

Local densities for a class of degenerate diffusions

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Abstract. We study a class of \mathbb{R}^d -valued continuous strong Markov processes that are generated, only locally, by an ultra-parabolic operator with coefficients that are regular w.r.t. the intrinsic geometry induced by the operator itself and not w.r.t. the Euclidean one. The first main result is a local Itô formula for functions that are not twice-differentiable in the classical sense, but only intrinsically w.r.t. to a set of vector fields, related to the generator, satisfying the Hörmander condition. The second main contribution, which builds upon the first one, is an existence and regularity result for the local transition density.

Résumé. Dans cet article on étudie une classe de processus de type Markov fort continus à valeurs dans \mathbb{R}^d qui sont engendrés, seulement localement, par un opérateur ultra-parabolique avec des coefficients réguliers par rapport à la géométrie intrinsèque induite par l'opérateur lui-même et non par rapport à la géométrie euclidienne. On obtient deux résultats importants. Le premier est une formule locale d'Itô valable pour des fonctions qui ne sont pas deux fois différentiables dans le sens classique, mais seulement intrinsèquement par rapport à un ensemble de champs de vecteurs, liés au générateur, satisfaisant à la condition d'Hörmander. La deuxième contribution, qui s'appuie sur la première, est un résultat d'existence et de régularité pour la densité de transition locale.

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1. Introduction

We study an \mathbb{R}^d -valued continuous strong Markov process X that is generated, in a way that will be specified later, *only locally* on a domain $D \subseteq \mathbb{R}^d$ by the degenerate operator

$$\mathcal{A}_t := \frac{1}{2} \sum_{i,j=1}^{p_0} a_{ij}(t, x) \partial_{x_i x_j} + \sum_{i=1}^{p_0} a_i(t, x) \partial_{x_i} + \langle Bx, \nabla_x \rangle \quad t \in [0, T_0[, x \in D. \quad (1.1)$$

Above, $p_0 \leq d$ and B is a $(d \times d)$ -matrix with constant real entries. In this paper, the focus is mainly on the case $p_0 < d$, which implies that no ellipticity condition on \mathcal{A}_t is satisfied (i.e. the second order part is fully degenerate). The main structural assumption on the local-generator \mathcal{A}_t is the following

Assumption 1.1. The matrix B is such that the Kolmogorov operator

$$\mathcal{K} := \frac{1}{2} \sum_{i=1}^{p_0} \partial_{x_i}^2 + \underbrace{\langle Bx, \nabla_x \rangle}_{=: Y} + \partial_t, \quad x \in \mathbb{R}^d, \quad (1.2)$$

is hypoelliptic on $\mathbb{R} \times \mathbb{R}^d$. Equivalently, the vector fields $\partial_{x_1}, \dots, \partial_{x_{p_0}}$ and Y satisfy the Hörmander condition

$$\text{rank Lie}(\partial_{x_1}, \dots, \partial_{x_{p_0}}, Y) = d.$$

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Assumption 1.1 is the only hypothesis required for the first main result of the paper, namely the *intrinsic* Itô’s formula. The second main result, about the local density of X , is stated under the following additional

Assumption 1.2. There exist $\alpha \in (0, 1]$, $N \in \mathbb{N} \cup \{0\}$ and $M > 0$ such that:

- (i) $a_{ij}, a_i \in C_B^{N,\alpha}]0, T_0[\times D$ for any $i, j = 1, \dots, p_0$, with all the (Lie) derivatives bounded by M ;
- (ii) the following coercivity condition holds on D :

$$M^{-1}|\xi|^2 \leq \sum_{i,j=1}^{p_0} a_{ij}(t, x)\xi_i\xi_j \leq M|\xi|^2, \quad t \in]0, T_0[, x \in D, \xi \in \mathbb{R}^{p_0}.$$

The spaces $C_B^{N,\alpha}]0, T_0[\times D$ appearing above are the intrinsic Hölder spaces induced by the vector fields $\partial_{x_1}, \dots, \partial_{x_{p_0}}$ and Y : their definition is recalled, for the reader’s convenience, in Section 2. Thanks to condition (ii) above, $(\partial_t + \mathcal{A}_t)$ can be seen as a perturbation of the hypoelliptic operator \mathcal{K} in (1.2).

Operators of the form (1.1) appear in various applicative fields; in physics, they describe the dynamics of some stochastic Hamiltonian systems (see e.g. [19] and [21]). In mathematical finance, a relevant prototype example that fits Assumptions 1.1 and 1.2 is the stochastic process $X = (X^1, X^2)$ defined by

$$\begin{cases} dX_t^1 = X_t^1 dW_t, \\ dX_t^2 = X_t^1 dt, \end{cases} \tag{1.3}$$

which is generated by the operator

$$\mathcal{A} = \frac{x_1^2}{2} \partial_{x_1 x_1} + x_1 \partial_{x_2}, \quad (x_1, x_2) \in \mathbb{R}_{>0}^2. \tag{1.4}$$

This operator is related to the valuation of a class of path-dependent financial derivatives known as arithmetic Asian options. The process X_t^1 is a geometric Brownian motion and represents the price of a risky asset, whereas X_t^2 represents its time-average. The operator fulfills Assumption 1.1 in that the commutator

$$[\partial_{x_1}, Y] := \partial_{x_1} Y - Y \partial_{x_1} = \partial_{x_2},$$

and also satisfies Assumption 1.2 for any $D =]a, \infty[$ with $a > 0$. Although more sophisticated models, with more flexible dynamics (local-stochastic volatility) for the price of the underlying asset, were proposed to price Asian options, the prototype process (1.3) is complex enough to exhibit some interesting mathematical properties. In fact, the problem of analytically characterizing its joint transition density is still partially open, and sharp upper/lower bounds were established only recently in [4].

We remark that, besides the fact that the partial derivative $\partial_{x_2 x_2}$ is missing in the generator \mathcal{A} in (1.4), the coefficient of the second order derivative $\partial_{x_1 x_1}$ also degenerates near zero. Nevertheless, the local nature of our study allows us to handle such diffusions and derive local results on a suitable domain. More generally (see Proposition 3.4 below for the precise statement), the class of stochastic processes that we consider includes locally-integrated diffusions of the form

$$dX_t = \mu(t, X_t) dt + \sigma(t, X_t) dW_t,$$

with $\mu :]0, T_0[\times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma :]0, T_0[\times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times n}$ such that, for any $(t, x) \in]0, T_0[\times D$,

$$\mu(t, x) = (a_1(t, x), \dots, a_{p_0}(t, x), 0, \dots, 0) + Bx, \tag{1.5}$$

$$\sigma \sigma^\top(t, x) = \begin{pmatrix} A(t, x) & 0_{p_0 \times (d-p_0)} \\ 0_{(d-p_0) \times p_0} & 0_{(d-p_0) \times (d-p_0)} \end{pmatrix}, \quad A = (a_{ij}(t, x))_{i,j=1,\dots,p_0}, \tag{1.6}$$

and with B and a_{ij}, a_i satisfying Assumptions 1.1 and 1.2 respectively. We mention that SDEs with a similar chained/blocks structure of the drift (in the sense of Lemma 2.1 below, but not necessarily linear) were studied in [5]. In this reference, the coercivity condition on the first p_0 components of Assumption 1.2(ii) is also common, and the Hölder-regularity of coefficients and their derivatives is similar. On the other hand, the norms used to account for the multiscale behavior of the process, due to the Hörmander condition, are different from those used in this paper.

We emphasize that no assumption is required on the generator of X outside the domain D , although the process X “lives” on \mathbb{R}^d , meaning that its trajectories are allowed to go in and out D .

1.1. Main results and comparison with the literature

Here we report and discuss the main results of the paper comparing them to the related literature. Granted that precise definitions will be given in the sequel, namely in Sections 2 and 3, we will provide here a heuristic explanation of all the objects that appear in the statements below.

The first main result of this paper is a local intrinsic Itô formula for X . In the following statement, $P_{t,x}$ represents the probability under which the process X starts from the point x at time t with probability one and \mathcal{F}^t is a filtration to which $(X_T)_{T \geq t}$ is adapted. Moreover, we denote by \mathcal{L} the differential operator

$$\mathcal{L} := \frac{1}{2} \sum_{i,j=1}^{p_0} a_{ij}(t, x) \partial_{x_i x_j} + \sum_{i=1}^{p_0} a_i(t, x) \partial_{x_i} + Y \tag{1.7}$$

where Y is the vector field as defined in (1.2).

Theorem 1.3 (Intrinsic Itô formula). *Let X be a local diffusion on \mathbb{R}^d generated by \mathcal{A}_t on D (in the sense of Definition 3.2) and let Assumption 1.1 be in force. Then, for any fixed $(t, x) \in]0, T_0[\times \mathbb{R}^d$, $\alpha \in]0, 1]$ and $f \in C_B^{2,\alpha}$ with compact support in $]0, T_0[\times D$, we have*

$$f(T, X_T) = f(t, X_t) + \int_t^T \mathcal{L}f(u, X_u) du + M_T^t, \quad t \leq T < T_0, \tag{1.8}$$

where M^t is a zero-mean \mathcal{F}^t -martingale under $P_{t,x}$, and

$$E_{t,x} [|M_T^t|^2] = E_{t,x} \left[\int_t^T \sum_{i,j=1}^{p_0} a_{ij}(s, X_s) \partial_{x_i} f(s, X_s) \partial_{x_j} f(s, X_s) ds \right]. \tag{1.9}$$

Formula (1.8) is a local result since no assumption is made on the generator of X outside D . Moreover, the Itô formula above is stronger than the classical one as it is proved for a class of functions that are not twice-differentiable in the classical sense, but only with respect to the non-Euclidean geometry induced by the vector fields $\partial_{x_1}, \dots, \partial_{x_{p_0}}$ and Y in Assumption 1.1. Roughly speaking, we say that $f \in C_B^{2,\alpha}(]0, T_0[\times D)$ if $\partial_{x_1} f, \dots, \partial_{x_{p_0}} f$ and Yf exist on $]0, T_0[\times D$ and they are α -Hölder continuous with respect to the semi-distance

$$|T - t|^{1/2} + |y - e^{(T-t)B}x|_B, \quad (t, x), (T, y) \in \mathbb{R} \times \mathbb{R}^d,$$

with $|\cdot|_B$ as in (2.2). Note that Yf is meant as a Lie derivative and not as a combination of Euclidean derivatives: in principle, $\partial_t f$ and $\partial_{x_i} f$ with $i > p_0$ do not exist; on the other hand, the Euclidean space $C^{2,\alpha}$ is included in $C_B^{2,\alpha}$. We also highlight the fact that Assumption 1.2 is not required in Theorem 1.3. Deferring the precise definition until the next section, $|\cdot|_B$ is an anisotropic quasi-norm on \mathbb{R}^d that accounts for multi-scale behavior of the diffusion. As an example, in the case of the model seen before for arithmetic Asian options, the quasi-norm reads as

$$|(x_1, x_2)|_B = |x_1| + |x_2|^{1/3}, \quad (x_1, x_2) \in \mathbb{R}^2,$$

and

$$u(t, x_1, x_2) = |x_2 - c|^{3/2}, \quad c \in \mathbb{R},$$

is an instance of function of class $C_B^{2,1}$, but only $C^{1,1/2}$ in the classical sense. We refer the reader to [13, Section 3.3] for more examples of functions whose intrinsic regularity is strictly higher than the Euclidean one, including functions of class $C_B^{2,\alpha}$ but only $C^{0,\alpha}$ in the classical meaning.

Just like the classical Itô formula is based on the standard Taylor expansion, the cornerstone of (1.8) is a non-Euclidean Taylor formula, proved in [13] and [14] for functions in $C_B^{2,\alpha}$, which roughly states that

$$f(T, y) = \mathbb{T}_{(t,x)}^{(2)} f(T, y) + O(|T - t|^{2+\alpha}) + O(|y - e^{(T-t)B}x|_B^{2+\alpha}) \quad \text{as } (T, y) \rightarrow (t, x) \in]0, T_0[\times D, \tag{1.10}$$

where

$$\begin{aligned} \mathbb{T}_{(t,x)}^{(2)} f(T, y) &= f(t, x) + (T - t)Yf(t, x) + \sum_{i=1}^{p_0} (y - e^{(T-t)B}x)_i \partial_{x_i} f(t, x) \\ &\quad + \frac{1}{2} \sum_{i,j=1}^{p_0} (y - e^{(T-t)B}x)_i (y - e^{(T-t)B}x)_j \partial_{x_i x_j} f(t, x). \end{aligned} \tag{1.11}$$

With (1.10)–(1.11) at hand, it is possible to outline the main arguments the proof of Theorem 1.3 is built upon. Analogously to the classical case, the key step is proving that

$$\frac{E_{t,x}[f(T, X_T)] - f(t, x)}{T - t} \longrightarrow \mathcal{L}f(t, x) \quad \text{as } T - t \rightarrow 0^+, \tag{1.12}$$

uniformly w.r.t. $x \in \mathbb{R}^d$, for any $f \in C_B^{2,\alpha}$ with compact support in $]0, T_0[\times D$. Applying (1.10) yields

$$\begin{aligned} \frac{E_{t,x}[f(T, X_T)] - f(t, x)}{T - t} &= \underbrace{\frac{E_{t,x}[\mathbb{T}_{(t,x)}^{(2)} f(T, X_T) - f(t, x)]}{T - t}}_{=:g_1(t,T)} + \underbrace{\frac{E_{t,x}[O(|X_T - e^{(T-t)B}x|^{2+\alpha})]}{T - t}}_{=:g_2(t,T)} \\ &\quad + O(|T - t|^{\frac{\alpha}{2}}). \end{aligned}$$

It is then clear that (1.12) holds true if

$$g_1(t, T) \longrightarrow \mathcal{L}f(t, x), \quad g_2(t, T) \longrightarrow 0, \quad \text{as } T - t \rightarrow 0^+,$$

uniformly w.r.t. $x \in \mathbb{R}^d$. Now, while the proof of the first limit above is quite straightforward and stems simply from the fact that X is locally generated by \mathcal{A}_t on D (see Definition 3.2), the second limit is a deeper result and represents the main element of novelty in the proof of Theorem 1.3. In particular, we note that $g_2(t, T) \rightarrow 0$ is a consequence of the fact that

$$\lim_{T-t \rightarrow 0^+} \frac{E_{t,x}[\mathbb{1}_{\{|X_T - x| < \delta\}} |X_T - e^{(T-t)B}x|^{2+\alpha}]}{T - t} = 0, \quad \delta > 0,$$

uniformly w.r.t. $x \in H$ compact subset of D , and we emphasize that the latter is stronger than the classical general estimate for diffusion processes (see [8] or [2])

$$\lim_{T-t \rightarrow 0^+} \frac{E_{t,x}[\mathbb{1}_{\{|X_T - x| < \delta\}} |X_T - x|^{2+\alpha}]}{T - t} = 0, \quad \delta > 0,$$

since the intrinsic quasi-norm $|\cdot|_B$ on \mathbb{R}^d is such that $|x| = o(|x|_B)$ as $x \rightarrow 0$ along the *degenerate* directions.

The second main result of the paper is the theorem below that states the existence of a local (on D) transition density $\Gamma(t, x; T, \xi)$ for X , reveals its intrinsic regularity w.r.t. both the forward and backward variables, and shows that it solves a forward and a backward Kolmogorov equation on $]t, T_0[\times D$ and $]0, T[\times D$, respectively. Before stating the result, we need to introduce the last additional assumption, which is only needed to prove the regularity w.r.t. the backward variables.

Assumption 1.4. X is a Feller process on D , i.e. for any $T \in]0, T_0[$ and bounded $\varphi \in C(\mathbb{R}^d)$ the function $(t, x) \mapsto E_{t,x}[\varphi(X_T)]$ is continuous on $]0, T[\times D$.

Note that, since the coercivity condition in Assumption 1.2(ii) only holds on D , the Feller property for the semigroup $\varphi \mapsto E_{t,\cdot}[\varphi(X_T)]$ is not ensured. This is due to the fact that the trajectories of X are allowed to leave and re-enter the domain D , but no assumption is made on the generator of X outside D . Had Assumption 1.2 been satisfied for $D = \mathbb{R}^d$, the Feller property would stem from PDEs arguments, namely the existence and regularity results for the fundamental solution of \mathcal{L} on \mathbb{R}^d that were proved in [17] and [6] by means of the so-called parametrix method.

Theorem 1.5. Let X be a local diffusion on \mathbb{R}^d generated by \mathcal{A}_t on D (in the sense of Definition 3.2) and let Assumption 1.1 be in force. Denoting by $p(t, x; T, \cdot)$ the transition distribution of X , we have:

- (a) if Assumption 1.2 with $N = 0$ is also in force, then X has a local transition density Γ on D , namely a non-negative measurable function $\Gamma(t, x; T, y)$ defined for any $0 < t < T < T_0$ and $x \in \mathbb{R}^d$, $y \in D$, such that

$$p(t, x; T, A) = \int_A \Gamma(t, x; T, y) dy, \quad A \in \mathcal{B}(D).$$

Furthermore, $\Gamma(t, x; T, \cdot)$ is continuous on D and locally bounded uniformly w.r.t. $x \in \mathbb{R}^d$;

- (b) if Assumption 1.2 with $N = 2$ is also in force, then for any $(t, x) \in]0, T_0[\times \mathbb{R}^d$ the function $\Gamma(t, x; \cdot, \cdot) \in C_B^{2,\alpha}(]t, T_0[\times D)$ and solves the forward Kolmogorov equation

$$\mathcal{L}^*u = 0, \quad \text{on }]t, T_0[\times D, \quad (1.13)$$

where \mathcal{L}^* is the formal adjoint of \mathcal{L} ;

- (c) if Assumption 1.2 with $N = 0$ and Assumption 1.4 are also in force, then for any $(T, y) \in]0, T_0[\times D$ the function $\Gamma(\cdot, \cdot; T, y) \in C_B^{2,\alpha}(]0, T[\times D)$ and solves the backward Kolmogorov equation

$$\mathcal{L}u = 0, \quad \text{on }]0, T[\times D. \quad (1.14)$$

This statement partially generalizes [9], Section 4, where analogous results were obtained under the assumption that the coefficients of the generator are smooth on D . In particular, the main assumption in the latter reference is a sort of local hypoellipticity condition for the generator, expressed in terms of Malliavin's matrix, on a given domain D of \mathbb{R}^d . So, if on the one hand the framework in [9] is more general, on the other hand our assumptions are weaker in that we assume the coefficients belonging to the intrinsic Hölder space $C_B^{0,\alpha}$ (or $C_B^{2,\alpha}$ for the regularity w.r.t. the forward variables). Again, we stress the fact that $C_B^{0,\alpha}$ ($C_B^{2,\alpha}$) not only does contain C^∞ , but also includes the standard Hölder space $C^{0,\alpha}$ ($C^{2,\alpha}$). Recent results on the local density assuming standard regularity of the coefficients were proved in [3], under local strong Hörmander-type conditions, and in [16], under local weak Hörmander-type conditions for two-dimensional diffusions.

The proof of Theorem 1.5 partially relies on some existence and regularity results (see [6] and [7] among others) obtained in a PDEs' context for the fundamental solution and the Green functions of Kolmogorov operators, as well as on some Schauder estimates (see again [7]). However, it is important to stress that the latter results alone are not enough to prove Theorem 1.5. This is due to the fact that our structural assumptions on the generator of X only hold on D , whereas there is no assumption on what is the behavior of the process outside D . For this reason, it will be necessary to combine the PDE results mentioned above with some probabilistic interlacing techniques and a crucial role will be played by the Itô formula of Theorem 1.3.

More in detail, we adapt and customize the localization technique introduced in [9], Section 4. We first prove a Feynman–Kac formula (Lemma 4.2) that allows to link the solutions of the Cauchy–Dirichlet problem for \mathcal{L} on suitable cylinders of $]0, T_0[\times D$ to the semigroup of the stopped diffusion. As this step is based on the application of Itô formula to the solution of the Cauchy–Dirichlet problem and the latter is not of class $C^{2,\alpha}$ in the classical sense but only intrinsically, it is clear that the Itô formula needed here is the one in Theorem 1.3 and not the classical one. The proof of the existence of the local density can be then completed by following closely the procedure employed by Kusuoka and Stroock, which makes use of a sequence of stopping times that keep track of when the process exits and re-enters the domain D .

Once Part (a) is proved, we need to depart from the latter procedure in order to prove part Part (b) and Part (c). In particular, to obtain the intrinsic regularity of the local density $\Gamma(t, x; T, y)$ w.r.t. the forward variables $(T, y) \in D$, it will be crucial to employ the Schauder internal estimates for the solutions of $\mathcal{L}u = 0$ proved in [7], combined with the Gaussian upper bounds for the Green function of \mathcal{L} proved in the same reference. Once we have proved that $\Gamma(t, x; \cdot, \cdot) \in C_B^{2,\alpha}(]t, T_0[\times D)$, then the fact that $\Gamma(t, x; \cdot, \cdot)$ solves (1.13) simply follows because the latter is satisfied by the transition probability kernel of X in the distributional sense (see Remark 3.7).

To prove that $\Gamma(\cdot, \cdot; T, y) \in C_B^{2,\alpha}(]0, T[\times D)$ and solves (1.14), we first show that the same holds true for the function $(t, x) \rightarrow E_{t,x}[\varphi(X_T)]$ for any $\varphi \in C_b(\mathbb{R}^d)$. This step is based again on a Feynman–Kac formula and a crucial role is played one more time by the intrinsic Itô formula of Theorem 1.3. Finally, Part (c) follows by proving that the same properties hold true for any bounded measurable function φ on \mathbb{R}^d . We remark that this last step is based on the fact that X actually enjoys the strong Feller property, namely the property in Assumption 1.4 extended to bounded measurable functions, which we prove in Lemma 4.6 by assuming the standard Feller property. Here we heavily rely again on the Schauder estimates in [7]. We point out that the latter result, i.e. proving the *strong* Feller property starting from the *standard* one, might enjoy an independent interest as it generalizes some previous results obtained in [18] under stronger assumptions, basically existence and uniform boundedness of the global transition density.

The rest of the paper is organized as follows. In Section 2 we recall the definition of B -quasi-norm, the B -intrinsic Hölder spaces and the related intrinsic Taylor formula. In Section 3 we give the precise definition of \mathcal{A}_t -local diffusion on \mathbb{R}^d and prove Theorem 1.3. In Section 3 we prove Theorem 1.5. In Appendix A we collect some useful PDE results for \mathcal{L} -like operators, and in Appendix B we recall some classic construction procedures for Markov processes.

2. Preliminaries: Hölder spaces and Taylor formula

We recall the following useful characterization of Assumption 1.1, proved in [10].

Lemma 2.1. *Assumption 1.1 is fulfilled if and only if B takes the block form*

$$B = \begin{pmatrix} * & * & \cdots & * & * \\ B_1 & * & \cdots & * & * \\ 0 & B_2 & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_r & * \end{pmatrix} \tag{2.1}$$

where B_j is a $(p_j \times p_{j-1})$ -matrix with full rank (equal to p_j) for $j = 1, \dots, r$, the $*$ -blocks are arbitrary, $p_0 \geq p_1 \geq \dots \geq p_r \geq 1$ and $p_0 + p_1 + \dots + p_r = d$.

We introduce the quasi-norm in \mathbb{R}^d

$$|x|_B := \sum_{j=0}^r \sum_{i=\bar{p}_{j-1}+1}^{\bar{p}_j} |x_i|^{1/(2j+1)}, \quad \bar{p}_j := \sum_{k=0}^j p_k, \quad \bar{p}_{-1} := 0, \tag{2.2}$$

that is homogeneous with respect to the dilations group

$$D_0(\lambda) = \text{diag}(\lambda I_{p_0}, \lambda^3 I_{p_1}, \dots, \lambda^{2r+1} I_{p_r}), \quad \lambda > 0. \tag{2.3}$$

For any $(t, x) \in \mathbb{R} \times \mathbb{R}^d$ and $i = 1, \dots, p_0$, we denote by

$$e^{\delta \partial_{x_i}}(t, x) = (t, x + \delta e_i), \quad e^{\delta Y}(t, x) = (t + \delta, e^{\delta B} x), \quad \delta > 0,$$

the integral curves of the vector fields $\partial_{x_1}, \dots, \partial_{x_{p_0}}, Y$ starting at (t, x) . Here e_i denotes the i -th element of the canonical basis of \mathbb{R}^d . Now, let Q be a domain in $\mathbb{R} \times \mathbb{R}^d$. For any $(t, x) \in Q$ we set

$$\delta_{(t,x)} := \sup\{\bar{\delta} \in]0, 1] \mid e^{\delta \partial_{x_1}}(t, x), \dots, e^{\delta \partial_{x_{p_0}}}(t, x), e^{\delta Y}(t, x) \in Q \text{ for any } \delta \in [-\bar{\delta}, \bar{\delta}]\}.$$

If V is compactly contained in Q (hereafter we write $V \Subset Q$), we set $\delta_V = \inf_{(t,x) \in V} \delta_{(t,x)}$. Note that $\delta_V \in]0, 1]$.

Definition 2.2. Let $\alpha_1 \in]0, 1]$, $\alpha_2 \in]0, 2]$ and $i = 1, \dots, p_0$. We say that $f \in C_{\partial_{x_i}}^{\alpha_1}(Q)$ and $g \in C_Y^{\alpha_2}(Q)$ if the following semi-norms are finite

$$\|f\|_{C_{\partial_{x_i}}^{\alpha_1}(V)} := \sup_{\substack{(t,x) \in V \\ 0 < |\delta| < \delta_V}} \frac{|f(e^{\delta \partial_{x_i}}(t, x)) - f(t, x)|}{|\delta|^{\alpha_1}}, \quad \|g\|_{C_Y^{\alpha_2}(V)} := \sup_{\substack{(t,x) \in V \\ 0 < |\delta| < \delta_V}} \frac{|g(e^{\delta Y}(t, x)) - g(t, x)|}{|\delta|^{\frac{\alpha_2}{2}}},$$

for any $V \Subset Q$.

We can now define the so-called B -Hölder spaces. We point out that slightly different versions of such spaces were previously adopted in several works (see [7] and [11] among others). Here we use the definition given [13], which is basically the one required in order to prove an intrinsic Taylor formula where the remainder is in terms of the intrinsic quasi-norm.

Definition 2.3. Let Q a domain of $\mathbb{R} \times \mathbb{R}^d$ and let $\alpha \in]0, 1]$, then:

- (i) $f \in C_B^{0,\alpha}(Q)$ if $f \in C_Y^\alpha(Q)$ and $f \in C_{\partial_{x_i}}^\alpha(Q)$ for any $i = 1, \dots, p_0$;

- (ii) $f \in C_B^{1,\alpha}(Q)$ if $f \in C_Y^{1+\alpha}(Q)$ and $\partial_{x_i} f \in C_B^{0,\alpha}(Q)$ for any $i = 1, \dots, p_0$;
 (iii) for $n \in \mathbb{N}$ with $n \geq 2$, $f \in C_B^{n,\alpha}(Q)$ if $Yf \in C_B^{n-2,\alpha}(Q)$ and $\partial_{x_i} f \in C_B^{n-1,\alpha}(Q)$ for any $i = 1, \dots, p_0$.

Moreover, for $f \in C_B^{n,\alpha}(Q)$ and $V \Subset Q$, we set

$$\|f\|_{C_B^{n,\alpha}(V)} := \begin{cases} \|f\|_{C_Y^\alpha(V)} + \sum_{i=1}^{p_0} \|f\|_{C_{\partial_{x_i}}^\alpha(V)}, & n = 0 \\ \|f\|_{C_Y^{\alpha+1}(V)} + \sum_{i=1}^{p_0} \|\partial_{x_i} f\|_{C_B^{0,\alpha}(V)}, & n = 1 \\ \|Yf\|_{C_B^{n-2,\alpha}(V)} + \sum_{i=1}^{p_0} \|\partial_{x_i} f\|_{C_B^{n-1,\alpha}(V)}, & n \geq 2. \end{cases}$$

If $f \in C_B^{n,\alpha}(Q)$ and has compact support then we write $f \in C_{0,B}^{n,\alpha}(Q)$.

The next result was proved in [13] in the particular case when the $*$ -blocks in (2.1) are null and then extended to the general case in [14].

Theorem 2.4. *Let Q be a domain of $\mathbb{R} \times \mathbb{R}^d$, $\alpha \in]0, 1]$ and $n \in \mathbb{N}_0$. If $f \in C_B^{n,\alpha}(Q)$ then we have:*

(1) *there exist*

$$Y^k \partial_x^\beta f \in C_B^{n-2k-[\beta]_B,\alpha}(Q), \quad 0 \leq 2k + [\beta]_B \leq n,$$

where $[\beta]_B$ denotes the height of the multi-index β defined as

$$[\beta]_B := \sum_{j=0}^r \sum_{i=\bar{p}_{j-1}+1}^{\bar{p}_j} (2j+1)\beta_i;$$

(2) *for any $(t_0, x_0) \in Q$, there exist two bounded domains U, V , such that $(t_0, x_0) \in U \subseteq V \subseteq Q$ and*

$$|f(t, x) - \mathbb{T}_{(s,y)}^{(n)} f(t, x)| \leq c_{B,U} \|f\|_{C_B^{n,\alpha}(V)} (|t-s|^{1/2} + |x - e^{(t-s)B} y|_B)^{n+\alpha}, \quad (t, x), (s, y) \in U, \quad (2.4)$$

where $c_{B,U}$ is a positive constant and $\mathbb{T}_{(s,y)}^{(n)}$ is the n -th order intrinsic Taylor polynomial of f centered at (s, y) given by

$$\mathbb{T}_{(s,y)}^{(n)} f(t, x) = \sum_{\substack{k \in \mathbb{N}_0, \beta \in \mathbb{N}_0^d \\ 0 \leq 2k + [\beta]_B \leq n}} \frac{1}{k! \beta!} (Y^k \partial_y^\beta f(s, y)) (t-s)^k (x - e^{(t-s)B} y)^\beta, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d.$$

Corollary 2.5. *If $f \in C_{0,B}^{n,\alpha}(Q)$, then (2.4) holds true with $U = \text{supp}(f)$ and $V = Q$.*

3. Local diffusions and intrinsic Itô formula

For a given $T_0 > 0$ we consider a continuous \mathbb{R}^d -valued strong Markov process $X = (X_t)_{t \in [0, T_0[}$ (in the sense of [8] as it is recalled in Appendix B) with transition probability function $p = p(t, x; T, d\xi)$, defined on a space

$$(\Omega, \mathcal{F}, (\mathcal{F}_T^t)_{0 \leq t \leq T < T_0}, (P_{t,x})_{0 \leq t < T_0, x \in \mathbb{R}^d}).$$

For any bounded Borel measurable function φ , we denote by

$$E_{t,x}[\varphi(X_T)] := (\mathbf{T}_{t,T} \varphi)(x) := \int_{\mathbb{R}^d} p(t, x; T, d\xi) \varphi(\xi), \quad 0 \leq t < T \leq T_0, x \in \mathbb{R}^d,$$

the $P_{t,x}$ -expectation and the semigroup associated with the transition probability function p , respectively (cf. Chapter 2.1 in [8]). Hereafter we also fix a domain D , which is an open and connected subset of \mathbb{R}^d .

Notation 3.1. For a given function $f(t, T)$ with $T \in]0, T_0[$ and $t \in [0, T[$, we set

$$\lim_{T \rightarrow t \rightarrow 0^+} f(t, T) := \lim_{h \rightarrow 0^+} f(t, t+h) = \lim_{h \rightarrow 0^+} f(t-h, t), \quad t \in [0, T_0[,$$

when the second and the third limits exist and coincide with each other.

The following two sets of limits will be used to give the definition of local diffusion.

(Lim-i) For any $t \in [0, T_0[$, $\delta > 0$, and H compact subset of D , there exist the limits

$$\lim_{T-t \rightarrow 0^+} \int_{\{|\xi-x|>\delta\} \cap H} \frac{p(t, x; T, d\xi)}{T-t} = 0, \quad \text{uniformly w.r.t. } x \in \mathbb{R}^d, \tag{3.1}$$

$$\lim_{T-t \rightarrow 0^+} \int_{|\xi-x|>\delta} \frac{p(t, x; T, d\xi)}{T-t} = 0, \quad \text{uniformly w.r.t. } x \in H. \tag{3.2}$$

(Lim-ii) For any $t \in [0, T_0[$, $\delta > 0$, and H compact subset of D , and for any $i = 1, \dots, d$, there exist the limits

$$\lim_{T-t \rightarrow 0^+} \int_{|\xi-x|<\delta} (\xi-x)_i \frac{p(t, x; T, d\xi)}{T-t} = \begin{cases} a_i(t, x) + (Bx)_i & \text{if } i = 1, \dots, p_0 \\ (Bx)_i & \text{if } i = p_0 + 1, \dots, d, \end{cases} \tag{3.3}$$

$$\lim_{T-t \rightarrow 0^+} \int_{|\xi-x|<\delta} (\xi-x)_i (\xi-x)_j \frac{p(t, x; T, d\xi)}{T-t} = \begin{cases} a_{ij}(t, x) & \text{if } i, j = 1, \dots, p_0 \\ 0 & \text{if } i \text{ or } j = p_0 + 1, \dots, d, \end{cases} \tag{3.4}$$

uniformly w.r.t. $x \in H$, for some B as in (2.1) and $a_{ij}, a_i \in L_{\text{loc}}^\infty([0, T_0[\times D)$.

Definition 3.2. Let \mathcal{A}_t an operator as in (1.1). We say that X is a *local diffusion generated by \mathcal{A}_t on D* (an \mathcal{A}_t -local diffusion in short) if (Lim-i) and (Lim-ii) hold. In case they hold with $D = \mathbb{R}^d$ then we call X a *global diffusion generated by \mathcal{A}_t* (an \mathcal{A}_t -global diffusion in short).

The following proposition is useful for the applications because several models are defined in terms of solutions to stochastic differential equations. It shows that (stopped) solutions of SDEs are local diffusions in the sense of Definition 3.2.

Remark 3.3. Since we are dealing with stopping times, we point that we did not impose any right-continuity assumption on the filtrations \mathcal{F}^t . This is justified by the fact that, in the next proposition as well as in the rest of the paper, we will only consider hitting times of closed sets, which appear to be stopping times even if the the filtration is not right-continuous (see [8], Theorem 2.2 p. 25).

Proposition 3.4. Let $(X_t)_{t \in [0, T_0[}$ be a continuous Markov process defined as $X_t = \hat{X}_{t \wedge \tau}$, where:

(i) \hat{X} is a solution of the SDE

$$d\hat{X}_t = \mu(t, \hat{X}_t) dt + \sigma(t, \hat{X}_t) dW_t$$

where W is a n -dimensional Brownian motion and the coefficients μ and σ are continuous and as in (1.5)–(1.6);

(ii) τ is the first exit time of \hat{X} from a domain D' containing D .

Then X is a \mathcal{A}_t -local diffusion on \mathbb{R}^d in the sense of Definition 3.2.

Proof. The statement is a particular case of Lemma 2.3 in [12], which proves that (Lim-i) and (Lim-ii) hold for the kernel of X . □

We have the first key result, whose proof is deferred to Section 3.3.

Proposition 3.5. Let Assumptions 1.1 be in force. Then X is an \mathcal{A}_t -local diffusion on \mathbb{R}^d if and only if (Lim-i) holds and for any $t \in [0, T_0[$, $\delta > 0$, and H compact subset of D , we have

$$\lim_{T-t \rightarrow 0^+} \int_{|\xi-x|<\delta} (\xi - e^{(T-t)Bx})_i \frac{p(t, x; T, d\xi)}{T-t} = a_i(t, x), \tag{3.5}$$

$$\lim_{T-t \rightarrow 0^+} \int_{|\xi-x|<\delta} (\xi - e^{(T-t)Bx})_i (\xi - e^{(T-t)Bx})_j \frac{p(t, x; T, d\xi)}{T-t} = a_{ij}(t, x), \tag{3.6}$$

$$\limsup_{T-t \rightarrow 0^+} \int_{|\xi-x|<\delta} |\xi - e^{(T-t)Bx}|_B^2 \frac{p(t, x; T, d\xi)}{T-t} < \infty, \tag{3.7}$$

for any $i, j = 1, \dots, p_0$, uniformly w.r.t. $x \in H$, for some B as in (2.1) and $a_{ij}, a_i \in L_{\text{loc}}^\infty([0, T_0[\times D)$.

The following proposition, whose proof is also deferred to Section 3.2, formalizes the fact that \mathcal{A}_t as in (1.1) is the generator of X on $[0, T_0[\times D$, and shows that the function space for which the semi-group is differentiable is $C_{0,B}^{2,\alpha}([0, T_0[\times D)$.

Proposition 3.6. *Let X be a \mathcal{A}_t -local diffusion on \mathbb{R}^d and Assumption 1.1 be in force. Then, for any $\varphi \in C_0([0, T_0[\times D)$ and $f \in C_{0,B}^{2,\alpha}([0, T_0[\times D)$ we have*

$$\lim_{T \rightarrow t^+} \sup_{x \in \mathbb{R}^d} |(\mathbf{T}_{t,T} \varphi(T, \cdot))(x) - \varphi(t, x)| = 0, \tag{3.8}$$

$$\lim_{T \rightarrow t^+} \sup_{x \in \mathbb{R}^d} \left| \frac{(\mathbf{T}_{t,T} f(T, \cdot))(x) - f(t, x)}{T - t} - \mathcal{L} f(t, x) \right| = 0, \tag{3.9}$$

for any $t \in]0, T_0[$. Moreover, it holds that

$$\frac{d}{dt} (\mathbf{T}_{t,T} f(T, \cdot))(x) = \mathbf{T}_{t,T} (\mathcal{L} f(T, \cdot))(x), \quad 0 < t < T < T_0, x \in \mathbb{R}^d. \tag{3.10}$$

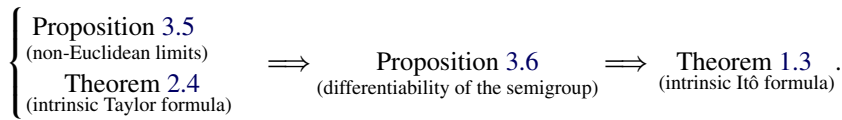
Remark 3.7. By Theorem 1.3 it is clear that, for any $(t, x) \in]0, T_0[\times \mathbb{R}^d$, $p(t, x; \cdot, \cdot)$ satisfies

$$\mathcal{L}^* u = 0 \quad \text{on }]t, T_0[\times D,$$

in the sense of distributions, with \mathcal{L}^* being the formal adjoint of \mathcal{L} , i.e.

$$\int_t^{T_0} \int_D p(t, x; T, d\xi) \mathcal{L} f(T, \xi) dT = 0, \quad f \in C_0^\infty([t, T_0[\times D). \tag{3.11}$$

The rest of this section is devoted to the proofs of Theorem 1.3, Proposition 3.6, and Proposition 3.5. The following scheme summarizes the logical implications among these statements:



3.1. Proof of Theorem 1.3 (Intrinsic It\^o formula)

We prove here the intrinsic It\^o formula in Theorem 1.3. The proof relies on the results of Proposition 3.6.

Proof of Theorem 1.3. First observe that the right-hand side term in (3.10) is bounded. This stems from the fact that the coefficients of \mathcal{L} are locally bounded on D (by Definition 3.2), $f \in C_{0,B}^{2,\alpha}([0, T_0[\times D)$, and $\mathbf{T}_{t,\tau}$ is a contraction. Therefore, we can integrate (3.10), and obtain the identity

$$(\mathbf{T}_{t,T} f(T, \cdot))(x) - f(t, x) = \int_t^T \mathbf{T}_{t,\tau} (\mathcal{L} f(\tau, \cdot))(x) d\tau, \quad T \in]t, T_0[. \tag{3.12}$$

Consider now the process M^t defined through (1.8). For $\tau \in [t, T]$ we have

$$E_{t,x} [M_T^t | \mathcal{F}_\tau^t] = M_\tau^t + E_{t,x} \left[f(T, X_T) - f(\tau, X_\tau) - \int_\tau^T \mathcal{L} f(u, X_u) du \mid \mathcal{F}_\tau^t \right] = M_\tau^t + \Phi(\tau, X_\tau)$$

where, by the Markov property,

$$\Phi(\tau, x) = E_{\tau,x} \left[f(T, X_T) - f(\tau, x) - \int_\tau^T \mathcal{L} f(u, X_u) du \right]$$

(by Fubini's theorem)

$$= (\mathbf{T}_{\tau,T} f(T, \cdot))(x) - f(\tau, x) - \int_\tau^T \mathbf{T}_{\tau,u} (\mathcal{L} f(u, \cdot))(x) du$$

which is 0 by (3.12).

To conclude we need to show that

$$Y_T^t := (M_T^t)^2 - \int_t^T \sum_{i,j=1}^{p_0} a_{ij}(s, X_s) \partial_{x_i} f(s, X_s) \partial_{x_j} f(s, X_s) ds,$$

has null $P_{t,x}$ -expectation. First note that

$$\sum_{i,j=1}^{p_0} a_{ij}(s, X_s) \partial_{x_i} f(s, X_s) \partial_{x_j} f(s, X_s) = \mathcal{L} f^2(s, X_s) - 2f(s, X_s) \mathcal{L} f(s, X_s),$$

which implies

$$Y_T^t = Y_T^{1,t} + Y_T^{2,t} + Y_T^{3,t},$$

with

$$Y_T^{1,t} = f^2(T, X_T) - \int_t^T \mathcal{L} f^2(u, X_u) du + f^2(t, X_t),$$

$$Y_T^{2,t} = 2f(t, X_t) \left(\int_t^T \mathcal{L} f(u, X_u) du - f(T, X_T) \right),$$

$$Y_T^{3,t} = -2 \int_t^T (f(T, X_T) - f(u, X_u)) \mathcal{L} f(u, X_u) du + \left(\int_t^T \mathcal{L} f(u, X_u) du \right)^2.$$

Now, by applying the first part of Theorem 1.3 to f^2 and f respectively, it is clear that $Y^{1,t}$ and $Y^{2,t}$ are martingales with null $P_{t,x}$ -expectation. Finally, the identity

$$\left(\int_t^T \mathcal{L} f(u, X_u) du \right)^2 = 2 \int_t^T \left(\int_u^T \mathcal{L} f(s, X_s) ds \right) \mathcal{L} f(u, X_u) du$$

along with (1.8) yields

$$Y_T^{3,t} = -2 \int_t^T (M_T^t - M_u^t) \mathcal{L} f(u, X_u) du,$$

which shows that $Y^{3,t}$ has null $P_{t,x}$ -expectation and thus concludes the proof. □

3.2. Proof of Proposition 3.6

We prove here Proposition 3.6. The key ingredients of the proof are the limits in Proposition 3.5, combined with the intrinsic Taylor formula reported in Section 2.

Proof of Proposition 3.6. The proof of (3.8) is identical to the proof of [12, Eq. (2.6)] as it is only based on the limits (3.1) and (3.2). The same goes for (3.10), which is a corollary of (3.8)–(3.9) and whose proof coincides with the proof of [12, Eq. (2.9)].

Therefore, we only need to prove (3.9). Set $f \in C_{0,B}^{2,\alpha}([0, T_0] \times D)$ whose support is contained in the interior of $]0, T_0[\times H$, with $H \subset D$ a compact subset. We have

$$\frac{\mathbf{T}_{t,T} f(T, x) - f(t, x)}{T - t} = I_{t,T,1}(x) + I_{t,T,2}(x)$$

where

$$I_{t,T,1}(x) = \int_H p(t, x; T, d\xi) \frac{f(T, \xi) - f(t, x)}{T - t}, \quad I_{t,T,2}(x) = -\frac{f(t, x)}{T - t} \int_{\mathbb{R}^d \setminus H} p(t, x; T, d\xi). \tag{3.13}$$

First note that, by (3.1), if $x \notin H$ it holds that

$$I_{t,T,1}(x) + I_{t,T,2}(x) = I_{t,T,1}(x) \longrightarrow 0 \quad \text{as } T - t \rightarrow 0^+, \text{ unif. w.r.t. } x \in \mathbb{R}^d \setminus H.$$

We now consider the case $x \in H$. By (3.2) the term $I_{t,T,2}(x)$ is negligible in the limit. As for $I_{t,T,1}(x)$, the intrinsic Taylor formula of Corollary 2.5 yields

$$\begin{aligned}
 f(T, \xi) - f(t, x) &= (T - t)Yf(t, x) + \sum_{i=1}^{p_0} (\xi - e^{(T-t)B}x)_i \partial_{x_i} f(t, x) \\
 &\quad + \frac{1}{2} \sum_{i,j=1}^{p_0} (\xi - e^{(T-t)B}x)_i (\xi - e^{(T-t)B}x)_j \partial_{x_i x_j} f(t, x) + R(t, x; T, \xi)
 \end{aligned}
 \tag{3.14}$$

with R such that

$$|R(t, x; T, \xi)| \leq c_{H,B} (|T - t|^{1/2} + |\xi - e^{(T-t)B}x|_B)^{2+\alpha}, \quad (t, x), (T, \xi) \in]0, T_0[\times H.
 \tag{3.15}$$

Next we prove that

$$\lim_{T-t \rightarrow 0^+} \int_H |\xi - e^{(T-t)B}x|_B^{2+\alpha} \frac{p(t, x; T, d\xi)}{T - t} = 0, \quad \text{unif. w.r.t. } x \in H.
 \tag{3.16}$$

For any $x \in H$ and $\delta > 0$ suitably small we have

$$\begin{aligned}
 \int_H |\xi - e^{(T-t)B}x|_B^{2+\alpha} \frac{p(t, x; T, d\xi)}{T - t} &\leq C \int_{H \setminus D_\delta(t, x, T)} \frac{p(t, x; T, d\xi)}{T - t} \\
 &\quad + \delta^\alpha \int_{D_\delta(t, x, T)} |\xi - e^{(T-t)B}x|_B^2 \frac{p(t, x; T, d\xi)}{T - t},
 \end{aligned}$$

where $D_\delta(t, x, T) = \{\xi \in \mathbb{R}^d \mid |\xi - e^{(T-t)B}x|_B \leq \delta\}$ and C is a positive constant. By (3.2) and (3.7) we obtain

$$\limsup_{T-t \rightarrow 0^+} \int_H |\xi - e^{(T-t)B}x|_B^{2+\alpha} \frac{p(t, x; T, d\xi)}{T - t} \leq C_1 \delta^\alpha, \quad \text{unif. w.r.t. } x \in H.$$

This proves (3.16) since δ is arbitrary.

Eventually, (3.9) follows by plugging (3.14) into (3.13) and passing to the limit using (3.20), (3.5), (3.6) and (3.15)–(3.16). This concludes the proof. \square

3.3. Proof of Proposition 3.5

The following preliminary result is necessary in order to carry on with the proof of Proposition 3.5. It states that the results in Theorem 1.3 hold true if the transformation f is twice differentiable in the Euclidean sense with compact support on $]0, T_0[\times D$.

Proposition 3.8 (Classical Itô formula). *Let X be a local diffusion on \mathbb{R}^d generated by \mathcal{A}_t on D (in the sense of Definition 3.2) and let Assumption 1.1 be in force. Then, for any fixed $(t, x) \in]0, T_0[\times \mathbb{R}^d$, and $f \in C^2(]0, T_0[\times D)$ with compact support in $]0, T_0[\times D$, Eq’s (1.8)–(1.9) hold true with M^t zero-mean \mathcal{F}^t -martingale under $P_{t,x}$.*

Proof. In [12, Lemma 2.2] it was proved that (3.8), (3.9) and (3.10) hold true for any $\varphi \in C_0(]0, T_0[\times D)$, $f \in C_0^2(]0, T_0[\times D)$. In particular, with (3.10) at hand, the proof of Proposition 3.8 is identical to the proof of Theorem 1.3. Note that no circular argument is employed here, because, for $\varphi \in C_0(]0, T_0[\times D)$ and $f \in C_0^2(]0, T_0[\times D)$, the proof of (3.8)–(3.9)–(3.10) does not rely on Proposition 3.5. \square

The proof of Proposition 3.5 also relies on the following

Lemma 3.9. *Under the hypothesis (Lim-i), for any $H \Subset D$ and for any $0 \leq t < T < T_0$, $\delta < \bar{\delta} := \text{dist}(H, \partial D)$, we have*

$$\lim_{T-t \rightarrow 0^+} \int_{\{|\xi - x| \geq \delta\} \cap H} \frac{p(t, x; T, d\xi)}{(T - t)^m} = 0, \quad m \geq 1,
 \tag{3.17}$$

uniformly w.r.t. $x \in H$.

Proof. Let $\varphi^{(x)} \in C_0^\infty(D)$ be a family of functions with partial derivatives uniformly bounded w.r.t. $x \in H$, and such that $\varphi^{(x)}(x) = 0$ and

$$\varphi = \varphi^{(x)} = 1 \quad \text{on } H \setminus B(x, \delta).$$

Note that Proposition 3.8 gives

$$\varphi(X_T) = \varphi(X_t) + \int_t^T \mathcal{A}_s \varphi(X_s) ds + M_T^t,$$

where M^t is an \mathcal{F}^t -martingale under $P_{t,x}$ with

$$E_{t,x}[|M_T^t|^2] = E_{t,x} \left[\int_t^T \sum_{k,l=1}^{p_0} a_{kl}(s, X_s) (\partial_{x_k} \varphi(X_s)) (\partial_{x_l} \varphi(X_s)) ds \right]. \tag{3.18}$$

Thus we obtain

$$\begin{aligned} & \int_{\{|\xi-x| \geq \delta\} \cap H} p(t, x; T, d\xi) \\ & \leq E_{t,x} [(\varphi(X_T))^{2(m+1)}] \leq 2^{2m+1} E_{t,x} \left[\left| \int_t^T \mathcal{A}_s \varphi(X_s) ds \right|^{2(m+1)} + |M_T^t|^{2(m+1)} \right] \\ & \leq 2^{2m+1} \left(E_{t,x} \left[\left| \int_t^T \mathcal{A}_s \varphi(X_s) ds \right|^{2(m+1)} \right] + \left(\frac{2(m+1)}{2m+1} \right)^{2(m+1)} E_{t,x} [|M_T^t|^2]^{m+1} \right), \end{aligned}$$

where we used [8, Lemma 3.8, p. 71] and Jensen’s inequality in the last step. Eventually, (3.17) stems from (3.18) combined with the fact that $\varphi \in C_0^\infty(D)$ and the coefficients of \mathcal{A} are in $L_{\text{loc}}^\infty([0, T_0] \times D)$. \square

We are now in the position to prove Proposition 3.5.

Proof of Proposition 3.5.

Part 1: *if.* We assume (Lim-i) to be satisfied together with the limits (3.5)–(3.6)–(3.7), and prove that (Lim-ii) holds. Let $t \in [0, T_0]$, $\delta > 0$, and H compact subset of D be fixed. All the following limits are uniform w.r.t. $x \in H$.

We first prove (3.3). For any $i = 1, \dots, d$, it holds that:

$$\begin{aligned} & \lim_{T-t \rightarrow 0^+} \int_{|x-\xi| < \delta} (\xi - x)_i \frac{p(t, x; T, d\xi)}{T-t} \\ & = \lim_{T-t \rightarrow 0^+} \int_{|x-\xi| < \delta} (\xi - e^{(T-t)B}x + e^{(T-t)B}x - x)_i \frac{p(t, x; T, d\xi)}{T-t} \\ & = \lim_{T-t \rightarrow 0^+} \int_{|x-\xi| < \delta} (\xi - e^{(T-t)B}x)_i \frac{p(t, x; T, d\xi)}{T-t} + (Bx)_i. \end{aligned} \tag{3.19}$$

Here we used the property

$$\lim_{T-t \rightarrow 0^+} \int_{|x-\xi| < \delta} p(t, x; T, d\xi) = 1, \tag{3.20}$$

which follows from (Lim-i). Now, for $i = 1, \dots, p_0$, (3.3) stems from (3.5) and (3.19). On the other hand, for $i = p_0 + 1, \dots, d$, note that condition (3.7) is equivalent to

$$\limsup_{T-t \rightarrow 0^+} \int_{|\xi-x| < \delta} |(\xi - e^{(T-t)B}x)_i|^{\frac{2}{q_i}} \frac{p(t, x; T, d\xi)}{T-t} < \infty$$

for a certain $q_i \geq 3$. Therefore, fixing $\varepsilon > 0$, for any $\rho \in]0, \delta]$ we obtain

$$\begin{aligned} & \limsup_{T-t \rightarrow 0^+} \int_{|\xi-x| < \delta} |(\xi - e^{(T-t)B}x)_i|^{\frac{2}{q_i} + \varepsilon} \frac{p(t, x; T, d\xi)}{T-t} \\ &= \limsup_{T-t \rightarrow 0^+} \int_{|\xi-x| < \rho} |(\xi - e^{(T-t)B}x)_i|^{\frac{2}{q_i} + \varepsilon} \frac{p(t, x; T, d\xi)}{T-t} \\ & \quad + \int_{\rho \leq |\xi-x| < \delta} |(\xi - e^{(T-t)B}x)_i|^{\frac{2}{q_i} + \varepsilon} \frac{p(t, x; T, d\xi)}{T-t} \\ &\leq \limsup_{T-t \rightarrow 0^+} \int_{|\xi-x| < \rho} |(\xi - e^{(T-t)B}x)_i|^{\frac{2}{q_i} + \varepsilon} \frac{p(t, x; T, d\xi)}{T-t} \\ & \quad + (\delta + |x| |I_d - e^{(T-t)B}|)^{\frac{2}{q_i} + \varepsilon} \int_{\rho \leq |\xi-x| < \delta} \frac{p(t, x; T, d\xi)}{T-t} \end{aligned}$$

(by (3.2))

$$\begin{aligned} &= \limsup_{T-t \rightarrow 0^+} \int_{|\xi-x| < \rho} |(\xi - e^{(T-t)B}x)_i|^{\frac{2}{q_i} + \varepsilon} \frac{p(t, x; T, d\xi)}{T-t} \\ &\leq \limsup_{T-t \rightarrow 0^+} (\rho + |x| |I_d - e^{(T-t)B}|)^\varepsilon \int_{|\xi-x| < \rho} |(\xi - e^{(T-t)B}x)_i|^{\frac{2}{q_i}} \frac{p(t, x; T, d\xi)}{T-t} \\ &\leq \rho^\varepsilon \limsup_{T-t \rightarrow 0^+} \int_{|\xi-x| < \delta} |(\xi - e^{(T-t)B}x)_i|^{\frac{2}{q_i}} \frac{p(t, x; T, d\xi)}{T-t}, \end{aligned}$$

which in turn implies

$$\limsup_{T-t \rightarrow 0^+} \int_{|\xi-x| < \delta} |(\xi - e^{(T-t)B}x)_i|^{\frac{2}{q_i} + \varepsilon} \frac{p(t, x; T, d\xi)}{T-t} = 0, \quad (3.21)$$

and thus, as ε is arbitrary,

$$\lim_{T-t \rightarrow 0^+} \int_{|x-\xi| < \delta} (\xi - e^{(T-t)B}x)_i \frac{p(t, x; T, d\xi)}{T-t} = 0.$$

The latter, combined with (3.19), proves (3.3) for $i = p_0 + 1, \dots, d$.

We now prove (3.4). By using (3.3) it is straightforward to show that

$$\begin{aligned} & \lim_{T-t \rightarrow 0^+} \int_{|\xi-x| < \delta} (\xi - x)_i (\xi - x)_j \frac{p(t, x; T, d\xi)}{T-t} \\ &= \lim_{T-t \rightarrow 0^+} \int_{|\xi-x| < \delta} (\xi - e^{(T-t)B}x)_i (\xi - e^{(T-t)B}x)_j \frac{p(t, x; T, d\xi)}{T-t}. \end{aligned} \quad (3.22)$$

Now, for $i, j = 1, \dots, p_0$, (3.4) simply stems from (3.6). In the case $i > p_0$ or $j > p_0$, by the Cauchy–Schwarz inequality we get

$$\begin{aligned} \int_{|\xi-x| < \delta} |(\xi - e^{(T-t)B}x)_i (\xi - e^{(T-t)B}x)_j| \frac{p(t, x; T, d\xi)}{T-t} &\leq \left(\int_{|\xi-x| < \delta} (\xi - e^{(T-t)B}x)_i^2 \frac{p(t, x; T, d\xi)}{T-t} \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_{|\xi-x| < \delta} (\xi - e^{(T-t)B}x)_j^2 \frac{p(t, x; T, d\xi)}{T-t} \right)^{\frac{1}{2}}, \end{aligned}$$

and the limits of the right-hand side integrals are both finite, at least one of each being zero. Indeed, by (3.6) and (3.21) we obtain

$$\int_{|\xi-x| < \delta} (\xi - e^{(T-t)B}x)_k^2 \frac{p(t, x; T, d\xi)}{T-t} = \begin{cases} a_{kk}(t, x) & \text{if } k = 1, \dots, p_0 \\ 0 & \text{if } k = p_0 + 1, \dots, d. \end{cases}$$

Therefore, we can conclude that for $i > p_0$ or $j > p_0$

$$\lim_{T \rightarrow 0^+} \int_{|\xi - x| < \delta} (\xi - e^{(T-t)B}x)_i (\xi - e^{(T-t)B}x)_j \frac{p(t, x; T, d\xi)}{T - t} = 0,$$

which, combined with (3.22), yields (3.4).

Part 2: only if. We now assume (Lim-i)–(Lim-ii) and prove the limits (3.5)–(3.6)–(3.7) to be true. Fix $t \in [0, T_0]$, $\delta > 0$, and H compact subset of D . Again, all the following limits are uniform w.r.t. $x \in H$.

The limits (3.5) and (3.6) stem again from (3.19)–(3.22) with $i, j = 1, \dots, p_0$. Thus to conclude we only need to prove (3.7). We remark that, by (3.2), it is not restrictive to assume $\delta < \text{dist}(H, \partial D)$. By (3.6) and by Jensen’s inequality, it is sufficient to prove

$$\limsup_{T \rightarrow 0^+} \frac{1}{T - t} E_{t,x} [|(X_T - e^{(T-t)B}x)_j| \mathbb{1}_{\{|X_T - x| < \delta\}}]^{\frac{2}{2i+1}} < \infty, \tag{3.23}$$

for any fixed $i \in \{1, \dots, r\}$ and $j \in \{\sum_{k=0}^{i-1} p_k + 1, \dots, d\}$. We prove (3.23) in two different steps:

Step 1. We prove that

$$\begin{aligned} & \limsup_{T \rightarrow 0^+} \frac{1}{T - t} E_{t,x} [|(X_T - e^{(T-t)B}x)_j| \mathbb{1}_{\{|X_T - x| < \delta\}}]^{\frac{2}{2i+1}} \\ & \leq \limsup_{T \rightarrow 0^+} \frac{1}{T - t} \left(\int_t^T E_{t,x} [|(B^{(j)}, X_s - e^{(s-t)B}x)| \mathbb{1}_{\{|X_s - x| < \delta\}}] ds \right)^{\frac{2}{2i+1}}, \end{aligned} \tag{3.24}$$

for any $i \geq 1$ and $j = \{p_0 + 1, \dots, d\}$. Here and further on, $B^{(j)}$ denotes the j -th row of B .

Let $\varphi_j^{(t,x)} \in C_0^\infty([t, T_0] \times D)$, $j = p_0 + 1, \dots, d$, be a family of functions with partial derivatives uniformly bounded w.r.t. $(t, x) \in [0, T_0] \times H$ and such that

$$\varphi_j(s, \xi) = \varphi_j^{(t,x)}(s, \xi) = \xi_j - (e^{(s-t)B}x)_j, \quad |\xi - x| < \delta. \tag{3.25}$$

Note that we have

$$\begin{cases} \partial_s \varphi_j(s, \xi) = -\langle B^{(j)}, e^{(s-t)B}x \rangle, \\ \partial_{\xi_j} \varphi_j(s, \xi) = 1, \\ \partial_{\xi_k} \varphi_j(s, \xi) = 0 \quad \text{for } k \neq j, \\ \partial_{\xi_k \xi_l} \varphi_j(s, \xi) = 0, \end{cases} \quad s \in [t, T_0], |\xi - x| < \delta. \tag{3.26}$$

By Proposition 3.8 we have

$$\varphi_j(T, X_T) = \varphi_j(t, X_t) + \int_t^T (\partial_s + \mathcal{A}_s) \varphi_j(s, X_s) ds + M_T^t$$

(by (3.25)–(3.26))

$$\begin{aligned} & = \varphi_j(t, X_t) + \int_t^T \langle B^{(j)}, X_s - e^{(s-t)B}x \rangle \mathbb{1}_{\{|X_s - x| < \delta\}} ds \\ & \quad + \int_t^T (\partial_s + \mathcal{A}_s) \varphi_j(s, X_s) \mathbb{1}_{\{|X_s - x| \geq \delta\}} ds + M_T^t, \end{aligned}$$

where M^t is an \mathcal{F}^t -martingale under $P_{t,x}$ with

$$E_{t,x} [|M_T^t|^2] = E_{t,x} \left[\int_t^T \sum_{k,l=1}^{p_0} a_{kl}(s, X_s) (\partial_{x_k} \varphi_j(s, X_s)) (\partial_{x_l} \varphi_j(s, X_s)) ds \right]. \tag{3.27}$$

Therefore, for any $i \geq 1$ and $j = p_0 + 1, \dots, d$ one obtains

$$E_{t,x} [|(X_T - e^{(T-t)B}x)_j| \mathbb{1}_{\{|X_T - x| < \delta\}}]^{\frac{2}{2i+1}} \leq E_{t,x} [|\varphi_j(T, X_T)|]^{\frac{2}{2i+1}} \leq \sum_{k=1}^3 I_k^{\frac{2}{2i+1}}(t, x; T),$$

where

$$I_1(t, x; T) = E_{t,x} \left[\left| \int_t^T (\partial_s + \mathcal{A}_s) \varphi_j(s, X_s) \mathbb{1}_{\{|X_s-x| \geq \delta\}} ds \right| \right], \quad I_2(t, x; T) = E_{t,x} [|M_T^t|],$$

$$I_3(t, x; T) = E_{t,x} \left[\left| \int_t^T \langle B^{(j)}, X_s - e^{(s-t)B} x \rangle \mathbb{1}_{\{|X_s-x| < \delta\}} ds \right| \right],$$

where we used that $\varphi_j(t, X_t) = 0$ $P_{t,x}$ -almost surely. Since the coefficients of \mathcal{A} are locally bounded on $[0, T_0] \times D$ and $\varphi_j \in C_0^\infty([t, T_0] \times D)$, we obtain

$$I_1(t, x; T) \leq C \int_t^T E_{t,x} [\mathbb{1}_{\{|X_s-x| \geq \delta\}} \mathbb{1}_{\{X_s \in \text{supp}(\varphi_j)\}}] ds$$

$$= C \int_t^T (s-t)^{\frac{2i+1}{2}} \int_{(\{\xi-x| \geq \delta\} \cap \text{supp}(\varphi_j))} \frac{p(t, x; s, d\xi)}{(s-t)^{\frac{2i+1}{2}}} ds$$

(by (3.17) with $m = \frac{2i+1}{2}$)

$$\leq C \int_t^T (s-t)^{\frac{2i+1}{2}} ds \leq C(T-t)^{\frac{2i+1}{2}+1},$$

for any $i \geq 1$, which proves

$$\lim_{T-t \rightarrow 0^+} \frac{1}{T-t} I_1^{\frac{2}{2i+1}}(t, x; T) = 0, \quad i \geq 1.$$

Similarly, Jensen’s inequality yields

$$I_2(t, x; T) \leq E_{t,x} [|M_T^t|^2]^{\frac{1}{2}}$$

(by (3.27) combined with (3.26))

$$= E_{t,x} \left[\int_t^T \mathbb{1}_{\{|X_s-x| \geq \delta\}} \sum_{k,l=1}^{p_0} a_{kl}(s, X_s) (\partial_{x_k} \varphi_j(s, X_s)) (\partial_{x_l} \varphi_j(s, X_s)) ds \right]^{\frac{1}{2}}$$

(by using again the local boundedness of a_{kl} on $[0, T_0] \times D$ and $\varphi_j \in C_0^\infty([0, T_0] \times D)$)

$$\leq C E_{t,x} \left[\int_t^T \mathbb{1}_{\{|X_s-x| \geq \delta\}} \mathbb{1}_{\{X_s \in \text{supp}(\varphi_j)\}} ds \right]^{\frac{1}{2}},$$

and by proceeding as we did to estimate $I_1(t, x; T)$ it easily follows that

$$\lim_{T-t \rightarrow 0^+} \frac{1}{T-t} I_2^{\frac{2}{2i+1}}(t, x; T) = 0, \quad i \geq 1.$$

Finally,

$$\limsup_{T-t \rightarrow 0^+} \frac{1}{T-t} I_3^{\frac{2}{2i+1}}(t, x; T) \leq \limsup_{T-t \rightarrow 0^+} \frac{1}{T-t} \left(\int_t^T E_{t,x} [|\langle B^{(j)}, X_s - e^{(s-t)B} x \rangle| \mathbb{1}_{\{|X_s-x| < \delta\}}] ds \right)^{\frac{2}{2i+1}}$$

yields (3.24) for any $i \geq 1$.

Step 2. We prove (3.23) by induction on i . To start the inductive procedure, set $i = 0$ and prove (3.23) for any $j \in \{1, \dots, d\}$. If $j \in \{1, \dots, \bar{p}_0\}$, then (3.23) stems from (3.6) by applying Jensen’s inequality. If $j \in \{\bar{p}_0 + 1, \dots, d\}$, then (3.23) follows trivially from (3.24) by observing that

$$|\langle B^{(j)}, X_s - e^{(s-t)B} x \rangle| \mathbb{1}_{\{|X_s-x| < \delta\}}$$

is uniformly bounded.

Set now $\bar{i} \in \{0, \dots, r - 1\}$, assume (3.23) true for $i = \bar{i}$ and $j \in \{\sum_{k=0}^{\bar{i}-1} p_k + 1, \dots, d\}$ and prove it true for $i = \bar{i} + 1$ and $j \in \{\sum_{k=0}^{\bar{i}} p_k + 1, \dots, d\}$. Now, by the block structure of B (2.1), we obtain

$$\int_t^T E_{t,x} [|(B^{(j)}, X_s - e^{(s-t)B}x)| \mathbb{1}_{\{|X_s - x| < \delta\}}] ds \leq C \sum_{k=\sum_{k=0}^{\bar{i}-1} p_k + 1}^d \int_t^T E_{t,x} [|(X_s - e^{(s-t)B}x)_k| \mathbb{1}_{\{|X_s - x| < \delta\}}] ds$$

(by inductive hypothesis)

$$\leq C \int_t^T (s - t)^{\frac{2\bar{i}+1}{2}} ds \leq C(T - t)^{\frac{2(\bar{i}+1)+1}{2}},$$

which, combined with (3.24), yields exactly (3.23) for $i = \bar{i} + 1$ and concludes the proof. □

4. Local densities

This section is devoted to the proof of Theorem 1.5. We will adapt and customize a localization procedure first introduced in [9]. From now on, throughout the rest of this section, we will always assume the structural Assumption 1.1 for the matrix B to be in force, and that the coefficients a_{ij}, a_i satisfy Assumption 1.2 for some α, M and with $N = 0$. Assumption 1.2 with $N = 2$ and Assumption 1.4 will be only required in the proof of Part (b) and Part (c) of Theorem 1.5, respectively.

4.1. Proof of Theorem 1.5, Part (a), (b)

We start by observing that the nature of Theorem 1.5(a), (b) is strictly local for what concerns the existence of the transition density and its regularity w.r.t. the forward space-variable. In other words, it is enough to prove it for $\Gamma(t, x; T, \xi)$ defined for any $0 < t < T < T_0$ and $x \in \mathbb{R}^d, \xi \in D'$ and for any sub-domain $D' \Subset D$. Therefore, it is not restrictive to assume that there exists a family $\tilde{a}_{ij}, \tilde{a}_i :]0, T_0[\times \mathbb{R}^d \rightarrow \mathbb{R}, i, j = 1, \dots, p_0$, such that $(\tilde{a}_{ij}, \tilde{a}_i)$ coincide with (a_{ij}, a_i) on $]0, T_0[\times D$ and, if Assumption 1.2 is satisfied for a certain N , then:

- (i) $\tilde{a}_{ij}, \tilde{a}_i \in C_B^{N,\alpha}(]0, T_0[\times \mathbb{R}^d)$ for any $i, j = 1, \dots, p_0$, with all the (Lie) derivatives bounded by M ;
- (ii) the following coercivity condition holds on \mathbb{R}^d

$$M^{-1}|\xi|^2 \leq \sum_{i,j=1}^{p_0} \tilde{a}_{ij}(t, x)\xi_i\xi_j \leq M|\xi|^2, \quad t \in]0, T_0[, x \in \mathbb{R}^d, \xi \in \mathbb{R}^{p_0}.$$

Let us denote by $\tilde{\mathcal{A}}_t$ the operator defined as

$$\tilde{\mathcal{A}}_t := \frac{1}{2} \sum_{i,j=1}^{p_0} \tilde{a}_{ij}(t, x)\partial_{x_i x_j} + \sum_{i=1}^{p_0} \tilde{a}_i(t, x)\partial_{x_i} + \langle Bx, \nabla_x \rangle \quad t \in [0, T_0[, x \in \mathbb{R}^d,$$

where B is as in (2.1). We consider an auxiliary \mathbb{R}^d -valued continuous strong Markov process $\tilde{X} = (\tilde{X}_t)_{t \in [0, T_0[}$ defined on a space $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t^X)_{0 \leq t \leq T < T_0}, (\tilde{P}_{t,x})_{0 \leq t < T_0, x \in \mathbb{R}^d})$, with transition probability function $\tilde{p} = \tilde{p}(t, x; T, d\xi)$, such that \tilde{X} is an $\tilde{\mathcal{A}}_t$ -global diffusion on \mathbb{R}^d in the sense of Definition 3.2; in Appendix B we briefly recall the standard construction of such \tilde{X} . In particular, it will result in $\tilde{p}(t, x; T, d\xi)$ having a density $\tilde{\Gamma}(t, x; T, \xi)$, which coincides with the fundamental solution of the operator

$$\tilde{\mathcal{L}} = \tilde{\mathcal{A}}_t + \partial_t = \frac{1}{2} \sum_{i,j=1}^{p_0} \tilde{a}_{ij}(t, x)\partial_{x_i x_j} + \sum_{i=1}^{p_0} \tilde{a}_i(t, x)\partial_{x_i} + Y, \quad t \in [0, T_0[, x \in \mathbb{R}^d. \tag{4.1}$$

In Theorem A.1 in Appendix A, some previous results are reported about the existence of $\tilde{\Gamma}$, its regularity and some sharp Gaussian upper bounds for $\tilde{\Gamma}$ and its derivatives.

Notation 4.1. For any $x \in \mathbb{R}^d, t < T$ and $\varepsilon \in]0, 1[$, we set the cylinder

$$H_\varepsilon(t, x; T) :=]t, T[\times S_\varepsilon(x), \quad S_\varepsilon(x) := B_1(x - \varepsilon e_1) \cap B_1(x + \varepsilon e_1),$$

where $B_r(x) \subset \mathbb{R}^d$ is the Euclidean (open) ball with radius r centered at x and $e_1 = (1, 0, \dots, 0)$ is the first vector of the canonical basis of \mathbb{R}^d . We define the lateral boundary and parabolic boundary of $H_\varepsilon(t, x; T)$, respectively, as

$$\partial_\Sigma H_\varepsilon(t, x; T) :=]t, T[\times \partial S_\varepsilon(t, x), \quad \partial_P H_\varepsilon(t, x; T) := \partial_\Sigma H_\varepsilon(t, x; T) \cup (\{T\} \times \overline{S_\varepsilon(x)}).$$

We also denote by $G = G(t, x; T, \xi)$ the Green function of $(\partial_t + \mathcal{A}_t)$ for $H_\varepsilon(0, x_0; T_0)$, which is defined for any $0 < t < T < T_0$ and $x, \xi \in \overline{S_\varepsilon(x_0)}$ and enjoys the properties listed in Lemma A.2. In the latter, we report some preliminary existence (and uniqueness) and regularity results for the solution of the Cauchy–Dirichlet problem on $H_\varepsilon(0, x_0; T_0)$ and in particular for the Green function G .

Roughly speaking, the following result shows that, prior to the exit time from $S_\varepsilon(x_0)$, X and \tilde{X} have the same law whose density coincides with the Green function G . The proof is based on the crucial fact that the Itô formula (1.8) is valid for functions that are C^2 in the intrinsic sense, i.e. C^2_B , and not only for functions that are C^2 in the Euclidean sense.

Lemma 4.2. *Let $x_0 \in D$ and $\varepsilon \in]0, 1[$ such that $\overline{S_\varepsilon(x_0)} \subset D$. Then, for any $(t, x) \in H_\varepsilon(0, x_0; T_0)$ we have*

$$P_{t,x}(X_T \in A, \tau^{(t)} > T) = \int_A G(t, x; T, \xi) d\xi = \tilde{P}_{t,x}(\tilde{X}_T \in A, \tilde{\tau}^{(t)} > T), \quad T \in]t, T_0[, A \in \mathcal{B}(S_\varepsilon(x_0)), \tag{4.2}$$

where $\tau^{(t)}$ and $\tilde{\tau}^{(t)}$ are the t -stopping times defined, respectively, as

$$\tau^{(t)} := \inf\{s \geq t : X_s \notin S_\varepsilon(x_0)\}, \quad \tilde{\tau}^{(t)} := \inf\{s \geq t : \tilde{X}_s \notin S_\varepsilon(x_0)\}. \tag{4.3}$$

Before proving Lemma 4.2 we want to stress the following

Remark 4.3. By (4.2) we obtain

$$\int_A G(t, x; T, \xi) d\xi \leq \tilde{P}_{t,x}(\tilde{X}_T \in A) = \int_A \tilde{\Gamma}(t, x; T, \xi) d\xi, \quad A \in \mathcal{B}(S_\varepsilon(x_0)),$$

which implies

$$G(t, x; T, \xi) \leq \tilde{\Gamma}(t, x; T, \xi), \quad 0 < t < T < T_0, x, \xi \in S_\varepsilon(x_0). \tag{4.4}$$

Proof of Lemma 4.2. Throughout the proof we will set $\tau := \tau^{(t)}$ to shorten notation. Note that (4.2) is equivalent to

$$E_{t,x}[\varphi(X_T)\mathbb{1}_{\tau > T}] = \int_{S_\varepsilon(x_0)} G(t, x; T, \xi)\varphi(\xi) d\xi = \tilde{E}_{t,x}[\varphi(\tilde{X}_T)\mathbb{1}_{\tilde{\tau} > T}], \quad \varphi \in C_0^\infty(S_\varepsilon(x_0)).$$

Denote by f the unique solution in $C_B^{2,\alpha}(H_\varepsilon(0, x_0; T)) \cap C((H_\varepsilon \cup \partial_P H_\varepsilon)(t, x_0; T))$ (see Lemma A.2) of

$$\begin{cases} \mathcal{L}f = 0 & \text{on } H_\varepsilon(0, x_0; T), \\ f = 0 & \text{on } \partial_\Sigma H_\varepsilon(0, x_0; T), \\ f(T, \cdot) = \varphi & \text{on } S_\varepsilon(x_0) \end{cases} \tag{4.5}$$

which is given by

$$f(t, x) = \int_{S_\varepsilon(x_0)} G(t, x; T, \xi)\varphi(\xi) d\xi.$$

Now, by Corollary 1.3 combined with Optional Sampling Theorem, the process $M_{t \wedge \tau}^f$ with M^t as in (1.8) and f as in (4.5), is an \mathcal{F}^t -martingale under $P_{t,x}$: we notice explicitly that, even if f is not defined on $[0, T_0[\times D$ as required by Theorem 1.3, a standard extension-truncation argument can be employed. Thus

$$E_{t,x}[\varphi(X_T)\mathbb{1}_{T < \tau}] = E_{t,x}[f(T \wedge \tau, X_{T \wedge \tau})] = f(t, x).$$

On the other hand, since $\mathcal{A} = \tilde{\mathcal{A}}$ on $]0, T_0[\times D$, \tilde{X} is also an \mathcal{A}_t -local diffusion on D and thus it holds

$$\tilde{E}_{t,x}[\varphi(\tilde{X}_T)\mathbb{1}_{T < \tilde{\tau}}] = \tilde{E}_{t,x}[f(T \wedge \tilde{\tau}, X_{T \wedge \tilde{\tau}})] = f(t, x),$$

which proves (4.2) and concludes the proof. □

Proof of Theorem 1.5(a), (b). Fix $(t, x) \in]0, T_0[\times \mathbb{R}^d$ and $x_0 \in D$ and let $\varepsilon \in]0, 1[$ such that $\overline{S_\varepsilon(x_0)} \subseteq D$. Let also U and V be two non-empty open subsets such that $x_0 \in U \subseteq V \subseteq S_\varepsilon(x_0)$.

Define now $\tau_0^{(t)} \equiv t$ and the families $(\sigma_n^{(t)})_{n \in \mathbb{N}}$ and $(\tau_n^{(t)})_{n \in \mathbb{N}}$ through the following recursion:

$$\sigma_n^{(t)} := \inf\{s \geq \tau_{n-1}^{(t)} : X_s \in \overline{V}\},$$

and $\tau_n^{(t)} := \tau^{(\sigma_n^{(t)})}$ according to notation (4.3), which is

$$\tau_n^{(t)} = \inf\{s \geq \sigma_n^{(t)} : X_s \notin S_\varepsilon(x_0)\}.$$

Hereafter, whenever it is clear from the context, we will drop the suffix (t) in $\tau_n^{(t)}, \sigma_n^{(t)}$ to ease the notation.

Part (a): note that for any $T \in]t, T_0[$ the event $X_T \in U$ is included in the disjoint union $\bigcup_{n \in \mathbb{N}} (\sigma_n^{(t)} < T < \tau_n^{(t)})$. Therefore, for any $A \in \mathcal{B}(U)$ one obtains

$$\begin{aligned} p(t, x; T, A) &= \sum_{n=1}^{\infty} P_{t,x}(X_T \in A, \sigma_n < T < \tau_n) = \sum_{n=1}^{\infty} P_{t,x}(X_T \in A, \sigma_n < T < \tau^{(\sigma_n)}) \\ &= \sum_{n=1}^{\infty} E_{t,x}[P_{t,x}(X_T \in A, \sigma_n < T < \tau^{(\sigma_n)} | \mathcal{F}_{\sigma_n})] \\ &= \sum_{n=1}^{\infty} E_{t,x}[P_{t,x}(X_T \in A, T < \tau^{(\sigma_n)} | \mathcal{F}_{\sigma_n}) \mathbb{1}_{\sigma_n < T}] \end{aligned}$$

(by the strong Markov property)

$$\begin{aligned} &= \sum_{n=1}^{\infty} E_{t,x}[P_{s,y}(X_T \in A, \tau^{(s)} > T) |_{s=\sigma_n, y=X_{\sigma_n}} \mathbb{1}_{\sigma_n < T}] \\ &= \sum_{n=1}^{\infty} E_{t,x}\left[\int_A G(\sigma_n, X_{\sigma_n}; T, \xi) d\xi \mathbb{1}_{\sigma_n < T}\right], \end{aligned} \tag{4.6}$$

where we used (4.2) in the last equality. Our derivation of (4.6) follows closely the original argument by [9] even if here we go a step further using the representation in terms of the Green kernel (4.2), which is crucial in the subsequent study of the regularity properties of the local density of X .

From (4.6) and since x_0 is arbitrary, it follows that $p(t, x; T, \cdot)$ is absolutely continuous w.r.t. the Lebesgue measure on D and therefore admits a density $\Gamma(t, x; T, \xi)$. Moreover, for any $x_0 \in D$ and $\varepsilon \in]0, 1[$ such that $\overline{S_\varepsilon(x_0)} \subset D$, we have the local representation

$$\Gamma(t, x; T, \xi) = \sum_{n=1}^{\infty} E_{t,x}[G(\sigma_n, X_{\sigma_n}; T, \xi) \mathbb{1}_{\sigma_n < T}], \quad T \in]t, T_0[, \xi \in S_\varepsilon(x_0). \tag{4.7}$$

Assume now that there exists $C_1 > 0$, independent of t, x and T , such that

$$\sum_{n=1}^{\infty} P_{t,x}(\sigma_n^{(t)} < T) \leq C_1, \quad T \in]t, T_0[. \tag{4.8}$$

Then, by the continuity of $G(t, x; T, \cdot)$ on $S_\varepsilon(x_0)$ combined with (4.4) and the estimates (A.1), it follows that $\Gamma(t, x; T, \cdot)$ is continuous and bounded on $S_\varepsilon(x_0)$, uniformly w.r.t. $x \in \mathbb{R}^d$. Therefore, to conclude the proof of Part (a) we only need to prove (4.8).

Start by observing that

$$\sum_{n=2}^{\infty} P_{t,x}(\sigma_n < T) \leq \sum_{n=1}^{\infty} P_{t,x}(\tau_n < T), \tag{4.9}$$

and that, by classical maximal estimates (e.g. [15], p. 296), it holds that

$$P_{s,y}(\tau^{(s)} < T) \leq C e^{-\frac{1}{C(T-s)}}, \quad 0 < s < T < T_0, y \in \partial V, \tag{4.10}$$

where $C > 0$ only depends on T_0 and \mathcal{A}_t , but not on s, T and y . Therefore, for any $n \geq 1$ we have

$$P_{t,x}(\tau_n < T) = E_{t,x}[E_{t,x}[\mathbb{1}_{\tau^{(\sigma_n)} < T} | \mathcal{F}_{\sigma_n}]] = E_{t,x}[E_{s,y}[\mathbb{1}_{\tau^{(s)} < T}] | s=\sigma_n, y=X_{\sigma_n}]$$

(by (4.10))

$$\leq C e^{-\frac{1}{C(T-t)}} P_{t,x}(\sigma_n < T) \leq C e^{-\frac{1}{C(T-t)}} P_{t,x}(\tau_{n-1} < T),$$

which yields $P_{t,x}(\tau_n < T) \leq (C e^{-\frac{1}{C(T-t)}})^n$. This combined with (4.9) proves (4.8) for $T - t \leq T^*$ and for a positive T^* suitably small only dependent on T_0 and \mathcal{A}_t . To prove (4.8) for a generic $T \in]t, T_0[$ consider a partition $t = t_0 < t_1 < \dots < t_{\bar{N}} = T$, such that $t_{k+1} - t_k < T^*$. Define $i_k := \inf\{n \in \mathbb{N} : \sigma_n^{(t)} \geq t_k\}$. We first observe that

$$\sum_{n=1}^{\infty} \mathbb{1}_{t_k \leq \sigma_n^{(t)} < t_{k+1}} = \sum_{n=i_k}^{\infty} \mathbb{1}_{\sigma_n^{(t)} < t_{k+1}} = \sum_{m=0}^{\infty} \mathbb{1}_{\sigma_{i_k+m}^{(t)} < t_{k+1}}$$

(since we have $\sigma_{i_k}^{(t)} \in \{\sigma_1^{(t_k)}, \sigma_2^{(t_k)}\}$ and thus, by induction, $\sigma_{i_k+m}^{(t)} \geq \sigma_{1+m}^{(t_k)}$)

$$\leq \sum_{m=1}^{\infty} \mathbb{1}_{\sigma_m^{(t_k)} < t_{k+1}}, \quad k = 0, \dots, \bar{N} - 1.$$

Hence

$$\begin{aligned} \sum_{n=1}^{\infty} P_{t,x}(\sigma_n^{(t)} < T) &= \sum_{k=0}^{\bar{N}-1} \sum_{n=1}^{\infty} P_{t,x}(t_k \leq \sigma_n^{(t)} < t_{k+1}) \leq \sum_{k=0}^{\bar{N}-1} \sum_{m=1}^{\infty} P_{t,x}(\sigma_m^{(t_k)} < t_{k+1}) \\ &= \sum_{k=0}^{\bar{N}-1} \sum_{m=1}^{\infty} E_{t,x}[P_{t,x}(\sigma_m^{(t_k)} < t_{k+1} | \mathcal{F}_{t_k}^t)] = \sum_{k=0}^{\bar{N}-1} E_{t,x} \left[\sum_{m=1}^{\infty} P_{t_k,y}(\sigma_m^{(t_k)} < t_{k+1}) \Big|_{y=X_{t_k}} \right], \end{aligned}$$

which proves (4.8).

Part (b): We assume here Assumption 1.2 to be in force for $N = 2$. By combining the representation of G in [7], p. 36, with the internal estimates in the same reference that are reported in Theorem A.3(b) below, it follows that

$$|\partial_{\xi_i} G(s, y; T, \xi)| + |\partial_{\xi_i \xi_j} G(s, y; T, \xi)| + |Y_{T,\xi} G(s, y; T, \xi)| \leq C_2, \quad 0 < s < T < T_0, y \in \partial V, \xi \in U, \tag{4.11}$$

for any $i, j = 1, \dots, p_0$. This and (4.8) allow us to employ bounded convergence theorem and differentiate twice under the sign of expectation the right-hand side of (4.7) w.r.t. ξ . For any $T \in]t, T_0[, \xi \in U$ we obtain:

$$\begin{aligned} \partial_{\xi_i} \Gamma(t, x; T, \xi) &= \sum_{n=1}^{\infty} E_{t,x}[\partial_{\xi_i} G(\sigma_n, X_{\sigma_n}; T, \xi) \mathbb{1}_{\sigma_n < T}], \\ \partial_{\xi_i \xi_j} \Gamma(t, x; T, \xi) &= \sum_{n=1}^{\infty} E_{t,x}[\partial_{\xi_i \xi_j} G(\sigma_n, X_{\sigma_n}; T, \xi) \mathbb{1}_{\sigma_n < T}], \end{aligned}$$

for $i, j = 1, \dots, p_0$. As for $Y\Gamma(t, x; \cdot, \cdot)$, we have

$$\begin{aligned} Y_{T,\xi} \Gamma(t, x; T, \xi) &= \lim_{h \rightarrow 0} \frac{\Gamma(t, x; T + h, e^{hB} \xi) - \Gamma(t, x; T, \xi)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \sum_{n=1}^{\infty} (E_{t,x}[G(\sigma_n, X_{\sigma_n}; T + h, e^{hB} \xi) \mathbb{1}_{\sigma_n < T+h}] - E_{t,x}[G(\sigma_n, X_{\sigma_n}; T, \xi) \mathbb{1}_{\sigma_n < T}]) \end{aligned}$$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{1}{h} \sum_{n=1}^{\infty} E_{t,x} \left[\underbrace{G(\sigma_n, X_{\sigma_n}; T+h, e^{hB}\xi) \mathbb{1}_{T \leq \sigma_n < T+h}}_{=: I_{1,n}(h)} \right] \\
 &\quad + \lim_{h \rightarrow 0} \frac{1}{h} \sum_{n=1}^{\infty} E_{t,x} \left[\underbrace{(G(\sigma_n, X_{\sigma_n}; T+h, e^{hB}\xi) - G(\sigma_n, X_{\sigma_n}; T, \xi)) \mathbb{1}_{\sigma_n < T}}_{I_{2,n}(h)} \right].
 \end{aligned}$$

Remark 4.3 together with (A.1) on one hand, and mean value theorem on the other, yield

$$\begin{aligned}
 \frac{1}{h} I_{1,n}(h) &\leq C \mathbb{1}_{T \leq \sigma_n < T+h}, \\
 \frac{1}{h} I_{2,n}(h) &= Y_{T,\xi} G(\sigma_n, X_{\sigma_n}; T + \tilde{h}, e^{\tilde{h}B}\xi) \mathbb{1}_{\sigma_n < T}, \quad \text{with } |\tilde{h}| \leq h.
 \end{aligned}$$

Here the low index in $Y_{T,\xi}$ is meant to stress that Y is computed with respect to the variables (T, ξ) . By (4.8) and (4.11) we can apply bounded convergence theorem and obtain

$$Y_{T,\xi} \Gamma(t, x; T, \xi) = \sum_{n=1}^{\infty} E_{t,x} [\mathbb{1}_{\sigma_n < T} Y_{T,\xi} G(\sigma_n, X_{\sigma_n}; T, \xi)], \quad T \in]t, T_0[, \xi \in U.$$

Proceeding analogously, by employing again the Schauder estimates reported in Theorem A.3(b), one also proves

$$\partial_{\xi_i} \Gamma(t, x; \cdot, \cdot) \in C_Y^{1+\alpha}([t, T_0] \times U), \quad \partial_{\xi_i \xi_j} \Gamma(t, x; \cdot, \cdot), Y \Gamma(t, x; \cdot, \cdot) \in C_B^{0,\alpha}([t, T_0] \times U), \quad i, j = 1, \dots, p_0,$$

which is $\Gamma(t, x; \cdot, \cdot) \in C_B^{2,\alpha}([t, T_0] \times U)$. Eventually, $\Gamma(t, x; \cdot, \cdot) \in C_B^{2,\alpha}([t, T_0] \times D)$ follows from the fact that x_0 is arbitrary. The fact that $\Gamma(t, x; \cdot, \cdot)$ solves (1.13) is now a straightforward consequence of Remark 3.7, by integrating by parts the left-hand side of (3.11) and since f is arbitrary. \square

4.2. Proof of Theorem 1.5, Part (c)

Hereafter throughout this section we assume Assumption 1.4 to be in force as well.

Notation 4.4. For any $T \in]0, T_0[$ and any φ bounded and Borel-measurable on \mathbb{R}^d (in short $\varphi \in m\mathcal{B}_b$), let $u_{\varphi,T} :]0, T[\times D \rightarrow \mathbb{R}$ be the function defined as $u_{\varphi,T}(t, x) := (\mathbf{T}_{t,T} \varphi)(x)$.

Lemma 4.5. Let $T \in]0, T_0[$ and $\varphi \in m\mathcal{B}_b$ such that $u_{\varphi,T} \in C(]0, T[\times D)$. Then, $u_{\varphi,T} \in C_B^{2,\alpha}(]0, T[\times D)$ and solves the backward Kolmogorov equation (1.14).

Again, the proof is based on the crucial fact that the Itô formula (1.8) is valid for functions that are C^2 in the intrinsic sense, i.e. C_B^2 , and not only for functions the are C^2 in the Euclidean sense.

Proof. Let $\delta > 0$, $x_0 \in D$, $\varepsilon \in]0, 1[$ such that $\overline{S_\varepsilon(x_0)} \subset D$ and denote by f the unique solution in $C_B^{2,\alpha}(H_\varepsilon(0, x_0; T - \delta)) \cap C((H_\varepsilon \cup \partial_P H_\varepsilon)(0, x_0; T - \delta))$ (see Theorem A.2) of

$$\begin{cases} \mathcal{L}f = 0 & \text{on } H_\varepsilon(0, x_0; T - \delta), \\ f = u_{\varphi,T} & \text{on } \partial_P H_\varepsilon(0, x_0; T - \delta). \end{cases} \tag{4.12}$$

For any $t \in]0, T - \delta[$ let now $\tau = \tau^{(t)}$ be the t -stopping time as defined in (4.3). By Theorem 1.3 combined with Optional Sampling Theorem, the process $M^t_{\wedge \tau}$, with M^t as in (1.8) and f as in (4.12), is an \mathcal{F}^t -martingale under $P_{t,x}$. Thus

$$f(t, x) = E_{t,x} [u_{\varphi,T}((T - \delta) \wedge \tau, X_{(T-\delta) \wedge \tau})] = E_{t,x} [E_{s,y} [\varphi(X_T)]_{|s=(T-\delta) \wedge \tau, y=X_{(T-\delta) \wedge \tau}}]$$

(by Strong Markov property)

$$= E_{t,x} [E_{t,x} [\varphi(X_T) | \mathcal{F}_{(T-\delta) \wedge \tau}]] = E_{t,x} [\varphi(X_T)] = u_{\varphi,T}(t, x).$$

Since x_0 and δ are arbitrary, then $u_{\varphi,T} \in C_B^{2,\alpha}(]0, T[\times D)$ and solves (1.14). \square

Lemma 4.6 (Strong Feller property). For any $T \in]0, T_0[$ and any $\varphi \in m\mathcal{B}_b$, $u_{\varphi,T} \in C(]0, T[\times D)$.

Proof. First note that, by Assumption 1.4 combined with Lemma 4.5, $u_{\psi,T} \in C_B^{2,\alpha}(]0, T[\times D)$ and solves the backward Kolmogorov equation (1.14), for any $\psi \in C_b(\mathbb{R}^d)$. Thus, by the internal Schauder estimates reported in Theorem A.3(a), for any bounded domain $V \Subset]0, T[\times D$ we have

$$\|u_{\psi,T}\|_{C_B^{0,\alpha}(V)} \leq C \sup_{]0, T[\times D} |u_{\psi,T}| \leq C \|\psi\|_\infty,$$

where C is a positive constant independent of ψ . In particular, by Theorem 2.4, for any $(t_0, x_0) \in]0, T[\times D$ there exists a neighborhood $U_{(t_0,x_0)}$ such that

$$|u_{\psi,T}(t, x) - u_{\psi,T}(t', x')| \leq C \|\varphi\|_\infty \|(t', x')^{-1} \circ (t, x)\|_B, \quad (t, x), (t', x') \in U_{(t_0,x_0)},$$

for any $\psi \in C_b(\mathbb{R}^d)$ such that $\|\psi\|_\infty \leq \|\varphi\|_\infty$. Therefore, in order to prove that $u_{\varphi,T}$ is continuous in (t_0, x_0) it suffices to prove that, for any $(t, x) \in U_{(t_0,x_0)}$, there exist a sequence of functions $\psi_n \in C_b(\mathbb{R}^d)$ with $\|\psi_n\|_\infty \leq \|\varphi\|_\infty$ such that

$$u_{\psi_n,T}(t, x) \longrightarrow u_{\varphi,T}(t, x), \quad u_{\psi_n,T}(t_0, x_0) \longrightarrow u_{\varphi,T}(t_0, x_0) \quad \text{as } n \rightarrow \infty. \tag{4.13}$$

To see this, let μ be the measure on $\mathcal{B}(\mathbb{R}^d)$ defined as

$$\mu(dz) = p(t, x; T, dz) + p(t_0, x_0; T, dz).$$

Note that we have

$$p(t, x; T, dz), p(t_0, x_0; T, dz) \ll \mu(dz). \tag{4.14}$$

Moreover, μ is a finite measure and thus, by Proposition 3.16 in [1], there exists a sequence of $\psi_n \in C_b(\mathbb{R}^d)$ with $\|\psi_n\|_\infty \leq \|\varphi\|_\infty$ such that

$$\|\psi_n - \varphi\|_{L^1(\mathcal{B}(\mathbb{R}^d), \mu)} \longrightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Therefore, by (4.14), $\psi_n \rightarrow \varphi$ both $p(t, x; T, dz)$ - and $p(t_0, x_0; T, dz)$ -almost everywhere. Thus bounded convergence theorem yields (4.13) and concludes the proof. \square

Proposition 4.7. For any $T \in]0, T_0[$ and any $\varphi \in m\mathcal{B}_b$, $u_{\varphi,T} \in C_B^{2,\alpha}(]0, T[\times D)$ and solves the backward Kolmogorov equation (1.14).

Proof. It is an immediate consequence of Lemmas 4.5 and 4.6. \square

Lemma 4.8. For any $0 < t < T < T_0$ and $\xi \in D$, the function $\Gamma(t, \cdot; T, \xi) \in m\mathcal{B}_b$, and

$$\Gamma(t, x; T, \xi) = E_{t,x}[\Gamma(s, X_s; T, \xi)] = (\mathbf{T}_{t,s}\Gamma(s, \cdot; T, \xi))(x), \quad t < s < T, x \in \mathbb{R}^d. \tag{4.15}$$

Proof. The boundedness is already contained in Part (a) of Theorem 1.5. We first prove the measurability of $\Gamma(t, \cdot; T, \xi)$. Let $(\varphi_n)_{n \in \mathbb{N}}$ a family of functions in $C_0(D)$ such that $\varphi_n \rightarrow \delta_\xi$, i.e.

$$\int_D f(y)\varphi_n(y) dy \rightarrow f(\xi) \quad \text{as } n \rightarrow \infty, f \in C(D).$$

Therefore, since $\Gamma(t, x; T, \cdot) \in C(D)$ (again by Part (a) of Theorem 1.5), we have

$$\Gamma(t, x; T, \xi) = \lim_{n \rightarrow \infty} \int_D \varphi_n(y)\Gamma(t, x; T, y) dy = \lim_{n \rightarrow \infty} u_{\varphi_n,T}(t, x), \quad x \in \mathbb{R}^d, \tag{4.16}$$

and since $u_{\varphi_n,T}(t, \cdot)$ is continuous (by Assumption 1.4), $\Gamma(t, \cdot; T, \xi)$ is measurable as it is the pointwise limit of a sequence of measurable functions.

We now prove (4.15). By (4.16) along with Markov property, it holds that

$$\begin{aligned} \Gamma(t, x; T, \xi) &= \lim_{n \rightarrow \infty} E_{t,x}[\varphi_n(X_T)] = \lim_{n \rightarrow \infty} E_{t,x}[E_{t,x}[\varphi_n(X_T)|\mathcal{F}_s^t]] = \lim_{n \rightarrow \infty} E_{t,x}[E_{s,y}[\varphi_n(X_T)]|_{y=X_s}] \\ &= \lim_{n \rightarrow \infty} E_{t,x}[u_{\varphi_n, T}(s, X_s)] = E_{t,x}[\Gamma(s, X_s; T, \xi)], \end{aligned}$$

where, in the last equality, we employed again (4.16) with $t = s$ and $x = X_s$ along with bounded convergence theorem (it is not restrictive to assume $\|\varphi_n\|_{L^1(D)} = 1$ and thus, since $\Gamma(s, x; T, \cdot)$ is locally bounded on D uniformly w.r.t. $x \in \mathbb{R}^d$, $u_{\varphi_n, T}(s, X_s)$ is bounded uniformly w.r.t. n). □

Proof of Theorem 1.5(c). It is a straightforward consequence of Proposition 4.7 and Lemma 4.8. □

Appendix A: Preliminary PDE results

In this appendix we collect some useful results about the operators \mathcal{L} in (1.7) and $\tilde{\mathcal{L}}$ in (4.1).

Theorem A.1. *Let Assumption 1.1 be in force and let the coefficients $\tilde{a}_{ij}, \tilde{a}_i$ in (1.7) satisfy Assumptions (i) with $N = 0$ and (ii) at the beginning of Section 4.1. There exists a unique fundamental solution for $\tilde{\mathcal{L}}$, namely a continuous non-negative function $\tilde{\Gamma} = \tilde{\Gamma}(t, x; T, \xi)$ defined for any $0 < t < T < T_0$ and $x, \xi \in \mathbb{R}^d$ enjoying the following properties:*

(a) *for any $(T, \xi) \in]0, T_0[\times \mathbb{R}^d$, the function $\tilde{\Gamma}(\cdot, \cdot; T, \xi) \in C_B^{2,\alpha}(]0, T[\times \mathbb{R}^d)$ and is a solution of*

$$\begin{cases} \tilde{\mathcal{L}}u = 0 & \text{on }]0, T[\times \mathbb{R}^d, \\ u(T, \cdot) = \delta_\xi, \end{cases}$$

where the terminal condition is in the distributional sense, i.e.

$$\lim_{t \rightarrow T^-} \int_{\mathbb{R}^d} \tilde{\Gamma}(t, x; T, \xi) \varphi(x) dx = \varphi(\xi), \quad \varphi \in C_0(\mathbb{R}^d);$$

Moreover, for any $0 < t < T < T_0$, $x, \xi \in \mathbb{R}^d$ and $i, j = 1, \dots, p_0$, we have

$$\begin{aligned} \tilde{\Gamma}(t, x; T, \xi) &\leq C \bar{\Gamma}_M(t, x; T, \xi), \\ |\partial_{x_i} \tilde{\Gamma}(t, x; T, \xi)| &\leq \frac{C}{\sqrt{T-t}} \bar{\Gamma}_M(t, x; T, \xi), \\ |\partial_{x_i x_j} \tilde{\Gamma}(t, x; T, \xi)| + |Y_{t,x} \tilde{\Gamma}(t, x; T, \xi)| &\leq \frac{C}{T-t} \bar{\Gamma}_M(t, x; T, \xi), \end{aligned} \tag{A.1}$$

where C is a positive constant that depends only on B, M, T_0 and where

$$\bar{\Gamma}_M(t, x; T, \xi) = \frac{1}{\sqrt{(2\pi)^d \det \mathbf{C}(T-t)}} \exp\left(-\frac{1}{2} \langle \mathbf{C}^{-1}(T-t)(y - e^{(T-t)B}x), (y - e^{(T-t)B}x) \rangle\right)$$

is the fundamental solution of a constant coefficient Kolmogorov operator of the form (1.2) with covariance matrix

$$\mathbf{C}(s) = \int_0^s e^{rB} \begin{pmatrix} M I_{p_0} & 0_{p_0 \times (d-p_0)} \\ 0_{(d-p_0) \times p_0} & 0_{(d-p_0) \times (d-p_0)} \end{pmatrix} e^{rB^*} dr.$$

(b) *if Assumption (i) with $N = 2$ is also in force, then for any $(t, x) \in]0, T_0[\times \mathbb{R}^d$, the function $\tilde{\Gamma}(t, x; \cdot, \cdot) \in C_B^{2,\alpha}(]t, T_0[\times \mathbb{R}^d)$ and is solution to*

$$\begin{cases} \tilde{\mathcal{L}}^*u = 0 & \text{on }]t, T_0[\times \mathbb{R}^d, \\ u(t, \cdot) = \delta_x, \end{cases}$$

where $\tilde{\mathcal{L}}^*$ is the formal adjoint of $\tilde{\mathcal{L}}$. Moreover, for any $0 < t < T < T_0$, $x, \xi \in \mathbb{R}^d$ and $i, j = 1, \dots, p_0$, we have

$$|\partial_{\xi_i} \tilde{\Gamma}(t, x; T, \xi)| \leq \frac{C}{\sqrt{T-t}} \bar{\Gamma}_M(t, x; T, \xi),$$

$$|\partial_{\xi_i \xi_j} \tilde{\Gamma}(t, x; T, \xi)| + |Y_{T,\xi} \tilde{\Gamma}(t, x; T, \xi)| \leq \frac{C}{T-t} \bar{\Gamma}_M(t, x; T, \xi).$$

Proof. See Theorems 1.4 and 1.5 in [6]. □

In the next result, the sets $S_\varepsilon(x_0) \subseteq \mathbb{R}^d$ are as defined in Notation 4.1.

Theorem A.2. Let $x_0 \in D$ and $\varepsilon \in]0, 1[$ such that $\overline{S_\varepsilon(x_0)} \subseteq D$. Then, under Assumptions 1.1 and 1.2 with $N = 0$, for any $T \in]0, T_0[$ and $h \in C(\partial_P H_\varepsilon(0, x_0; T))$, there exists a unique solution in $C_B^{2,\alpha}(H_\varepsilon(0, x_0; T)) \cap C((H_\varepsilon \cup \partial_P H_\varepsilon)(0, x_0; T))$ to

$$\begin{cases} \mathcal{L}f = 0, & \text{on } H_\varepsilon(0, x_0; T), \\ f = h, & \text{on } \partial_P H_\varepsilon(0, x_0; T). \end{cases}$$

Moreover, if $h|_{\partial_\Sigma H_\varepsilon(0, x_0; T)} \equiv 0$, then the following representation holds:

$$f(t, x) = \int_{S_\varepsilon(x_0)} G(t, x; T, \xi) h(T, \xi) d\xi, \quad (t, x) \in (H_\varepsilon \cup \partial_P H_\varepsilon)(0, x_0; T),$$

where G denotes the Green function of \mathcal{L} for $H_\varepsilon(0, x_0; T_0)$, namely a continuous non-negative function $G(t, x; T, \xi)$ defined for any $0 < t < T < T_0$ and $x, \xi \in \overline{S_\varepsilon(x_0)}$ enjoying the following properties:

(a) for any $(T, \xi) \in H_\varepsilon(0, x_0; T_0)$, the function $G(\cdot, \cdot; T, \xi) \in C_B^{2,\alpha}(H_\varepsilon(0, x_0; T)) \cap C((H_\varepsilon \cup \partial_\Sigma H_\varepsilon)(0, x_0; T))$ and solves

$$\begin{cases} \mathcal{L}f = 0 & \text{on } H_\varepsilon(0, x_0; T), \\ f = 0 & \text{on } \partial_\Sigma H_\varepsilon(0, x_0; T), \\ f(T, \cdot) = \delta_\xi & \text{on } S_\varepsilon(x_0); \end{cases}$$

(b) if Assumption 1.2 with $N = 2$ is also in force, then G is also the Green function of the formal adjoint \mathcal{L}^* for $H_\varepsilon(0, x_0; T_0)$. In particular, for any $(t, x) \in H_\varepsilon(0, x_0; T_0)$, the function $G(t, x; \cdot, \cdot) \in C_B^{2,\alpha}(H_\varepsilon(t, x_0; T_0)) \cap C((H_\varepsilon \cup \partial_\Sigma H_\varepsilon)(t, x_0; T_0))$ and solves

$$\begin{cases} \mathcal{L}^* f = 0 & \text{on } H_\varepsilon(t, x_0; T_0), \\ f = 0 & \text{on } \partial_\Sigma H_\varepsilon(t, x_0; T_0), \\ f(t, \cdot) = \delta_x & \text{on } S_\varepsilon(x_0). \end{cases}$$

Proof. See [11] and [7, Section 4]. □

Theorem A.3. Let $Q \subset \mathbb{R} \times \mathbb{R}^d$ be a bounded domain. Then, under Assumptions 1.1 and 1.2 with $N = 0$, for any domain $Q_0 \Subset Q$ there exists $c > 0$ such that:

(a) if u is a bounded function in $C_B^{2,\alpha}(Q)$ such that $\mathcal{L}u = 0$ on Q , we have

$$\sup_{Q_0} \left(\sum_{i=1}^{p_0} |\partial_{x_i} u| + \sum_{i,j=1}^{p_0} |\partial_{x_i x_j} u| + |Yu| \right) + \|u\|_{C_B^{0,\alpha}(Q_0)} + \|u\|_{C_B^{1,\alpha}(Q_0)} + \|u\|_{C_B^{2,\alpha}(Q_0)} \leq c \sup_Q u, \tag{A.2}$$

(b) if Assumption 1.2 with $N = 2$ is also in force, and u is a bounded function in $C_B^{2,\alpha}(Q)$ such that $\mathcal{L}^*u = 0$ on Q , then (A.2) holds.

Proof. It is a particular case of [7, Theorem 1.3]. □

Appendix B: Fundamental solutions and Markov processes

We recall some basic notions about Markov processes as given in [8] and [20]. A *transition distribution* is a kernel $p(t, x; T, \cdot)$ that satisfies:

- (1) $p(t, x; T, \cdot)$ is a probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ for all $0 \leq t < T < T_0$ and $x \in \mathbb{R}^d$;
- (2) $p(t, \cdot; T, A)$ is $\mathcal{B}(\mathbb{R}^d)$ -measurable for any $0 \leq t < T < T_0$ and $A \in \mathcal{B}(\mathbb{R}^d)$;
- (3) if $0 \leq t < s < T < T_0$, $x \in \mathbb{R}^d$ and $A \in \mathcal{B}(\mathbb{R}^d)$, the following Chapman–Kolmogorov identity holds:

$$p(t, x; T, A) = \int_{\mathbb{R}^d} p(s, \xi; T, A)p(t, x; s, d\xi).$$

A *Markov process with transition distribution* p is a stochastic process $X = (X_t)_{0 \leq t < T_0}$ defined on the quartet $(\Omega, \mathcal{F}, (\mathcal{F}_t^0)_{0 \leq t < T_0}, (P_{t,x})_{0 \leq t < T_0, x \in \mathbb{R}^d})$ such that:

- (a) (Ω, \mathcal{F}) is a measurable space and $(\mathcal{F}_t^t)_{0 \leq t < T_0}$ is a family of filtrations satisfying $\mathcal{F}_t^t \subseteq \mathcal{F}_{t'}^{t'}$, for $t \leq t'$, $T \leq T'$ and $\mathcal{F} = \mathcal{F}_{T_0}^0$ (i.e. \mathcal{F} is the smallest σ -algebra containing all \mathcal{F}_T^t);
- (b) $(X_t)_{t \leq T_0}$ is adapted to \mathcal{F}^t for any $t \in [0, T_0[$;
- (c) for any $(t, x) \in [0, T_0[\times \mathbb{R}^d$, $P_{t,x}$ is a probability measure on $(\Omega, \mathcal{F}_{T_0}^t)$ satisfying

$$P_{t,x}(X_t = x) = 1,$$

$$P_{t,x}(X_T \in A | \mathcal{F}_s^t) = p(s, X_s; T, A), \quad t \leq s < T < T_0, A \in \mathcal{B}(\mathbb{R}^d).$$

Theorem 2.2.2 in [20] guarantees that for any transition distribution p there exists a Markov process $X = (X_t)_{t \in [0, T_0[}$ having p as transition distribution.

A transition distribution can be defined from a differential operator $\tilde{\mathcal{L}}$ of the form (4.1) satisfying conditions (\tilde{i}) – (\tilde{ii}) at the beginning of Section 4.1. Indeed, if $\tilde{\Gamma}$ denotes the fundamental solution of $\tilde{\mathcal{L}}$ in Theorem A.1 then

$$\tilde{p}(t, x; T, A) := \int_A \tilde{\Gamma}(t, x; T, \xi) d\xi, \quad 0 \leq t < T < T_0, x \in \mathbb{R}^d, A \in \mathcal{B}(\mathbb{R}^d) \tag{B.1}$$

defines a transition distribution. In virtue of the properties of Γ in Theorem A.1, if \tilde{p} is as in (B.1) then the associated Markov process admits a continuous version and is an \mathcal{A}_t -global diffusion on \mathbb{R}^d in the sense of Definition 3.2. This is a consequence of Kolmogorov continuity theorem (see, for instance, Theorems 2.1.6 and 2.2.4 in [20]) and the estimate given in the following

Lemma B.1. *Let $\tilde{X} = (\tilde{X}_t)_{t \in [0, T_0[}$ be a Markov process with transition distribution \tilde{p} in (B.1). Then for any $q \geq 1$ there exists a positive constant C such that*

$$E_{t,x}[|\tilde{X}_T - \tilde{X}_s|^q] \leq C(1 + |x|^q)|T - s|^{\frac{q}{2}}, \quad t \leq s < T < T_0, x \in \mathbb{R}^d. \tag{B.2}$$

Proof. We recall the definition of D_0 in (2.3) and notice that

$$|z| = |D_0(\lambda)D_0(\lambda^{-1})z| \leq C\lambda|D_0(\lambda^{-1})z|, \quad z \in \mathbb{R}^d, 0 < \lambda < 1. \tag{B.3}$$

Now we have

$$E_{t,x}[|\tilde{X}_T - \tilde{X}_s|^q] = E_{t,x}[E_{s, \tilde{X}_s}[|\tilde{X}_T - \tilde{X}_s|^q]] = \int_{\mathbb{R}^d} \tilde{\Gamma}(t, x; s, \xi) \int_{\mathbb{R}^d} \tilde{\Gamma}(s, \xi; T, y)|y - \xi|^q dy d\xi$$

$$\leq C \int_{\mathbb{R}^d} \bar{\Gamma}_M(t, x; s, \xi) \int_{\mathbb{R}^d} \bar{\Gamma}_M(s, \xi; T, y)(|y - e^{(T-s)B}\xi|^q + |(e^{(T-s)B} - I)\xi|^q) dy d\xi$$

(applying estimate (B.3) with $z = y - e^{(T-s)B}\xi$, $\lambda = (T - s)^{\frac{q}{2}}$ and by Proposition 3.5 in [6], for some $M' > M$)

$$\leq C(T - s)^{\frac{q}{2}} \int_{\mathbb{R}^d} \bar{\Gamma}_M(t, x; s, \xi) \int_{\mathbb{R}^d} \bar{\Gamma}_{M'}(s, \xi; T, y) dy d\xi$$

$$+ C(T - s)^q \int_{\mathbb{R}^d} \bar{\Gamma}_M(t, x; s, \xi) \int_{\mathbb{R}^d} |\xi|^q \bar{\Gamma}_M(s, \xi; T, y) dy d\xi$$

that yields (B.2). □

We finally observe that Theorem A.1 also implies that \tilde{X} is a *Feller process on D* in the sense of Assumption 1.4, and as such it is a strong Markov process (see [8], Corollary 2.6, p. 28).

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