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Abstract. Expressively equivalent logical languages can enunciate logical notions in notationally diversified ways. Frege's Begriffsschrift, Peirce's Existential Graphs, and the notations presented by Wittgenstein in the *Tractatus* all express the sentential fragment of classical logic, each in its own way. In what sense do expressively equivalent notations differ? According to recent interpretations, Begriffsschrift and Existential Graphs differ from other logical notations because they are capable of 'multiple readings'. We refute this interpretation by showing that there are at least three different kinds of such multiple readings. While readings of the first kind do not capture any essential difference among notations but only among vocabularies, corresponding to readings of the second and the third kind two general parameters according to which notations may differ are defined: linearity vs. non-linearity, and tabularity vs. non-tabularity. This answers the question of how there can be substantially different but expressively equivalent logical notations.

Keywords: Begriffsschrift; Existential Graphs; truth-tables; Frege; Peirce; Wittgenstein; philosophy of notation; vocabulary; linearity; tabularity.

1. Introduction

Expressively equivalent logical languages can enunciate logical concepts and operations in notationally diversified ways. We know what it means for two or more logical languages to be expressively equivalent, but we do not know what it means for languages that are expressively equivalent to differ from each other in some meaningful and substantial ways. This paper investigates the meaning of such 'notational differences'.[1]

Peirce and Frege, two pioneers of modern logic, invented logical notations that differed in significant ways from the notations of their predecessors as well as from their followers. Frege invented the Begriffsschrift (BS) in 1879 (Frege 1879). Peirce created the system of Existential Graphs (EGs) (Peirce 1896, 1903,

[1] To pre-empt misinterpretations from the get-go: one must not conflate expressive equivalence with notational or syntactic equivalence. Two notations are expressively equivalent if they characterize the same logic (anything expressible in one is also expressible in the other; in model-theoretic terms, have the same models). But two expressively equivalent notations, such as BS and EGs, may clearly differ in terms of not being notationally equivalent. For the sake of clarity and conciseness, the present paper is concerned only with languages expressively equivalent to propositional calculus PC in the sense of being two-element Boolean algebras (Ma & Pietarinen 2018b).

1906, 2019) after having devoted his best efforts to the development of a number of algebraical systems of logic (Peirce 1880, 1883, 1885). Wittgenstein presented in the *Tractatus Logico-Philosophicus* a notation for sentential logic in which truth-tables are propositional signs, and later a notation with one truth-operation 'N', equivalent to a generalization of the joint denial. All these notations can express classical propositional logic and so are Boolean algebras. Yet it is evident that each of these systems represents propositional logic in ways that significantly differ from standard languages we find in typical definitions of logic. Different notations may have the same logic. The question thus arises: In which senses are these notations different?

According to the proposal recently advanced by Macbeth (2005),[2] BS is peculiar in the sense that its formulas can have multiple equivalent analyses or readings. A similar interpretation concerning EGs has been proposed by Shin (2002).[3] According to these authors, the fact that in BS and EGs a formula can have multiple, logically equivalent readings is a characteristic feature of both BS and EGs. They take BS and EGs to be qualitatively different from other, expressively equivalent notations that define languages of propositional logic. For other reasons but with similar results, Landini (2007) proposed an interpretation of what Wittgenstein's notations could be thought to achieve, namely that logically equivalent formulas have exactly one and the same representation. The present paper analyses these views and subjects them under critical scrutiny.

We show that the fact that formulas of BS and EG are capable of multiple readings is insuffient to characterize either of these notations. The Macbeth–Shin claim that formulas of BS and EGs are capable of multiple readings boils down to the interdefinability of primitive sentential operators. Other notations exist which differ as to how many and which primitive sentential operators they adopt. The claim that *only* BS and EGs manifest multiple readings cannot hold, because there are notations which share with BS and EGs the set of primitive operators and whose formulas have the same multiple readings as formulas of BS and EGs. On the other hand, the claim that BS and EGs do manifest, like other notations, such multiple readings, though true, ceases to be of theoretical interest, because what one wants to know is *how* BS and EGs *differ* from expressively equivalent notations, not what properties they share with them.

Landini's analysis allows us to differentiate between what in section 4 we label 'tabular' and 'non-tabular' notations. As it will be shown, Landini's argument only holds for Wittgenstein's truth-tabular notation presented at *TLP* 4.442 (which is, in our terminology, a tabular notation), but it fails for the N-notation introduced at *TLP* 5.502 (which is a non-tabular notation).[4]

From the Macbeth-Shin point of view, formulas in tabular notations have multiple readings in formulas of non-tabular notations. But, as we show here, the sense in which the formulas of BS and EGs have multiple readings is different from the sense in which Wittgenstein's truth-tabular notation has multiple readings in non-

[2] See also Thiel 1995, Shin & Moktefi 2012, and Cook 2013. Bellucci, Moktefi & Pietarinen (2018) is a recent study of Frege, Peano's and Peirce as pioneers of philosophy of logical notations.

[3] See also Shin & Moktefi 2012, Macbeth 2012, Shin & Hammer 2014, and Shin 2015; for a rebuttal, see Bellucci & Pietarinen 2016.

[4] Landini holds that it was Wittgenstein's mistaken view that his N-operator captures what the *ab*-notation captures. In our terminology, Landini thinks that Wittgenstein thought his N-notation, which is non-tabular, to be tabular. Whether Landini is right or not, nothing in our arguments depends on this concern.

tabular notations. There thus emerge the necessity of differentiating between different *kinds* of multiple readings. We also show that a third kind of multiplicity of readings, irreducible either to the kind identified by Macbeth and Shin or to the kind discussed by Landini, depends on whether the notation is 'linear' or 'non-linear', in the sense to be specified in Section 3.

In this manner, three different *senses* in which a formula can be said to have the alleged 'multiple readings' can be recognized, so that we are obliged to recognize (at least) three different kinds of 'multiple readings'. We identify readings of the first, second, and third kind, related to the *vocabulary*, *linearity*, and *tabularity* of the notation, respectively. Corresponding to the second and the third kind, we define two general parameters according to which notations may differ: linearity vs. non-linearity, and tabularity vs. non-tabularity. Corresponding to the readings of the first kind, only a difference among the vocabulary, or sets of truth-functional operators, can be defined.[5] In conclusion, this argument provides a new answer to the question of how there can be subtantially different logical notations of languages that are expressively equivalent.

2. Vocabulary: Multiple readings of the first kind

The sentential fragment of BS is based on two primitive sentential operators, negation and material implication. The rules of construction of complex sentential formulas given by Frege in the *Begriffsschrift* are generally known. According to Macbeth (2005), BS formulas can be 'read' or 'analyzed' in various, provably equivalent ways. For example, the BS formula in Fig. 1 can be read both as (1a) and as (1b). Likewise, the BS formula in Fig. 2 can be read as (2a–d).

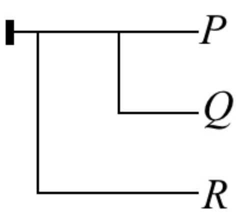

Figure 1

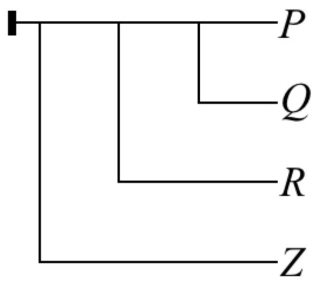

Figure 2

(1a) $R \supset (Q \supset P)$
(1b) $(R \,\&\, Q) \supset P$

(2a) $Z \supset (R \supset (Q \supset P))$
(2b) $(Z \,\&\, R) \supset (Q \supset P)$
(2c) $Z \supset ((R \,\&\, Q) \supset P)$
(2d) $(Z \,\&\, R \,\&\, Q) \supset P$

[5] We do not claim that logical notations can differ according to our parameters *only*. There can be other parameters, aspects or undiscovered phenomena with respect to which logical notations may differ substantially. What we claim is that our parameters *do* qualitatively differentiate between logical notations.

According to Macbeth, '[t]he equivalence of these […] formulae, though it must be proven in standard (one-dimensional) notation, is a given of Frege's two-dimensional notation. […] Frege's one formula corresponds to an equivalence class of formulae in standard one-dimensional notation' (2005: 51). That is, several equivalent formulas in standard one-dimensional notation correspond to one and the same BS formula, and each of those equivalent formulas is a distinct 'analysis' (2005: 50) of the BS formula in question. This, Macbeth argues, constitutes a 'fundamental difference' between the BS and standard notations:

> Frege's logical language functions as a fundamentally different kind of language from that of quantificational logic. Rather than directly saying something about something, *Begriffsschrift* sentences display objects, concepts, and truth-values in logical relations of various kinds and can be analyzed in various ways. (2005: 72)

It is true that one and the same BS formula can correspond to distinct, provably equivalent formulas in standard notation. But it does not follow from this that BS is a 'fundamentally different kind of language'. In the *Begriffsschrift* Frege explains how to express conjunctive or disjunctive sentences with his notational apparatus containing only negation and implication. For example, Fig. 3 represents '*Q* or *P*' and Fig. 4 represents '*Q* and *P*' (Frege 1879, §7):

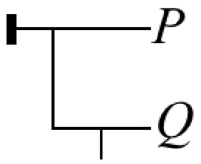

Figure 3

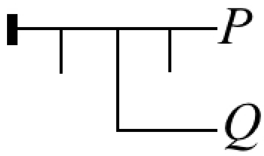

Figure 4

This indicates that Frege recognizes that BS formulas do have multiple readings, given that other connectives can be defined in terms of its primitives. But he was far from claiming that such a feature is only present in his notation. In fact, the same phenomenon is present in every expressively adequate sentential language which does not have a distinct sign for each of its truth-functions.

Mutatis mutandis, the same argument applies to EGs. The language of the Alpha department of EGs, which agrees with the sentential calculus and Boolean algebra (Ma & Pietarinen 2017), has two primitives: conjunction, represented by juxtaposing propositional variables on the sheet of assertion, and negation, represented by an oval encircling negated sentence(s).[6] According to Shin (2002), the fact that one and the same Alpha graph can be read in multiple ways in the standard notation of propositional logic makes Alpha a 'fundamentally different kind of language'.

As an example, the graph in Fig. 5 can be multiply read as (5a–c).

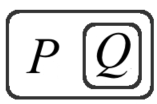

Figure 5

[6] We offer a more detailed account of Alpha's syntax in Section 3.

(5a) $\sim (P \,\&\, {\sim}Q)$
(5b) $P \supset Q$
(5c) ${\sim}P \vee Q$

This depends, Shin argues, on the possibility of 'carving up' a graph in multiple ways (2002: 77–80) — a possibility which is excluded in standard notations. However, the possibility of 'carving up' one and the same formula in multiple ways refers to the possibility of translating one configuration of signs of one language (call it the *source language*, Ls) into multiple configurations of signs in another, provably logically equivalent language (call it the *target language*, Lt). Shin's condition for having multiple translations of this kind is that Lt *has a greater number of primitive operators than* Ls. In the case of BS, Ls has two primitive operators (negation and implication), while the Lt in which (1a–b) and (2a–d) are written has at least one more (conjunction). Likewise, in the case of EGs, Ls has two primitive operators (negation and conjunction), while the Lt in which (5a-c) are written has at least two more (implication and disjunction). As a matter of fact, had Lt the same primitives as Ls, we would be unable to distinguish (1a) from (1b), (2a) from (2b–d) and (5a) from (5b–c).

In general, given a Ls with *n* primitive operators, a formula of Ls can be 'multiply analyzed' — i.e., can correspond via translation — to a class of equivalent formulas of Lt only if Lt has (at least) n+1 primitive operators ($n \leq 15$). What Macbeth calls 'multiple analyses' and Shin 'multiple readings' of a formula are merely *translations* of a formula of a Ls into several equivalent formulas in a Lt, where Lt has a richer logical vocabulary than Ls, so that one single configuration of primitive signs in Ls can correspond to several equivalent configurations of primitive signs in Lt. In general, since notations exist which have the same set of primitive operators as BS and EGs, it follows that the possibility of multiple readings is no specific feature of either BS or EGs.[7]

Take the language with only the Sheffer stroke in its vocabulary, for example. It is expressively equivalent to sentential logic.[8] Following Macbeth and Shin, one should now say that several, equivalent formulas in Lt correspond to one and the same formula of the Ls with the Sheffer stroke as its sole logical operation. The equivalence class of formulas in (1a–b) corresponds to (6), and the equivalence class of formulas in (2a–d) corresponds to (7). In Macbeth's and Shin's terms, formula (6) is 'multiply analyzed' as (1a–b), and (7) is 'multiply analyzed' as (2a–d).

(6) $(P \mid ((Q \mid (R \mid R)) \mid (Q \mid (R \mid R))))$

(7) $(Z \mid ((P \mid ((Q \mid (R \mid R)) \mid (Q \mid (R \mid R)))) \mid (P \mid ((Q \mid (R \mid R)) \mid (Q \mid (R \mid R))))))$

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[7] In a critical review of Macbeth's book, Sullivan (2009) argues that the fact that a notation is capable of multiple readings says nothing of the notation itself. According to Sullivan, "[d]ifferent "readings" here are simply different translations into English" (2009: 93) or, as in Macbeth's and Shin's examples, into a notation richer than the BS or EGs in the amount of the primitive symbols those notations consist of.

[8] More appropriately speaking the Peirce Stroke, as it was Peirce who discovered the functional completeness of the NOR and NAND operators in 1880 (see Peirce 1989: 218–221).

It is obvious that, by the same token, one and the same formula of such Shefferian notation can be ‘multiply analyzed’, for example, in BS formulas. For example, (8) can be multiply read into Fig. 6 and Fig. 7.

(8) $P \mid P$

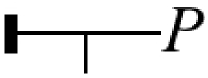

Figure 6

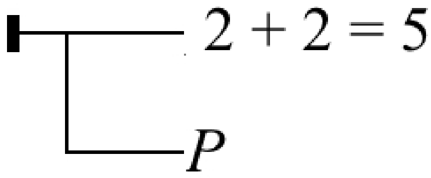

Figure 7

Or, as another example, take a notation with only implication as the primitive operator, such as positive implicational logic. A sentence in the notation of such logic, such as (9), would be multiply analysed into (1a-b) exactly as the formula in Fig. 1 is in Macbeth’s account.

(9) $R \supset (Q \supset P)$

We present these examples only to highlight the fact that what Macbeth and Shin call ‘multiple analyses’ or ‘multiple readings’ of a formula are translations of a formula of a Ls into several, provably logically equivalent formulas, in a Lt which has a richer logical vocabulary than Ls has: one single configuration of primitive signs in Ls corresponds to several equivalent configurations of primitive signs in Lt. But then the possibility of multiple readings captures no essential character of either BS or EGs, because these systems do not depend on any specific feature of Ls. They depend on the *relation* between Ls and Lt in terms of their vocabularies.

In other words, multiple readings in the Shin-Macbeth sense concern *differences in the vocabularies of languages that do not differ in their expressive power*. The ‘multiple readings’ discussed by Macbeth and Shin does not distinguish between logical *notations*, but between *sets of primitive operations*. If multiple readings in this sense really distinguished among notations, any language with negation and conjunction as its set of primitive operations would be in this sense *the same notation* as EGs, and any language with negation and conditional would be in this sense *the same notation* as BS. This consequence needs to be avoided.

Macbeth suggests that the reason why the BS is capable of multiple readings in a language Lt is that BS is two-dimensional while Lt is one-dimensional. The example above concerning the Sheffer-stroke notation and some Lt shows that being two-dimensional is unnecessary for multiple readings to emerge. But how, then, is the two-dimensional character of BS linked to the possibility of multiple readings?

One could naturally argue that BS makes multiple readings *easier* to obtain than would happen in other notations. For example, one may argue that the formula in Fig. 1 is more easily read as (1a–b) than (6) or (9) are. However, this is an appeal to psychological convenience, which is of no concern in logic. That multiple readings are easier in some notations and harder in others does not affect the fact that such readings (in the Macbeth-Shin sense) can in principle occur among notations that differ in their logical vocabularies, independently of any cognitive, perceptual or empirical matters.

Shin has applied the possibility of multiple readings to the demarcation of 'diagrammatic' from 'non-diagrammatic' (or 'symbolic') notations. She situates EGs and the BS on the diagrammatic side and the notations into which EGs and BS formulas are 'multiply read' on the symbolic side.[9] The problem with this use of the multiple-readings phenomenon is that other notations, which we might also consider 'symbolic', do allow multiple readings: as remarked above, in principle any sentence in the standard notation for sentential logic with only negation and conjunction can have the same multiple readings as EGs have, and any sentence in the standard notation for sentential logic with only negation and conditional can have the same multiple readings as BS has.

On the other hand, there are certain weakenings of classical Alpha that involve modifications to the rules of transformations and hence give rise to non-standard logical behaviour (Ma & Pietarinen 2018). Some of them, such as semi-De Morgan graphs, cannot be 'multiply read' at all. Yet the graphs of the logic of semi-De Morgan algebras have forms that 'look' exactly the same as the forms of the classical Alpha graphs that agree with Boolean algebras. If Shin's argument were correct, these non-classical systems of graphs that are generalizations of Boolean algebra would not count as diagrammatic at all, while graphs of the language of a stronger logic that 'look exactly the same' would. In brief, an appeal to the 'visual' features of a language cannot be the solution.

Shin's explanation of what differentiates the multiple readings of 'diagrammatic' notations from the multiple readings of 'non-diagrammatic' ones is that, when we 'multiply read' a formula of a 'non-diagrammatic' language, what we actually do is *extend* the vocabulary of the language (cf. Shin 2002: 84). Thus, for example, in order to differentiate the multiple readings of Fig. 5 into (5a–c) from those of (6) into (1a–b), Shin would say that while Fig. 5 is multiply read into (5a–c), (6) is in fact not multiply read as (1a) and as (1b). Rather, the language of (6) is extended into the language of (1a) and (1b) (i.e., further primitive operators are added to the vocabulary of the language). That is, we actually do not read the formula differently but we *extend* the vocabulary of the language.

But this is a *non sequitur*. No language can extend its own vocabulary without producing a *different*, often an equally expressive, language.[10] And since what Shin means with 'multiple readings' are *translations* among languages, the possibility of taking one language as an extension of another is irrelevant. In fact, to impose such a condition — *that Lt should not be an extension of Ls* — is to significantly modify the definition of 'multiple readings'. Under the new condition Shin's 'multiple readings' have to be defined as '*multiple translations of formulas of Ls into formulas of Lt in which Lt is not an extension of Ls'*. But this new definition transforms a question *de facto* into a question *de iure*: *de facto*, a given Ls is poorer than Lt in its primitives. But according to the new condition, in order to speak of 'multiple readings' it is requisite not just that Ls be poorer than Lt (*de facto*) but that it could not possibly be enriched (*de iure*). In other words, instead of simply differentiating languages by the set of their primitive operators, the revised definition would differentiate languages that can be extended from those that cannot. It is wholly unclear how one could pass from the notion of a 'language incapable of being extended' to the notion of a 'diagrammatic language'. Nor is it clear why and to what

[9] See Shin 2002: 4; Moktefi & Shin 2012: 664; Shin 2015: 56.

[10] Cf. e.g. Church 1956: 48, 48n111.

extent EGs are a non-extendible language. The situation, however, seems to be this: either we accept the new condition, classifying EGs as a *sui generis* system whose set of primitives cannot be enriched (thus falling short of demarcating between diagrammatic and non-diagrammatic systems), or we refuse to accept the new condition, in which case we can only conclude that the whole multiple-readings argument (in the Macbeth-Shin sense) is only a roundabout way of saying that when two languages differ in their primitive truth-functional operations, one single configuration of primitive signs in one language corresponds to several equivalent configurations of primitive signs in the other.

Hence the 'multiple readings' argument fails to shed light on the question of notational difference. Two languages may be expressively equivalent and yet they may differ in the set of primitive operators. If two languages with different sets of primitive operators were really two *different* languages, as the proponents of the 'multiple readings' argument seem to hold, then one might be led to think that languages with the same set of primitive operators should be the *same* notation. But of course, languages with the same set of primitive operators may be different under several, other respects. In the sequel we will propose a more fine-grained analysis in order to capture the gist of the idea erroneously termed 'multiple readings'.

Furthermore, if the multiple-readings argument is intended (as it is according to Shin) to isolate either or both BS and EGs from other major classes of notations, such as 'diagrammatic' from 'symbolic', it fails to do so because any notation equivalent to either BS or EGs in the set of primitives allows precisely those multiple readings that BS and EGs allow. If we wish the multiple readings phenomenon to capture some essential property of either or both the BS and EGs but not, say, of the Sheffer stroke notation, then we should — as Shin tacitly does — add the condition that the language into which we multiply translate (Lt) should not be an extension of the language from which we translate (Ls). This condition does not solve the original problem, however, because then an argument has to be provided why BS and EGs cannot be extended in the same manner in which the Sheffer notation can be extended, namely by introducing further primitives by definition. No argument for the unextendability of these notations has been provided in Shin's account.

3. Linearity: Multiple Readings of the second kind

Not all 'multiple readings' are of the kind just described. Some of Macbeth's examples of multiple readings of a BS formula depend on certain other conditions, and the same holds of another kind of multiple readings that seem to characterize Peirce's Alpha EGs. We now show that while the multiple readings per vocabulary characterized above depend on the difference between primitive operators in Ls and Lt, the multiple readings of this second kind depend on a difference between Ls and Lt with respect to what can be called their 'abstract syntax'.

Macbeth says:

> Once we have learned to read Frege's notation in this way, it is easy to see, in Frege's notation, that interchanging subcomponents is permissible. Changing the order of subcomponents does need to be justified, which is why interchange of subcomponents is given as a rule in *Grundgesetze* and is proved as a theorem in *Begriffsschrift*; but in Frege's notation one such rule can cover all cases of this form of embedding. In our standard notations, by contrast, having proved that, say, '(P & Q) ⊃ R' is equivalent to '(Q & P) ⊃ R' will not save one the trouble of having also to prove that, say, 'P ⊃ ((Q & R) ⊃ S)' is equivalent to 'Q ⊃ ((R & P) ⊃ S)'. Indeed, these look, in standard notation, to be quite different sorts of cases. Where Frege has one rule and a two-dimensional notation to fix the equivalence of the four linear sentences discussed earlier, as well as all twenty variants with 'Q', 'R',

and ‘S’ in different orders, a standard linear notation brings with it the demand that one prove, for each pair of the twenty-four sentences involved, that they are equivalent. (Macbeth 2005: 52–53)

The equivalencies (2a–d) above, which are multiple readings per vocabulary in the sense specified, are insufficient to guarantee the equivalence of (10a) and (10b), however:

(10a) $P \supset ((Q \,\&\, R) \supset S)$
(10b) $Q \supset ((R \,\&\, P) \supset S)$

A further principle is required which combines with (2a–d). This principle, obviously enough, is the principle of the *transposition of the antecedents* (TA) in a conditional. That is, the following logical equivalence holds in PC (where ‘=’ is the symbol of the meta-language that denotes logical equivalence):

(TA) $A \supset (B \supset C) = B \supset (A \supset C)$.

In the BS, TA allows us to pass from Fig. 8 to Fig. 9:

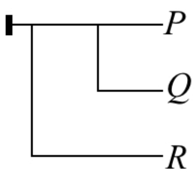

Figure 8

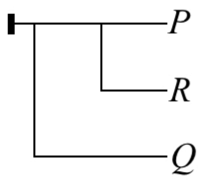

Figure 9

Angelelli (2008) has objected that Macbeth’s account leaves it unclear in what sense the application of TA in BS is different from its application in standard linear notations. Axiom 8 of the *Begriffsschrift*, here reproduced in Fig. 10, expresses precisely this:

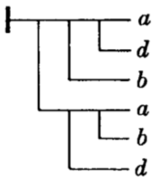

Figure 10

In order to establish the logical equivalence of Fig. 8 and Fig. 9, TA or some equivalent of it has to be applied. Just as in linear notation, the condition-stroke *R* in Fig. 8 *precedes* the condition-stroke *Q* on the horizontal stroke, so we need a permission to swap the two. That is, given the difference in the linear arrangement of the conditions *P* and *Q*, Fig. 8 is *not* in itself equivalent to Fig. 9: their equivalence needs a proof. In this respect, as Angelelli observes, nothing distinguishes BS from standard, linear notation.

Macbeth acknowledges that TA is a rule of the system. But even so, we need an explanation of how this distinguishes BS from standard notations. If TA is an axiom of the BS, then BS cannot have this kind of multiple readings. If, on the contrary, we consider ‘$R \supset (Q \supset P)$’ and ‘$Q \supset (R \supset P)$’ as two syntactically distinct but logically equivalent readings of Fig. 8, then the point must be that the interchangeability of the conditions is a ‘given’ of the system. In other words, in saying that the equivalence between formulas is ‘given’ in Frege’s notation while it is ‘proved’ in standard linear notation, Macbeth must mean that the rule (TA) that justifies the equivalence is already provided at the syntactical level of the notation and need not be added as an axiom of the system.

In other words, what Macbeth should be saying is that the equivalence of formulas is *embedded in the notation of the system itself.* In order for Fig. 8 to be multiply analyzed in linear notation both as ‘$R \supset (Q \supset P)$’ and as ‘$Q \supset (R \supset P)$’ and without any intervention of TA, we need to read Fig. 8 in such a way that the fact that the condition R is to the left of, and thus precedes, the condition Q, is *not syntactically significant*. The reason is that if R’s preceding Q in Fig. 8 is considered syntactically significant, the only ‘analysis’ of Fig. 8 is ‘$R \supset (Q \supset P)$’, from which we can obtain Fig. 9 only by TA. But to say that we multiply analyze Fig. 8 at once as ‘$R \supset (Q \supset P)$’ and as ‘$Q \supset (R \supset P)$’ can only mean that we do not consider the *ordering* on the horizontal stroke as a syntactically significant feature of the notation. That is, we read Fig. 8 as *being the same formula* as Fig. 9. Only in this sense can we say that Fig. 8 is ‘multiply analyzed’ by those sentences without the intervention of TA, as TA only applies if Fig. 8 and Fig. 9 are syntactically distinct formulas. It would make no sense to apply it to one and the same formula.

In order to be exact about the sense of this ‘sameness’ we distinguish between *token* and *type* formulas. The distinction type/token was introduced by Peirce in his *Syllabus of Logic* for the Lowell Lectures delivered in autumn 1903, and it has since become canonic. Consider the following sentences:[11]

(11a) $(P \wedge Q)$

(11b) $\mathbf{(P \wedge Q)}$

(11c) $(\ P \wedge \qquad Q \qquad)$

(12a) $(P \wedge Q)$

(12b) $(Q \wedge P)$

The difference between (11a–c) is a typographical difference, which is irrelevant at the syntactic level simply because qualities such as font and size are not taken to be representing facts. Therefore, we say that (11a–c) are sentence tokens of the same sentence type. In contrast, the difference between (12a) and (12b) is a difference in the ordering, which is syntactically relevant because order is a representing fact. (Rules of adjunction, commutativity and associativity need to be explicitly provided.) Therefore, (12a) and (12b) are not two sentence tokens of the same sentence type, but two sentence types logically equivalent in standard sentential logic. Given these

[11] Example adapted from Howse et. al. 2002. Let us ignore the special cases of non-associative languages, for example, in which some such differences may be non-typographic. Clearly in all cases in which typographical differences aquire relevant new meanings in any language or meta-language of logic, those differences come to life not by changing the typography but by definitions and conventions associated with those changes.

senses of ‘type’ and ‘token’, it is easy to see that in a linear language *permutation always produces types and never tokens*.[12]

To have ‘$R \supset (Q \supset P)$’ and ‘$Q \supset (R \supset P)$’ as alternative and equivalent ‘analyses’ of Fig. 8, one has to consider Fig. 8 as the *same type* as Fig. 9. Compare Euler diagrams that represent classes. The sentence ‘No *X* is *Y*’ is represented in Euler diagrams as in Fig. 11:

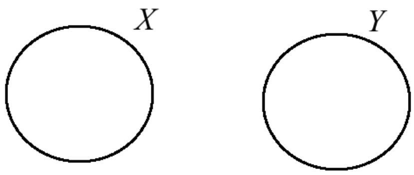

Figure 11

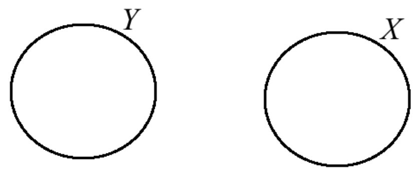

Figure 12

In Euler’s system, size, shape and orientation of the circles are synctactically irrelevant. Only the partial or total overlaps and non-overlaps of circles are synctactially relevant. Thus, Fig. 11 also represents the sentence ‘No *Y* is *X*’, logically equivalent to the former. For two different sentences of the linear language, Euler’s system has just one formula: the Eulerian formula represented in Fig. 12 is the same Eulerian formula type as that represented in Fig. 11, and each is a distinct Eulerian token of the same Eulerian type.

To save Macbeth’s ‘multiple analyses’ of the second kind, we could suppose that Fig. 8 and Fig. 9 are different tokens of the same type, just as Fig. 11 and Fig. 12 are different Eulerian formula tokens of the same type. If Fig. 8 and Fig. 9 are to be different tokens of the same type, permuting the conditions *R* and *Q* should not produce a different sentence type. Consider the BS formula in Fig. 13:

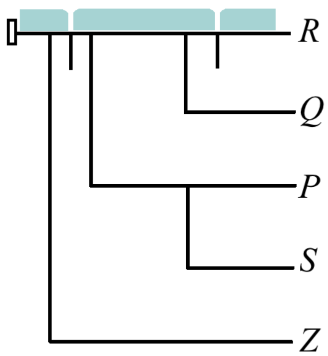

Figure 13

In the horizontal dimension we have the upper horizontal stroke divided by the sign of negation into three fragments or areas. Since conditions attached to the same horizontal without any negation occurring between them can be interchanged without altering the logical value, then in order to have Fig. 8 and Fig. 9 as tokens of the same type the convention must be adopted that *any segment of a BS horizontal in which no*

[12] In Hjelmslev’s largely forgotten terminology (Hjelmslev 1961), a ‘permutation’ is a mutation in the ordering of the elements of a sentence, which mutation produces distinct types. When any two elements do not mute, they ‘substitute’, and substitution produces distinct tokens of the same type.

negation sign occurs is unordered. Under this notational convention (let us call it the Convention of Permutation of Conditions, CPC), Fig. 8 and Fig. 9 are not two different formulas that need an equivalence proof by the application of TA. Under this notational convention, Fig. 8 and Fig. 9 are the *same formula* (i.e. the same formula type), and TA loses its applicability. TA states a rule of *logical equivalence*, while CPC states a rule of *syntactical equivalence*.

According to CPC, conditions attached to such an area can always be interchanged, while conditions attached to different areas cannot always be interchanged: switching within areas is permitted, switching across areas is not. This perfectly corresponds to the fact that in any composite conditional of the form '$A \supset (B \supset C)$', A and B can be switched without producing a logically non-equivalent sentence. However, the horizontal dimension cannot be completely unordered, because negation introduces asymmetry in the syntactic structure of the formula. The horizontal is thus not totally (i.e., linearly) but only *partially* ordered. In the vertical dimension, by contrast, conditions can never be interchanged unless the switch is justified on the horizontal dimension. This perfectly corresponds to the fact that in any composite conditional of the form '$(A \supset B) \supset C$', A and B *cannot* be switched without altering the semantic value. Permutation in the vertical dimension produces types, while premutation in the horizontal dimension (within areas delimited by negation signs) produces tokens. Since in a linear language permutation always produces types and never tokens, the conclusion is that BS is not linear: the vertical dimension is totally ordered, but the horizontal dimension is only partially ordered.

In saying that the equivalence of formulas like those in Fig. 8 and Fig. 9 is 'proven' in standard notations while it is 'given' in BS, Macbeth may be following the tendency which we find in Frege's own presentation of the BS, namely to consider TA as a convention of the notation rather than as an axiom of the system (that is, as a rule of syntactical equivalence rather than a rule of logical equivalence). For example, in *Grundgesetze*, just after having remarked the necessity of proving some form of TA, Frege adds: "as not to become tied up in excessive complexity, I here wish to assume this interchangeability generally granted, and to make use of it in future without further explicit mention" (Frege 2013, §12). Likewise, in the notes from the lectures held in 1910-1911 and in 1913 taken by Carnap, Frege considers coordinate conditions to be interchangeable without treating this convention as an axiom of the system (see Reck & Awodey 2004: 52, 56, 71). However, insofar as TA is adopted as a rule of *logical* equivalence and not as a rule of *syntactical* equivalence (as CPC), Fig. 8 and Fig. 9 are types, not tokens of the notation.

There are non-linear languages based on some sort of 'generalized' CPC. The Alpha system of EGs is such language. Alpha graphs have two primitives: conjunction and negation. Assertion is represented as the placement of the sentential variable on the sheet at any position, and conjunction as the *unordered juxtaposition* of variables on the sheet. The sheet, termed the sheet of assertion (SA), is in itself a well-formed Alpha graph. Thus Fig. 14 is the Alpha graph for 'what is true' or 'the True', while Fig. 15 is the assertion that P is true. Fig. 16 is the assertion that both P and Q are true, that is, 'P & Q'; Fig. 17 is the assertion that P, Q and R are true, that is, 'P & Q & R', and so on.

Figure 14

P

Figure 15

P Q *P Q R*

Figure 16 Figure 17

Since the position of the letters on the sheet is not a representing fact (the sheet is unordered), not only all linear permutations of *P*, *Q*, and *R* but also all possible positions of them on the SA such as those in Fig. 18a–d must count as the same Alpha graph *type*:[13]

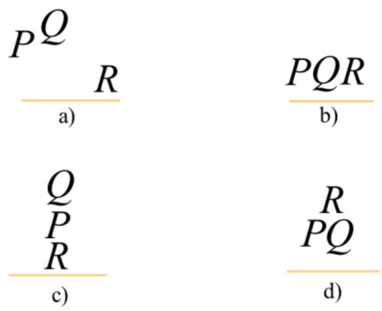

Figure 18

Each formula in Fig. 18a-d is a distinct graph token of the same graph type, and none of them is a different graph type. In linear notation, permutation always produces different sentence types; in EGs, mutations of subgraphs on the SA or within whatever area, just as any other movement on SA or the same area of a graph will not produce different graph types but different graph tokens of the same graph type. For this reason, EGs dispense with the standard rules of adjunction, commutativity and associativity.[14]

The other operation, negation, is represented by an oval that encircles the sentential variables that are denied. Each graph instance of Fig. 19a–d is a notational variant (token) of the assertion that *P* and *Q* are true while *R* is false, or '*P* & *Q* & ~*R*':

[13] We refer to Roberts (1973) for the staple presentation of the basics of the system of EGs, and Peirce (1911, 2019) for further details on his own original accounts. It is necessary to appreciate the system of conventions that underlie the way Peirce's graphical system of logic was set up. For example, given an area of a graph, such as the area of the Sheet of Assertion in the four examples of Fig. 23, a graph can be scribed on any position of that area. As noted above, properties such as commutativity of graphs on the same area at its different positions, as well as the properties of associativity and adjunction, are properties that fall from the properties of the space that the Sheet of Assertion represents and on which the graphs are scribed (see e.g. Pietarinen & Bellucci 2017). The role of the syntactic rules of transformation governing the system of permissible transformations is irrelevant.
[14] This was observed *en passant* by Hammer 1996 and Dipert 2006.

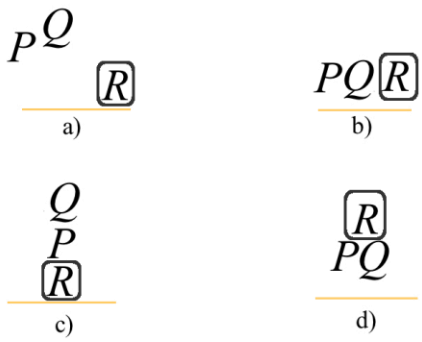

Figure 19

By means of conjunction and negation, all composite sentences can be constructed. For example, each graph instance of Fig. 20a–d is a notational variant (token) of the assertion that ‘it is not true that *P* and *Q* are true and *R* is false’, or ‘~(*P* & *Q* & ~*R*)’:

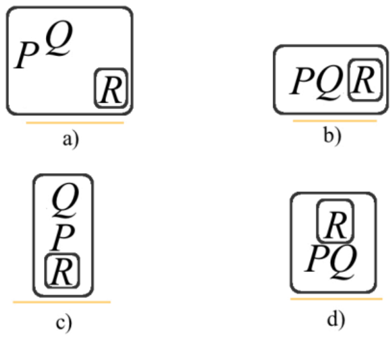

Figure 20

Here is how Peirce defined these key elements of the syntax of EGs:

> That fine oval line […] is called a *cut*, and is supposed to cut off the part of the paper inside of it, which is called “the *area* of the cut” from the part of the paper outside of it, which is called “the *place* of the cut”; so that the area of the cut is no part of the place of the cut. The cut, together with its area, and together with the entire graph on that area is called an *enclosure*. This enclosure is a graph which means that the entire graph on the area of the cut is cut off from the place. (Peirce 1911: 8–9)

To clarify, take the Alpha graph in Fig. 21:

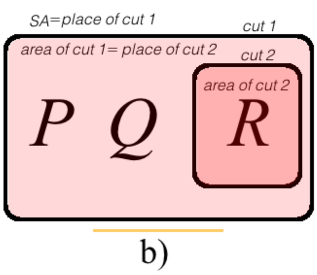

Figure 21[15]

[15] Color and shading are not part of the syntax but are may be used as a didactic aid to highlight distinct areas.

The 'place' of the outer cut (cut 1) is the SA, since the entire graph is scribed on the SA. The 'area' of the outer cut is the place of the inner cut (cut 2). The outer cut with its area and whatever is scribed on this area is an 'enclosure'. An Alpha graph is a *nested structure*, that is, any graph can be presented as a parsing tree of finite sequences of areas from the SA into the areas of cuts of increasing depth. A nest terminating on a cut-free area is a maximal nest. The interpretation of the nested structure has a *direction*. In fact, given a nested structure, we could interpret it either inside-out or outside-in. EGs are interpreted in the second way, which Peirce terms the 'endoporeutic' direction of interpreting EGs. It begins with the outermost area, the SA, and proceeds *inwardly* in the nested structure.[16]

With these definitions in hand, let us return to our original problem. In Alpha graphs, multiple readings per linearity are obtained thanks to the fact that any placement of two or more graphs on the same *area* counts as the same Alpha graph type and not as two distinct Alpha graph types obtained by permutation. In translating the content of an Alpha graph in a sentence of a linear notation we are forced to 'order' that stuff which in the Alpha graphs is not ordered, namely positions in the same area. Thus, the Alpha graph type of which those in Fig. 18a–d are graph tokens can be translated in a linear notation as any of the six linear permutations in (13a–f):

(13a) $P \,\&\, Q \,\&\, R$
(13b) $P \,\&\, R \,\&\, Q$
(13c) $Q \,\&\, P \,\&\, R$
(13d) $Q \,\&\, R \,\&\, P$
(13e) $R \,\&\, Q \,\&\, P$
(13f) $R \,\&\, P \,\&\, Q$

Likewise, the Alpha graph type, of which those in Fig. 19a–d are graph tokens, can be translated in linear notation as (14a–f). The Alpha graph type, of which those in Fig. 20a–d are graph tokens, can be translated in linear notation as (15a–f):

(14a) $P \,\&\, Q \,\&\, {\sim}R$
(14b) $P \,\&\, {\sim}R \,\&\, Q$
(14c) $Q \,\&\, P \,\&\, {\sim}R$
(14d) $Q \,\&\, {\sim}R \,\&\, P$
(14e) ${\sim}R \,\&\, Q \,\&\, P$
(14f) ${\sim}R \,\&\, P \,\&\, Q$

(15a) ${\sim}(P \,\&\, Q \,\&\, {\sim}R)$
(15b) ${\sim}(P \,\&\, {\sim}R \,\&\, Q)$
(15c) ${\sim}(Q \,\&\, P \,\&\, {\sim}R)$
(15d) ${\sim}(Q \,\&\, {\sim}R \,\&\, P)$
(15e) ${\sim}({\sim}R \,\&\, Q \,\&\, P)$
(15f) ${\sim}({\sim}R \,\&\, P \,\&\, Q)$

Just like Euler diagrams and BS conditionals (when CPC is applied), one and the same Alpha graph type admits of multiple readings of the second kind in linear

[16] All of these are Peirce's own definitions and terminology, see especially Peirce 1910. As regards the converse, 'exoporeutic' interpretation, the Alpha graph in Fig. 25 would mean 'It is true that *R* or it is false that *P* and *Q*'.

notation. These multiple readings are the effect of the convention that juxtaposition on an area is the sign of logical conjunction (Peirce 1903). Under this convention, the only syntactically significant fact concerning conjunction is that the *conjuncts are placed on the same area*, that is, at any position of an area.

Alpha EGs have a symmetric abstract syntax, because the basic mode of junction is juxtaposition on the sheet, and juxtaposition is a symmetric mode of junction: graphs juxtaposed on the same area are conjunctively asserted yet nothing is implied about their ordering. Thus the abstract syntax of Alpha EGs perfectly corresponds to the fact that in any composite conjunctive formula the conjuncts can be switched without thereby producing a logically distinct formula. The sheet, however, cannot be completely unordered, because otherwise the syntax could not represent asymmetric relations, that is, because otherwise *logically distinct sentences would not be syntactically distinct*, which is obviously crucial to maintain expressiveness. The ordering is effected by cuts that divide the sheet into non-overlapping areas. The sheet is thus not totally (i.e., linearly) ordered, but forms a parsing tree with the entire graph as the unique root. Only the cuts bring asymmetry to the system. The relation between '(*P* and *Q*)' and '*R*' in the Alpha graph in Fig. 21 is asymmetric and is expressed by an asymmetric mode of junction (juxtaposition on different areas). In contrast, the relation between '*P*' and '*Q*' is symmetric and is expressed by a symmetric mode of junction (juxtaposition on the same area). In other words, EGs represent order only where it is necessary (asymmetric relations), and leave the rest to be represented as unordered (symmetric relations). This is precisely what a BS (with CPC) does with compound conditionals, representing some as linearly ordered — those of the form '$(A \supset B) \supset C$' — and some as partially ordered — those of the form '$A \supset (B \supset C)$'.

4. Tabularity: Multiple readings of the third kind

In this section we propose a third mode of notational variability in terms of a third kind of multiplicity of readings. In brief, we call *tabular notations* those in which only the *result* of a truth-operation on elementary propositions is represented, while in *non-tabular notations* truth-operations are represented as operating on elementary propositions.

One can introduce the idea behind multiple readings of this third kind with a reference to the *Tractatus*. Wittgenstein famously claimed that an operation is different from a function (*TLP* 5.25). The operators of the sentential calculus represent truth-operations, not truth-functions. An *operation* is the 'expression of a relation between the structures of its result and its base' (*TLP* 5.22). The *sense* of a proposition, moreover, is 'its agreement and disagreement with the possibilities of the existence and non-existence of the atomic facts' (*TLP* 4.2), that is, with the truth-possibilities of the elementary propositions (*TLP* 4.3), and two propositions have the same sense if they agree and disagree with the same truth-possibilities of the elementary propositions. From these definitions of operation and sense it follows that the occurrence of an operation in a proposition does not characterize its sense (*TLP* 5.25). For example, the occurrence of the operation '&' in '$\sim(P \mathbin{\&} \sim Q)$' does not characterize its sense, because this proposition has the same sense as '$P \supset Q$', and '&' does not occur in the latter. Likewise, the occurrence of negation '~' in '$\sim\sim P$' does not characterize its sense, because that sentence has the same sense as '*P*', and '~' does not occur in the latter (cf. *TLP* 5.44).

As a result, different truth-operations on elementary propositions can produce the same truth-function of elementary propositions, and 'all those results of truth-operations on truth-functions are identical which are one and the same truth-function of elementary propositions' (*TLP* 5.41). The sense of '*P* ⊃ *Q'* or '~(*P* & ~*Q*)' is not characterized by the occurrence of this or that operation in them. They represent distinct truth-operations on elementary propositions (*P*, *Q*) that have the same truth-function of those propositions as a result. Their sense is the same, and this is captured in the truth-table, represented by Wittgenstein's as (*P*, *Q: TFTT).* The sense of this proposition is unaffected by being presented either as '*P* ⊃ *Q'* or as '~(*P* & ~*Q*)'.

However controversial these Tractarian theses might seem,[17] it is nonetheless clear that they agree with Wittgenstein's idea that the truth-table that lays out the truth-conditions of a proposition is itself a propositional sign (*TLP* 4.442). For what the propositional sign has to represent or express is the proposition's sense, and since the proposition's sense is its agreement and disagreement with the truth-possibilities of elementary propositions, any sign that represents the coordination between the proposition in question and the truth-possibilities of elementary propositions is *ipso facto* a legitimate representation of that proposition. Accordingly, the notation resulting from taking the truth-table to be a propositional sign is a legitimate one.[18]

Landini (2007) has persuasively argued that Wittgenstein's goal in the *Tractatus* was to provide a notation in which all and only logical equivalent propositions should have the same representation:

> He hoped to demonstrate that a deductive calculus for logic can be supplanted by a representational system in which all and only logical equivalents have exactly one and the same expression. The representation of quantifier-free sentences in terms of their truth-conditions (or, alternatively, Venn's representation) offers just such a notation. As Wittgenstein sees matters, systems that employ different logical particles '&,' '∨,' '⊃,' '~,' etc., hide their formal ('internal') nature. Wittgenstein attempted to exploit the truth-table representation of propositions as evidence for his view that a proper representation would reveal that tautologies and contradictions are scaffolding. (Landini 2007: 124).

In a notation in which the truth-table is the propositional sign, all and only logically equivalent propositions will be represented by the same sign. In this notation, all and only the propositions logically equivalent to '*P* ⊃ *Q*', and thus not only '~(*P* & ~*Q*)' but also '(*P* ⊃ *Q*) & (*Q* ∨ ~*Q*)', '~~(*P* ⊃ *Q*)', etc. will have the same representation, namely the Wittgensteinian truth-table (*P*, *Q: TFTT*).[19]

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[17] On the Tractarian distinction between function and operation see Anscombe 1959: ch. 8 (criticized in Dummett 1981: 324–325); Hylton 1997; Landini 2007: 127–129; Potter 2008: ch. 18.

[18] We mean sentential logic, for full predicate logic is not decidable and not expressible in a truth-tabular form.

[19] A language in which all and only logical equivalents have exactly one and the same expression makes it easy to see that there is a decision procedure for the logic of such language. Propositional logic is decidable (e.g. by truth-tables), full quantification theory only semi-decidable. Landini argues that the undecidability of quantification theory undermined Wittgenstein's project of finding a notation in which all and only logical equivalents have one and the same representation (2007: 118). As far as the propositional fragment is concerned, decidability is not a problem. Landini is correct that Wittgenstein's notations (the truth-tabular one we are discussing) and the *ab*-notation (Landini 2007: 113–114), substantially equivalent to the former, are decision procedures for propositional logic. The failure of the decidability of suitably large fragments of first-order (and higher-order) logics need not have undermined this project of Wittgenstein's, however. For one, some useful fragments of first-order logic are decidable, and for the full language the set of inconsistent formulas can be effectively decided. On the other hand, Wittgenstein's N-operator defines an undecidable theory of predicate logic.

If the truth-tabular notation presented by Wittgenstein at *TLP* 4.442 is admitted as a legitimate notation for sentential logic, we are at once provided with a means for distinguishing two kinds of notations. We label them 'tabular' and 'non-tabular'. Tabular notations are those in which only the *result* of a truth-operation on elementary propositions is represented. Non-tabular notations are those in which truth-operations are represented as operating on elementary propositions. Whenever an operation is represented, there is a possibility of representing the result of that operation by means of other operations, while when only the result of an operation is represented, this possibility is excluded.

The distinction between tabular and non-tabular notations shows that, while it is correct to say that the truth-tabular notation that Wittgenstein presents at *TLP* 4.442 approximates the notational ideal of having all and only logically equivalent propositions represented by the same sign, it is not the case that the notation for the sentential fragment of classical logic that Wittgenstein introduces at *TLP* 5.502, namely the so called N-notation, conforms to that ideal. The N-notation does not conform to the notational ideal because it is non-tabular, and in non-tabular notations there is always the syntactic possibility of representing the result of one operation by means of some other operations, and therefore of representing logically equivalent propositions by *syntactically distinct* propositional signs.

The reason is this. In the N-notation every truth-function is a result of successive application of the N-operator to elementary propositions (*TLP* 5.5). The application of the N-operator to one propositional argument '*P*', represented as 'N(*P*)', results in its negation, '~*P*'. Applied to two propositional arguments '*P*' and '*Q*', represented as 'N(*P, Q*)', it results in their joint denial, '~*P* & ~*Q*'. Applied to an arbitrary number of propositional arguments, it results in the joint denial of all the arguments (*TLP* 5.502). The N-operator is more powerful than the Sheffer stroke in the sense that, whereas the Sheffer stroke is a binary connective, the N-operator has a finite number of propositional arguments, taking the joint denial of them all.

Landini takes Wittgenstein to have thought that the N-notation has a truth-tabular nature akin to the truth-tabular representation of propositional logic Wittgenstein had presented at *TLP* 4.442, because in the truth-tabular notation all and only logically equivalents are represented by the same sign. This thought attributed to Wittgenstein's cannot be quite right, however. In the N-notation, 'NNN(*P*)' and 'N(*P*)' are logically equivalent but syntactically distinct formulas. The same is true, for example, of 'NN(*P*,*Q*)' and 'NN(*Q*,*P*)', which are logically equivalent but syntactically distinct. In order to make justice to Wittgenstein's claim concerning the N-notation, Landini proposes — attributing the origins of these proposals to Wittgenstein himself — five equational 'rules of operation' (Landini 2007: 129–130) by means of which the equivalence of the representation of logically equivalent propositions is meant to be achieved:

(L1) $\mathrm{N}(\xi_1, \ldots \xi_n) = \mathrm{N}(\xi_i, \ldots, \xi_j)$, $1 \leq i \leq n$, and $1 \leq j \leq n$, $i \neq j$.
(L2) $\mathrm{N}(\ldots\xi, \ldots, \xi\ldots) = \mathrm{N}(\ldots\xi, \ldots)$.
(L3) $\mathrm{N}(\ldots \mathrm{NN}(\xi_1, \ldots, \xi_n)\ldots) = \mathrm{N}(\ldots\xi_1, \ldots, \xi_n, \ldots)$.
(L4) $\mathrm{N}(\ldots\mathrm{N}(\ldots\xi, \ldots, \mathrm{N}\xi, \ldots)\ldots) = \mathrm{N}(\ldots)$.
(L5) $\mathrm{NN}(\gamma, \mathrm{N}(\xi_1, \ldots, \xi_n)) = \mathrm{N}(\mathrm{N}(\gamma, \mathrm{N}\xi_1), \ldots, \mathrm{N}(\gamma, \mathrm{N}\xi_n))$.

Clause (L1) corresponds to a generalization of the commutation rule '$\xi\gamma = \gamma\xi$'. Clause (L2) is a rule of elimination of equivalents. Clause (L3) corresponds to the rule of insertion and omission of double negation in whatever context it occurs ('$\xi = {\sim}{\sim}\xi$').

Clause (L4) allows us to delete or insert any argument of the form ‘N(…ξ, …, Nξ, ...)’, which is a tautology, in whatever context it occurs. Clause (L5) is a distribution rule.

Landini also takes Wittgenstein to have thought that (L1—5) now make the N-notation tabular: ‘by application of (1)–(5) we can see how Wittgenstein thought that the N-operator recovers the features of truth-table representations’ (2007: 130). Yet, Landini avers that (L1—5) are not to be taken as rules in their proper sense: ‘[t]hese rules assert the sameness of certain *practices* of operation. They are not, therefore, identity statements’ (*ibid.*). It is far from clear in what sense (L1-5) are not rules of logical equivalence. A notation in which a rule of double negation is applicable is a notation in which in every context such as ‘ξ = ~~ξ’ syntactically distinct sentences appear on both sides of the equivalence sign ‘=’. These are sentences which the rule declares to be logically equivalent. On both sides of Landini’s ‘=’ in (L3), two syntactically distinct sentences in N-notation must appear. He states that ‘N(NN(*p*, *q*), *r*) is to be regarded, *in some sense*, as the same as N(*p*, *q*, *r*)’ (2007: 129). Now, the sense in which the former sentence is to be regarded as the same as the latter is that they are *logically* equivalent, not that they are *syntactically* equivalent. For were they syntactically equivalent, (L3) could not be applied, because (L3) states the logical equivalence of syntactically distinct sentences. One cannot apply that rule to sentences which are not syntactically distinct. By the same token, a notation in which a rule of commutation is applicable is a notation in which on both sides of the sign ‘=’ in ‘ξγ = γξ’ distinct sentences appear, which the rule declares to be logically equivalent. And therefore on both sides of ‘=’ in (L1) two syntactically distinct sentences in N-notation must appear. Whenever a rule of logical equivalence of syntactically distinct sentences applies, it is not true that all logical equivalents have the same representation. It is not sufficient to do what Landini thinks Wittgenstein should have done, namely to declare that these clauses are not rules of logical equivalence but rules of syntactical equivalence. Such a declaration is merely nominal and *cannot transform a rule of logical equivalence into a rule of syntactical equivalence*.[20]

Landini’s Wittgenstein lands here in the same position as Macbeth does regarding BS: Macbeth takes TA to be operative in the BS notation in order to have Fig. 1 and Fig. 11 as the ‘same’ sign. Landini takes Wittgenstein to have wanted these quasi-rules of (L1–5) to be operative in the N-notation in order to have all and only logical equivalents represented by the same sign. But just as TA is a rule of logical equivalence and not of syntactical equivalence (like CPC), so are (L1–5). Labelling them ‘rules of practices of operation’ rather than ‘statements of logical equivalence’ does not change the fact that they must succeed in capturing exactly when syntactically distinct sentences are also logically equivalent. Landini is correct in saying that Wittgenstein’s truth-tabular notation fulfils the ideal of having all and only logical equivalents represented by the same sign. But the same is not true of the N-notation. The reason is that the truth-tabular notation is, indeed, tabular, while the N-notation is non-tabular.

A notation in which TA or L1 is a ‘given’ of the notation, and in which for that reason nothing like the rules of TA or L1 can be applied, is Peirce’s Alpha fragment of EGs. In it, as we have seen, the graphs in Fig. 22a–d are not different

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[20] To do so would be to conflate what can be done syntactically with what the semantic constrains are. One cannot conjure a syntactic permutation-invariance up by *fiat*, because whether one is able to have permutation-invariance in the syntax of the language depends on the semantic facts of the matter.

graph types whose logical equivalence is stated by TA or L1. They are different graph tokens of the same graph type, whose syntactical equivalence is given by the abstract syntax of the notation (juxtaposition on the SA is a symmetric and permutation-invariant operation). A symptom of this is the fact that while Landini's (L2–5) exist in EGs as rules of logical equivalence, (L1) is conspicuously missing in that system. Of the remaining rules, they are similar to permissive rules of transformation in the theory of logical graphs. (L2) is in Peirce's graphs the iteration/deiteration rule as applied within the same area of a subgraph. (L3) is the introduction/elimination of the double cut. (L4) is iteration/deiteration across the cuts to/from the areas in the nests of cuts. (L5) is a derived rule of distributivity from the basic rules.[21] But (L1) is missing because the abstract syntax of the system is permutation-invariant and this renders (L1) inapplicable. Alpha graphs are non-linear. Yet, in Alpha graphs we do not have one and the same syntactic representation for 'ξ' and '$\sim\sim\xi$', and a special permissive rule of transformation, the rule of erasure/insertion of double cut, has to be employed.[22] This means that the system of Alpha graphs is non-tabular.

Using Macbeth's and Shin's terminology, one might say that formulas written in tabular notation are 'multiply read' in formulas written in non-tabular notations. We would thus have a *third* kind of multiple reading, per tabularity, which is distinct from the multiple readings of the first kind (vocabulary) and those of the second kind (linearity). The tabular propositional sign (*P*, *Q: TFTT*) can be 'multiply read' in linear non-tabular notations, such as in the standard linear notation for sentential logic, as '$P \supset Q$', '$\sim(P \,\&\, \sim Q)$', '$(P \supset Q) \,\&\, (Q \vee \sim Q)$', '$\sim\sim(P \supset Q)$', and so on. Multiple readings of this sort are nevertheless radically different from the multiple readings that Macbeth and Shin attribute to BS and EGs formulas. Multiple readings per vocabulary are only possible in non-tabular notations, because they depend on a difference in primitive operators between languages, and since tabular notations do not represent operations but the results of the operations, multiple readings of the first kind do not concern them.

Likewise, the second type of multiple readings, per linearity, is only possible in non-tabular notations, and thus only where the multiple readings of the third kind are not possible. As a brief example, the *ab*-notation contained in the 'Notes' dictated by Wittgenstein to Moore in Norway in 1914, and then presented in the *Tractatus* at *TLP* 6.1203, is tabular but not linear. The formula in Fig. 22 is the truth-table of a certain truth-function of the elementary propositions '*p*' and '*q*', but it is not linear. In this notation 'T*p*F' is the same formula type of 'F*p*T', because the order in which 'T*p*F' and 'F*p*T' are written down is syntactically irrelevant, and because it is syntactically irrelevant where or in what form the parentheses that connect the truth-poles are drawn. The only syntactically relevant fact is which of the

[21] On the derivation of distributivity, which requires Peirce's Rule (residuation), see Ma & Pietarinen 2017a. We note that the set of rules (L2–L5) is semantically incomplete. To have a complete system one also needs a rule of erasure as well as the rule of insertion, which are not equivalence rules, plus the axiom that the sheet of assertion represents tautology.

[22] The rule of erasure/insertion of double cut is itself a rule of transformation which Peirce derived from certain more primitive considerations that have to do with a conditionals (the scroll) as the primitive sign and observations that some graphs are blanks (SA). The rule of erasure/insertion is the only rule that is needed to demonstrate how residuation (Peirce's Rule), in turn, emerges, showing the essentially observational nature of logical rules. The rule of erasure/insertion also achieves the same result as what TA does in linear languages.

truth-poles are connected and how. The propositional sign in Fig. 22 is not syntactically different from that in Fig. 23. They are tokens of the same type.

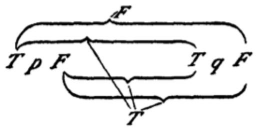

Figure 22

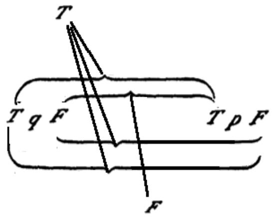

Figure 23

Likewise, the ordinary two-dimensional truth-tables are non-linear. The truth-tables in Fig. 24 and 25 are not two different types but distinct tokens of one and the same type. Order of neither rows nor columns is relevant in the truth-table notation in so far as the correspondence between the truth-values of the bases and the truth-values of the result is preserved.

P	*Q*	
T	T	T
T	F	F
F	T	T
F	F	T

Figure 24

	Q	*P*
T	T	T
F	F	T
T	T	F
T	F	F

Figure 25

In the same way, Wittgenstein’s truth-tabular representations at *TLP* 5.101 are non-linear. To see this, take the propositional signs in (16) and (17):

(16) (*P*, *Q*: TFTT)
(17) (*Q*, *P*: TFTT)

In Wittgenstein’s notation, the correspondence of the truth-value of the bases and that of the result is supposedly fixed. Swapping ‘*P*’ and ‘*Q*’ in (16) does not affect the result. What one obtains is (17) — simply another way of writing (16). Once the correspondence between the values of the bases (the ‘T’ and ‘F’ of *P* and *Q*) and the values of the result is fixed, all permutations of the propositional variables and of ‘T’ and ‘F’ in the result are permissible; they do not change a formula type into another type. Such permutations only produce distinct tokens of the same formula type.[23] Tabular languages cannot be linear. Linearity only concerns non-tabular languages.

[23] Cf. Potter 2008, 161-162,

5. Conclusion

How can there be substantially different notations — such as fragments of BS, EGs, N-notation, truth-tables, *ab*-notation, a system of Venn diagrams, etc. — that are expressively equivalent to classical propositional logic?[24] Our proposed answer comes, in summary, in three parts.

The first kind of multiple readings, per vocabulary, does not delineate any special type of notation. To say that certain notations have multiple readings per vocabulary is only a disguised way of saying that there are expressively equivalent notations that differ in their logical vocabularies. The possibility of multiple readings per vocabulary is not peculiar to BS or EGs: all multiple readings per vocabulary possible in BS are also possible in a sentential language with '~' and '⊃', and all those readings that are possible in EGs are also possible in a sentential language with '~' and '&'. Such multiple readings can only occur in non-tabular notations, both linear and non-linear. For this reason, and unlike the other two kinds of multiple readings (per linearity and per tabularity), multiple readings per vocabulary do not differentiate between qualitatively different logical notations but only among classes of sets of primitive logical operators.

The second kind of multiple readings, per linearity, distinguishes linear from non-linear notations. An Alpha graph can have multiple translations into a standard linear language for sentential logic. Euler diagrams are another kind of non-linear notation for the logic of classes and syllogisms. If we take TA to apply in BS as a rule of logical equivalence then BS is linear. This is not in conflict with BS being two-dimensional; it is only that both of its two dimensions are linearly ordered (permutation always produces sentence types). Hence, while BS might differ from a linear language with '~' and '⊃' under some other respects,[25] yet BS does not differ from linear languages under the parameter of linearity, namely that of producing different types by permutation. If, on the contrary, we take CPC to be operative in the BS as a rule of syntactical equivalence, then BS is non-linear, because its horizontals are partially ordered by the negation sign and because permutation does not always produce distinct formula types.

The possibility of having multiple readings of the third kind, per tabularity, distinguishes tabular from non-tabular notations. One and the same sign in a language that represents the result of truth-operations (tabular language), without representing those operations themselves, has multiple translations into a language in which the truth-operations themselves are represented (non-tabular languages). Wittgenstein's *ab*-notation and all truth-tabular notations are tabular (and so are Venn

[24] The present paper is not a review of all possible notational varieties equivalent to sentential logic, let alone the countless other logics. Our claims are limited to what we have justifiably taken to come from the pool of fundamental parameters of notational variability.

[25] Landini (2012) draws attention to the fact that the *Grundgezetze* have a rule of the 'amalgamation of the horizontals' which is absent in the 1879 presentation of the notation, and that this difference is crucial. According to Landini, the amalgamation of horizontals plays an essential role in the proof of a theorem from Basic Law IV, and there seems to be no way of representing this operation without Frege's BS (2012: 12–13, 52–57). In Landini's view, Frege's *Grundgesetze* notation is a two dimensional notation because only in this way it can allow for the rule of amalgamation. Whether Landini's claim is correct does not affect the fact that BS *is* linear in our sense, although linear on both its dimensions, given that TA is accepted as a rule of logical equivalence, and BS is non-linear if CPC is accepted as a rule of syntactical equivalence.

diagrams, see Landini 2007). In contrast, Wittgenstein's N-notation, as well as BS and EGs, are non-tabular. But unlike EGs, the N-notation is linear. Tabular notations, in contrast, are non-linear and can be multiply read in linear languages. Non-tabular notations can be either linear or non-linear.[26]

These, in sum, are our reasons for what constitute and what fail to constitute significant and meaningful notational differences between expressively equivalent logical languages.

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Conflict of interest
The authors declare that they have no conflict of interest.

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[26] Notations can be non-tabular along different types of linearity and non-linearity. We do not delve further into such further distinctions in the present paper. We also leave aside the important topic of whether expressively identical notations differ in their simplicity, iconicity and analyticity (R 300; CP 4.561n), or whether an emphasis on iconicity would count as a criterion for the development of certain graphical notations such as EGs, over others notations such as the algebraic ones. This topic has been covered in the recent literature and it necessitates introducing the quantificational Beta part of the logic of EGs over and above the propositional Alpha for its adequate investigation (Pietarinen 2015a,b). It is only the notational differences prompted by the Alpha part that the present paper is concerned with.

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