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Hyperbolicity of first and second order extended thermodynamics theory of polyatomic rarefied gases

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Abstract

The balance laws of Rational Extended Thermodynamics describe well the evolution of rarefied gases in non-equilibrium. Usually, it is necessary to approximate the theory in a neighborhood of an equilibrium state and consequently, its hyperbolicity property remains valid only in a neighborhood of the equilibrium state of the field variables, called *hyperbolicity region*. The goal of this paper is first to determine the differential system with 14 fields for a rarefied polyatomic polytropic gas, approximated at the second-order in the non-equilibrium variables. Then, we investigate and compare the hyperbolicity property of the first-order and second-order systems. In particular, we analyze the role played by the dynamic pressure and the molecular degrees of freedom. Finally, we also show that in the monatomic singular limit the quadratic theory for a polyatomic gas converges to the corresponding quadratic theory for a monatomic gas.

Keywords: Rational extended thermodynamics, Non-equilibrium thermodynamics, Hyperbolicity region, Rarefied polytropic gases
2020 MSC: 35L40, 76N10, 76N15

1. Introduction

Rational Extended Thermodynamics (RET) is a well-established macroscopic theory, aimed to describe non-equilibrium thermodynamic phenomena in both monatomic [1] and polyatomic gases [2]. Unlike the usual classical thermodynamics theory (CT), RET involves as independent field variables not only mass density, velocity and temperature, but also non-equilibrium quantities as the viscous tensor, the dynamic pressure and the heat flux, referring to the mathematical structure of truncated moment systems of the kinetic theory. The corresponding field equations turn out to be balance laws supplemented

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by local and instantaneous constitutive equations that satisfy universal physical principles, like, the entropy principle, the relativity principle and the thermodynamical stability. RET constitutes a dynamic and active research field and the theory was successfully applied to several physical problems with good agreement with experimental data [1, 2].

From the mathematical point of view, RET models are expected to be hyperbolic PDE systems with a convex extension and can be rewritten in a symmetric form using the *main field* as set of independent variables [3, 4, 5, 2]. In this way the well-posedness of the Cauchy problem is guaranteed and under suitable conditions global smooth solutions exists for all time. The hyperbolicity property is also essential for a realistic physical description of non-stationary phenomena, since it ensures the finiteness of the disturbance speeds, in contrast to the infinite speed predicted by parabolic models like the Thermodynamics of Irreversible Processes (TIP), and in particular the Navier-Stokes-Fourier theory.

Normally, the balance equations are approximated through a Taylor expansion in the neighborhood of a local equilibrium. Such an expansion gives rise to a theory that can be easily constructed [6], but unfortunately, it destroys the global hyperbolicity property of the PDE system. In fact, it turns out that the hyperbolicity remains valid only in a domain of the phase space, called *hyperbolicity region*. The first study of a hyperbolicity region dated back to 1996 [5], when Müller and Ruggeri determined the domain of hyperbolicity in the case of the one-dimensional model with 13 moments for monatomic gases, linearized with respect to the non-equilibrium variables (that coincides with the Grad's one [7]). After that, the study of the hyperbolicity region was carried on by different authors always in the framework of monatomic gases [8, 9, 10, 11]. Recently Brini and Ruggeri [12] have considered the 13-moment RET equations of rarefied monatomic gases both in the case of one-dimensional and three-dimensional field variables. In this framework, they have compared the effect of different orders of approximation of the theory, proving that a quadratic theory presents two different advantages with respect to the usual linearized one. On one hand, the hyperbolicity region of the quadratic one-dimensional equations is "larger" than the corresponding one of the first-order theory. On the other hand, when three-dimensional field variables are considered, the quadratic theory is capable of overcoming the singularity reported by Cai, Fan and Li [10, 11]. In fact, the equilibrium point is at the boundary of the hyperbolicity region of the three-dimensional linear RET 13-moment theory, while for the corresponding quadratic theory there exists a neighborhood of the equilibrium state that is included in the hyperbolicity domain. Thanks to these results, RET theories and, in particular, the second-order approximation with respect to the equilibrium state are seen in a new light.

As far as the RET of polyatomic gases is concerned, the hyperbolicity region does not seem to us to have been studied in the literature before. This observation motivates the present work. Here we construct for the first time a complete three-dimensional second-order RET theory with 14 moments for a rarefied polytropic gas and analyze systematically its hyperbolicity properties, comparing such results with those for the usual linear model. The role of the number of degrees of freedom D and of the dynamic pressure Π is particularly investigated, while the case of a monatomic gas is always recovered as singular limit when $D \to 3$.

The paper is organized as follows. The differential system of a RET theory for polyatomic gases is described in Section 2, while Section 3 contains the three-dimensional equations for the second-order theory in the case of a rarefied polytropic gas. The general questions about hyperbolicity property and hyperbolicity domains are treated in Section 4. Sections 5-8 are devoted to the cases of one-dimensional and three-dimensional variables. Section 9 explores the monatomic singular limit of the second-order theory for polyatomic gases. After the conclusions and final remarks of Section 10, some detailed calculations are reported in Appendix A.

2. The $\mathbf{ET}^{\alpha}_{n,P}$ theory for a rarefied polyatomic gas

The RET theories for rarefied polyatomic gases can be obtained referring to the kinetic theory introduced by Borgnakke and Larsen [13], as described in [2]. In fact, the distribution function $f \equiv f(t, \mathbf{x}, \mathbf{c}, I)$ for a polyatomic gas is assumed to depend on the time t, on the space variable $\mathbf{x} = (x_1, x_2, x_3)$, on the microscopic velocity $\mathbf{c} = (c_1, c_2, c_3)$ and on the continuous variable $I \in [0, \infty)$ that represents the energy of the internal modes of the molecules, taking into account the exchange of energy between translational and internal modes. So, the distribution function $f(t, \mathbf{x}, \mathbf{c}, I)$ is defined on the extended domain $[0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3 \times [0, \infty)$. This model for the distribution function was applied by Bourgat, Desvillettes, Le Tallec and Perthame [14] to the derivation of the generalized Boltzmann equation for a polyatomic gas that exhibit the same form as the one for a monatomic gas:

$$\partial_t f + c_j \partial_j f = \mathcal{Q}, \tag{1}$$

where ∂_t is the partial derivative with respect to t, ∂_j $(j = 1, 2, 3)$ is the partial derivative with respect to x_j and we omit, from now on, the symbol of sum for repeated indices Moreover, $\mathcal{Q}$ denotes the collisional term that has to account for the internal degrees of freedom of the molecules [13, 14].

From the Boltzmann equation (1) a double infinite hierarchy of balance laws (the moment system) is commonly derived [15, 16, 2]. Such a hierarchy was first postulated by Arima, Taniguchi, Ruggeri and Sugiyama [17] at a phenomenological level and in the case of 14 fields. To obtain the moment system, one has to define initially the momentum-like F moments and the energy-like G

moments as

$$
\begin{aligned}
&F = m \int_{\mathbb{R}^3} \int_0^{+\infty} f I^a dI \, d\mathbf{c} \\
&F_{k_1 k_2 \ldots k_j} = m \int_{\mathbb{R}^3} \int_0^{+\infty} f c_{k_1} c_{k_2} \ldots c_{k_j} I^a dI d\mathbf{c} \quad \text{with } j = 1, 2, \ldots, \\
&G_{ll} = m \int_{\mathbb{R}^3} \int_0^{+\infty} \left(c^2 + 2\frac{I}{m} \right) f I^a dI d\mathbf{c}, \\
&G_{llk_1 k_2 \ldots k_s} = m \int_{\mathbb{R}^3} \int_0^{+\infty} f \left(c^2 + 2\frac{I}{m} \right) c_{k_1} c_{k_2} \ldots c_{k_s} I^a dI d\mathbf{c} \quad s = 1, 2, \ldots \\
&P_{k_1 k_2 \ldots k_j} = m \int_{\mathbb{R}^3} \int_0^{+\infty} \mathcal{Q}(f) \, c_{k_1} c_{k_2} \ldots c_{k_j} I^a dI d\mathbf{c} \\
&Q_{llk_1 k_2 \ldots k_s} = m \int_{\mathbb{R}^3} \int_0^{+\infty} \mathcal{Q}(f) \, c_{k_1} c_{k_2} \ldots c_{k_s} \left(c^2 + 2\frac{I}{m} \right) I^a dI d\mathbf{c}
\end{aligned}
$$

where $k_h = 1, 2, 3 \; \forall h \in \mathbb{N} \setminus \{0\}$. The measure I^a is introduced in such a way that the internal energy at equilibrium coincides with the one of a polyatomic polytropic gas [14, 18]:

$$
\varepsilon = \frac{D}{2} \frac{k_{\mathcal{B}}}{m} T, \tag{2}
$$

provided that the constant a is related to D by the relation

$$
a = (D - 5)/2.
$$

D represents the degrees of freedom of the gas molecule, so that, for example, a monatomic molecule presents only $D = 3$ translational degrees of freedom [2] and we remark that the monatomic gas is a singular limit because $a > -1$, see [2]. In the previous relations $k_{\mathcal{B}}$, m and T denote respectively the Boltzmann constant, the atomic mass and the absolute equilibrium temperature.

The two infinite hierarchies present a peculiarly elegant structure [2]: the flux component of one equation becomes the density component of the following one:

$$
\begin{array}{ll}
\partial_t F + \partial_i F_i = 0, & \\
\quad\swarrow & \\
\partial_t F_{k_1} + \partial_i F_{ik_1} = 0, & \\
\quad\swarrow & \\
\partial_t F_{k_1 k_2} + \partial_i F_{ik_1 k_2} = P_{k_1 k_2}, & \partial_t G_{ll} + \partial_i G_{lli} = 0, \\
\quad\swarrow & \quad\swarrow \\
\partial_t F_{k_1 k_2 k_3} + \partial_i F_{ik_1 k_2 k_3} = P_{k_1 k_2 k_3}, & \partial_t G_{llk_1} + \partial_i G_{llik_1} = Q_{k_1}, \\
\vdots & \vdots \\
\partial_t F_{k_1, k_2 \dots k_j} + \partial_i F_{ik_1 k_2 \dots k_j} = P_{k_1 k_2 \dots k_j}, & \partial_t G_{llk_1 k_2 \dots k_j} + \partial_i G_{llik_1 k_2 \dots k_j} = Q_{k_1 k_2 \dots k_j}. \\
\vdots & \vdots
\end{array}
$$

We recall that the first two equations of the F-hierarchy and the first scalar equation of the G-hierarchy represent the usual conservation laws of mass, momentum and energy [2].

The set of balance laws is usually truncated at some index of truncation N for the F-series and at the index M for the G-series [2, 15]. In [16] it was proved that the requirement of Galilean invariance and the conditions that the characteristic velocities depend on D imply necessarily that $M = N - 1$. This relation is also confirmed by the classical limit of the relativistic moments [19]. Moreover, in [19] Pennisi and Ruggeri proved that the simple structure of the binary hierarchy considered here is a particular case of a more complex hierarchy structure of the moments.

In RET there are more concise ways to denote the density, flux and production terms of the previous equations. In fact, you can refer to the vectorial notation:

$$
\begin{aligned}
&\mathbf{F} = (F, F_{k_1}, F_{k_1 k_2}, \dots F_{k_1 k_2 \dots k_N})^T, \\
&\mathbf{F}^i = (F_i, F_{ik_1}, F_{ik_1 k_2}, \dots F_{ik_1 k_2 \dots k_N})^T, \\
&\mathbf{G}_{ll} = (G_{ll}, G_{llk_1}, G_{llk_1 k_2}, \dots G_{llk_1 k_2 \dots k_M})^T, \\
&\mathbf{G}^i_{ll} = (G_{lli}, G_{llik_1}, G_{llik_1 k_2}, \dots G_{llik_1 k_2 \dots k_M})^T \\
&\mathbf{P} = (0, 0, P_{k_1 k_2}, \dots, P_{k_1 k_2 \dots k_N})^T \\
&\mathbf{Q} = (0, Q_{llk_1}, Q_{llk_1 k_2}, \dots Q_{llk_1 \dots k_M}),
\end{aligned}
$$

and the two truncated hierarchies can be rewritten in a very concise way

$$
\partial_t \mathbf{F} + \partial_j \mathbf{F}^j = \mathbf{P}, \quad \partial_t \mathbf{G}_{ll} + \partial_j \mathbf{G}^j_{ll} = \mathbf{Q}. \tag{3}
$$

In addition, also a multi-index notation is used to recap the previous equations, starting from the symbols

$$
F_A = \begin{cases} F & \text{if } A = 0 \\ F_{k_1 \dots k_A} & \text{if } 1 \le A \le N \end{cases}, \quad F_{iA} = \begin{cases} F_i & \text{if } A = 0 \\ F_{ik_1 \dots k_A} & \text{if } 1 \le A \le N \end{cases},
$$

$$G_{llA'} = \begin{cases} G_{ll} & \text{if } A' = 0 \\ G_{llk_1 \dots k'_A} & \text{if } 1 \le A' \le M \end{cases}, \quad G_{illA'} = \begin{cases} G_{lli} & \text{if } A' = 0 \\ G_{llik_1 \dots k_{A'}} & \text{if } 1 \le A' \le M \end{cases},$$

and

$$\begin{aligned} P_A &= P_{k_1 \dots k_A} && 2 \le A \le N, \\ Q_{llA'} &= Q_{llk_1 \dots k'_A} && 1 \le A' \le M. \end{aligned}$$

The truncation of the two hierarchies implies a question about how to close the system and, in particular, how to express the last fluxes and the productions in terms of the independent field variables that in the present case are the density components.

In particular, the Maximum Entropy Principle (MEP) is commonly used to this aim for a theory with many moments [20, 2]. In the case of a polyatomic gas the actual distribution function $f_{N,M}$ is the one that maximizes the entropy defined by

$$h = -k_{\mathcal{B}} \int_{\mathbb{R}^3} \int_0^\infty f \ln f I^a dI d\mathbf{c}$$

under the constraints that the moments F_A and $G_{llA'}$ are prescribed:

$$\begin{aligned} F_A &= \int_{\mathbb{R}^3} \int_0^\infty m f c_A I^a \, dI d\mathbf{c}, && 0 \le A \le N \\ G_{llA'} &= \int_{\mathbb{R}^3} \int_0^\infty m f \left(c^2 + \frac{2I}{m} \right) c_{A'} I^a \, dI d\mathbf{c}, && 0 \le A' \le M. \end{aligned}$$

The distribution function $f_{N,M}$ is obtained from the variational principle, connected to the functional [16]:

$$\begin{aligned} \mathcal{L}_{(N,M)}(f) = -k_{\mathcal{B}} \int_{\mathbb{R}^3} \int_0^\infty f \log f \, I^a \, dI d\mathbf{c} + \sum_{A=0}^{N} u'_A \left(F_A - \int_{\mathbb{R}^3} \int_0^\infty m f c_A I^a \, dI d\mathbf{c} \right) \\ + \sum_{A'=0}^{M} v'_{A'} \left(G_{llA'} - \int_{\mathbb{R}^3} \int_0^\infty m f \left(c^2 + \frac{2I}{m} \right) c_{A'} I^a dI d\mathbf{c} \right), \end{aligned}$$

where u'_A and $v'_{A'}$ are the Lagrange multipliers (the main field vectors) corresponding to the $F-$ and the $G-$ hierarchy respectively, and it holds

$$u'_A = \begin{cases} u' & \text{if } A = 0 \\ u'_{k_1 \dots k_A} & \text{if } 1 \le A \le N \end{cases}, \quad c_A = \begin{cases} 1 & \text{if A} = 0 \\ c_{k_1} \dots c_{k_A} & \text{if } 1 \le A \le N. \end{cases}$$

$$v'_{A'} = \begin{cases} v'_{ll} & \text{if } A' = 0 \\ v'_{llk_1 \dots k_{A'}} & \text{if } 1 \le A' \le M \end{cases}, \quad c_{A'} = \begin{cases} 1 & \text{if A}' = 0 \\ c_{k_1} \dots c_{k_{A'}} & \text{if } 1 \le A' \le M. \end{cases}$$

It is possible to show that the phase density that maximizes the entropy under prescribed moments, reads [16]:

$$f_{N,M} = \exp\left(-1 - \frac{m}{k_{\mathcal{B}}} \chi_{N,M} \right) \quad \text{with} \quad \chi_{N,M} = u'_A c_A + \left(c^2 + \frac{2I}{m} \right) v'_{A'} c_{A'}, \quad (4)$$

where the sums in repeated multiindex $\sum_{A=0}^{N}$ and $\sum_{A'=0}^{M}$ are omitted in $(4)_2$, At equilibrium the main field components vanish, except for the first four of u'_A and the first one of v'_A, while the distribution function becomes [14, 15, 2]

$$f^{(E)} = \frac{\rho}{mA(T)} \left(\frac{m}{2\pi k_{\mathcal{B}} T}\right)^{3/2} \exp\left[-\frac{1}{k_{\mathcal{B}}T}\left(\frac{mC^2}{2} + I\right)\right], \tag{5}$$

where ρ is the gas density, T the temperature, $\mathbf{v}$ the velocity and $\mathbf{C} = \mathbf{c} - \mathbf{v}$ the peculiar velocity, while

$$A(T) = \int_0^\infty \exp\left(-\frac{I}{k_{\mathcal{B}}T}\right) I^a dI.$$

Likewise for monatomic rarefied gases, the moments constructed by the previous distribution function (4) presents in general problems of convergence. To overcome this problem, the distribution function is usually approximated through the Taylor expansion of order $\alpha > 0$ ($\alpha \in \mathbb{N}$) in the neighborhood of a local equilibrium; f is approximated as $f^{(\alpha)}$ [2, 15]

$$f^{(\alpha)}_{N,M} = f^{(E)}\Big[1 - \frac{m}{k_{\mathcal{B}}}\bar{\chi}_{N,M} + \frac{m^2}{2k_{\mathcal{B}}^2}\bar{\chi}^2_{N,M} + \cdots + (-1)^\alpha \frac{m^\alpha}{\alpha! k_{\mathcal{B}}^\alpha}\bar{\chi}^\alpha_{N,M}\Big], \tag{6}$$

where

$$\bar{\chi}_{N,M} = \bar{u}'_A c_A + \left(c^2 + \frac{2I}{m}\right)\bar{v}'_{A'} c_{A'}$$

and

$$\bar{u}'_A = u'_A - u'^{(E)}_A, \qquad \bar{v}'_{A'} = v'_{A'} - v'^{(E)}_{A'},$$

if $u'^{(E)}_A$ and $v'^{(E)}_{A'}$ denote the main field components at equilibrium.

The moments derived from (6) are always convergent since the equilibrium distribution function (5) dominates any polynomial. Unfortunately, the approximated distribution function (6) is not always positive and this is the reason why the differential system loses the global convexity of the entropy and the corresponding equation system is hyperbolic only in some neighborhood of the equilibrium state.

Also the last flux components that are not in the list of the densities are approximated as

$$F^{(\alpha)}_{ik_1k_2\ldots k_N} = m\int_{\mathbb{R}^3}\int_0^{+\infty} f^{(\alpha)} c_{k_1}c_{k_2}\ldots c_{k_N} I^a dI d\mathbf{c},$$
$$G^{(\alpha)}_{llk_1k_2\ldots k_M} = m\int_{\mathbb{R}^3}\int_0^{+\infty} f^{(\alpha)}\left(c^2 + 2\frac{I}{m}\right) c_{k_1}c_{k_2}\ldots c_{k_M} I^a dI d\mathbf{c}.$$

We will denote this system as $\mathrm{ET}^\alpha_{n,P}$ where α denotes the degree of approximation, n the number of scalar equations of the system and we add a P both to indicate that we consider a polyatomic gas and to distinguish the present theory from our previous one [6] ET_n^α, valid for a monatomic gas.

This procedure is often implemented under the assumption of zero velocity. The most general case with $\mathbf{v} \neq \mathbf{0}$ if obtained from the previous one thanks to the Galilean invariance requirement [21]. In this manner, one determines the non-convective part of densities, fluxes , productions and main fields and then the dependence on $\mathbf{v}$ is determined thanks to [2, 21, 6].

3. $\mathbf{ET}^{\mathbf{2}}_{\mathbf{14},\boldsymbol{P}}$ balance equations for a rarefied polytropic gas

In the present work we will focus on a rarefied polytropic gas, taking into account the thermal and the caloric equations of state given by

$$p = \frac{k_{\mathcal{B}}}{m}\rho T,$$

and (2). As usual, p denotes the equilibrium pressure and ρ the mass density. In the following we define also

$$\theta = \frac{k_{\mathcal{B}}}{m}T.$$

To describe such a gas, we refer to the quadratic theory $\mathrm{ET}^2_{14,P}$. The corresponding $\mathrm{ET}^1_{14,P}$ theory was proposed as a first example of an extended thermodynamics theory for a polyatomic gas by Arima, Ruggeri, Sugiyama and Taniguchi [17] and then applied successfully to several physical problems [2]. In the present work the density and flux components for the second-order theory was determined following the procedures introduced and described in [6, 12]. In particular, the RET system (3) with 14 moments for a polytropic gas involves the following F- and G-type density components:

$$\mathbf{F} \equiv (F, F_i, F_{ik}), \qquad \mathbf{G}_{ll} \equiv (G_{ll}, G_{ill}) \quad \text{with } i,k = 1,2,3 \tag{7}$$

and the following flux components

$$\mathbf{F}^j \equiv \left(F_j, F_{ji}, F^{(\alpha)}_{jik}\right), \qquad \mathbf{G}^j_{ll} \equiv \left(G_{jll}, G^{(\alpha)}_{jill}\right) \quad \text{with } i,j,k = 1,2,3. \tag{8}$$

The densities (7) play the role of the field variables, while in the fluxes (8) we indicate with an apex α the last fluxes that are not in the list of densities and, therefore, represent constitutive equations to be determined. In the present paper we will consider the cases $\alpha = 1$ and $\alpha = 2$, corresponding respectively to the first-order and to the second-order approximation in the non-equilibrium variables. Such constitutive equations will be determined starting from the method introduced in [6, 12].

Besides the already cited mass density ρ and internal energy ε, in what follows we consider as field variables related to the moments: the components of the velocity v_i, the components of the heat flux q_i and the components of the viscous stress tensor σ_{ij} that we can divide in the deviatoric part $\sigma_{<ij>}$ and in the isotropic part $-\Pi\delta_{ij}$:

$$\sigma_{ij} = \sigma_{<ij>} - \Pi\delta_{ij},$$

$\Pi = -\sigma_{ll}/3$ is called dynamic pressure.

Using the procedure indicated in [6, 12] after cumbersome calculations that, for the sake of brevity we omit here, the balance equations (3) for $\mathrm{ET}^2_{14,P}$ can be explicitly written as[1]

$$\begin{aligned}
&\partial_t\rho + \partial_k\rho v_k = 0,\\
&\partial_t(\rho v_i) + \partial_k(\rho v_i v_k + p\delta_{ik} - \sigma_{ik}) = 0\\
&\partial_t(\rho v_i v_j + p\delta_{ij} - \sigma_{ij}) + \partial_k\Bigg\{\rho v_i v_j v_k + 3pv_{(i}\delta_{jk)} - 3\sigma_{(ij}v_{k)} + \frac{6}{D+2}q_{(i}\delta_{jk)} + \\
&+ \underline{\frac{24}{(D+2)^2 p}\delta_{(ij}\sigma_{k)l}q_l - \frac{12}{(D+2)p}q_{(i}\sigma_{jk)}}\Bigg\} = P_{ij},\\
&\partial_t(2\rho\varepsilon + \rho v^2) + \partial_k(\rho v^2 v_k + 2(p + \rho\varepsilon)v_k - 2\sigma_{kl}v_l + 2q_k) = 0,\\
&\partial_t(\rho v^2 v_i + 2(\rho\varepsilon + p)v_i - 2\sigma_{il}v_l + 2q_i) + \partial_{x_k}\Bigg\{\rho v^2 v_i v_k + 2\rho\varepsilon v_i v_k + \\
&+ p(v^2\delta_{ik} + 4v_i v_k) - \sigma_{ik}v^2 - 4v_{(k}\sigma_{i)l}v_l +\\
&+ \frac{4}{D+2}q_l v_l\delta_{ik} + \frac{2(8+2D)}{D+2}q_{(i}v_{k)} + (D+2)\frac{p^2}{\rho}\delta_{ik} - (D+4)\frac{p}{\rho}\sigma_{ik} +\\
&+ \underline{\frac{2}{\rho}\sigma_{il}\sigma_{kl} + \frac{4}{(D+2)^2 p}(4(D+4)q_i q_k + (D+6)q^2\delta_{ik} +}\\
&\underline{+4(2v_{(i}\sigma_{k)l}q_l + \sigma_{lm}q_l v_m\delta_{ik}) - (2D+4)(2q_{(i}\sigma_{k)l}v_l + \sigma_{ik}q_l v_l))}\Bigg\} = P_{ill},
\end{aligned} \tag{9}$$

where the underlined terms correspond to the quadratic ones with respect to the non-equilibrium quantities and are ignored in the case of $\mathrm{ET}^1_{14,P}$ equations [17, 2].

4. Hyperbolicity region

As already mentioned in the introduction, the validity of the hyperbolicity property is a pivot of RET theories, both from the mathematical and from the

---

[1] In what follows we will denote the symmetric part of a tensor $T_{i_1 i_2 \dots i_k}$ with the parentheses $T_{(i_1 i_2 \dots i_k)} = \sum_{\text{permutation of indexes}} T_{i_1 i_2 \dots i_k}/N_P$, where N_P represents the number of index permutations.

physical point of view. So, to test at first the second-order model that we have presented just now, it is natural to start from the hyperbolicity.

If we consider a generic vector of d field independent variables $\mathbf{u} \in \mathbb{R}^d$ and we assume that they all depend only on t and on x_1, the RET equation system can be reduced to the following form

$$\mathbf{A}\partial_t \mathbf{u} + \mathbf{B}\partial_1 \mathbf{u} = \mathbf{P}, \qquad \mathbf{A} = \mathbf{A}(\mathbf{u}),\ \mathbf{B} = \mathbf{B}(\mathbf{u}),$$

where $\mathbf{A}$ and $\mathbf{B}$ are $d \times d$ square matrices, and $\mathbf{P}$ is the production vector. The previous PDE system is hyperbolic only if $\mathbf{A}$ is a non-singular matrix ($\det(\mathbf{A}) \neq 0$) and the generalized eigenvalue problem $(\mathbf{B} - \lambda \mathbf{A})\mathbf{r} = \mathbf{0}$ (with $\mathbf{r} \in \mathbb{R}^d$) presents all real eigenvalues λ (characteristic velocities) and a complete set of eigenvectors $\mathbf{r}$. Unfortunately, often the degree of the corresponding characteristic polynomial is too high to allow the analytical direct calculation of its roots. Several approaches can be introduced to overcome this difficulty, at least at a numerical standpoint. In what follows we will consider two of them. The first one is in principle an analytical technique and is aimed to identify the boundaries of the hyperbolicity region and was introduced in [12] . The starting point is the idea that at the boundaries the characteristic polynomial $\mathcal{P}(\lambda)$ presents at least a double root. So, if at least two characteristic velocities have the same value λ_* necessarily we have:

$$\mathcal{P}(\lambda_*) = 0, \qquad \frac{d\mathcal{P}}{d\lambda}(\lambda_*) = 0. \tag{10}$$

As the coefficients of the characteristic polynomial depend on the field $\mathbf{u}$, relations (10) give the equations of the boundaries of the hyperbolicity domain in parametric form, in terms of the parameter λ_*. Unfortunately, in some cases such equations cannot be treated analytically and a numerical final step could be required.

Alternatively, the hyperbolicity property can be verified through a numerical approach, identifying the zone in which the characteristic polynomial presents all real roots.

5. The hyperbolicity region of $\mathbf{ET}^1_{14,P}$ for one-dimensional field variables

Here, we assume the one-dimensional variables $\mathbf{u} = (\rho, v_1, \theta, \sigma_{11}, \Pi, q_1) \in \mathbb{R}^6$, depending on t and only the x_1 space variable. The one-dimensional density components (7) read

$$\begin{aligned}
\mathbf{F} &= \Big(\rho,\ \rho v_1,\ 3p + 3\Pi + \rho v_1^2,\ p - \sigma_{11} + \rho v_1^2\Big),\\
\mathbf{G} &= \Big(Dp + \rho v_1^2, 2q_1 - 2\sigma_{11} v_1 + (2 + D) p v_1 + \rho v_1^3\Big),
\end{aligned} \tag{11}$$

while the one-dimensional flux components $\mathbf{F}^1, \mathbf{G}^1_{ll}$ with constitutive equations at first order $\alpha = 1$ in the nonequilibrium variables are:

$$\mathbf{F}^{1,(1)} = \Bigg(\rho v_1,\ p - \sigma_{11} + \rho v_1^2,\ \frac{10 q_1}{D+2} + 5 p v_1 + 3\Pi v_1 - 2\sigma_{11} v_1 + \rho v_1^3,$$
$$\frac{6 q_1}{D+2} + 3 p v_1 - 3\sigma_{11} v_1 + \rho v_1^3 \Bigg),$$
$$\mathbf{G}^{1,(1)} = \Bigg(2 q_1 - 2\sigma_{11} v_1 + (D+2) p v_1 + \rho v_1^3,\ (D+2)\frac{p^2}{\rho} - (D+4)\frac{p}{\rho}\sigma_{11} +$$
$$+ \frac{4(D+5) q_1 v_1}{D+2} + (D+5) p v_1^2 - 5\sigma_{11} v_1^2 + \rho v_1^4 \Bigg).$$

The explicit expression of matrices $\mathbf{A}$ and $\mathbf{B}$ are easily deduced from the previous expressions.

To simplify the following calculations, we introduce the dimensionless quantities:

$$\tilde{\lambda} = \frac{\lambda - v_1}{c_0}, \quad \tilde{\sigma}_{ij} = \frac{\sigma_{ij}}{\rho c_0^2}, \qquad \tilde{\Pi} = \frac{\Pi}{\rho c_0^2}, \qquad \tilde{q}_i = \frac{q_i}{\rho c_0^3}, \tag{12}$$

where $c_0 = \sqrt{k_{\mathcal{B}} T/m} = \sqrt{\theta}$.

Matrix $\mathbf{A}$ is non-singular ($\det(\mathbf{A}) = -6 D k_{\mathcal{B}} \rho^2/m$) and referring to the dimensionless "non-convective" characteristic velocities $\tilde{\lambda}$ and to the dimensionless quantities introduced in (12), the characteristic polynomial turns out to be

$$\mathcal{P}^{(1)}_{1D}(\tilde{\lambda}) = -6D\frac{k_{\mathcal{B}}}{m}\theta p^2 \tilde{\lambda}^2 (a_4^{(1)}\tilde{\lambda}^4 + a_3^{(1)}\tilde{\lambda}^3 + a_2^{(1)}\tilde{\lambda}^2 + a_1^{(1)}\tilde{\lambda} + a_0^{(1)}) \tag{13}$$

with

$$a_4^{(1)} = 1; \quad a_3^1 = 0; \quad a_2^{(1)} = \frac{2((2D^2+3D+4)\tilde{\sigma}_{11} - (2D^2+7D))}{D(D+2)},$$
$$a_1^{(1)} = -\frac{12(D+5)\tilde{q}_1}{(D+2)^2}, \quad a_0^{(1)} = \frac{3((D+4)\tilde{\sigma}_{11}^2 - (2D+4)\tilde{\sigma}_{11} + D + 2)}{D+2};$$

if 1D stands for the case of one-dimensional field variables, while the apex $^{(1)}$ corresponds to the linear theory.

We remark that in this case the characteristic polynomial depends only on two non-equilibrium variables q_1 and $\sigma_{11} = \sigma_{<11>} - \Pi$. Therefore, the shear viscous component $\sigma_{<11>}$ and the dynamic pressure Π do not appear separately. The same will happen to the equations of the hyperbolicity region boundaries. At equilibrium the roots of the characteristic polynomial are [2]

$$\tilde{\lambda}^E_{1,2} = 0, \qquad \tilde{\lambda}^E_{3,4} = \pm\sqrt{\frac{2D + 7 - \sqrt{(2D+7)^2 - 3(D+2)^2}}{D+2}},$$
$$\tilde{\lambda}^E_{5,6} = \pm\sqrt{\frac{2D + 7 + \sqrt{(2D+7)^2 - 3(D+2)^2}}{D+2}}.$$

Since all the characteristic velocities are real and it is possible to verify that there exists a basis of eigenvectors, the hyperbolicity requirement is satisfied at equilibrium. By continuity, it is possible to show that the hyperbolicity property is valid at least in a neighborhood of the equilibrium state.

Due to the decomposition of the characteristic polynomial (13), an explicit expression of the boundary equation is obtained through (10) (for the sake of simplicity we omit the apex $^{(1)}$ in the coefficients $a_l^{(1)}$):

$$\tilde{q}_1 = \pm \frac{(D+2)^2(2a_2 + \sqrt{a_2^2 + 12a_0})\sqrt{-a_2 + \sqrt{a_2^2+12a_0}}}{18\sqrt{6}(D+5)} \tag{14}$$

where for a prescribed value of D the maximum value of the 11-component of the dimensionless viscous tensor belonging to the hyperbolicity region reads[2]

$$\tilde{\sigma}_{11} \le s_{11} = \frac{b_1 - \sqrt{6b_2}}{(D+1)^2(D-4)^2}, \tag{15}$$

with

$$b_1 = D^4 + 8D^3 + 17D^2 + 28D, \quad b_2 = D^7 + 26D^6 + 85D^5 + 50D^4 - 32D^3 + 32D^2.$$

The hyperbolicity domain in the $(\tilde{q}_1, \tilde{\sigma}_{11})$-plane is bounded by the two curves given by (14) and is represented for different values of D in Figure 1. The star indicates the local equilibrium point. We observe that the hyperbolicity region enlarges when D increases and the smallest region is the one of the monatomic gas.

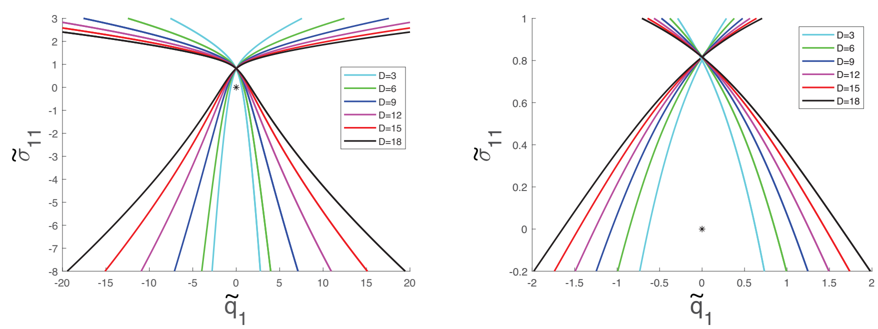

Figure 1: Hyperbolicity region for the $\mathrm{ET}^1_{14,P}$ theory in the case of one-dimensional field variables, for different values of the number of degrees of freedom, D. The figure on the right shows a zoom of the figure on the left in the neighborhood of the equilibrium point (symbolized by a star).

[2]The ostensible singularity for $D = 4$ in (15) can be easily overcome. In fact, it holds that $\lim_{D\to 4} s_{11} = 13/16$.

6. The hyperbolicity region of $\mathbf{ET}^2_{14,P}$ for one-dimensional field variables

We pass now to the quadratic theory $\mathrm{ET}^2_{14,P}$, still in the case of one-dimensional field variables. The flux components with $\alpha = 2$ are in the present case:

$$\begin{aligned}
\mathbf{F}^{1,(2)} =& \mathbf{F}^{1,(1)} + \left(0, 0, \frac{4q_1(3(D+2)\Pi - 2(D-3)\sigma_{11})}{(D+2)^2 p}, -\frac{12Dq_1\sigma_{11}}{(D+2)^2 p}\right), \\
\mathbf{G}^{1,(2)} =& \mathbf{G}^{1,(1)}_{ll} + \left(0, \frac{2\sigma_{11}^2}{\rho} + \frac{4q_1((5D+22)q_1 - 6D\sigma_{11}v_1)}{(D+2)^2 p}\right).
\end{aligned} \tag{16}$$

Referring to (11), (12) and (16), the corresponding characteristic polynomial is easily determined

$$\mathcal{P}^{(2)}_{1D}(\tilde{\lambda}) = -\frac{k_{\mathcal{B}}}{m}\theta p^2 \frac{6}{(D+2)^5}((D+2)\tilde{\lambda} - 4\tilde{q}_1)(\sum_{j=0}^{5} a_j^{(2)}\tilde{\lambda}^j)$$

with

$$\begin{aligned}
a_5^{(2)} &= D(2+D)^4, \qquad a_4^{(2)} = -8D(D+2)^2(11+4D)\tilde{q}_1, \\
a_3^{(2)} &= -12D^2(D+2)^2\tilde{\sigma}_{11}^2 + 2D(2+D)^2(14+17D+2D^2)\tilde{\sigma}_{11} \\
&- 2D(D+2)^3(7+2D) + 4(22+5D)(4+4D+13D^2)\tilde{q}_1^2, \\
a_2^{(2)} &= -12D\tilde{q}_1(8(D^2-2D-2)\tilde{\sigma}_{11}^2 + (2+D)(60+28D+5D^2)\tilde{\sigma}_{11} \\
&+ 4(22+5D)\tilde{q}_1^2 - 3(2+D)^2(7+2D)) \\
a_1^{(2)} &= 12D\tilde{q}_1^2(-4(D+2)(4D+17) + (268+84D+11D^2)\tilde{\sigma}_{11} \\
&+ 4(-22+D)\tilde{\sigma}_{11}^2) + 3D(D+2)^4(\tilde{\sigma}_{11}-1)^2, \\
a_0^{(2)} &= -12D(D+2)^2\tilde{q}_1(2+D-2\tilde{\sigma}_{11})(\tilde{\sigma}_{11}-1)^2.
\end{aligned}$$

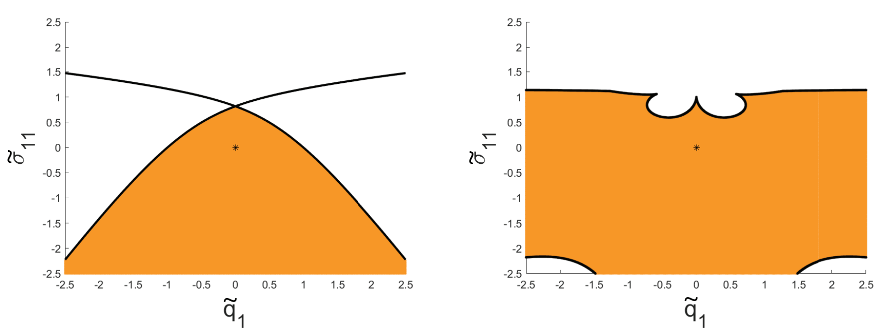

Figure 2: The hyperbolicity regions of the $\mathrm{ET}^1_{14,P}$ theory (on the left) and of $\mathrm{ET}^2_{14,P}$ theory (on the right) in the case of one-dimensional field variables when $D = 8$, the equilibrium point is symbolized by a star.

At equilibrium the roots of $\mathcal{P}^{(2)}_{1D}(\tilde{\lambda}) = 0$ coincide with those of the first-order expansion and the same considerations about hyperbolicity at equilibrium holds even in this case. Beyond the equilibrium state, the characteristic polynomial can be decomposed in the product of a first-degree polynomial and a fifth-degree polynomial that cannot be factorized analytically. Therefore, for the quadratic theory it is not possible to derive the explicit equations of the boundaries of the hyperbolicity domain. Here, we refer to the analytical technique already introduced in (10), equipped with some numerical steps and also to the direct numerical calculation described in Section 4. Figure 2 compares the hyperbolicity domain when $D = 8$ for $\mathrm{ET}^1_{14,P}$ and $\mathrm{ET}^2_{14,P}$ theories. It is evident that the hyperbolicity regions of the two theories exhibit a completely different topology. Since the linear and the quadratic theories are both approximated in the neighborhood of an equilibrium state, it is natural to start from such a neighborhood. However, we remark that the most relevant difference between the two hyperbolicity domains is that no boundaries can be found along the q_1-axis (when σ_{11}=0) for the second-order theory, while the first-order one presents boundaries plotted in Figure 2. Therefore, we can conclude somehow that $\mathrm{ET}^2_{14,P}$ presents always "larger" neighborhoods of the equilibrium point with respect to $\mathrm{ET}^1_{14,P}$, as it was already observed for monatomic gases ($D = 3$) described by 13-moment RET theory [12].

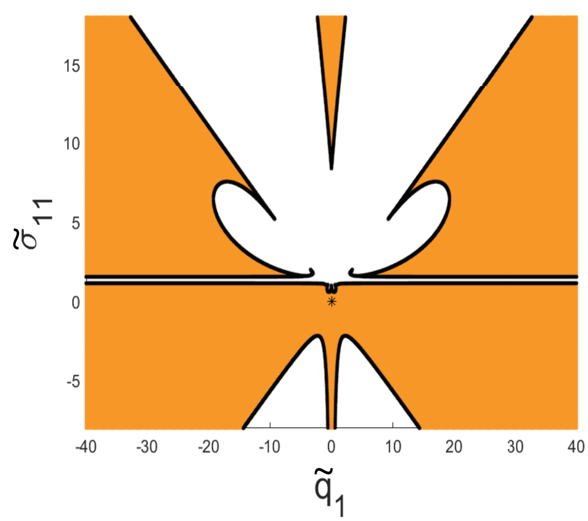

Figure 3: An overall view of the hyperbolicity region of the $\mathrm{ET}^2_{14,P}$ theory in the case of one-dimensional field variables when $D = 8$.

Just to study the hyperbolicity region of the system beyond the validity of the expansion in the non-equilibrium variables, we present in Figure 3 an overall view of the hyperbolicity region of $\mathrm{ET}^2_{14,P}$ with $D = 8$, for large values of the non-equilibrium variables.

Concerning the asymptotic behavior of the hyperbolicity domain, when $D \to \infty$, the hyperbolicity regions of the linear and the quadratic expansions tend both to the half space $\tilde{\sigma}_{11} \leq 1$, but of course this is a mathematical more than a physical curiosity.

7. The hyperbolicity region of $\mathbf{ET}^1_{14,P}$ for three-dimensional field variables

In this section we consider all the 14 field components $\mathbf{u} = (\rho, v_j, T, \Pi, \sigma_{<ij>}, q_i)$ for $i, j = 1, 2, 3$ assuming that such quantities still depend on time and only on the space variable x_1.

In the previous sections we have observed that in the case of one-dimensional variables the characteristic polynomial depends on $\sigma_{<11>}$ and Π only in the combination $\sigma_{<11>} - \Pi$, that is σ_{11}. Therefore, in order to compare the present results with the previous ones it is convenient to refer to the following 14 variables $\mathbf{u} = (\rho, v_j, T, \Pi, \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{22}, \sigma_{23}, q_i)$. In σ_{ij} we have omitted σ_{33}, since it is not independent, due to the constraint $\sigma_{ll} = -3\Pi$ that implies:

$$\sigma_{33} = -\left(\sigma_{11} + \sigma_{22} + 3\Pi\right).$$

The density vectors are:

$$\begin{aligned} &\mathbf{F} \equiv \left(F, F_1, F_2, F_3, F_{ll}, F_{11}, F_{22}, F_{12}, F_{13}, F_{23}, \right), \\ &\mathbf{G}_{ll} \equiv \left(G_{ll}, G_{1ll}, G_{2ll}, G_{3ll}\right), \end{aligned} \tag{17}$$

and the corresponding flux vectors in the linear theory $\mathrm{ET}^1_{14,P}$

$$\begin{aligned} &\mathbf{F}^{1,(1)} \equiv \left(F_1, F_{11}, F_{12}, F_{13}, F^{(1)}_{1ll}, F^{(1)}_{111}, F^{(1)}_{122}, F^{(1)}_{112}, F^{(1)}_{113}, F^{(1)}_{123}, \right), \\ &\mathbf{G}^{1,(1)}_{ll} = \left(G_{1ll}, G^{(1)}_{11ll}, G^{(1)}_{12ll}, G^{(1)}_{13ll}\right), \end{aligned}$$

if the following relations are taken into account

$$\begin{aligned} &F = \rho; \quad F_i = \rho v_i; \quad F_{ll} = 3p + 3\Pi + \rho v^2; \quad F_{ij} = p\delta_{ij} - \sigma_{ij} + \rho v_i v_j; \\ &G_{ll} = Dp + \rho v^2; \quad G_{ill} = 2q_i + (D+2)p v_i - 2\sigma_{il} v_l + \rho v_i v^2; \\ &F^{(1)}_{1ll} = \frac{10 q_1}{(D+2)} + (5p + 3\Pi)v_1 - 2\sigma_{1l} v_l + \rho v_1 v^2; \\ &F^{(1)}_{1ij} = \frac{2(q_1\delta_{ij} + q_i\delta_{1j} + q_j\delta_{1i})}{D+2} - [(p\delta_{ij} - \sigma_{ij})v_1 + (p\delta_{1i} - \sigma_{1i})v_j + \\ &+ (p\delta_{1j} - \sigma_{1j})v_i] + \rho v_1 v_i v_j; \\ &G^{(1)}_{1ill} = -\frac{(D+4)p\sigma_{1i}}{\rho} + \frac{(D+2)p^2}{\rho}\delta_{1i} + \frac{4q_l v_l \delta_{1i} + 2(D+4)(q_1 v_i + q_i v_1)}{D+2} + \\ &+ p v^2 \delta_{1i} + (D+4)p v_1 v_i - \sigma_{1i} v^2 - 2(\sigma_{1l} v_i v_l + \sigma_{il} v_1 v_l) + \rho v^2 v_1 v_i. \end{aligned}$$

Also in the present case, the non-singularity of matrix $\mathbf{A}$ is easily verified: $\det(\mathbf{A}) = -24 D k_{\mathcal{B}} \rho^4 / m$. So, one has to focus on the generalized eigenvalue problem. We omit here the analytic complete expression of the characteristic polynomial, due to its complicated structure, but we recall that the eigenvalues at equilibrium are all real, although not all distinct [2]:

$$\tilde{\lambda}^E_{1-6} = 0, \quad \tilde{\lambda}^E_{7,8} = -\sqrt{\tfrac{D+4}{D+2}}, \quad \tilde{\lambda}^E_{9,10} = \sqrt{\tfrac{D+4}{D+2}},$$

$$\tilde{\lambda}^E_{11,12} = \pm\sqrt{\tfrac{2D+7-\sqrt{D^2+16D+37}}{2D}}, \qquad \tilde{\lambda}^E_{13,14} = \pm\sqrt{\tfrac{2D+7+\sqrt{D^2+16D+37}}{2D}}.$$

The hyperbolicity property at equilibrium is satisfied, since there exists a basis of independent eigenvectors. For polyatomic gases described by 14 moments, the hyperbolicity region is a domain in the 9-dimensional space of the non-equilibrium variables. If we consider its 2-dimensional sections, obtained under the assumption of only two non-vanishing non-equilibrium variables, we deal with 36 different sections overall. Here, we will present only some peculiar cases.

7.1. The $(\tilde{q}_1, \tilde{\sigma}_{12})$-section in $ET^1_{14,P}$

We start with the $(\tilde{q}_1, \tilde{\sigma}_{12})$-section, already studied for rarefied monatomic gases in [10, 11, 12]. When the only non-zero variables are $\hat{q}_1$ and $\hat{\sigma}_{12}$, the characteristic polynomial associated to $\mathrm{ET}^1_{14,P}$ reads

$$\mathcal{P}^{(1)}_{3D}|_{q_1,\sigma_{12}} = -\frac{24 k_{\mathcal{B}} p^4 \vartheta^3}{m(D+2)^6}\tilde{\lambda}^3\Big[(D+2)^2\tilde{\lambda}^3 - (D+2)(D+4)\tilde{\lambda} - 2(D+4)\tilde{q}_1\Big]\Big[\sum_{j=0}^{8} c^{(1)}_j \tilde{\lambda}^j\Big],$$

where

$$c^{(1)}_8 = D(D+2)^4, \quad c^{(1)}_7 = 0, \quad c^{(1)}_6 = -D(D+2)^3(5D+18),$$
$$c^{(1)}_5 = -2D(D+2)^2(7D+34)\tilde{q}_1,$$
$$c^{(1)}_4 = D(D+2)^2(68+42D+7D^2+12(1+D)\tilde{\sigma}^2_{12}),$$
$$c^{(1)}_3 = 4D(D+2)(D+4)(22+5D)\tilde{q}_1,$$
$$c^{(1)}_2 = 3D(D+4)(-(D+2)^3+8(D+5)\tilde{q}^2_1) - 12(D+2)^2(-8+(D-1)D)\tilde{\sigma}^2_{12},$$
$$c^{(1)}_1 = -6(2+D)(4+D)\tilde{q}_1(D(2+D)-4(5+D)\tilde{\sigma}^2_{12}),$$
$$c^{(1)}_0 = -12(D+2)^2(D+4)\tilde{\sigma}^2_{12}.$$

Cai, Fan and Li [10, 11] have noticed that in the case of a monatomic gas described by ET^1_{13} theory, the equilibrium point belongs to the boundary of the hyperbolicity domain. The same result holds also in the case of $\mathrm{ET}^1_{14,P}$ theory for polytropic gases. Thus, the stability of the equation system is not guaranteed even locally. In Figure 4, on the left, the $(\tilde{q}_1, \tilde{\sigma}_{12})$-section is shown when $D = 8$.

7.2. The $(\tilde{\sigma}_{12}, \tilde{\sigma}_{13})$-section in $ET^1_{14,P}$

For a simple analytical study of this phenomenon we consider the case in which all the non-equilibrium field components vanish except $\tilde{\sigma}_{12}$ and $\tilde{\sigma}_{13}$ and introduce, as for the case of monatomic gases [12], the quantity $\tilde{\sigma}^2 = \tilde{\sigma}_{12}^2 + \tilde{\sigma}_{13}^2$. Then, the characteristic polynomial reduces to the form

$$\mathcal{P}^{(1)}_{3D}|_{\sigma_{12},\sigma_{13}}(\tilde{\lambda}) = -\tfrac{24k_{\mathcal{B}}\theta^3 p^4}{(2+D)^3 m}\tilde{\lambda}^4\Big[(D+2)\tilde{\lambda}^2 - 4 - D\Big]\Big[d^{(1)}_8\tilde{\lambda}^8 + d^{(1)}_6\tilde{\lambda}^6 + d^{(1)}_4\tilde{\lambda}^4 + d^{(1)}_2\tilde{\lambda}^2 + d^{(1)}_0\Big], \tag{18}$$

where

$$\begin{aligned}
d^{(1)}_8 &= D(2+D)^2\\
d^{(1)}_6 &= -D(2+D)(18+5D)\\
d^{(1)}_4 &= 68D + 42D^2 + 7D^3 + 12D\tilde{\sigma}^2(1+D),\\
d^{(1)}_2 &= -3D(D+2)(D+4) + 12(8+D-D^2)\tilde{\sigma}^2\\
d^{(1)}_0 &= -12(4+D)\tilde{\sigma}^2.
\end{aligned}$$

Taking into account that $d^{(1)}_0 \leq 0$, the polynomial contained in the second square brackets presents real roots if and only if $d^{(1)}_0$ vanishes, that is to say $\tilde{\sigma}^2 = 0$. In this way, the section of the hyperbolicity region in the $(\tilde{\sigma}_{12}, \tilde{\sigma}_{13})$-plane reduces only to the equilibrium point. Hence, there does not exist a neighborhood of the equilibrium state enclosed in the hyperbolicity domain.

The singularity is also observed in some bi-dimensional sections of the hyperbolicity domains that involve the dynamic pressure.

7.3. The $(\tilde{\sigma}_{12}, \Pi)$-section in $ET^1_{14,P}$

As an example, we focus here on the $(\tilde{\sigma}_{12}, \tilde{\Pi})$-section. Referring to the $\mathrm{ET}^1_{14,P}$ theory and assuming that all the non-equilibrium variables vanish except Π and σ_{12}, the characteristic polynomial reduces to

$$\mathcal{P}^{(1)}_{3D}|_{\Pi,\sigma_{12}} = -\frac{24k_{\mathcal{B}}p^4\vartheta^3}{(D+2)^3 m}\tilde{\lambda}^4\mathcal{P}_2(\tilde{\lambda})\mathcal{P}_8(\tilde{\lambda})$$

with

$$\begin{aligned}
\mathcal{P}^{(1)}_2(\tilde{\lambda}) &= (2+D)\tilde{\lambda}^2 - (D+4-6\tilde{\Pi}),\\
\mathcal{P}^{(1)}_8(\tilde{\lambda}) &= g^1_8\tilde{\lambda}^8 + g^{(1)}_6\tilde{\lambda}^6 + g^{(1)}_4\tilde{\lambda}^4 + g^{(1)}_2\tilde{\lambda}^2 + g^{(1)}_0,
\end{aligned}$$

and

$$\begin{aligned}
g^{(1)}_8 &= D(D+2)^2, \qquad g^{(1)}_6 = -D(D+2)(5D+18),\\
g^{(1)}_4 &= D(12\tilde{\sigma}_{12}^2(D+1)) + 68 + 42D + 7D^2),\\
g^{(1)}_2 &= -3(D^3 + 6D^2 + 8D + \tilde{\sigma}_{12}^2(4D^2 - 4D - 32)), \quad g^{(1)}_0 = -12(4+D)\tilde{\sigma}_{12}.
\end{aligned}$$

The necessary condition in order to have real roots of the quadratic polynomial $\mathcal{P}_8^{(1)}$ is that $g_0^{(1)} \geq 0$ and this requirement is satisfied only if $\tilde{\sigma}_{12} = 0$. Moreover, the roots of $\mathcal{P}_2^{(1)}$ are real only for $\tilde{\Pi} \leq (D+4)/6$. In other words, the (Π, σ_{12})-section of the hyperbolicity region for $\mathrm{ET}^1_{14,P}$ theory coincide with a half line containing the equilibrium point. This fact confirms once more the instability of the $\mathrm{ET}^1_{14,P}$ balance system.

8. The hyperbolicity region of $\mathbf{ET}^2_{14,P}$ for three-dimensional field variables

To overcome the singularity described in the previous examples, we pass to the quadratic theory $\mathrm{ET}^2_{14,P}$, as already done for a rarefied monatomic gas in [12]. In the case of $\mathrm{ET}^2_{14,P}$ the density vector is (17), while for the flux vector it holds

$$
\begin{aligned}
\mathbf{F}^{1,(2)}\Big|_{3D} &= \mathbf{F}^{1,(1)}\Big|_{3D} + \left(0,0,0,0,\Delta F^{(2)}_{1ll}, \Delta F^{(2)}_{111}, \Delta F^{(2)}_{122}, \Delta F^{(2)}_{112}, \Delta F^{(2)}_{113}, \Delta F^{(2)}_{123}\right),\\
\mathbf{G}^{1,(2)}\Big|_{3D} &= \mathbf{G}^{1,(1)}\Big|_{3D} + \left(0, \Delta G^{(2)}_{11ll}, \Delta G^{(2)}_{12ll}, \Delta G^{(2)}_{13ll}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta F^{(2)}_{1ll} &= \frac{-8(D-3)\sigma_{1l}q_l + 12(D+2)\Pi q_1}{(D+2)^2 p};\\
\Delta F^{(2)}_{1ij} &= \frac{8(\sigma_{1l}q_l\delta_{ij} + \sigma_{il}q_l\delta_{1j} + \sigma_{jl}q_l\delta_{1i})}{(D+2)^2 p} - \frac{4(q_1\sigma_{ij} + q_i\sigma_{1j} + q_j\sigma_{1i})}{(D+2)p};\\
\Delta G^{(2)}_{1ill} &= \frac{2\sigma_{1l}\sigma_{il}}{\rho} + \frac{16(D+4)q_1q_i + 4(D+6)q^2\delta_{1i}}{(D+2)^2 p} + \frac{16}{(D+2)^2 p}(\sigma_{il}q_l v_1 + \sigma_{1l}q_l v_i +\\
&\quad + \sigma_{lm}q_l v_m \delta_{1i}) - \frac{8}{(D+2)p}(\sigma_{1i}q_l v_l + \sigma_{1l}v_l q_i + \sigma_{il}v_l q_1).
\end{aligned}
$$

The calculation of matrices $\mathbf{A}$ and $\mathbf{B}$ and the use of the dimensionless variables (12) allows us to analyze the generalized eigenvalue problem in a rather simple way. Nevertheless, as for the linear approximation, we deal with a polynomial that presents a very complicated structure, so we do not report here its complete expression. We have analyzed all the 36 bi-dimensional sections using both the semi-analytical and numerical approaches. Some of these bi-dimensional sections are shown in the following figures. Also in this case, the existence of a basis of eigenvalues was numerically verified.

8.1. The $(\tilde{q}_1, \tilde{\sigma}_{12})$-section in $ET^2_{14,P}$

In particular, we present the section (q_1, σ_{12}) of the hyperbolicity region of $\mathrm{ET}^2_{14,P}$ in Figure 4 (notice that different axis scales are used in the two images of Figure 4). For the second-order model there exists a large neighborhood of the equilibrium point, completely contained in the hyperbolicity region. The complete expression of the characteristic polynomial in the case that the unique

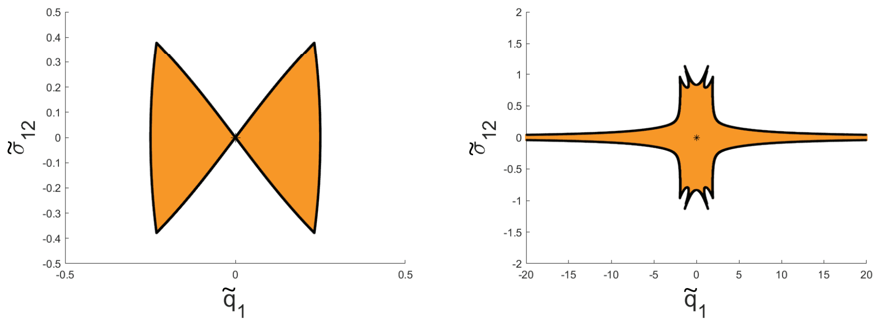

Figure 4: The $(\tilde{q}_1, \tilde{\sigma}_{12})$-section of the hyperbolicity regions of the $\mathrm{ET}^1_{14,P}$ theory (figure on the left) and of the $\mathrm{ET}^2_{14,P}$ theory (figure on the right) in the case of three-dimensional field variables when $D = 8$. Notice that the axis scale of the two figures is very different.

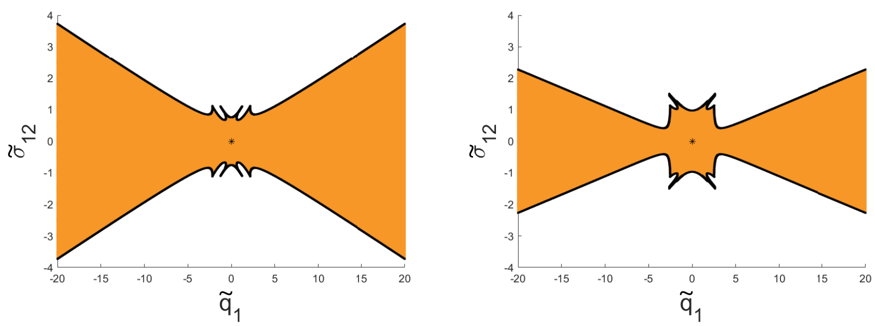

Figure 5: The $(\tilde{q}_1, \tilde{\sigma}_{12})$-section of the hyperbolicity region of the $\mathrm{ET}^2_{14,P}$ theory in the case of three-dimensional field variables when $D = 6$ (figure on the left) and $D = 12$ (figure on the right).

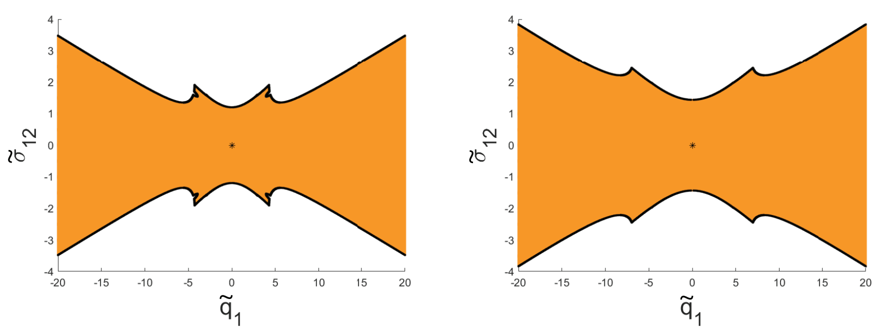

Figure 6: The $(\tilde{q}_1, \tilde{\sigma}_{12})$-section of the hyperbolicity region of the $\mathrm{ET}^2_{14,P}$ theory in the case of three-dimensional field variables when $D = 20$ (figure on the left) and $D = 30$ (figure on the right).

non-equilibrium variables are (q_1, σ_{12}) can be found in Appendix Appendix A.1.

The dependence of the hyperbolicity region on D is analyzed in Figures 5 and 6, that show the bi-dimensional section in the plane $(\tilde{q}_1, \tilde{\sigma}_{12})$ for different values of D. We remark that in order to analyze the role of D, we have chosen an axis scale that goes beyond the validity of the Taylor expansion. Far from the equilibrium state the region seems to restrict passing from $D = 6$ to $D = 8$, but for further increasing values of D the region begins to enlarge indefinitely.

8.2. The $(\tilde{\sigma}_{12}, \tilde{\sigma}_{13})$-section in $ET^2_{14,P}$

To consider again a case that can be treated analytically and compared with the results of (18) and those in [12], we focus on the characteristic polynomial obtained under the assumption that only $\tilde{\sigma}_{12}, \tilde{\sigma}_{13}$ do not vanish. Thanks to its simple mathematical structure (see Appendix A.2), it is possible to show that the polynomial presents all real roots if and only if

$$\tilde{\sigma}^2 = \tilde{\sigma}_{12}^2 + \tilde{\sigma}_{13}^2 \leq \tilde{\sigma}_0^2, \quad \text{with } \tilde{\sigma}_0^2 = \frac{(2+D)^2}{16(1+D)}.$$

Therefore, the hyperbolicity domain is a circle of increasing radius σ_0 for increasing value of D. The minimum value od σ_0 is $5/8$, that corresponds to the monatomic limit $D \to 3$ described by [12], while for an indefinite number of degrees of freedom $\tilde{\sigma}_0 \to \infty$. Hence an increasing number of degrees of freedom coincide with an enlarging $(\tilde{\sigma}_{12}, \tilde{\sigma}_{12})$-section of the hyperbolicity region.

8.3. The $(\tilde{\sigma}_{12}, \tilde{\Pi})$-section in $ET^2_{14,P}$

As already observed for the previous cases, the hyperbolicity region of the quadratic theory turns out to be larger with respect to the corresponding one of the linear theory, even for what concerns the sections involving Π . In fact, keeping in mind the $(\tilde{\sigma}_{12}, \tilde{\Pi})$-section studied in the previous section, we deal with the following analytic equation of the section of the hyperbolicity domain, that was deduced in Appendix A.3:

$$\begin{aligned} &-\frac{D+2}{12} \leq \tilde{\Pi} \leq \tilde{\Pi}^*, \qquad \text{with} \quad -\frac{D+2}{4\sqrt{D+1}} \leq \tilde{\sigma}_{12} \leq \frac{D+2}{4\sqrt{D+1}}, \\ &\tilde{\Pi}^* = \frac{1}{24}\left[(D^2+6D+8)(1+2\tilde{\sigma}_{12}^2) - 2\sqrt{(D+2)^3(D+6)(\tilde{\sigma}_{12}^4+\tilde{\sigma}_{12}^2)}\right]. \end{aligned} \tag{19}$$

Moreover we have also the half line

$$\tilde{\sigma}_{12} = 0, \qquad \text{and} \quad \tilde{\Pi} \leq -\frac{D+2}{12}. \tag{20}$$

We observe the great difference of the hyperbolicity region in the second order with respect the linear case where we have only the half line $\tilde{\Pi} \leq (D+4)/6$. Moreover the size of the domain at second order increases with D. Figure 7 shows the domain of the $(\tilde{\sigma_{12}}, \tilde{\Pi})$-section of the hyperbolicity domain for $\mathrm{ET}^2_{14,P}$ when $D = 12$. The half line including the dashed line corresponds to the linear case.

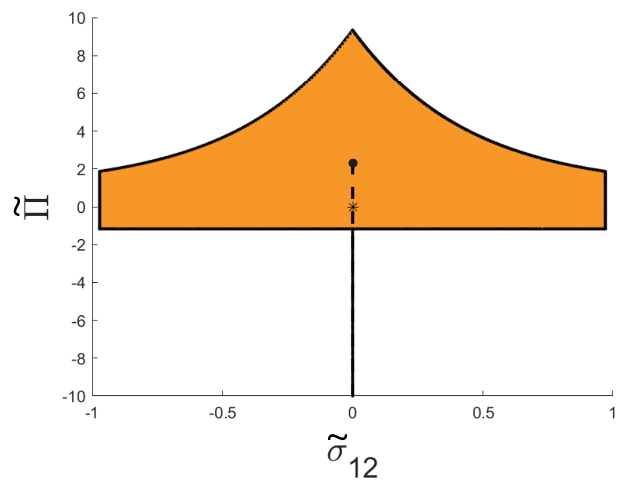

Figure 7: The $(\tilde{\sigma}_{12}, \tilde{\Pi})$-section of the hyperbolicity region of the $\mathrm{ET}^2_{14,P}$ theory in the case of three-dimensional field variables when $D = 12$, the $*$ symbolizes the equilibrium point. The domain of hyperbolicity in the linear case is the half line starting from the black circle.

9. Monatomic singular limit

For the linear theory $\mathrm{ET}^1_{14,P}$ it was proved in [22] that when $D \to 3$ the equation system converges to the corresponding system of a monatomic gas. In truth, the equation for the dynamic pressure Π still exists if $D = 3$, but it admits as unique solution $\Pi = 0$ for any time if initial data compatible with a monatomic gas are imposed, i. e. $\Pi(0, x_i) = 0$. The remaining 13 equations converge to the ET^1_{13} balance laws for a monatomic gas (Grad's equations).

In what follows we are going to prove the same property also for the second-order theory. $\mathrm{ET}^2_{14,P}$. In fact, if we consider the trace of equation $(9)_3$, introducing the material derivative and proceeding in the same way as in [22, 1] it is easly verified that the equation $(9)_3$ reduces to:

$$\begin{aligned} \dot{\Pi} + &\left(\frac{2(D-3)}{3D} p + \frac{5D-6}{3D}\Pi \right) \frac{\partial v_k}{\partial x_k} - \frac{2(D-3)}{3D} \frac{\partial v_{\langle i}}{\partial x_{k\rangle}} \sigma_{\langle ik \rangle} \\ &+ \frac{4(D-3)}{3D(D+2)} \frac{\partial q_k}{\partial x_k} + \underline{\frac{4}{3} \frac{1}{(D+2)^2} \frac{\partial}{\partial x_k} \left\{ \frac{2(3-D)}{p} \sigma_{<kl>} q_l + \frac{5D}{p} \Pi q_k \right\}} \\ &= -\frac{\Pi}{\tau}, \end{aligned} \tag{21}$$

where we have chosen as $P_{ll} = -3\Pi/\tau$, and moreover, the dot indicates the material derivative, τ is a relaxation time and the underlined terms are the ones of second order in nonequilibrium variables. In particular, when we assume

$D = 3$ the previous equation (21) becomes:

$$\dot{\Pi} + \underline{\frac{4q_k}{5p}\frac{\partial \Pi}{\partial x_k}} = -\Pi \left\{ \frac{1}{\tau} + \frac{\partial v_k}{\partial x_k} + \underline{\frac{4}{5}\frac{\partial}{\partial x_k}\left(\frac{q_k}{p}\right)} \right\}. \tag{22}$$

As already recalled, in the monatomic singular limit we have to prescribe initial data compatible with the monatomic gas, that is to say $\Pi(0, x_k) = 0$. So, under the hypothesis of the uniqueness of the solution, the equation (22) admits only the solution $\Pi(t, x_k) = 0$ for any $t > 0$. Besides the equation (22) that is surely satisfied, the remaining 13 equations of (9) converge for $D = 3$ to the ones of second order theory of monatomic gas that we have deduced in a previous paper [12].

9.1. Monatomic limit of the hyperbolicity domain

The question about the monatomic limit concerns also the hyperbolicity regions. We start recalling that in the case of one-dimensional field variables, the characteristic polynomial does not depend explicitly con Π but only on the (1,1)-component of viscous tensor $\sigma_{11} = \sigma_{<11>} - \Pi$ that in the monatomic limit converges to $\sigma_{<11>}$., This property makes the monatomic limit very simple. In fact, referring to [1, 12] it is easily verified that for the characteristic polynomial (13) of the linear theory $\mathrm{ET}^1_{14,P}$ in the case of one-dimensional variables, it holds

$$\lim_{D\to 3} \mathcal{P}^{(1)}_{1D}(\tilde{\lambda}) = 3\theta^{1/2}\tilde{\lambda}\mathcal{P}^{(1)}_{1D,ET_{13}}(\tilde{\lambda}),$$

where $\mathcal{P}^{(1)}_{1D,ET_{13}}$ denotes the corresponding characteristic polynomial for the ET^1_{13} theory of monatomic gases. Obviously, also (14) reduces to the corresponding equations of the boundaries of the hyperbolicity region of the linear 13-moment system.

An analogous result can be found also for the quadratic theory in the case of one-dimensional field variables, since the monatomic limit gives:

$$\lim_{D\to 3} \mathcal{P}^{(2)}_{1D}(\tilde{\lambda}) = 3\left(\tilde{\lambda} - \frac{4\tilde{q}_1}{5}\right)\mathcal{P}^{(2)}_{1D,ET_{13}}(\tilde{\lambda});$$

denoting by $\mathcal{P}^{(2)}_{1D,ET_{13}}$ the corresponding characteristic polynomial for ET^2_{13} theory [12]. We remark that the above mentioned hyperbolicity regions present their minimum when $D = 3$.

The results can be extended to the hyperbolicity domains of the complete three-dimensional set of field variables, but now we deal with characteristic polynomials that depend explicitly on Π. Comparing the results for monatomic gases [12] and those obtained in the previous Sections, we conclude that the limit $D \to 3$ when $\Pi \to 0$ is in agreement with the previuos results except for the bi-dimensional sections that involve the Π variable. In fact, such sections gain meaning only when $D > 3$.

10. Conclusions

In this paper we derived a RET model of rarefied polytropic gases with 14 moments, approximated at the second-order in the neighborhood of an equilibrium state. The hyperbolicity property of such balance laws is examined starting from a general analysis of the hyperbolicity regions of $\mathrm{ET}^{\alpha}_{14,P}$ (with $\alpha = 1, 2$). Some results obtained in this paper for a polytropic rarefied gas can be viewed as an extension and a generalization of those obtained in [12] for a monatomic rarefied gas. In particular, it was possible to verify that the hyperbolicity domain of $\mathrm{ET}^{2}_{14,P}$ is larger than the corresponding one for $\mathrm{ET}^{1}_{14,P}$, at least in the neighborhood of the equilibrium state. Moreover, in the case of three-dimensional variables, the first-order theory presents a singularity similar to that detected by Cai, Fan and Li for a monatomic gas. Such a singularity is overcome in a natural way by the second-order theory, without loss of validity of the entropy principle and keeping the balance law structure of the PDE system.

Beside these outcomes, we obtained results that are strictly connected to polytropic gases. On one hand, we recognize the relevance of the role played by the number of degrees of freedom of the gas molecules. On the other hand, we found out that the dynamic pressure influences directly the hyperbolicity property only in the case of three-dimensional variables.

The monatomic singular limit of the polyatomic theory is also successfully investigated together with a comparative analysis of monatomic and polyatomic hyperbolicity regions.

In the case of three-dimensional field variables, the hyperbolicity region of $\mathrm{ET}^{\alpha}_{14,P}$ is a nine-dimensional domain. To compare its amplitude for different values of D, one could define the maximum radius of the hypersphere centered in the equilibrium state and enclosed in such a domain. Unfortunately, the determination of the radius and of its dependence on D requires very heavy numerical calculations and for this reason it will be postponed to a future work.

It will be interesting to analyze the domain of hyperbolicity in the non-polytropic case too, starting from the recent paper by Ruggeri that obtained the differential system at first-order referring to the MEP procedure [23]. This will be the subject of a successive paper.

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Appendix A. Characteristic polynomial of $\mathbf{ET}^{2}_{14,P}$ in the case of three-dimensional variables

The characteristic polynomial of $\mathrm{ET}^{2}_{14,P}$ for three dimensional field components exhibits a very complicated structure. In this Appendix we present its explicit expression in some special cases for only two non-vanishing non-equilibrium variables.

Appendix A.1. Characteristic polynomial in the $(\tilde{q}_1, \tilde{\sigma}_{12})$-section in $ET^2_{14,P}$

First of all, let us focus on the case when only the variables q_1 and σ_{12} are not zero, in this case the polynomial reduces to

$$\mathcal{P}^{(2)}_{3D}|_{q_1,\sigma_{12}} = -\frac{24k_{\mathcal{B}}p^4\vartheta^3}{(D+2)^{15}m}[(D+2)\tilde{\lambda} - 4\tilde{q}_1]\mathcal{P}_4(\tilde{\lambda})\mathcal{P}_9(\tilde{\lambda})$$

with

$$\begin{aligned}
\mathcal{P}_4(\tilde{\lambda}) =& \sum_{k=0}^{4} f_k^{(2)}\tilde{\lambda}^k,\\
f_4^{(2)} =& -(2+D)^5; \quad f_3^{(2)} = -4(D+2)^3(12+5D)\tilde{q}_1;\\
f_2^{(2)} =& 32(D+2)(18+D(19+4D))\tilde{q}_1^2 - (D+2)^4(4\tilde{\sigma}_{12}^2 + D + 4);\\
f_1^{(2)} =& -256(D+1)(D+4)\tilde{q}_1^3 + [10(D+2)^3(D+4) + 32(D+2)(D+1)\tilde{\sigma}_{12}^2]\tilde{q}_1;\\
f_0^{(2)} =& -24(D+2)^2(D+4)\tilde{q}_1^2 + 2(D+2)^4\tilde{\sigma}_{12}^2;
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{P}_9(\tilde{\lambda}) =& \sum_{j=0}^{9} c_j^{(2)},\\
c_9^{(2)} =& D(D+2)^9, \quad c_8^{(2)} = -4D(D+2)^7(34+13D)\tilde{q}_1,\\
c_7^{(2)} =& (D+2)^5\left[-D(2+D)^3(18+5D) + 4(88+1308D+\right.\\
&\left. + 1282D^2 + 257D^3)\tilde{q}_1^2 - 16(1+D)(2+D)^2\tilde{\sigma}_{12}^2,\right]\\
c_6^{(2)} =& -2(D+2)^3\tilde{q}_1\left[-D(2+D)^3(804+572D+97D^2) + \right.\\
&+ 32(264+1324D+2314D^2+1117D^3+153D^4)\tilde{q}_1^2 -\\
&\left.- 16(2+D)^2(4+40D+57D^2+15D^3)\tilde{\sigma}_{12}^2\right]\\
c_5^{(2)} =& (D+2)\left[D(2+D)^6(68+42D+7D^2) - \right.\\
&- 4(2+D)^3(352+11256D+14396D^2+5610D^3+687D^4)\tilde{q}_1^2 +\\
&+ 64(3168+13216D+26984D^2+22220D^3+6976D^4+723D^5)\tilde{q}_1^4 +\\
&+ (6D(2+D)^4(128+156D+52D^2+7D^3) -\\
&- 64(2+D)^2(160+484D+996D^2+543D^3+75D^4)\tilde{q}_1^2)\tilde{\sigma}_{12}^2 +\\
&\left.+ 64D(1+D)^2(2+D)^4\tilde{\sigma}_{12}^4\right],
\end{aligned}$$

$$
\begin{aligned}
c_4^{(2)} =& -8\tilde{q}_1\left[D(2+D)^6(238+144D+23D^2)+\right.\\
&+64(22+5D)(32+180D+320D^2+219D^3+38D^4)\tilde{q}_1^4+\\
&+(2+D)^4(288+1824D+1988D^2+804D^3+103D^4)\tilde{\sigma}_{12}^2+\\
&+64(1+D)(2+D)^3(2+3D+5D^2+D^3)\tilde{\sigma}_{12}^4+\\
&+\tilde{q}_1^2(-(2+D)^3(1760+35496D+48412D^2+18726D^3+2211D^4)-\\
&\left.-16(2+D)(1152+5480D+8884D^2+6170D^3+1643D^4+131D^5)\tilde{\sigma}_{12}^2)\right];
\end{aligned}
$$

$$
\begin{aligned}
c_3^{(2)} =& -3D(2+D)^8(4+D)+12288D(1+D)(4+D)(22+5D)\tilde{q}_1^6-\\
&-2(2+D)^7(50+11D)\tilde{\sigma}_{12}^2+8D(2+D)^5(10+D)(14+11D)\tilde{\sigma}_{12}^4+\\
&+\tilde{q}_1^4(-192(2+D)^2(176+4492D+6320D^2+2381D^3+269D^4)-\\
&-512(1+D)(1104+4712D+3928D^2+766D^3+17D^4)\tilde{\sigma}_{12}^2)+\\
&+\tilde{q}_1^2(24D(2+D)^5(732+441D+68D^2)+16(2+D)^3(2440+3780D+\\
&+4090D^2+1935D^3+253D^4)\tilde{\sigma}_{12}^2-\\
&-256(1+D)(2+D)(-72-340D-290D^2-35D^3+11D^4)\tilde{\sigma}_{12}^4);
\end{aligned}
$$

$$
\begin{aligned}
c_2^{(2)} =& 6\tilde{q}_1\left[7D(D+2)^7(D+4)\ +2(D+2)^5(48-12D+52D^2+19D^3)\tilde{\sigma}_{12}^2-\right.\\
&-16(2+D)^4(20+268D+169D^2+32D^3)\tilde{\sigma}_{12}^4+\\
&+\tilde{q}_1^4(62D(2+D)(4+D)(676+768D+143D^2)+2048(1+D)(44+\\
&+126D+32D^2+D^3)\tilde{\sigma}_{12}^2)+\tilde{q}_1^2(-32D(2+D)^4(348+208D+31D^2)+\\
&+32(2+D)^2(-784+76D+96D^2+51D^3+14D^4)\tilde{\sigma}_{12}^2+\\
&\left.-512(1+D)(32+156D+104D^2+15D^3+D^4)\tilde{\sigma}_{12}^4)\right]
\end{aligned}
$$

$$
\begin{aligned}
c_1^{(2)} =& 12\left[32D(2+D)^3(4+D)\tilde{q}_1^2(-16D(2+D)^3+(59+20D)\tilde{q}_1^2)-\right.\\
&-(2+D)(-D(1+D)(2+D)^6-2(2+D)^3(16+596D+240D^2+17D^3)\tilde{q}_1^2+\\
&+64(-280+504D+526D^2+232D^3+31D^4)\tilde{q}_1^4)\tilde{\sigma}_{12}^2+\\
&+2(-D(2+D)^5(4+20D+13D^2)+32(2+D)^3(84+328D+169D^2+27D^3)\tilde{q}_1^2+\\
&\left.+2048(1+D)^2(18+D)\tilde{q}_1^4)\tilde{\sigma}_{12}^4-48(1+D)(2+D)^5\tilde{\sigma}_{12}^6\right];
\end{aligned}
$$

$$
\begin{aligned}
c_0^{(2)} =& 48(2+D)^2\tilde{q}_1\left[D(D+2)^2(-2(D+2)^2(D+1)\tilde{\sigma}_{12}^2+2(13D^2+20D+4)\tilde{\sigma}_{12}^4+\right.\\
&+48(D+1)\tilde{\sigma}_{12}^6)+\tilde{q}_1^2(D+4)(6D(D+2)^3-2(D+2)(5D+14)(7D+2)\tilde{\sigma}_{12}^2-\\
&\left.-128(D+1)(D+4)\tilde{\sigma}_{12}^4)\right].
\end{aligned}
$$

Appendix A.2. Characteristic polynomial in the $(\tilde{\sigma}_{12}, \tilde{\sigma}_{13})$-section in $ET^2_{14,P}$

For three-dimensional field variables, when $\tilde{\sigma}_{12}$ and $\tilde{\sigma}_{13}$ are the only non-vanishing non-equilibrium variables, the characteristic polynomial of $\mathrm{ET}^2_{14,P}$

presents the following quadratic form

$$\mathcal{P}^{(2)}_{3D}|_{\sigma_{12},\sigma_{13}}(\tilde{\lambda}) = -\frac{24Dk_{\mathcal{B}}p^4\theta^3}{(2+D)^5 m}\tilde{\lambda}^2\left[(D+2)\tilde{\lambda}^4 - (4\tilde{\sigma}^2 + D + 4)\tilde{\lambda}^2 + \right. \\ \left. +2\tilde{\sigma}^2\right]\left[d_8^{(2)}\tilde{\lambda}^8 + d_6^{(2)}\tilde{\lambda}^6 + d_4^{(2)}\tilde{\lambda}^4 + d_2^{(2)}\tilde{\lambda}^2 + d_0^{(2)}\right],$$

where $\tilde{\sigma}^2 = \tilde{\sigma}_{12}^2 + \tilde{\sigma}_{13}^2$ and

$$d_8^{(2)} = (2+D)^4,$$

$$d_6^{(2)} = -(2+D)^2(36 + 28D + 5D^2 + 16(D+1)\tilde{\sigma}^2),$$

$$d_4^{(2)} = 64(D+1)^2\tilde{\sigma}^4 + (768 + 936D + 312D^2 + 42D^3)\tilde{\sigma}^2 + \\ + (D+2)^2(68 + 42D + 7D^2),$$

$$d_2^{(2)} = -12(2+D)^3 - 3D(2+D)^3 - 2(2+D)^2(50 + 11D)\tilde{\sigma}^2 + \\ + 8(10+D)(14+11D)\tilde{\sigma}^2,$$

$$d_0^{(2)} = +12(1+D)(2+D)^2\tilde{\sigma}^2 - 12(4 + D(20 + 13D))\tilde{\sigma}^4 - 576(1+D)\tilde{\sigma}^6.$$

Thanks to its simple mathematical structure, it is possible to show that the polynomial presents all real roots if and only if $\tilde{\sigma}^2 \leq \tilde{\sigma}_0^2$ where $\tilde{\sigma}_0^2 = \frac{(2+D)^2}{16(1+D)}$.

Appendix A.3. Characteristic polynomial in the $(\tilde{\sigma}_{12}, \tilde{\Pi})$-section in $ET^2_{14,P}$

Finally, for three-dimensional field variables, when $\tilde{\Pi}$ and $\tilde{\sigma}_{12}$ are the only non-vanishing non-equilibrium variables, the characteristic polynomial of $\mathrm{ET}^2_{14,P}$ reduces to

$$\mathcal{P}^{(2)}_{3D}\Big|_{\Pi,\sigma_{12}} = -\frac{24k_{\mathcal{B}}p^4\vartheta^3}{(D+2)^6 m}\tilde{\lambda}^2\mathcal{P}_4^{(2)}(\tilde{\lambda})\mathcal{P}_8^{(2)}(\tilde{\lambda}),$$

where

$$\mathcal{P}_4^{(2)} = f_4^{(2)}\tilde{\lambda}^4 + f_2^{(2)}\tilde{\lambda}^2 + f_0^{(2)},$$
$$\mathcal{P}_8^{(2)} = g_8^{(2)}\tilde{\lambda}^8 + g_6^{(2)}\tilde{\lambda}^6 + g_4^{(2)}\tilde{\lambda}^4 + g_2^{(2)}\tilde{\lambda}^2 + g_0^{(2)},$$

with

$$
\begin{aligned}
&f_4^{(2)} = (D+2)^2,\\
&f_2^{(2)} = \left[24\tilde{\Pi} - (4D+8)\tilde{\sigma}_{12}^2 - (D^2+6D+8)\right],\\
&f_0^{(2)} = \left[(24\tilde{\Pi} + 2D + 4)\tilde{\sigma}_{12}^2\right],\\
&g_8^{(2)} = (D+2)^4, \quad g_6^{(2)} = -(2+D)^2((2+D)(18+5D) + 16(1+D)\tilde{\sigma}_{12}^2),\\
&g_4^{(2)} = (2+D)^2(68+7D(6+D)) + 6(128+156D+52D^2+7D^3)\tilde{\sigma}_{12}^2+\\
&+64(1+D)^2\tilde{\sigma}_{12}^4,\\
&g_2^{(2)} = -3(2+D)^3(4+D) - 2(2+D)^2(50+11D)\tilde{\sigma}_{12}^2+\\
&+8(10+D)(14+11D)\tilde{\sigma}_{12}^4,\\
&g_0^{(2)} = 12\tilde{\sigma}_{12}^2(1+D+3\tilde{\sigma}_{12}^2)((2+D)^2 - 16(1+D)\tilde{\sigma}_{12}^2).
\end{aligned}
$$

All the coefficients depend on even powers of $\tilde{\sigma}_{12}$, this fact implies a symmetry of the results with respect to the Π-axis. Taking into account that $f_4^{(2)} > 0$, in order to present all real roots, the coefficients of the quadratic polynomial $\mathcal{P}_4^{(2)}$ have to satisfy the inequalities:

$$
f_2^{(2)} \leq 0, \quad f_0^{(2)} \geq 0, \quad (f_2^{(2)})^2 - 4f_4^{(2)}f_0^{(2)} \geq 0,
$$

that correspond to the following conditions for $\tilde{\Pi}$

$$
-\frac{(D+2)}{12} \leq \tilde{\Pi} \leq \tilde{\Pi}^* \quad \text{if } \tilde{\sigma}_{12} \neq 0, \tag{A.1}
$$

$$
\tilde{\Pi}^* = \frac{1}{24}\left[D^2+6D+8+\tilde{\sigma}_{12}^2(2D^2+12D+16) - 2\sqrt{(D+2)^3(D+6)(\tilde{\sigma}_{12}^4+\tilde{\sigma}_{12}^2)}\right],
$$

$$
\tilde{\Pi} \leq \frac{D^2+6D+8}{24} \quad \text{if } \tilde{\sigma}_{12} = 0. \tag{A.2}
$$

As the domain (A.2) is in part contained in (A.1), we can conclude that the domain of hyperbolicity is given by the inequalities (A.1) for any $\tilde{\sigma}_{12}$ (see (19)) and the half line given in (20).

The requirement concerning the reality of the roots of $\mathcal{P}_8^{(2)}$ implies some restrictions on $\tilde{\sigma}_{12}$: since the polynomial presents all real and distinct roots when $\tilde{\sigma}_{12}$ vanishes, there should be a neighborhood of the equilibrium point that fits the hyperbolicity requirement. Such a region exists in particular if

$$
g_2^{(2)} \leq 0 \quad \text{and} \quad g_0^{(2)} \geq 0,
$$

that is to say when

$$
-\frac{D+2}{4\sqrt{D+1}} \leq \tilde{\sigma}_{12} \leq \frac{D+2}{4\sqrt{D+1}};
$$

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