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Orientable arithmetic matroids

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Abstract

The theory of matroids has been generalized to oriented matroids and, recently, to arithmetic matroids. We want to give a definition of "oriented arithmetic matroid" and prove some properties like the "uniqueness of orientation".

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The aim of this paper is to relate two different generalizations of matroids: the oriented matroids and the arithmetic matroids.

Oriented matroids have a large use in mathematics and science (for a general reference see [1, 2]); they are related to the simplex method for linear programming, to the chirality of molecules in theoretical chemistry, and to knot theory. For instance, the Jones polynomial of a link is a specialization of the signed Tutte polynomial (see [3]) of an oriented graphic matroid [4, 5]. Another interesting fact is the correspondence between oriented matroids and arrangements of pseudospheres [6] that generalizes the correspondence between realizable matroids and central hyperplane arrangements.

Arithmetic matroids appear as the combinatorial object for the cohomology module of the complement of a toric arrangement [7, 8, 9]. The study of toric arrangements is related to zonotopes, partition functions, box splines, and Dahmen-Micchelli spaces (see [10, 11, 8]). The obvious correspondence between realizable arithmetic matroids and central toric arrangements has not been generalized to the non-realizable cases, so far.

With the aim of filling this gap, we define a class of well-behaved arithmetic matroids which we call *orientable arithmetic matroids* (see Definition 1.6) hoping that these correspond to "arrangements of pseudo-tori".

An $r \times n$ matrix with integer coefficients describes at the same time a central toric arrangement, an oriented matroid, and an arithmetic matroid. It comes natural to say that two matrices are equivalent if they describe the same toric arrangement. Geometrically, the group $\mathrm{GL}_r(\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})^n$ acts on the set of all $r \times n$ matrices with integer coefficients by left multiplication and sign reverse

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of the columns. Two realizations (i.e. matrices) of an arithmetic matroid are equivalent if and only if they belong to the same $\mathrm{GL}_r(\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})^n$-orbit.

The space $\mathrm{M}(r, n; \mathbb{Z})$ is included in $\mathrm{M}(r, n; \mathbb{Q})$ and the action of $\mathrm{GL}_r(\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})^n$ extends naturally to the one of $\mathrm{GL}_r(\mathbb{Q}) \times (\mathbb{Z}/2\mathbb{Z})^n$. Theorem 3.12 in [12] shows that all representations of an arithmetic matroid belong to the same $\mathrm{GL}_r(\mathbb{Q}) \times (\mathbb{Z}/2\mathbb{Z})^n$-orbit. From this fact, it can be easily deduced that representable arithmetic matroids have a unique orientation. We extend this result to the non-representable case, showing (Theorem 6.1) that orientable arithmetic matroids have a unique orientation (up to re-orientation).

We start by recalling some standard definitions and giving the compatibility condition, eq. (GP), between the orientation and the multiplicity function of an oriented arithmetic matroid. The condition (GP) coincides with the Plücker relation for the Grassmannian. We prove that oriented arithmetic matroids are closed under deletion, contraction, and duality. Next, we show that the condition (GP) implies a generalization of the Leibniz rule for the determinant. We state and prove a result about the uniqueness of orientation so that it makes sense to speak of orientable arithmetic matroids instead of oriented arithmetic matroids. Finally, we show that orientable arithmetic matroids, upon forgetting the multiplicity function, are realizable matroids (see Proposition 7.3). Moreover, we state the condition "strong GCD" (see Definition 7.1) implying the realizability of orientable arithmetic matroids.

All the discussion can be generalized to quasi-arithmetic matroids and so to matroids over $\mathbb{Z}$ (see [13]). It is not clear to the author how arithmetic matroids and orientable arithmetic matroids are related to matroids over hyperfields (see [14]).

1. Definitions

Let E be a finite totally ordered set. We will frequently make use of r-tuples of elements of E, so with an abuse of notation for any set $A = \{a_1, \ldots, a_r\} \subset E$ we will write A for the increasing tuple $(a_1, \ldots, a_r)$.

We give the definition of a matroid in terms of its basis, since [2, Theorem 1.2.3] shows that it is equivalent to the one given in terms of independent sets.

Definition 1.1. A matroid over a finite set E is a non-empty set $\mathcal{B} \subset \mathcal{P}(E)$ such that

$$\forall\, B_1, B_2 \in \mathcal{B}\, \forall\, x \in B_1 \setminus B_2\, \exists\, y \in B_2 \setminus B_1 \text{ such that } B_1 \setminus \{x\} \cup \{y\} \in \mathcal{B}. \quad (1)$$

Since this definition of matroids is cryptomorphic to the one involving the *rank function* (see [2, Theorem 1.3.2]), we denote a matroid with the pair (E, rk).

Throughout this paper we will denote the r-tuples $(y_i, x_2, \ldots, x_r)$ by $\underline{x}_i$ and $(y_0, \ldots, y_{i-1}, y_{i+1}, \ldots, y_r)$ by $\underline{y}^i$, for $i = 0, \ldots, r$, where $\underline{x} = (x_2, \ldots, x_r)$ and $\underline{y} = (y_0, \ldots, y_r)$.

Definition 1.2 ([1, Definition 3.5.3])**.** A *chirotope* is a function $\chi\colon E^r \to \{-1, 0, 1\}$ such that:

(B0) it is not identically zero, i.e. $\chi \not\equiv 0$,

(B1) it is alternating, i.e. $\chi(\sigma \underline{x}) = \operatorname{sgn}(\sigma)\chi(\underline{x})$ for all $\sigma \in \mathfrak{S}_r$,

(B2) for all $x_2, \dots, x_r$ and all $y_0, \dots, y_r \in E$ such that

$$\chi(\underline{x}_i)\chi(\underline{y}^i) \geq 0,$$

for all $i > 0$, then we have

$$\chi(\underline{x}_0)\chi(\underline{y}^0) \geq 0.$$

Definition 1.3 ([1, p. 134])**.** The *re-orientation* with respect to $A \subseteq E$ of a chirotope χ is the chirotope χ' defined by

$$\chi'(\underline{x}) = (-1)^{|A \cap \{x_1, \dots, x_r\}|}\chi(\underline{x}).$$

Two chirotopes are equivalent if one is a re-orientation of the other one.

The set $\{\{b_1, \dots, b_r\} \subset E \mid \chi(b_1, \dots, b_r) \neq 0\}$ is the matroid over E associated with the chirotope χ. A well-known cryptomorphism of Lawrence [15] between *oriented matroids* and chirotopes is stated in [1, Theorem 3.5.5].

For every matroid $\mathcal{M} = (E, \mathrm{rk})$ and every subset $A \subseteq E$ we denote by $\mathcal{M}/A$ the *contraction* of A and with $\mathcal{M} \setminus A$ the *deletion* of A.

Let us recall the definition of "arithmetic matroid" introduced in [16, 17].

Definition 1.4. A *molecule* (A, B) of the matroid is a pair of sets $A \subset B \subseteq E$ such that the matroid $(\mathcal{M}/A) \setminus B^c$ has a unique basis.

For $A \subseteq E$ we denote the maximal subset $S \supseteq A$ of rank equal to $\mathrm{rk}(A)$ by $\overline{A}$.

Definition 1.5. An *arithmetic matroid* is (E, rk, m) such that (E, rk) is a matroid and $m\colon \mathcal{P}(E) \to \mathbb{N}_+ = \{1, 2, \dots\}$ a function satisfying:

1. if $A \subseteq E$ and $x \in E$ is dependent on A, then $m(A \cup \{x\}) | m(A)$;
2. if $A \subseteq E$ and $x \in E$ is independent on A, then $m(A) | m(A \cup \{x\})$;
3. if (A, B) is a molecule then
$$m(A)m(B) = m(B \cap \overline{A})m((B \setminus \overline{A}) \cup A);$$
4. if (A, B) is a molecule then
$$\rho(A, B) \overset{\text{def}}{=} \sum_{A \subseteq S \subseteq B} (-1)^{|\overline{A} \cap B| - |S|} m(S) \geq 0.$$

We call m the *multiplicity function.*

Definition 1.6. An *oriented arithmetic matroid* $(E, \mathrm{rk}, m, \chi)$ is a matroid (E, rk) of rank r together with two structures: a chirotope $\chi\colon E^r \to \{-1, 0, 1\}$ and a multiplicity function $m\colon \mathcal{P}(E) \to \mathbb{N}_+$ such that:

1. The unoriented matroid associated with the chirotope χ is the matroid (E, rk).
2. The triple (E, rk, m) is an arithmetic matroid.
3. For all $x_2, \ldots, x_r$ and all $y_0, \ldots, y_r \in E$ the following equality holds

$$\sum_{i=0}^{r} (-1)^i \chi(\underline{x}_i) m(\underline{x}_i) \chi(\underline{y}^i) m(\underline{y}^i) = 0, \tag{GP}$$

   where $\underline{x}_i = (y_i, x_2 \ldots, x_r)$ and $\underline{y}^i = (y_0, \ldots y_{i-1}, y_{i+1} \ldots y_r)$.

Since the rank function rk is completely determined by the chirotope χ, we omit rk and write (E, χ, m) for an oriented arithmetic matroid.

Remark 1.7. Our property (GP), related to the Grassmannian-Plücker relations, implies the properties (GP_r), for all r, defined in [18, Definition 10.3].

Notice that the compatibility condition (GP) involves only the values of the multiplicity function on the basis of (E, rk).

Remark 1.8. The condition (GP) implies (B2) of Definition 1.2.

Example 1.9. Consider the matrix

$$X = \begin{pmatrix} 1 & 0 & 0 & -4 & 0 & 3 & 0 \\ 0 & 2 & 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 3 & 0 & 1 & -1 & -1 \end{pmatrix}$$

and let $E = \{1, 2, \ldots, 7\}$. The set $\mathcal{B} \subset \mathcal{P}(E)$ of indexes $B \subset E$ such that the corresponding columns of X are a basis of $\mathbb{Q}^3$ is a matroid.

Let $\chi\colon E^3 \to \{-1, 0, 1\}$ be the function defined by $(i, j, k) \mapsto \mathrm{sgn}(\det X[i, j, k])$ where $X[i, j, k]$ is the square matrix whose columns are the ith, jth and kth columns of X (in this order). This function χ is a chirotope whose underlying matroid is $\mathcal{B}$.

The matrix X defines an arithmetic matroid (E, rk, m) (see [16, 17]), for any base $B = \{i, j, k\} \subset E$ we have $m(B) = |\det X[i, j, k]|$.

2. Deletion

The deletion of $A \subset E$ is an operation defined for matroids [2, p. 22], for oriented matroids [1, p. 133], and for arithmetic matroids [16, section 4.3] [17, section 3]. We now define a deletion operation for oriented arithmetic matroids.

The triple $(E \setminus A, \chi \setminus A, m \setminus A)$ satisfies the first two conditions of Definition 1.6.

Proposition 2.1. *The triple* $(E \setminus A, \chi \setminus A, m \setminus A)$ *is an oriented arithmetic matroid.*

Proof. Let s be the rank of $E \setminus A$ and $\underline{f} = (a_1, \ldots, a_{r-s}) \subseteq A$ such that $\mathrm{rk}((E \setminus A) \cup \underline{f}) = r$. Consider the elements $x_2, \ldots, x_s$ and $y_0, \ldots, y_s$ in $E \setminus A$. For all $0 \leq i \leq s$, the triples $(\underline{x}_i, \underline{y}^i, \underline{f})$ and $(\underline{y}^i, \underline{x}_i, \underline{f})$ are molecules. The equality

$$m(\underline{x}_i \cup \underline{y}^i)^2 m(\underline{x}_i \cup \underline{f}) m(\underline{y}^i \cup \underline{f}) = m(\underline{x}_i \cup \underline{y}^i \cup \underline{f})^2 m(\underline{x}_i) m(\underline{y}^i)$$

follows from condition (3) applies to the two molecules. Notice that $\underline{x}_i \cup \underline{y}^i$ does not depend on i so we can denote it $\underline{x} \cup \underline{y}$. We have

$$m(\underline{x} \cup \underline{y} \cup \underline{f})^2 \sum_{i=0}^{s} (-1)^i \chi(\underline{x}_i \cup \underline{f}) m(\underline{x}_i) \chi(\underline{y}^i \cup \underline{f}) m(\underline{y}^i) =$$
$$= m(\underline{x} \cup \underline{y})^2 \sum_{i=0}^{s} (-1)^i \chi(\underline{x}_i \cup \underline{f}) m(\underline{x}_i \cup \underline{f}) \chi(\underline{y}^i \cup \underline{f}) m(\underline{y}^i \cup \underline{f}).$$

The right side is, up to a non-zero scalar, the equation (GP) applied to $x_2, \ldots, x_s$, $a_1, \ldots, a_{r-s}$ and $y_0, \ldots, y_s, a_1, \ldots, a_{r-s}$ for the oriented arithmetic matroid (E, χ, m). Therefore, we have proven the claimed equality

$$\sum_{i=0}^{s} (-1)^i \chi(\underline{x}_i \cup \underline{f}) m(\underline{x}_i) \chi(\underline{y}^i \cup \underline{f}) m(\underline{y}^i) = 0,$$

and this completes the proof. □

3. Contraction

The contraction (or restriction) of $A \subset E$ is an operation defined for matroids [2, p. 22], for oriented matroids [1, p. 134], and for arithmetic matroids [16, section 4.3] [17, section 3]. We now define a contraction operation for oriented arithmetic matroids.

Let A be a subset of E and call $r - s$ its rank. We choose an independent list $\underline{f} = (a_1, \ldots, a_{r-s})$ of elements in A. Define $\chi/A\colon (E \setminus A)^s \to \{-1, 0, 1\}$ as $\chi/A(\underline{z}) = \chi(\underline{z} \cup \underline{f})$ and $m/A(S) = m(A \cup S)$.

Proposition 3.1. *The triple* $(E \setminus A, \chi/A, m/A)$ *is an oriented arithmetic matroid.*

Proof. We call $T = A \setminus \underline{f}$ and fix the elements $x_2, \ldots, x_s$ and $y_0, \ldots, y_s$ of $E \setminus A$. Observe that $(\underline{f}, T, \underline{x}_i)$ and $(\underline{f}, T, \underline{y}^i)$ are molecules of (E, rk). Thus

$$m(A)^2 m(\underline{x}_i \cup \underline{f}) m(\underline{y}^i \cup \underline{f}) = m(\underline{f})^2 m(\underline{x}_i \cup A) m(\underline{y}^i \cup \underline{f}).$$

Since $m(A)$ and $m(f)$ are nonzero, then condition (GP) for $\underline{x}$ and $\underline{y}$ in the contracted matroid is equivalent to condition (GP) for $\underline{x} \cup \underline{f}$ and $\underline{y} \cup \underline{f}$ in the original matroid. □

4. Duality

The duality is an operation defined for matroids [2, chapter 2], for oriented matroids [1, p. 135], and for arithmetic matroids [16, p. 339] [17, p. 5526]. We now define duality for oriented arithmetic matroids.

Recall that the set E is ordered. For every $\underline{z} = (z_1, \dots, z_k) \subseteq E$ we call $\underline{z}'$ the complement of $\underline{z}$ in E with some arbitrary order and let $\sigma(\underline{z}, \underline{z}')$ be the sign of the permutation that reorders the list $(\underline{z}, \underline{z}')$ as they appear in E. We define $\chi^* \colon E^{n-r} \to \{-1, 0, 1\}$ as

$$\chi^*(\underline{z}) = \chi(\underline{z}')\sigma(\underline{z}, \underline{z}')$$

and the multiplicity function $m^* \colon \mathcal{P}(E) \to \mathbb{N}_+$ as $m^*(\underline{z}) = m(\underline{z}')$.

Proposition 4.1. *The triple (E, χ^*, m^*) is an oriented arithmetic matroid.*

Proof. Let $\underline{x} = (x_2, \dots, x_{n-r})$ and $\underline{y} = (y_0, \dots, y_{n-r})$ be two sublists of E. Coherently with the notation above, let $\underline{x}' = (x'_0, \dots, x'_r)$ and $\underline{y}' = (y'_2, \dots, y'_r)$ be their complements. For every $0 \leq i \leq n-r$ the element y_i is equal to x_k or x'_j. In the first case $\chi^*(\underline{x}_i) = 0$ and in the second case

$$\chi^*(\underline{x}_i) = \chi(\underline{x}'^j)\sigma(\underline{x}_i, \underline{x}'^j) = (-1)^{n-r+1+j}\chi(\underline{x}'^j)\sigma(\underline{x}, \underline{x}').$$

Analogously, if $y_i = x'_j$ then

$$\chi^*(\underline{y}^i) = \chi(\underline{y}'_j)\sigma(\underline{y}^i, \underline{y}'_j) = (-1)^{n-r+i}\chi(\underline{y}'_j)\sigma(\underline{y}, \underline{y}')$$

where $\underline{y}'_j = (x'_j, y'_2, \dots, y'_r)$. If $y_i = x'_j$, then $m^*(\underline{x}_i) = m(\underline{x}'^j)$ and $m^*(\underline{y}^i) = m(\underline{y}'_j)$. Thus, up to a sign, the condition (GP) for $\underline{y}'$ and $\underline{x}'$ in the original matroid implies condition (GP) for $\underline{x}$ and $\underline{y}$ in the dual matroid. □

5. GP-functions

We now study functions satisfying a relation that looks like the Plücker relation for the Grassmannian. A posteriori all these functions are nothing else that the determinant $\det \colon V^r \to \mathbb{Q}$ restricted to a finite (multi-)set $E \subset V$.

Definition 5.1. A map $f \colon E^r \to \mathbb{Q}$ is a *GP-function* if it is alternating and for all $\underline{x} \in E^{r-1}$ and all $\underline{y} \in E^{r+1}$ the following equality holds

$$\sum_{i=0}^{r} (-1)^i f(y_i, x_2 \dots, x_r) f(y_0, \dots y_{i-1}, y_{i+1} \dots y_r) = 0.$$

We denote the point-wise product of two function χ and m with

$$\chi m(\underline{b}) \overset{\text{def}}{=} \chi(\underline{b}) \cdot m(\underline{b}).$$

Example 5.2. The main examples of GP-function are the functions χm for every oriented arithmetic matroid. Another example is given by a map $i\colon E \to V$, where V is a $\mathbb{Q}$-vector space of dimension n, and consider the function $\det\colon V^n \to \mathbb{Q}$. The composition of the natural inclusion $E^n \to V^n$ with the determinant det is a GP-function.

The following theorem is a generalization of the Leibniz formula for the determinant.

Theorem 5.3. *Let $f\colon E^r \to \mathbb{Q}$ be a GP-function. Then for all $(a_1, \dots, a_r)$ in E^r and $(b_1, \dots, b_r) \in E^r$ the following formula holds:*

$$\sum_{\sigma \in \mathfrak{S}_r} (-1)^{\operatorname{sgn}\sigma} \prod_{i=1}^{r} f(a_1, \dots, b_{\sigma(i)}, \dots, a_r) = f(a_1, \dots, a_r)^{r-1} f(b_1, \dots, b_r), \quad (2)$$

where $b_{\sigma(i)}$ substitutes a_i.

Proof. We prove the lemma by induction, the base case $r = 2$ being trivial. We fix $(a_1, \dots, a_r) \in E^r$ and $(b_1, \dots, b_r) \in E^r$. Let $g\colon E^{r-1} \to \mathbb{Q}$ be the GP-function defined by

$$g(x_2, \dots, x_r) = f(a_1, x_2, \dots, x_r).$$

By induction we have

$$\sum_{\sigma \in \mathfrak{S}_{r-1}} (-1)^{\operatorname{sgn}\sigma} \prod_{i=2}^{r} g(a_2, \dots, c_{\sigma(i)}, \dots, a_r) = g(a_2, \dots, a_r)^{r-2} g(c_2, \dots, c_r). \quad (3)$$

The left hand side of the eq. (2) can be rewritten as:

$$\sum_{j=1}^{r} f(b_j, a_2, \dots, a_r) \sum_{\sigma \in \mathfrak{S}_{r-1}} (-1)^{\operatorname{sgn}\sigma + \operatorname{sgn}\tau_j} \prod_{i=2}^{r} f(a_1, \dots, b_{\sigma(\tau_j(i))}, \dots, a_r), \quad (4)$$

where $\tau_j = (1, j)$ and $\mathfrak{S}_{r-1}$ is the subgroup of $\mathfrak{S}_r$ of permutations that fix the element 1. Now, for every j, we use eq. (3) with $c_i = b_{\tau_j(i)}$ to manipulate expression (4):

$$f(a_1, \dots, a_r)^{n-2} \Big[f(b_1, a_2, \dots, a_r) f(a_1, b_2, \dots, b_r) - \sum_{j=1}^{r} f(b_j, a_2, \dots, a_r) \cdot f(a_1, b_2, \dots, b_1, \dots, b_r) \Big]$$

that is equal to the left hand side of (2) since f is a GP-function. $\square$

Lemma 5.4. *Let f and g be two GP-functions and $B = (b_1, \dots, b_r) \in E^r$. Suppose that $f(B) = g(B) \neq 0$ and $f(C) = g(C)$ for all $C = (c_1, \dots, c_r) \in E^r$ such that $|\{i \mid c_i \neq b_i\}| = 1$. Then $f = g$.*

Proof. We use Theorem 5.3 for the function f and g. Considering eq. (2) for f and g, the left hand sides for f and g are equal, so

$$f(b_1, \ldots, b_r)^{r-1} f(d_1, \ldots, d_r) = g(b_1, \ldots, b_r)^{r-1} g(d_1, \ldots, d_r),$$

for all $d_1, \ldots, d_r \in E$. By hypothesis $f(b_1, \ldots, b_r) = g(b_1, \ldots, b_r) \neq 0$, thus we have $f(d_1, \ldots, d_r) = g(d_1, \ldots, d_r)$ for all $(d_1, \ldots, d_r) \in E^r$. □

6. Uniqueness of the orientation

This section is dedicated to prove the following theorem:

Theorem 6.1. *Let (E, χ, m) and (E, χ', m) be two oriented arithmetic matroids over the same matroid (E, rk). Then χ' is a re-orientation of χ.*

We fix a total order on $E \simeq [n]$ such that $[r]$, the first r elements, are a basis of the matroid.

The basis graph of a matroid was first studied in [19] and [20].

Definition 6.2. The *basis graph* $\mathcal{BG}$ of a matroid $(E, \mathcal{B})$ is the graph on the set $\mathcal{B}$ of vertices with an edge between two vertices B_1 and B_2 if $|B_1 \setminus B_2| = 1$.

Once chosen a basis B_0 of a matroid, we define $\mathcal{BG}_1$ to be the induced subgraph of $\mathcal{BG}$ whose vertices are all vertices adjacent to B_0. Define $\mathcal{BG}_{\leq 1}$ the induced subgraph whose vertices are the ones adjacent to B_0 and B_0 itself.

Our strategy in proving Theorem 6.1 is the following: suppose that $\chi([r]) = \chi'([r])$, Lemma 6.7 proves that, up to reorientation, χ and χ' coincides on all vertices of distance one from $[r]$. Lemma 6.8 proves that $\chi(B) = \chi'(B)$ using Theorem 5.3.

Definition 6.3. Consider a matroid (E, rk). Let $\mathcal{G}$ be the bipartite graph on vertices E and an edge between $i \in B_0$ and $j \in E \setminus B_0$ if $B_0 \setminus \{i\} \cup \{j\}$ is a basis. We call this graph the B_0-*fundamental circuit graph* of the matroid (E, rk).

Definition 6.4. The *Line graph* $L(G)$ of a graph $G = (V, E)$ is the graph whose set of vertices is the set E of edges in G. The graph $L(G)$ has an edge between e_1 and $e_2 \in E$ if and only if the edges e_1 and e_2 are incident in G.

The Line graph of $\mathcal{G}$ is the graph $\mathcal{BG}_1$. A *coordinatizing path* in $\mathcal{G}$ is a spanning forest of the graph $\mathcal{G}$. We choose a coordinatizing path P of the graph $\mathcal{G}$ and its Line graph $L(P)$ is an induced subgraph of $\mathcal{BG}_1$.

Example 6.5. We continue the Example 1.9: we chose as B_0 the basis $\{1, 2, 3\}$. The B_0-fundamental circuit graph is shown in Example 6.5. The six highlighted edge are a choice of a coordinatizing path P, and its Line graph $L(P)$ has vertices the six bases $\{4, 2, 3\}, \{1, 4, 3\}, \{1, 5, 3\}, \{1, 2, 5\}, \{1, 2, 6\}$, and $\{1, 2, 7\}$. The graph $\mathcal{BG}_1$ has two more vertices given by the bases $\{6, 2, 3\}$ and $\{1, 7, 3\}$.

The following lemma is essentially proven in [21, Lemma 6].

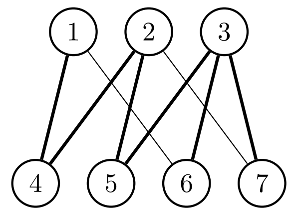

Figure 1: The B_0-fundamental circuit graph and a coordinatizing path P highlighted.

Lemma 6.6. *Let (E, χ, m) be an oriented arithmetic matroid with basis graph $\mathcal{BG}$, B_0 be a vertex of $\mathcal{BG}$ and P be a coordinatizing path in a graph $\mathcal{G}$ (i.e. the B_0-fundamental circuit graph of the matroid). Then there exists a re-orientation χ' of χ such that $\chi'(B) = \chi'(B_0)$ for all vertices $B \in L(P)$.*

We prove in our setting the equivalent of [21, Lemma 9].

Lemma 6.7. *Let (E, rk, m) be an arithmetic matroid with basis graph $\mathcal{BG}$, B_0 be a vertex of $\mathcal{BG}$ and P be a coordinatizing path in the graph $\mathcal{G}$. Let χ and χ' be two orientations of the arithmetic matroid (E, rk, m) such that $\chi(B) = \chi(B_0)$ and $\chi'(B) = \chi'(B_0)$ for all vertices $B \in L(P)$. If $\chi(B_0) = \chi'(B_0)$, then $\chi(B') = \chi'(B')$ for all $B' \in \mathcal{BG}_{\leq 1}$.*

Proof. Consider the subgraph $\mathcal{H}$ of $\mathcal{G}$ with the same set of vertices and with an edge between $i \in B_0$ and $j \in E \setminus B_0$ if and only if $\chi(B_0 \setminus \{i\} \cup \{j\}) = \chi'(B_0 \setminus \{i\} \cup \{j\}) \neq 0$. The graph $\mathcal{H}$ contains the chosen coordinatizing path P by hypothesis. Suppose that $\mathcal{H} \neq \mathcal{G}$ and let T ($T \neq \emptyset$) be the set of edges of $\mathcal{G}$ not contained in $\mathcal{H}$. For each $(i, j) \in T$ we can consider $l(i, j)$ the length of the minimal path in $\mathcal{H}$ connecting the vertices i and j. Obviously, $l(i, j)$ is a odd number greater than 2. Let us fix $(h, k) \in T$ with $l(h, k)$ minimal among all $l(i, j)$ for $(i, j) \in T$ and a minimal path $Q = (h = i_0, j_0, i_1, \dots, i_t, j_t = k)$ in $\mathcal{H}$ between h and k, where t is defined by the equality $2t + 1 = l(h, k)$. By minimality of (h, k), two vertices i_a and j_b are connected in $\mathcal{G}$ if and only if $a = b$, $a = b + 1$ or $b = t$ and $a = 0$.

Without loss of generality, we suppose $i_v = v+1$ for $0 \leq v \leq t$, $B_0 = [r]$, and $j_v = r+v+1$ for $0 \leq v \leq t$. Apply Theorem 5.3 with $a_i = i$ and $b_j = t+j+2$ to the GP-functions χm and $\chi' m$. The product $\prod_{i=1}^{r} \chi m(a_1, \dots, b_{\sigma(i)}, \dots, a_r)$ is non zero if and only if $(a_i, b_{\sigma(i)})$ is an edge in the path Q or if $\{a_i, b_{\sigma(i)}\} = \{h, k\}$ for all $i \leq t + 1$ and $b_{\sigma(i)} = a_i$ for all $t + 1 < i \leq r$. The same implication holds for the function $\chi' m$. This happens only for two different permutations τ and

η, say that $\tau(h) = k$ and $\eta(h) = j_0$. We define

$$\begin{aligned}
x &\stackrel{\text{def}}{=} \chi m(a_1, \ldots, a_{h-1}, b_k, a_{h+1}, \ldots, a_r),\\
a &\stackrel{\text{def}}{=} \prod_{i \neq h} \chi m(a_1, \ldots, b_{\tau(i)}, \ldots, a_r),\\
b &\stackrel{\text{def}}{=} \prod_{i=1}^{r} \chi m(a_1, \ldots, b_{\eta(i)}, \ldots, a_r),\\
c &\stackrel{\text{def}}{=} \chi m(a_1, \ldots, a_r)^{r-1} \chi m(b_1, \ldots, b_r).
\end{aligned}$$

Thus, eq. (2) can be reduced to $ax + b = c$. The equivalent relation for χ' is $ax' + b = c'$ with $x' = \pm x$ and $c' = \pm c$. Since a, b, c and x are non-zero, then $x = x'$ and so

$$\chi(a_1, \ldots, a_{h-1}, b_k, a_{h+1}, \ldots, a_r) = \chi'(a_1, \ldots, a_{h-1}, b_k, a_{h+1}, \ldots, a_r).$$

This equality contradicts the assumption $\mathcal{H} \neq \mathcal{G}$. □

Lemma 6.8. *Let (E, rk, m) be an arithmetic matroid and χ and χ' two orientations of the arithmetic matroid (E, rk) that coincide on the elements of $\mathcal{BG}_{\leq 1}$. Then $\chi = \chi'$.*

Proof. By hypothesis both χm and $\chi' m$ are GP-functions, so by Lemma 5.4 they are equal. □

Theorem 6.1 follows from Lemmas 6.6 to 6.8.

We show an example of an orientable arithmetic matroid that is not representable.

Example 6.9. Let $([3], \mathrm{rk}, m)$ be the orientable arithmetic matroid associated with the matrix $\left(\begin{smallmatrix} 1 & 1 & 2 \\ 0 & n & n \end{smallmatrix}\right)$ for a integer $n > 1$. Let m' be the multiplicity function defined by $m'([3]) = 1$ and $m'(A) = m(A)$ for all $A \subsetneq [3]$. The triple $([3], \mathrm{rk}, m')$ is a non-representable arithmetic matroid, since the multiplicity function does not have the GCD property. This matroid is orientable, indeed any orientation χ of $([3], \mathrm{rk}, m)$ is an orientation of $([3], \mathrm{rk}, m')$. Figure 2 represents an arrangement of hypersurfaces of T^2, the compact two dimensional torus, whose pattern of intersections coincides with the arithmetic matroid $([3], \mathrm{rk}, m')$ for $n = 3$.

7. Representability

Definition 7.1. An arithmetic matroid (E, rk, m) has the *strong GCD property* if

$$m(A) = \gcd\{m(B) \mid B \text{ basis and } |B \cap A| = \mathrm{rk}\, A\}$$

for all $A \subseteq E$.

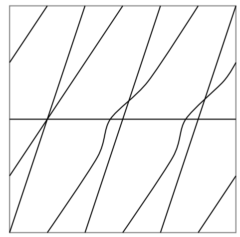

Figure 2: An arrangement of hypersurfaces in the compact torus.

Notice that arithmetic matroids with the strong GCD property are uniquely determined by their values on the basis of the underlying matroid. The strong GCD property is equivalent to the statement that both (E, rk, m) and (E, rk^*, m^*) are GCD arithmetic matroids.

Proposition 7.2. *Let (E, rk, m) be an orientable arithmetic matroid. Then the underlying matroid (E, rk) is representable over $\mathbb{Q}$.*

Proof. We choose an orientation χ of the arithmetic matroid (E, rk, m) and a basis $B_0 = (b_1, \dots b_r)$ of the matroid. For each $e \in E$, consider in $\mathbb{Q}^r$ the vector

$$v_e \stackrel{\text{def}}{=} (\chi m(b_1, \dots, b_{i-1}, e, b_{i+1}, \dots, b_r))_{1 \le i \le r}.$$

We choose a total order on $E = [n]$ such that $B_0 = [r]$. Let N be the matrix that represent the vectors v_i, for $i = 1, \dots, n$, in the canonical basis of $\mathbb{Q}^r$. We claim that, for each $A \subseteq [n]$ of cardinality r, the functions $\det N[A]$ and $\chi m(B_0)^{r-1} \chi m(A)$ coincide. The claimed equality holds if $A = B_0$. If $A = \{1, \dots, i-1, i+1, \dots, r, j\}$, then

$$\begin{aligned} \det N[A] &= (-1)^{r-i} \frac{\chi m(1, \dots, i-1, j, i+1, \dots, r)}{\chi m([r])} \det N[[r]] \\ &= \frac{\chi m(A)}{\chi m(B_0)} \chi m(B_0)^r = \chi m(B_0)^{r-1} \chi m(A) \end{aligned}$$

The GP-function $\chi m(B_0)^{r-1} \chi m(\cdot)$ and $\det N[\cdot]$ coincide on $\mathcal{BG}_{\le 1}$, thus by Lemma 5.4 $\chi m(B_0)^{r-1} \chi m(B) = \det N[B]$ for all $B \subset E$, $|B| = r$. The matroid defined by N is (E, rk) since they have the same set of basis. □

Proposition 7.3. *Let (E, rk, m) be an orientable arithmetic matroid with the strong GCD property. Then (E, rk, m) is representable.*

Proof. Consider an orientation of (E, rk, m), the vectors $v_e \in \mathbb{Q}^r$ for $e \in E$ defined in the proof of Proposition 7.2, and let Λ the lattice generated by $\{v_e\}_{e \in E}$. Let G be a finite abelian group of cardinality $m(\emptyset) = m(E)$. We claim that the elements $(v_e, 0)$ in $\Lambda \times G$ are a representation of the arithmetic matroid (E, rk, m). Observe that the index $[\mathbb{Z}^r : \Lambda]$ is equal to $m(B_0)^{r-1} m(E)$. Let (E, rk, m') be the arithmetic matroid described by the vectors $(v_e, 0)$. The multiplicity functions m and m' coincides on all basis of the matroid (E, rk), hence by the GCD property $m = m'$. □

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