta_2)\}] \, \nu(\mathbf{dx})$$

$$= \int_{\mathbb{G}^T} (\hat{\theta} - a_2) \, [\mathrm{tr}\{\tilde{\rho}_T(\theta_2)\}^{1/2}] \frac{[\mathrm{tr}\{\tilde{\rho}_T(\theta_1)\} - \mathrm{tr}\{\tilde{\rho}_T(\theta_2)\}]}{\mathrm{tr}\{\tilde{\rho}_T(\theta_2)\}^{1/2}} \, \nu(\mathbf{dx}).$$

By the Cauchy–Schwarz inequality, we get

$$\mathbb{V}[\hat{\theta} \,|\, \theta_0 = \theta_2] \geq \frac{(a_1 - a_2)^2}{\displaystyle\int_{\mathbb{G}^T} \frac{[\mathrm{tr}\{\tilde{\rho}_T(\theta_1)\} - \mathrm{tr}\{\tilde{\rho}_T(\theta_2)\}]^2}{\mathrm{tr}\{\tilde{\rho}_T(\theta_2)\}} \, \nu(\mathbf{dx})}. \tag{7}$$

The denominator of the previous expression can be rearranged as follows

$$\frac{[\mathrm{tr}\{\tilde{\rho}_T(\theta_1)\} - \mathrm{tr}\{\tilde{\rho}_T(\theta_2)\}]^2}{\mathrm{tr}\{\tilde{\rho}_T(\theta_2)\}} = \frac{[\mathrm{tr}\{\tilde{\rho}_T(\theta_1)\}]^2}{\mathrm{tr}\{\tilde{\rho}_T(\theta_2)\}} - 2\mathrm{tr}\{\tilde{\rho}_T(\theta_1)\} + \mathrm{tr}\{\tilde{\rho}_T(\theta_2)\}. \tag{8}$$

By integrating equation (8) over $\mathbb{G}^T$ gives -2 and $+1$, for the second and third term, respectively. Since the reasoning followed so far holds for all values in Θ, we can rewrite equation (7) as follows

$$\mathbb{V}[\hat{\theta}] \geq \sup_{\theta \neq \theta_0} \frac{(\theta - \theta_0)^2}{\displaystyle\int_{\mathbb{G}^T} \frac{\mathrm{tr}\{\tilde{\rho}_T \,|\, \theta\}^2}{\mathrm{tr}\{\tilde{\rho}_T \,|\, \theta_0\}} \, \nu(\mathbf{dx}) - 1} \tag{9}$$

where we allow θ to vary over the whole parameter space Θ except for θ_0, and an unbiased

estimator is considered. Equation (9) represents the extension of the HCR inequality to the case of discrete parameter models in repeatedly observed quantum systems □

The denominator of equation (4) in Theorem 1, i.e. the Fisher information quantity of $\theta \in \Theta$ for T repeated measurements, is a Neyman χ^2_N divergence, see the left hand side of equation (8). An interesting research topic, worth of investigation, is concerned with the existence of quantum analogs of Fisher information in the discrete-parameter setting, that play the role of Helstrom information or Wigner-Yanase information in the case when the parameter space is not continuous. These analogs are usually derived based on the specification of operators that are non-commutative versions of second order derivatives, see Tsuda and Matsumoto (2005). In the present setting, similar results might be obtained for the time sequence of effect operators $O(x_t)$ based on results of Hansen (2008) and Cai (2018).

3. Application

Nielsen and Mølmer (2008) derive the equation describing the time evolution of a quantum system placed inside a cavity by monitoring the state of the electromagnetic radiations transmitted through the cavity. Such indirect measurement turns out to be very useful to prepare a system in a certain quantum mechanical state and to achieve the so called quantum non-demolition measurements of an observable (Nielsen and Mølmer, 2008; Nielsen and Chuang, 2010). In this regard, in order to prepare the system in the so called Dicke state (Dicke, 1954), they consider a quantum system with N different photon-number states whose measurement modifies the probability distribution over the possible states without directly disturbing the system dynamics.

Specifically, following the notation of Nielsen and Mølmer (2008), section V.C.1, the joint probability, at time t, to observe the quantum state ρ_t, along with the observed signal y, is

$$P_{\rho_t} = \sum_{n=1}^{N} \frac{C_n(0)}{\sqrt{2\pi t}} \exp\left(-\frac{(y + \theta_n t)^2}{2t}\right). \tag{10}$$

Here, $C_n(t) = C_n(0) \exp\left(-\theta_n \hat{y} - \theta_n^2 t/2\right)$, $n = 1, \ldots, N$, the time dependent probability assigned to each photon-number state n, is a function of the initial condition $C_n(0)$ and the variable

y, having mean, $-\theta_n t$, defined in equation (38) of Nielsen and Mølmer (2008). Note that, assuming that the true value θ_0 is an element of the parameter space $\Theta = \{\theta_1, \ldots, \theta_N\}$, equation (10) consists of a sum in which each term is proportional to a Gaussian density function with variance t and mean $-\theta_n t$. The latter quantity is the object of our inferential problem since it conveys all the available information about the parameter of interest, i.e. the photon number state.

Following the same reasoning adopted in the proof of Theorem 1, we select one element of equation (10) at the time, in order to compare signals obtained at different values of the parameters θ_n. Let us rewrite the integral in the HCR (9) as follows

$$\int_{\mathbb{G}} \frac{\text{tr}\{\tilde{\rho}_t \, | \, \theta_n\}^2}{\text{tr}\{\tilde{\rho}_t \, | \, \theta_0\}} \, \nu(dx) - 1 = \int_{\mathbb{R}} \frac{\left[\frac{C_n(0)}{\sqrt{2\pi t}} \exp\left(-\frac{(y+\theta_n t)^2}{2t} \right) \right]^2}{\frac{C_0(0)}{\sqrt{2\pi t}} \exp\left(-\frac{(y+\theta_0 t)^2}{2t} \right)} \, dy - 1. \tag{11}$$

By considering the right hand side of equation (11), taking the square and completing it with the term $\pm t^2 (2\theta_n - \theta_0)^2$, we get:

$$\frac{C_n(0)^2}{C_0(0)} \exp\left[-\frac{t^2}{2t} (2\theta_n^2 - \theta_0^2) \right] \exp\left[\frac{t^2}{2t} (2\theta_n - \theta_0)^2 \right] \frac{1}{\sqrt{2\pi t}} \exp\left[-\frac{1}{2t} (y + t(2\theta_n - \theta_0))^2 \right].$$

Taking the integral over $\mathbb{R}$ gives

$$\frac{C_n(0)^2}{C_0(0)} \exp\left[-\frac{t^2}{2t} (2\theta_n^2 - \theta_0^2) \right] \exp\left[\frac{t^2}{2t} (2\theta_n - \theta_0)^2 \right].$$

Finally, rearranging terms, the right hand side of equation (11) becomes

$$\frac{C_n(0)^2}{C_0(0)} \exp\left[t(\theta_n - \theta_0)^2 \right] - 1. \tag{12}$$

Since it is reasonable to assume that the differences in the mean of the integrated signals are of the form $\theta_n = \theta_0 + \alpha_n$, where α_n is a non zero real number, then the HCR bound in

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equation (9) becomes

$$\mathbb{V}[\hat{\theta}] \geq \max_{\theta_n \neq \theta_0} \frac{(\theta_n - \theta_0)^2}{\frac{C_n(0)^2}{C_0(0)} \exp\left[t(\theta_n - \theta_0)^2\right] - 1} = \max_{\alpha_i \neq 0} \frac{C_n(0)\,\alpha_n^2}{C_0(0)^2 \exp\left[t\alpha_n^2\right] - 1}.$$

The previous expression reaches its maximum when α tends to 0, thus leading to

$$\lim_{\alpha \to 0} \frac{C_0(0)\,\alpha^2}{C_n(0)^2 \exp\left[t\alpha^2\right] - 1} = \frac{C_0(0)\,2\alpha}{C_n(0)^2\,t\,2\alpha \exp\left[t\alpha^2\right]} = \frac{C_0(0)}{C_n(0)^2 t}$$

where the last term represents the lower bound for the variance of the mean estimator, which is linearly dependent on t.

4. Conclusions

A discrete counterpart of the Cramér–Rao bound in parametric quantum models is derived. Without relying on classical regularity assumptions, the extension sets the lower bound for the variance of an unbiased estimator of the parameter of interest. By means of an illustration related to a quantum optics problem, in which the classical Fisher information quantity is obtained, we emphasise how the derived results find applications time-dependent sequences of quantum systems.

Acknowledgements

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