HÖLDER BEHAVIOR OF VISCOSITY SOLUTIONS OF SOME FULLY NONLINEAR EQUATIONS IN THE HEISENBERG GROUP

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Abstract. In this paper we prove the $C^{0,\alpha}$ regularity of bounded and uniformly continuous viscosity solutions of some degenerate fully nonlinear equations in the first Heisenberg group.

1. Introduction

In this paper we prove $C^{0,\alpha}$ regularity of bounded and uniformly continuous viscosity solutions of some degenerate fully nonlinear elliptic equations in the first Heisenberg group $\mathbb{H}$. It is known that the theory of viscosity solutions is very flexible and that the existence of viscosity solutions of second order PDEs is not strictly related to the degeneracy of the elliptic operator, see [15], [13]. The regularity of viscosity solutions of second order elliptic, possibly nonlinear, PDEs is also well established: one of the key ingredients is given by the Harnack inequality which, in turn, is based on the Alexandroff-Bakelman-Pucci inequality, ABP in short, and the consequent maximum principle. We refer to the books [21, 10] for a comprehensive introduction to the subject. Despite several attempts, see e.g. [22, 23, 18, 19, 2], a sub-elliptic version of the ABP inequality is not yet available. Nevertheless, another approach to prove regularity results is known in the literature of viscosity solutions, it relies on the so called *Theorem of the Sums* or *maximum principle for semicontinuous functions*, see e.g. [13, 12, 14, 25, 15]. See also the very interesting improvement obtained in [29].

We are interested in the regularity of viscosity solutions of fully nonlinear equations that are not uniformly elliptic in the classical sense. To be more precise, in this note we deal with nonlinear PDEs that are modeled on the vector fields belonging to the first layer of a stratified algebra. The simplest example of this kind of geometric structure is provided by the first Heisenberg group $\mathbb{H}$: let us spend few words about it in order to properly

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describe both the class of equations and the regularity results in which we are interested in.

The first Heisenberg group $\mathbb{H}$ is a non-abelian, homogeneuos, connected and simply connected Lie group modeled on $\mathbb{R}^3$, whose group law is given by

$$(y_1, y_2, y_3) \cdot (x_1, x_2, x_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + 2(x_1 y_2 - x_2 y_1)) \, .$$

The corresponding Lie algebra of left-invariant vector fields admits a 2-step stratification, $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$, where $\mathfrak{h}_1 = \text{span}\{X_1, X_2\}$ and $\mathfrak{h}_2 = \text{span}\{X_3\}$ for $X_1 = \partial_{x_1} + 2x_2 \partial_{x_3}$, $X_2 = \partial_{x_2} - 2x_1 \partial_{x_3}$ and $X_3 = \partial_{x_3} = -\frac{1}{4}[X_1, X_2]$. Given a sufficiently smooth function $u : \mathbb{H} \to \mathbb{R}$ and a point $x \in \mathbb{H}$, the *horizontal gradient* of u at x is denoted by $\nabla_{\mathbb{H}} u$ and is defined as $\nabla_{\mathbb{H}} u(x) = (X_1 u(x), X_2 u(x))$. The *symmetrized horizontal Hessian* of u at $x = (x_1, x_2, x_3) \in \mathbb{H}$ is represented by $D^{2,*}_{\mathbb{H}} u(x)$ and is defined as the 2×2 matrix

$$D^{2,*}_{\mathbb{H}} u(x) := \begin{bmatrix} X_1^2 u(x) & \frac{1}{2}(X_1 X_2 u(x) + X_2 X_1 u(x)) \\ \frac{1}{2}(X_1 X_2 u(x) + X_2 X_1 u(x)) & X_2^2 u(x) \end{bmatrix}.$$

We refer to the books [9] and [11] for a detailed introduction to the subject.

We are now ready to introduce the class of intrinsic fully nonlinear equations we will deal with.

Definition 1.1. Let $\lambda, \Lambda > 0$ be real positive numbers and let S^2 the set of 2×2 real symmetric matrices. Let $F : S^2 \to \mathbb{R}$ be a continuous function such that for every $H_1, H_2 \in S^2$, if $H_1 \geq H_2$, then

$$\lambda \text{Tr}(H_1 - H_2) \leq F(H_1) - F(H_2) \leq \Lambda \text{Tr}(H_1 - H_2).$$

We consider the following class of equations

$$F(D^{2,*}_{\mathbb{H}} u(x)) - c(x) u(x) = f(x), \quad \text{in } \mathbb{H}, \tag{1}$$

where $\beta, \beta' \in (0, 1]$, $f \in C^{0,\beta'}(\mathbb{H})$, $c \in C^{0,\beta}(\mathbb{H})$, and $c \geq 0$ for every $x \in \mathbb{H}$.

At first sight, the former class of equations seems to fit within the definition given for the classical fully nonlinear equations (see e.g. [15]), but it has to be noted that, despite we work in $\mathbb{H} \equiv \mathbb{R}^3$, $\tilde{A} \in S^2$ and not in S^3. Therefore, the class of operators F are not uniformly elliptic operator in the classical sense.

We want to highlight an equivalent way to look at the class of equations coming from (1). For every $A \in S^3$ and for every $x \in \Omega \subseteq \mathbb{H}$, we first define

$$\tilde{A}_x := \begin{bmatrix} \langle A X_1(x), X_1(x) \rangle, & \langle A X_1(x), X_2(x) \rangle \\ \langle A X_1(x), X_2(x) \rangle, & \langle A X_2(x), X_2(x) \rangle \end{bmatrix},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean scalar product in $\mathbb{R}^3$. Now, for every $A \in S^3$ and for every $x \in \mathbb{H}$, we can define $\tilde{F}(A, x) := F(\tilde{A}_x)$. Therefore, (1) can be equivalently written as

$$\tilde{F}(D^2 u(x), x) - c(x) u(x) = f(x), \quad \text{in } \mathbb{H}. \tag{2}$$

In this way we can also profit of the classical definition of viscosity solution, without no need to work with the intrinsic one proposed in [5], see Section 2.

We want now to give a brief account of the operators falling within the framework of Definition 1.1. As a very particular case, that class includes the real part of the Kohn-Laplace operator in $\mathbb{H}$, namely

$$\Delta_{\mathbb{H}} u := \left(\frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_3}\right)^2 u + \left(\frac{\partial}{\partial x_1} - 2x_1 \frac{\partial}{\partial x_3}\right)^2 u,$$

that is a degenerate elliptic operator at every point $x \in \mathbb{H}$. Another interesting class of operators that falls into this framework is provided by the Pucci-Heisenberg operators, see e.g. [8, 16, 17]. Given $0 < \lambda \leq \Lambda$ real constants, we define

$$\mathcal{P}^{+}_{\mathbb{H},\lambda,\Lambda}(D^{2,*}_{\mathbb{H}} u(x)) := \max_{A \in \mathcal{A}_{\lambda,\Lambda}} \mathrm{Tr}(A D^{2,*}_{\mathbb{H}} u(x)) = \Lambda \sum_{e>0} e - \lambda \sum_{e<0} e$$

and

$$\mathcal{P}^{-}_{\mathbb{H},\lambda,\Lambda}(D^{2,*}_{\mathbb{H}} u(x)) := \min_{A \in \mathcal{A}_{\lambda,\Lambda}} \mathrm{Tr}(A D^{2,*}_{\mathbb{H}} u(x)) = \lambda \sum_{e>0} e - \Lambda \sum_{e<0} e,$$

where

$$\mathcal{A}_{\lambda,\Lambda} := \{A \in S^2 : \quad \lambda|\xi|^2 \leq \langle A\xi, \xi\rangle_{\mathbb{R}^2} \leq \Lambda|\xi|^2, \ \xi \in \mathbb{R}^2 \setminus \{0\}\},$$

and e denotes the generic eigenvalue of the symmetrized horizontal Hessian matrix of u at x. As a last example, we also want to mention the sub-elliptic Monge-Ampére-type operators, see e.g. [3, 4, 32].

We state our main result.

Theorem 1.2. *Let $u \in C(\mathbb{H})$ be a bounded and uniformly continuous viscosity solution of the equation*

$$F(D^{2,*}_{\mathbb{H}} u(x)) - c(x)u(x) = f(x), \quad \textit{in } \mathbb{H},$$

for F as in Definition 1.1. Let L_c, L_f, β, β' be positive constants such that $\beta, \beta' \in (0, 1]$ and

$$|c(x) - c(y)| \leq L_c|x - y|^{\beta}, \quad |f(x) - f(y)| \leq L_f|x - y|^{\beta'}$$

for every $x, y \in \mathbb{H}$. If

$$\inf_{x \in \mathbb{H}} c(x) := c_0 > 0,$$

then there exist $\alpha := \alpha(c_0, L_c, L_f, \Lambda) \in (0, 1]$, $\alpha \leq \min\{\beta, \beta'\}$, and $L := L(c_0, L_c, L_f, \Lambda) > 0$ such that

$$|u(x) - u(y)| \leq L|x - y|^{\alpha}, \quad \textit{for every } x, y \in \mathbb{H},$$

that is $u \in C^{0,\alpha}(\mathbb{H})$.

A few comments about Theorem 1.2 are now in order. The proof follows an argument based on the Theorem of the sums as it appears in the literature, e.g. [28, 27]. The main ingredient is given by a *duplication of variables* and consists in the choice of a suitable function by ψ (see (5)). This technique is quite classical in the context of fully nonlinear PDE's (see [28, Section VII.1]) and it has already been used by Ishii [27] even in the context of degenerate linear second order operators.
Unfortunately, using this approach, we are not able to address the case in which $\inf_{x\in\mathbb{H}} c = 0$ nor to relax the hypothesis on u. At the present stage we are not even able to deal with equations definite only on bounded sets. We want to stress further that we cannot weaken our assumptions even in

the linear case i.e. $F(D^{2,*}_{\mathbb{H}}u(x)) = \Delta_{\mathbb{H}}u(x)$, which is well known to be a hypoelliptic operator, see [24]. We think that these problems could represent only a technical difficulty that we hope to solve in a future research.

We are principally interested in a Hölder regularity property of viscosity solutions of (1), without using a Harnack inequality (we refer to [1] for the proof of a Harnack inequality in a similar setting). Moreover, at this stage, we are not interested in finding a sharp Hölder exponent as we shall explain in few rows. For this reason we do not use neither the Carnot-Charathéodory distance nor the equivalent distance coming from the so called Kóranyi norm, (see [11] for further details about these distances). It has to be noted that our result implies a local Hölderianity in terms of any intrinsic distance. Indeed, denoting by d_{CC} the Carnot-Charathéodory distance in the Heisenberg group and $K \subset \mathbb{H}$ any compact set, it is well known that there exist positive constants C_1, C_2 such that for every $x, y \in K$

$$C_1|x-y| \leq d_{CC}(x,y) \leq C_2|x-y|^{\frac{1}{2}}.$$

It would be certainly interesting to find a suitable technique to obtain global Hölderianity in terms of an intrinsic distance.

Regularity of viscosity solutions is a subject that attracts the interests of many researchers. Thus we like to point out the following more or less recent results about some properties of the solutions of nonlinear equations in the degenerate elliptic case: [26, 6, 35, 33, 7, 1] and, concerning the evolutive framework, [30].

We conclude this introduction recalling that in [27] the Lipschitz regularity, in the classical Euclidean sense, to viscosity solutions of linear smooth second order elliptic operators, even possibly degenerate elliptic, has been proved in all of $\mathbb{R}^n$. In this perspective, our result could be seen as an extension of his result to the fully nonlinear setting.

The paper is organized as follows, in Section 2 we introduce the notation and the basic definitions in the Heisenberg group. Section 3 is devoted to the proof of Theorem 1.2.

2. Preliminaries

In this section we will fix the notation we will use throughout the paper. We will also introduce the basics facts about the first Heisenberg group $\mathbb{H}$, including the notion of viscosity solution as introduced in [5].

First of all, for every $m \in \mathbb{N}$ we will denote by S^m the set of $m \times m$ real symmetric matrices.

2.1. **The first Heisenberg group.** The first Heisenberg group $\mathbb{H}$ is a homogeneous, non-abelian, connected and simply connected Lie group modeled on $\mathbb{R}^3$. It is endowed with the following non-commutative group law

$$x \cdot y = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + 2(y_1x_2 - y_2x_1)),$$

where $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{H}$. The inverse of a point $x \in \mathbb{H}$ is given by $-x = (-x_1, -x_2, -x_3)$. To simplify the notation in future computations, we will also denote a point $x \in \mathbb{H}$ as $x = (x', x_3)$, for $x' = (x_1, x_2)$.

The Heisenberg group is also endowed with a homogeneous semigroup of dilation

$$\delta_\lambda(x) = (\lambda x_1, \lambda x_2, \lambda^2 x_3), \quad \text{for every } \lambda > 0 \text{ and for every } x \in \mathbb{H}.$$

Its Lie algebra $\mathfrak{h}$ admits a step 2 stratification, namely $\mathfrak{h} = \mathfrak{h}_1 \bigoplus \mathfrak{h}_2$, where

$$\mathfrak{h}_1 = \operatorname{span}\{X_1, X_2\} \quad \text{and} \quad \mathfrak{h}_2 = \operatorname{span}\{X_3\},$$

for $X_1 = \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_3}$, $X_2 = \frac{\partial}{\partial x_2} - 2x_1 \frac{\partial}{\partial x_3}$ and $X_3 = \frac{\partial}{\partial x_3}$. In particular $[X_1, X_2] = -4X_3$. The vector fields X_1 and X_2 span the so called *horizontal distribution* and are usually addressed as *horizontal vector fields.* They can be identified with the vectors $(1, 0, 2x_2)$ and $(0, 1, -2x_1)$ respectively, so that we can also write $X_1(x) = (1, 0, 2x_2)$ and $X_2(x) = (0, 1, -2x_1)$. Given a sufficiently smooth function $u : \mathbb{H} \to \mathbb{R}$, the horizontal vector fields are homogeneous of degree 1 with respect to the family of dilation δ_λ, that means

$$X_1 u(\delta_\lambda x) = \lambda (X_1 u)(\delta_\lambda x) \quad \text{and } X_2 u(\delta_\lambda x) = \lambda (X_2 u)(\delta_\lambda x).$$

Continuing with the notation, we denote by $\nabla_{\mathbb{H}} u(x) = X_1 u(x) X_1(x) + X_2 u(x) X_2(x) = (X_1 u(x), X_2 u(x))$ the intrinsic gradient of u, and by $D^{2,*}_{\mathbb{H}} u(x)$ its symmetrized horizontal Hessian at the point $x \in \mathbb{H}$, which is given by

$$D^{2,*}_{\mathbb{H}} u(x) = \begin{bmatrix} X_1^2 u(x), & \frac{(X_1X_2 + X_2X_1)u(x)}{2} \\ \frac{(X_1X_2 + X_2X_1)u(x)}{2}, & X_2^2 u(x) \end{bmatrix}.$$

By a straightforward computation we obtain the following

Lemma 2.1. *Let $\Omega \subseteq \mathbb{H}$ be an open set and let $u \in C^2(\Omega)$. Then*

$$\begin{aligned} D^{2,*}_{\mathbb{H}} u(x) &= \begin{bmatrix} X_1^2 u(x), & \frac{X_1X_2u(x) + X_2X_1u(x)}{2} \\ \frac{X_1X_2u(x) + X_2X_1u(x)}{2}, & X_2^2 u(x) \end{bmatrix} \\ &= \begin{bmatrix} \langle D^2u(x)X_1(x), X_1(x)\rangle, & \langle D^2u(x)X_1(x), X_2(x)\rangle \\ \langle D^2u(x)X_1(x), X_2(x)\rangle, & \langle D^2u(x)X_2(x), X_2(x)\rangle \end{bmatrix}. \end{aligned}$$

Moreover, for every $\alpha, \beta \in \mathbb{R}$ and for every $x \in \Omega$

$$\langle D^2u(\alpha X_1 + \beta X_2)^T, (\alpha X_1 + \beta X_2)\rangle = \langle D^{2,*}_{\mathbb{H}} u(\alpha, \beta)^T, (\alpha, \beta)\rangle.$$

Analogously a simple computation yields

Lemma 2.2. *Let $A, B \in S^3$ and let $\Omega \subseteq \mathbb{H}$ be an open set. Assume that $\phi \in C^2(\Omega \times \Omega)$. If*

$$\begin{bmatrix} A, & 0 \\ 0, & -B \end{bmatrix} \le D^2\phi(x, y) + \frac{1}{\mu}(D^2\phi(x, y))^2$$

then

$$\begin{aligned} &\langle A(\alpha_1 X_1 + \beta_1 X_2), (\alpha_1 X_1 + \beta_1 X_2)\rangle - \langle B(\alpha_2 X_1 + \beta_2 X_2), (\alpha_2 X_1 + \beta_2 X_2)\rangle \\ &\le \langle D^2\phi(x, y) \begin{bmatrix} \alpha_1 X_1 + \beta_1 X_2, \\ \alpha_2 X_1 + \beta_2 X_2 \end{bmatrix}, [\alpha_1 X_1 + \beta_1 X_2, \alpha_2 X_1 + \beta_2 X_2]\rangle_{\mathbb{R}^2} \\ &+ \frac{1}{\mu}\langle (D^2\phi(x, y))^2 \begin{bmatrix} \alpha_1 X_1 + \beta_1 X_2 \\ \alpha_2 X_1 + \beta_2 X_2, \end{bmatrix}, [\alpha_1 X_1 + \beta_1 X_2, \alpha_2 X_1 + \beta_2 X_2]\rangle_{\mathbb{R}^2}. \end{aligned}$$

2.2. **Viscosity solutions.** We recall the definition of viscosity solution for our particular equation written in the form (2), as it is given in [15].

Definition 2.3. We say that $u \in C(\Omega)$ is a sub-solution of (2) if for every $\phi \in C^2$ and for every $x_0 \in \Omega$ if $u - \phi$ realizes a maximum at x_0 in a open neighborhood U_{x_0} of x_0 then

$$F(D^2\phi(x_0), x_0) - c(x_0)\phi(x_0) \geq f(x_0).$$

Analogously we shall say that $u \in C(\Omega)$ is a super-solution of (2) if for every $\phi \in C^2$ and for every $x \in \Omega$ if $u - \phi$ realizes a minimum at x_0 in a open neighborhood U_{x_0} of x_0 then

$$F(D^2\phi(x_0), x_0) - c(x_0)u(x_0) \leq f(x_0).$$

If $u \in C(\Omega)$ is both a sub-solution and a super-solution of (2), then u is a viscosity solution of the equation (2).

Since we can also work with the equation written in the form (1), we recall also that there exists an intrinsic definition of viscosity solution concerning sub-elliptic semi-jets, see [5], [31].

2.3. **Theorem on Sums.** This result is described with different names, for instance, in the papers [14] and [28] see also [34]. In the following part $J^{2,+}u(\hat{x})$ and $J^{2,-}u(\hat{x})$ denote respectively the classical super-jet and sub-jet of u at the point $\hat{x} \in \Omega$ for the function $u \in C(\Omega)$.

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set. Let $u \in USC(\bar{\Omega})$ and $v \in LSC(\bar{\Omega})$. For $\phi \in C^2(\mathbb{R}^n \times \mathbb{R}^n)$. If there exists $(\hat{x}, \hat{y}) \in \Omega$ such that*

$$u(\hat{x}) - v(\hat{y}) - \phi(\hat{x}, \hat{y}) = \max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} \left(u(x) - v(y) - \phi(x, y)\right),$$

then for each $\mu > 0$, there are $A = A(\mu)$ and $B = B(\mu)$ such that

$$(D_x\phi(\hat{x}, \hat{y}), A) \in \overline{J}^{2,+}u(\hat{x}), \quad (-D_y\phi(\hat{x}, \hat{y}), B) \in \overline{J}^{2,+}u(\hat{y})$$

and

$$-\left(\mu + \|D^2\phi(\hat{x}, \hat{y})\|\right)\begin{bmatrix} I, & 0 \\ 0, & I \end{bmatrix} \leq \begin{bmatrix} A, & 0 \\ 0, & -B \end{bmatrix} \leq D^2\phi(\hat{x}, \hat{y}) + \frac{1}{\mu}(D^2\phi(\hat{x}, \hat{y}))^2,$$

where

$$D^2\phi(\hat{x}, \hat{y}) = \begin{bmatrix} D^2_{xx}\phi(\hat{x}, \hat{y}), & D^2_{yx}\phi(\hat{x}, \hat{y}) \\ D^2_{xy}\phi(\hat{x}, \hat{y}), & D^2_{yy}\phi(\hat{x}, \hat{y}) \end{bmatrix}$$

and $\|D\|$ is the norm given by the maximum, in absolute value, of the eigenvalues of the symmetric matrix D.

3. Proof of Theorem 1.2

In this section we prove our main result. In order to do this, we have to fix some notations. Let $\alpha \in (0, 1]$ and let $u : \mathbb{H} \to \mathbb{R}$ be a bounded and uniformly continuous function. We consider

$$w(x, y) = u(x) - u(y) - L|x - y|^\alpha - \delta|x|^2 - \epsilon.$$

For simplicity, let us denote by $\varphi(x, y) := L|x - y|^\alpha$.
Assume for the moment that we have satisfied the hypothesis requested to apply the Theorem of sums. This means that, still denoting with (x, y)

the point that realizes the maximum there exist two symmetric matrices $A, B \in S^3$ such that

$$(D_x\varphi(x,y) + 2\delta\, x, A + 2\delta I) \in \bar{J}^{2,+}u(x)$$

and

$$(-D_y\varphi(x,y), B) \in \bar{J}^{2,-}u(y),$$

with

$$\begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix} \leq D^2\varphi + \frac{1}{\mu}(D^2\varphi)^2 =: \begin{pmatrix} M & -M \\ -M & M \end{pmatrix} + \frac{2}{\mu}\begin{pmatrix} M^2 & -M^2 \\ -M^2 & M^2 \end{pmatrix},$$

where

$$M = L\alpha|x-y|^{\alpha-2}\left((\alpha-2)\frac{x-y}{|x-y|}\otimes\frac{x-y}{|x-y|} + I\right) \tag{3}$$

and

$$M^2 = L^2\alpha^2|x-y|^{2(\alpha-2)}\left(\alpha(\alpha-2)\frac{x-y}{|x-y|}\otimes\frac{x-y}{|x-y|} + I\right). \tag{4}$$

It is worth to remark that the matrix

$$(\alpha-2)\frac{x-y}{|x-y|}\otimes\frac{x-y}{|x-y|} + I$$

has smallest eigenvalue $\alpha - 1$ associated with the eigenvector $\frac{x-y}{|x-y|}$, while the largest eigenvalue is 1.

Remark 3.1. To improve the readability, we will denote by N the matrix $N := M + \frac{2}{\mu}M^2$. In particular recalling (3) and (4) we get

$$\|N\| \leq L\alpha|x-y|^{\alpha-2} + \frac{2}{\mu}L^2\alpha^2|x-y|^{2(\alpha-2)}.$$

Lemma 3.2. *Let $A, B, N \in S^3$ such that*

$$\begin{bmatrix} A, & 0 \\ 0, & -B \end{bmatrix} \leq \begin{bmatrix} N, & -N \\ -N, & N \end{bmatrix}.$$

Then, for every $[\xi, \eta] \in \mathbb{R}^6$

$$[\xi,\eta]\begin{bmatrix} A, & 0 \\ 0, & -B \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix} \leq [\xi,\eta]\begin{bmatrix} N, & -N \\ -N, & N \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix}$$

and

$$\langle A\xi, \xi\rangle - \langle B\eta, \eta\rangle \leq \langle N(\xi-\eta), (\xi-\eta)\rangle$$

In addition we get that, for every $a, b \geq 0$ and for every $\xi_1, \xi_2, \eta_1, \eta_2 \in \mathbb{R}^3$

$$\begin{aligned} &a\langle A\xi_1, \xi_1\rangle + b\langle A\xi_2, \xi_2\rangle - a\langle B\eta_1, \eta_1\rangle - b\langle B\eta_2, \eta_2\rangle \\ &\leq (a\langle N(\xi_1-\eta_1), (\xi_1-\eta_1)\rangle + b\langle N(\xi_2-\eta_2), (\xi_2-\eta_2)\rangle). \end{aligned}$$

Proof. The result follows by straightforward calculation recalling Lemma 2.2. □

Remark 3.3. By Lemma 3.2, we get that for every $A, B \in S^3$,

$$\langle AX_1(x), X_1(x)\rangle - \langle BX_1(y), X_1(y)\rangle$$
$$\leq \langle M(X_1(x) - X_1(y), X_1(x) - X_1(y)) + \frac{1}{\mu}\langle M^2(X_1(x) - X_1(y), X_1(x) - X_1(y))\rangle$$

and

$$\langle AX_2(x), X_2(x)\rangle - \langle BX_2(y), X_2(y)\rangle$$
$$\leq \langle M(X_2(x) - X_2(y), X_2(x) - X_2(y)) + \frac{1}{\mu}\langle M^2(X_2(x) - X_2(y), X_2(x) - X_2(y))\rangle.$$

Moreover summing term by term the previous inequalities we have

$$\langle AX_1(x), X_1(x)\rangle + \langle AX_2(x), X_2(x)\rangle - (\langle BX_1(y), X_1(y)\rangle + \langle BX_2(y), X_2(y)\rangle)$$
$$\leq \langle M(X_2(x) - X_2(y), X_2(x) - X_2(y)) + \frac{1}{\mu}\langle M^2(X_2(x) - X_2(y), X_2(x) - X_2(y))$$
$$+\langle M(X_1(x) - X_1(y), X_1(x) - X_1(y)) + \frac{1}{\mu}\langle M^2(X_1(x) - X_1(y), X_1(x) - X_1(y))$$

Since, $X_1(x) - X_1(y) = (0, 0, 2(y_1 - x_1))$ and $X_2(x) - X_2(y) = (0, 0, 2(x_2 - y_2))$, we get

$$\langle A_x X_1(x), X_1(x)\rangle + \langle A_x X_2(x), X_2(x)\rangle - (\langle B_y X_1(y), X_1(y)\rangle + \langle B_y X_2(y), X_2(y)\rangle)$$
$$\leq 4((x_1 - y_1)^2 + (x_2 - y_2)^2)\langle Me_3, e_3\rangle + \frac{4}{\mu}((x_1 - y_1)^2 + (x_2 - y_2)^2)\langle M^2 e_3, e_3\rangle.$$

Corollary 3.4. *Let $A, B, N \in S^3$ such that*

$$\begin{bmatrix} A, & 0 \\ 0, & -B \end{bmatrix} \leq \begin{bmatrix} N, & -N \\ -N, & N \end{bmatrix}.$$

Then, if $\xi_1 = X_1(x)$, $\eta_1 = X_1(y)$, $\xi_2 = X_2(x)$ and $\eta_2 = X_2(y)$, we get that for every $a, b \geq 0$

$$a\langle AX_1(x), X_1(x)\rangle + b\langle AX_2(x), X_2(x)\rangle - a\langle AX_1(y), X_1(y)\rangle - b\langle AX_2(y), X_2(y)\rangle$$
$$\leq \left(a(x_2 - y_2)^2 + b(x_1 - y_1)^2\right) n_{33},$$

where $n_{33} = \langle Ne_3, e_3\rangle$. In particular, if $a = 1 = b$

$$Tr(\tilde{A}) - Tr(\tilde{B}) \leq \left((x_2 - y_2)^2 + (x_1 - y_1)^2\right) n_{33},$$

Proof. Keeping in mind Lemma 3.2 and choosing $\xi_1 - \eta_1 = 2(0, 0, x_2 - y_2)$ and $\xi_2 - \eta_2 = -2(0, 0, x_1 - y_1)$, then the result immediately follows. □

We can now proceed with the proof of our main result. For sake of simplicity we denote by $\|u\|_\infty := \|u\|_{L^\infty(\mathbb{H})}$.

Proof of Theorem 1.2. Let $u : \mathbb{H} \to \mathbb{R}$ be a bounded and uniformly continuous viscosity solution of

$$F(D^{2*}u(x)) - c(x)u(x) = f(x), \quad x \in \mathbb{H}.$$

Let $\overline{L} > 0$ be a fixed constant and let $L > L_0$. Let δ and ϵ be positive constants, and let $\alpha \in (0, 1]$. We define the function $\psi : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ as

$$\psi(x, y) := u(x) - u(y) - L|x - y|^\alpha - \delta|x|^2 - \epsilon. \tag{5}$$

We also put $\theta := \sup_{\mathbb{H}\times\mathbb{H}} \psi(x, y)$.

Claim: there exist $L_0 > 0$ and $\alpha_0 = \alpha_0(\|u\|_\infty, \Lambda, \lambda, c_0) \in (0, 1]$ such that

$\theta \le 0$, for every $\delta > 0$ and for every $\epsilon > 0$. This would be enough to conclude the proof, indeed

$$\psi(x, y) \le u(\hat{x}) - u(\hat{y}) - L_0|\hat{x} - \hat{y}|^{\alpha_0} - \delta|\hat{x}|^2 - \epsilon \le 0$$

and letting $\delta \to 0$ and $\epsilon \to 0$ we get

$$u(x) - u(y) - L_0|x - y|^{\alpha_0} \le 0, \quad \text{for every } x, y \in \mathbb{H}.$$

In order to prove the claim, we argue by contradiction. Let us suppose that there exist δ_0 and ϵ_0 such that for every $\delta < \delta_0$ and for every $\epsilon < \epsilon_0$

$$0 < \theta = \sup_{\mathbb{H}\times\mathbb{H}} \{u(x) - u(y) - L|x - y|^\alpha - \delta|x|^2 - \epsilon\}.$$

Then, for every fixed $0 < \delta < \delta_0$ and $0 < \epsilon < \epsilon_0$, there exists a sequence $\{(x_j, y_j)\}_{j\in\mathbb{N}} \in \mathbb{H} \times \mathbb{H}$ such that

$$\lim_{j\to\infty} (u(x_j) - u(y_j) - L|x_j - y_j|^\alpha - \delta|x_j|^2 - \epsilon) = \theta.$$

Since $\theta > 0$, there exists $\bar{j} \in \mathbb{N}$ such that $u(x_j) - u(y_j) - L|x_j - y_j|^\alpha - \delta|x_j|^2 - \epsilon > \frac{\theta}{2} > 0$, for every $j > \bar{j}$. Therefore

$$\infty > 2\|u\|_\infty \ge u(x_j) - u(y_j) > L|x_j - y_j|^\alpha + \delta|x_j|^2 + \epsilon, \quad \text{for every } j > \bar{j}.$$

Thus

$$\delta|x_j|^2 < 2\|u\|_\infty, \quad \text{and} \quad |x_j - y_j|^\alpha < \frac{2||u||_\infty}{L}.$$

By compactness, possibly extracting a subsequence, we get that there exists $(\hat{x}, \hat{y}) \in \mathbb{H} \times \mathbb{H}$ such that

$$\lim_{k\to\infty} x_{j_k} = \hat{x} \quad \text{and} \quad \lim_{k\to\infty} y_{j_k} = \hat{y}.$$

Possibly taking L sufficiently large, by the uniformly continuity of u we would get a contradiction whenever $\left(\frac{2||u||_\infty}{L}\right)^{\frac{1}{\alpha}} < \eta(\epsilon)$, where η is the parameter independent of $\hat{x}, \hat{y} \in \mathbb{H}$ associated with the uniform continuity of u. Therefore there exists $\gamma > 0$ such that

$$\lim_{k\to\infty} |x_{j_k} - y_{j_k}|^\alpha = \gamma.$$

This implies that $|\hat{x} - \hat{y}|^\alpha = \gamma > 0$, independently of δ. Thus, we can assume that γ does not depend on δ. As a consequence, if $\theta > 0$ then there exists $(\hat{x}, \hat{y}) \in \mathbb{H}$ and $\gamma > 0$ such that

$$\theta := \sup_{\mathbb{H}\times\mathbb{H}} \psi(x, y) = \psi(\hat{x}, \hat{y}),$$

and there exists $\bar{\gamma}$ independent of δ such that

$$0 < \bar{\gamma} < |\hat{x} - \hat{y}|^\alpha < \frac{2\|u\|_\infty}{L}.$$

Recalling that u is a viscosity solution, and keeping in mind the hypotheses on c and f, we get by Theorem of the sums:

$$\begin{aligned} &c_0(L|\hat{x} - \hat{y}|^\alpha + \delta|\hat{x}|^2) \le c_0(u(\hat{x}) - u(\hat{y})) \le c(\hat{x})(u(\hat{x}) - u(\hat{y})) \\ &= c(\hat{x})u(\hat{x}) - c(\hat{y})u(\hat{y}) + u(\hat{y})(c(\hat{y}) - c(\hat{x})) \\ &\le F(\tilde{A} + 2\delta\tilde{I}) - F(\tilde{B}) + f(\hat{y}) - f(\hat{x}) + u(\hat{y})(c(\hat{y}) - c(\hat{x})). \end{aligned} \tag{6}$$

We consider now two cases.

If $\tilde{A}+2\delta\tilde{I}\geq\tilde{B}$, keeping in mind Definition 1.1, we get

$$\begin{aligned}c_0L|\hat{x}-\hat{y}|^\alpha+c_0\delta|\hat{x}|^2\leq\Lambda\mathrm{Tr}(\tilde{A}-\tilde{B})+2\delta(|X(\hat{x})|^2+|Y(\hat{x})|^2)\\+f(\hat{y})-f(\hat{x})+u(\hat{y})(c(\hat{y})-c(\hat{x})).\end{aligned}$$

To simplify the notation, let us denote

$$R:=|X_1(\hat{x})|^2+|X_2(\hat{x})|^2=2(1+2|\hat{x}'|^2).$$

Moreover, by the Theorem of the sums and recalling Corollary 3.4, we deduce that:

$$\mathrm{Tr}(\tilde{A}-\tilde{B})\leq\left((\hat{x}_2-\hat{y}_2)^2+(\hat{x}_1-\hat{y}_1)^2\right)n_{33}.$$

We recall also that keeping in mind the classical computation we get

$$\begin{aligned}n_{33}=L\alpha|\hat{x}-\hat{y}|^{\alpha-2}\left((\alpha-2)\frac{(\hat{x}_3-\hat{y}_3)^2}{|\hat{x}-\hat{y}|^2}+1\right)\\+\frac{2}{\mu}L^2\alpha^2|\hat{x}-\hat{y}|^{2(\alpha-2)}\left(\alpha(\alpha-2)\frac{(\hat{x}_3-\hat{y}_3)^2}{|\hat{x}-\hat{y}|^2}+1\right).\end{aligned}$$

Hence
(7)

$$\begin{aligned}&c_0\leq\frac{\Lambda}{L}|\hat{x}-\hat{y}|^{-\alpha}\left((\hat{x}_2-\hat{y}_2)^2+(\hat{x}_1-\hat{y}_1)^2\right)n_{33}\\&+\frac{L_f}{L}|\hat{x}-\hat{y}|^{\beta-\alpha}+\frac{L_c}{L}|\hat{x}-\hat{y}|^{\beta'-\alpha}+\frac{\delta\left(2R-c_0(|\hat{x}|^2)\right)}{L|\hat{x}-\hat{y}|^\alpha}\\&=\Lambda\left((\hat{x}_2-\hat{y}_2)^2+(\hat{x}_1-\hat{y}_1)^2\right)\alpha|\hat{x}-\hat{y}|^{-2}\left((\alpha-2)\frac{(\hat{x}_3-\hat{y}_3)^2}{|\hat{x}-\hat{y}|^2}+1\right)\\&+\Lambda L\alpha^2\frac{2}{\mu}\left((\hat{x}_2-\hat{y}_2)^2+(\hat{x}_1-\hat{y}_1)^2\right)|\hat{x}-\hat{y}|^{\alpha-4}\left(\alpha(\alpha-2)\frac{(\hat{x}_3-\hat{y}_3)^2}{|\hat{x}-\hat{y}|^2}+1\right)\\&+\frac{L_f}{L}|\hat{x}-\hat{y}|^{\beta-\alpha}+\frac{L_c}{L}|\hat{x}-\hat{y}|^{\beta'-\alpha}+\frac{\delta}{L}|\hat{x}-\hat{y}|^{-\alpha}\left(2R-c_0|\hat{x}|^2\right).\end{aligned}$$

We remark that if the factor

$$(\alpha-2)\frac{(\hat{x}_3-\hat{y}_3)^2}{|\hat{x}-\hat{y}|^2}+1$$

were negative then, possibly taking μ sufficiently large, we could assume that

$$\begin{aligned}&0>\Lambda\left((\hat{x}_2-\hat{y}_2)^2+(\hat{x}_1-\hat{y}_1)^2\right)\alpha|\hat{x}-\hat{y}|^{-2}\left((\alpha-2)\frac{(\hat{x}_3-\hat{y}_3)^2}{|\hat{x}-\hat{y}|^2}+1\right)\\&+\Lambda L\alpha^2\frac{2}{\mu}\left((\hat{x}_2-\hat{y}_2)^2+(\hat{x}_1-\hat{y}_1)^2\right)|\hat{x}-\hat{y}|^{\alpha-4}\left(\alpha(\alpha-2)\frac{(\hat{x}_3-\hat{y}_3)^2}{|\hat{x}-\hat{y}|^2}+1\right)\end{aligned}$$

concluding that

$$c_0\leq\frac{L_f}{L}|\hat{x}-\hat{y}|^{\beta-\alpha}+\frac{L_c}{L}|\hat{x}-\hat{y}|^{\beta'-\alpha}+\frac{\delta}{L}|\hat{x}-\hat{y}|^{-\alpha}\left(2R-c_0|\hat{x}|^2\right)\tag{8}$$

and obtaining a contradiction possibly taking a larger L as it explained in an analogous case later on. Otherwise,

$$(\alpha-2)\frac{(\hat{x}_3-\hat{y}_3)^2}{|\hat{x}-\hat{y}|^2}+1\geq0\quad\text{and}\quad\alpha(\alpha-2)\frac{(\hat{x}_3-\hat{y}_3)^2}{|\hat{x}-\hat{y}|^2}+1>0,$$

we can obtain from (7), possibly fixing μ in such a way that

$$\Lambda\left((\hat{x}_2-\hat{y}_2)^2+(\hat{x}_1-\hat{y}_1)^2\right)\alpha|\hat{x}-\hat{y}|^{-2}\left((\alpha-2)\frac{(\hat{x}_3-\hat{y}_3)^2}{|\hat{x}-\hat{y}|^2}+1\right)$$
$$=\Lambda L\alpha^2\frac{2}{\mu}\left((\hat{x}_2-\hat{y}_2)^2+(\hat{x}_1-\hat{y}_1)^2\right)|\hat{x}-\hat{y}|^{\alpha-4}\left(\alpha(\alpha-2)\frac{(\hat{x}_3-\hat{y}_3)^2}{|\hat{x}-\hat{y}|^2}+1\right),$$

the following inequality

$$\begin{aligned}c_0 \le\ & 2\Lambda\alpha\left((\hat{x}_2-\hat{y}_2)^2+(\hat{x}_1-\hat{y}_1)^2\right)|\hat{x}-\hat{y}|^{-2}\left((\alpha-2)\frac{(\hat{x}_3-\hat{y}_3)^2}{|\hat{x}-\hat{y}|^2}+1\right)\\ &+\frac{L_f}{L}|\hat{x}-\hat{y}|^{\beta-\alpha}+\frac{L_c}{L}|\hat{x}-\hat{y}|^{\beta'-\alpha}+\frac{\delta}{L}|\hat{x}-\hat{y}|^{-\alpha}\left(2R-c_0|\hat{x}|^2\right)\end{aligned} \tag{9}$$

holds. Notice that with respect to the inequality (8) the worst case is given by previous inequality (9). Thus we have to discuss the following term that appears in (9):

$$\delta\left(2R-c_0|\hat{x}|^2\right)=4\delta+(8-c_0)\delta|\hat{x}'|^2-c_0\delta\hat{x}_3^2.$$

We know that

$$\delta|\hat{x}|^2\le 2||u||_\infty^2,\quad \text{for every } \delta<\delta_0.$$

Moreover

$$\delta\left(2R-c_0|\hat{x}|^2\right)\le 4\delta+(8-c_0)\delta|\hat{x}|^2,$$

and, if $c_0>8$, we deduce that

$$\delta\left(2R-c_0|\hat{x}|^2\right)\le 4\delta.$$

Thus, from (9) it follows that

$$\begin{aligned}c_0 &\le 2\Lambda\alpha\left((\hat{x}_2-\hat{y}_2)^2+(\hat{x}_1-\hat{y}_1)^2\right)|\hat{x}-\hat{y}|^{-2}\left((\alpha-2)\frac{(\hat{x}_3-\hat{y}_3)^2}{|\hat{x}-\hat{y}|^2}+1\right)\\ &\quad+\frac{L_f}{L}|\hat{x}-\hat{y}|^{\beta-\alpha}+\frac{L_c}{L}|\hat{x}-\hat{y}|^{\beta'-\alpha}+\frac{8\delta}{L}|\hat{x}-\hat{y}|^{-\alpha}\\ &\le 2\Lambda\alpha+\frac{L_f}{L}|\hat{x}-\hat{y}|^{\beta-\alpha}+\frac{L_c}{L}|\hat{x}-\hat{y}|^{\beta'-\alpha}+\frac{8\delta}{L}|\hat{x}-\hat{y}|^{-\alpha}\\ &\le 2\Lambda\alpha+2^{\frac{\beta}{\alpha}-1}\left(\frac{L_f}{L^{\frac{\beta}{\alpha}}}||u||_\infty^{\frac{\beta}{\alpha}-1}+\frac{L_c}{L^{\frac{\beta'}{\alpha}}}||u||_\infty^{\frac{\beta'}{\alpha}-1}\right)+\frac{8\delta}{L}|\hat{x}-\hat{y}|^{-\alpha}.\end{aligned}$$

Now, letting $\delta\to 0$, we get

$$c_0\le 2\Lambda\alpha+2^{\frac{\beta}{\alpha}-1}\left(\frac{L_f}{L^{\frac{\beta}{\alpha}}}||u||_\infty^{\frac{\beta}{\alpha}-1}+\frac{L_c}{L^{\frac{\beta'}{\alpha}}}||u||_\infty^{\frac{\beta'}{\alpha}-1}\right). \tag{10}$$

Hence, taking L sufficiently large and α sufficiently small, we get a contradiction with the positivity of c_0. In particular we need that

$$\alpha<\frac{c_0}{2\Lambda}.$$

In case $\tilde{A}+2\delta\tilde{I}<B$, then

$$\lambda\mathrm{Tr}(\tilde{B}-\tilde{A}-2\delta\tilde{I})\le F(\tilde{B})-F(\tilde{A}+2\delta\tilde{I})\le\Lambda\mathrm{Tr}(\tilde{B}-\tilde{A}-2\delta\tilde{I}),$$

so that

$$-\Lambda\mathrm{Tr}(\tilde{B}-\tilde{A}-2\delta\tilde{I})\le F(\tilde{A}+2\delta\tilde{I})-F(\tilde{B})\le-\lambda\mathrm{Tr}(\tilde{B}-\tilde{A}-2\delta\tilde{I})$$

and, as a consequence, we get that

$$F(\tilde{A}+2\delta\tilde{I})-F(\tilde{B})\leq -\lambda \mathrm{Tr}(\tilde{B}-\tilde{A}-2\delta\tilde{I})<0.$$

The contradiction follows from (6) since we immediately obtain

$$\begin{aligned}
c_0L|\hat{x}-\hat{y}|^{\alpha} &\leq F(\tilde{A}+2\delta\tilde{I})-F(B)+f(\hat{y})-f(\hat{x})+u(\hat{y})(c(\hat{y})-c(\hat{x}))\\
&\leq -\lambda \mathrm{Tr}(\tilde{B}-\tilde{A}-2\delta\tilde{I})+L_f|\hat{x}-\hat{y}|^{\beta}+L_c|\hat{x}-\hat{y}|^{\beta'}\\
&\leq L_f|\hat{x}-\hat{y}|^{\beta}+L_c|\hat{x}-\hat{y}|^{\beta'}.
\end{aligned}$$

Moreover we obtain

$$c_0\leq \frac{L_f}{L}|\hat{x}-\hat{y}|^{\beta-\alpha}+\frac{L_c}{L}|\hat{x}-\hat{y}|^{\beta'-\alpha}.$$

For $\beta,\beta'\geq\alpha$, we reach a contradiction by sending $L\to+\infty$. This closes the proof. □

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