An introduction to the geometrical analysis of vector fields—with applications to maximum principles and Lie groups

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Preface

VECTOR fields, and many of their applications, are the main subjects of this book. Usually, students of scientific disciplines first meet vector fields during their undergraduate courses, either in connection with Ordinary Differential Equations (ODEs, in the sequel), or within physics, studying conservative forces and potentials. Later in their studies, many students encounter the geometric meaning of vector fields in the context of differential geometry and manifold theory. Vector fields reappear in more advanced studies, such as in Lie group theory or throughout the literature of Partial Differential Equations (PDEs, in the sequel). It is manifest, owing to their interdisciplinary nature, that vector fields play a remarkable role not only in mathematics, but also in physics and in applied mathematics.

The aim of this book is to provide the reader with a gentle path through the multifaceted theory of vector fields, starting from the definitions and the basic properties of vector fields and flows, and ending with some of their countless applications. The building blocks of rich background material (Chap.s 1-to-7) comprise the following topics, just to name a few of them:

- commutators and Lie derivatives; the semi-group property and the equation of variation; global vector fields; C^0, C^k, C^ω-dependence;
- relatedness, invariance and commutability; Hadamard-type formulas;
- composition of flows: Taylor approximations and exact formulas;
- the algebraic Campbell-Baker-Hausdorff-Dynkin (CBHD, in the sequel) Theorem; the CBHD series and its convergence; the CBHD operation and its local associativity; Poincaré-type ODEs;
- iterated commutators and the Hörmander bracket-generating condition; connectivity; sub-unit curves and the control distance.

Once the background material is established, the applications mainly deal, according to our choice, with the following three settings (Chap.s 8-to-18):

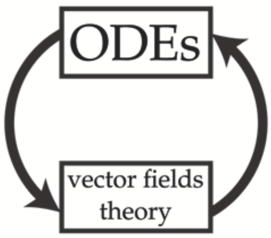
(I) *ODE theory*;
(II) *maximum principles* (weak, strong and propagation principles);
(III) *Lie groups* (with an emphasis on the construction of Lie groups).

Before describing the separate contents, we make some overall comments on these applications, thus providing a general motivation for the book.

(I). First, we would like to focus on the leading role played by ODEs in this monograph. This role is twofold:

- we shall obtain applications of our theory *to* ODEs; and
- we shall use ODEs *as a tool* for the derivation of many results of our theory.

Thus ODEs intervene, in a manner of speaking, in a "circular way": as an instrument and, at the same time, as an object of applications. On the one hand, it is not surprising that ODEs provide a major tool in the theories of vector fields or of Lie groups: just think that the flow of a vector field (hence, the exponential map for a Lie group) is the solution to an ODE. On the other hand, some applications of the analysis of vector fields to ODEs, especially concerning the CBHD Theorem, are less known in the literature, and are therefore amongst our main achievements.



An example may help clarify this double role of ODEs: since the advent of modern Lie group theory, the theorem expressing the group-multiplication of two group-exponentials as another group-exponential, say

$$\mathrm{Exp}(X)\mathrm{Exp}(Y) = \mathrm{Exp}(Z(X,Y)), \tag{P.1}$$

is known, quintessentially, as *the* CBHD Theorem.[1] Most of its proofs (see e.g., the one in [Varadarajan (1984)]) *rely on an ODE* valid in the associated Lie algebra (see (P.8)), which we may trace back to none other than Poincaré (in the context of Lie groups of transformations). It does not escape the notice of experts that this ODE is valid not only in the Lie algebra of a Lie group, but also in other more general settings; however, buried as it tends to be in Lie group theory, the beautiful ODE argument may easily escape the attention of a student, or of a non-expert.

Most importantly, the ODE tool used in the proof of (P.1) can be formulated as *an independent ODE theorem* in a completely autonomous way, and so can be

[1]The fact that this Lie group result has a predecessor in non-commutative algebra which should be considered as the CBHD Theorem *par excellence* has already been discussed in [Bonfiglioli and Fulci (2012)]. In the literature, Dynkin's name is not often used as a label for this theorem; our choice is instead consistent with the historical presentation in [Bonfiglioli and Fulci (2012)] (see also [Achilles and Bonfiglioli (2012)] for more details).

viewed independently of any Lie group context. It can be thought of (and taught) as a result of ODE theory; it can even be exploited in the construction of Lie groups (as we do in this book), a procedure that is possible only after the ODE result has been liberated from the Lie group framework.

As a novelty with respect to the existing literature, this is our point of view with what we shall name 'the CBHD Theorem *for ODEs*'. In the choice of the ODE prerequisites as well, we shall take the liberty to make some non-standard choices. For example, even within the ODE theory itself there are results which are presented more and more rarely in ODE textbooks, and we hope that this monograph may be a good occasion (due to their subsequent applications) to bring them back to the attention of students. For instance, this is the case with the C^ω-dependence results (see App. B), or with the integral version of the equation of variation in the non-autonomous case (Chap. 12).

(II). We should now disclose our intent in introducing four chapters devoted to *Maximum Principles* (as applications of the vector-fields-and-flows machinery) in their many declinations: Weak and Strong Maximum Principles; Maximum Propagation along principal or drift vector fields.

We clearly have in mind the possible applications of Maximum Principles *to PDEs*, for example in obtaining an exhaustive Potential Theory for classes of partial differential operators. The theory presented here is general enough to comprise wide classes of operators, more general (for example) than the sub-Laplacians on Carnot groups considered in [Bonfiglioli *et al.* (2007)]. As a consequence, Chap.s 8-to-11 of the book may be useful also to young researchers in PDEs.

(III). As regards the Lie group theory, our applications are based on the multifaceted use that one can make of the previously established machinery relating the CBHD Theorem and vector fields/flows:

(1) in the proof of (P.1), which reduces to a few lines once the ODE version of the CBHD Theorem is available;
(2) in the study of the CBHD series and of the CBHD operation on finite-dimensional Lie algebras;
(3) in implementing *Lie's Third Theorem* in some special but meaningful cases: in its local formulation, and also in the global one when dealing with nilpotent algebras; the tools mentioned in (2) prove to be particularly powerful and natural to use in solving this problem;
(4) in the *construction of Lie groups* starting from the exponentiation of Lie algebras of vector fields, without using the global version of Lie's Third Theorem;
(5) in a simple construction of Carnot groups (equipped with their homogeneous structure), starting from the stratified Lie algebras.

The Contents of the Book

With a view towards unveiling the unitary 'concept' of this monograph, let us describe more closely the topics it contains. In this introductory description, we shall keep the mathematical rigour at a minimum level, conveying the ideas rather than declaring the list of the assumptions behind any single result — there will be plenty of time for mathematical rigour throughout the main part of the book.

P.1. Basic facts on flows

Given a sufficiently regular vector field (v.f., for short) X on a domain Ω in $\mathbb{R}^N$, and fixing $x \in \Omega$, we denote by any of the following symbols

$$\gamma_{X,x}(t), \quad \gamma(t, X, x), \quad \Psi_t^X(x), \quad \exp(tX)(x)$$

the maximal solution (as a function of t), defined on its maximal domain $\mathcal{D}(X, x)$, of the Cauchy problem

$$\dot{\gamma}(t) = X(\gamma(t)), \qquad \gamma(0) = x.$$

As a function of x (or of (t, x), often in the literature), we shall refer to this function as *the flow of* X. We shall investigate $\Psi_t^X(x)$ as a function:

(A) of t (time),
(B) of x (the starting point), and
(C) of X (the vector field).

(A). *As a function of* t, the flow admits the Taylor series expansion

$$\sum_{k=0}^{\infty} \frac{X^k(x)}{k!}\, t^k, \tag{P.2}$$

motivating the "exponential-type" notation $\exp(tX)(x)$, and giving a meaningful tool for the differentiation of a function f along a v.f. X:

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}\Big\{ f(\gamma_{X,x}(t)) \Big\} = (X^k f)(\gamma_{X,x}(t)).$$

The degree-two expansion (wrt t) of the flow furnishes a first remarkable interpretation of the commutator $[X, Y]$ of two v.f.s X and Y:

$$\lim_{t \to 0} \frac{\Psi_t^{-Y} \circ \Psi_t^{-X} \circ \Psi_t^{Y} \circ \Psi_t^{X}(x) - x}{t^2} = [X, Y](x). \tag{P.3}$$

(B). *As a function of* x, particularly relevant is the Jacobian matrix

$$\mathcal{J}_{\Psi_t^X}(x) = \frac{\partial}{\partial x}\Big\{ x \mapsto \Psi_t^X(x) \Big\}.$$

For example, the Lebesgue measure of a set A evolves in the following way

$$\mathrm{meas}(\Psi_t^X(A)) = \int_A \exp\left(\int_0^t \mathrm{div}(X)(\Psi_\tau^X(x))\, \mathrm{d}\tau \right) \mathrm{d}x.$$

This result (Liouville's theorem for the flow of a v.f.), besides giving a meaningful interpretation of $\mathrm{div}(X)$ (the divergence of X), is a consequence of the *equation of variation* associated with the ODE defining the flow:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}_{\Psi_t^X}(x) = \mathcal{J}_X(\Psi_t^X(x))\,\mathcal{J}_{\Psi_t^X}(x).$$

Again as a function of x, the composition of two flow maps (for the same v.f. X) gives rise to one of the most important features of the flow:

$$\textit{the semi-group property}\quad \Psi_t^X \circ \Psi_s^X = \Psi_{t+s}^X.$$

Actually, the latter is so entrenched with v.f.s, that (roughly put) it characterizes the maps $F(t,\cdot):\Omega\to\Omega$ which are flows of some v.f. Another story is the composition of two flows related to *different vector fields* X and Y (see Sec. P.2), the same as a different story is the product of two matrix-exponentials $e^A e^B$ when $A \neq B$.

(C). The analysis of $\Psi_t^X(x)$ *as a function of* X requires a lot of work, which is another of the main goals of this book. This work starts with the *non-autonomous equation of variation* ruling the following general (non-autonomous) Cauchy problem, also depending on the parameter ξ,

$$\text{(CP)}:\qquad \begin{cases} \dot\gamma(t) = f(t,\gamma(t),\xi) \\ \gamma(t_0) = x. \end{cases}$$

If $t \mapsto \Phi^{\xi}_{t,t_0}(x)$ denotes the solution of (CP), one can obtain the following *integral and non-autonomous* equation of variation of (CP):

$$\frac{\partial}{\partial\xi}\Phi^{\xi}_{t,t_0}(x) = \int_{t_0}^{t}\Big(\frac{\partial}{\partial x}\Phi^{\xi}_{t,s}\Big)(\Phi^{\xi}_{s,t_0}(x))\cdot\Big(\frac{\partial f}{\partial\xi}\Big)(s,\Phi^{\xi}_{s,t_0}(x),\xi)\,\mathrm{d}s. \tag{P.4}$$

Since this result is generally not[2] presented in ODE textbooks, we shall provide all the details. When the parametric f on the rhs of the ODE comes (in an autonomous way) from a vector field of the form

$$f(t,x,\xi) = \xi_1 X_1(x) + \cdots + \xi_m X_m(x),$$

where $X_1,\ldots,X_m$ are v.f.s forming the basis of a finite-dimensional Lie algebra $\mathfrak{g}$, one can obtain from (P.4) a formula for the differential of the map

$$\mathfrak{g} \ni X \mapsto \Psi_t^X(x).$$

This is the case, e.g., when $\mathfrak{g}$ is the Lie algebra $\mathrm{Lie}(\mathbb{G})$ of some Lie group $\mathbb{G}$.

But nothing forces us to be anchored to a pre-existing Lie group: for instance, $\mathfrak{g}$ may be *any finite-dimensional Lie algebra of smooth v.f.s*, and we may use the formula for the differential of $X \mapsto \Psi_t^X(x)$ to ask ourselves whether $\mathfrak{g}$ is the Lie algebra of a Lie group, without knowing it in advance.

We shall return to this question in Sec. P.7.

[2]Occasionally, the autonomous case when f does not depend on t is used in Lie group textbooks; see e.g., [Duistermaat and Kolk (2000)].

P.2. Composition of flows: the CBHD Theorem

As (P.2) shows, it is reasonable to expect that the composition $\Psi_t^Y \circ \Psi_s^X$ of the flows of two *different* v.f.s may be ruled by a formula analogous to the one behind the product of two formal exponentials series

$$e^{tY}e^{sX} = \sum_{j\geq 0}\frac{(tY)^j}{j!}\sum_{i\geq 0}\frac{(sX)^i}{i!}. \tag{P.5}$$

As a matter of fact, when passing from the formal exponential series $e^{tY}e^{sX}$ to the composition of flows of vector fields $\Psi_t^Y \circ \Psi_s^X$, one unexpectedly discovers, by taking the partial derivatives

$$\frac{\partial^{i+j}}{\partial s^i \partial t^j}\Big|_{(s,t)=(0,0)}\Big\{f\big(\Psi_t^Y \circ \Psi_s^X(x)\big)\Big\} = (X^iY^jf)(x), \tag{P.6}$$

that the order of tY and sX in (P.5) has to be reversed; yet the brilliant idea of the shift to the formal-power-series setting turns to be very fruitful. This idea traces back to the years between the nineteenth and twentieth centuries, with the works of Campbell, Baker and Hausdorff (see [Achilles and Bonfiglioli (2012); Bonfiglioli and Fulci (2012)]). Adding the name of Dynkin to those of the aforementioned mathematicians, we shall study in detail *the CBHD Theorem*, which states that

$$e^a e^b = e^{Z(a,b)}, \text{ where}$$
$$Z(a,b) = a + b + \frac{1}{2}[a,b] + \frac{1}{12}\big([a,[a,b]] + [b,[b,a]]\big) + \cdots$$

is a series of Lie-polynomials in the non-commuting indeterminates a and b. The first explicit expression of the summands Z_h of $Z(a,b) = \sum_{h=1}^{\infty} Z_h(a,b)$ is due to [Dynkin (1947)] (whence our naming 'Dynkin polynomials').

In establishing the CBHD Theorem we make major use of *algebraic tools*; in this spirit, we follow the same pattern several times in the book:

- we establish an appropriate abstract algebraic setting;
- we derive the corresponding CBHD Theorem in this framework;
- we bring out an infinite family of identities by means of this theorem, identities that can profitably be used in many other different contexts.

Going back to the framework of the flows of v.f.s, and taking into account the needed reversing of X and Y visible in (P.6), we obtain the *Taylor approximation* of $\Psi_t^Y \circ \Psi_s^X$ up to arbitrary order n, in the following form

$$\exp(tY)(\exp(sX)(x)) = \exp\left(\sum_{h=1}^{n} Z_h(sX,tY)\right)(x) + \mathcal{O}\big(|s|+|t|\big)^{n+1}. \tag{P.7}$$

This type of formulas are particularly useful in the theory of linear PDEs; see e.g., [Bonfiglioli and Lanconelli (2012a); Christ *et al.* (1999); Citti and Manfredini (2006); Folland (1975); Magnani (2006); Morbidelli (2000); Nagel *et al.* (1985); Rothschild and Stein (1976); Varopoulos *et al.* (1992)].

Some natural questions arise: is it legitimate to let n go to ∞ in (P.7)? Is the series expected to converge? We shall answer these *non-trivial questions* in Sec. P.5, another goal of the book.

P.3. Our fundamental ally: ODEs ruling flows and the CBHD Theorem

A notable feature of the CBHD Theorem is that it rests on the validity (in an algebraic setting) of a suitable ODE, which traces back to [Poincaré (1900)], and for this reason we name it '*the Poincaré ODE*': setting $Z(t) := Z(ta, tb)$, this is the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t} Z(t) = \mathbf{b}\big(\mathrm{ad}\, Z(t)\big)(a) + \mathbf{b}\big(-\mathrm{ad}\, Z(t)\big)(b), \tag{P.8}$$

where $\mathbf{b}$ is the formal power series defined by

$$\mathbf{b}(z) = \frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!}\, z^n,$$

which is the generating function of the Bernoulli numbers B_n. In its turn, (P.8) follows from the formal PDEs, each having its own interest and applicability,

$$\begin{aligned} &\frac{\partial}{\partial s} C(s,t) = \mathbf{b}\big(\mathrm{ad}\, C(s,t)\big)(a) \quad \text{and} \\ &\frac{\partial}{\partial t} C(s,t) = \mathbf{b}\big(-\mathrm{ad}\, C(s,t)\big)(b), \end{aligned} \tag{P.9}$$

where $C(s,t) := Z(sa, tb)$. Just to give an idea of the depth of (P.8) and (P.9), we shall derive from them several applications to Lie group theory (Chap.s 14, 15, 17) and to ODE theory (Chap. 13).

Another indispensable tool in the analysis of v.f.s is the study of how a v.f. Y changes under the action of the flow of another v.f. X: this is given by $\mathrm{d}\Psi^X_{-t} Y$, the pushforward of the v.f. Y under the diffeomorphism Ψ^X_{-t}, the latter being the flow of X "running backward in time". We study this fundamental topic under the name of *Hadamard's Theorem for flows*. The reason for this (non-standard) naming is the analogy with the so-called Hadamard formula for formal power series in the indeterminates a, b:

$$e^a\, b\, e^{-a} = e^{\mathrm{ad}\, a} b := b + [a,b] + \tfrac{1}{2}\,[a,[a,b]] + \tfrac{1}{3!}\,[a,[a,[a,b]]] + \cdots .$$

This analogy will eventually become manifest, if we consider the formula

$$\mathrm{d}\Psi^X_{-t} Y = e^{\mathrm{ad}\, tX} Y := Y + t[X,Y] + \tfrac{t^2}{2}\,[X,[X,Y]] + \tfrac{t^3}{3!}\,[X,[X,[X,Y]]] + \cdots ;$$

here, the higher order commutators of the v.f.s X and Y must satisfy some growth assumption for the series to converge: for example, if X and Y belong to a finite-dimensional Lie algebra of v.f.s, then this growth assumption is fulfilled.

As important as Poincaré's ODE (P.8), another ODE-like formula plays the leading role here, which we name *the Hadamard ODE for flows*:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathrm{d}\Psi^X_{-t} Y\Big) = \mathrm{d}\Psi^X_{-t}[X,Y], \tag{P.10}$$

an identity of time-dependent vector fields. When $t = 0$, (P.10) gives a remarkable interpretation of the commutator $[X, Y]$ (complementary to (P.3)), via the notion of the so-called *Lie derivative*.

Due to the invariance of a v.f. X under its flow, (P.10) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathrm{d}\Psi^X_{-t} Y\Big) = [X, \mathrm{d}\Psi^X_{-t} Y].$$

In a finite-dimensional setting, this constant-coefficient, linear and homogeneous ODE can be integrated, to give the mentioned $\mathrm{d}\Psi^X_{-t} Y = e^{\mathrm{ad}\, tX} Y$.

P.4. Long commutators, connectivity and the control-distance

Let us consider a family $X = \{X_1, \dots, X_k\}$ of (not necessarily distinct) smooth v.f.s. Like in (P.3), giving an approximation of the flow of $t^2\,[X_1, X_2]$ by means of $\Psi^{-X_2}_t \circ \Psi^{-X_1}_t \circ \Psi^{X_2}_t \circ \Psi^{X_1}_t$, we can *iterate this procedure* in order to approximate the flow of the long commutator

$$t^k\,[\cdots[[X_1, X_2], X_3], \cdots X_k], \tag{P.11}$$

by means of suitable compositions of the flows of $\pm X_1, \dots, \pm X_k$ (usually referred to as the 'horizontal' directions wrt the family X).

The proof of the approximation of the flow of (P.11) is very laborious, but the "spirit" behind it is the same that we described in Sec. P.2:

- we first lift to an abstract level (formal power series in k indeterminates);
- we produce universal identities in this setting;
- we specialize these identities by putting v.f.s in place of the indeterminates.

This is the key ingredient for the remarkable *Connectivity Theorem* of Carathéodory, Chow, Hermann and Rashevskiĭ. This result states that, given a *Hörmander system* of smooth v.f.s X on a connected open set Ω, any pair of points $x, y \in \Omega$ can be connected by a continuous curve $\gamma_{x,y}$ in Ω, which piecewise is an integral curve of $\pm X_1, \dots, \pm X_k$; the curve $\gamma_{x,y}$ is the typical example of a *X-subunit path*.

By definition, Hörmander systems of v.f.s satisfy the so-called *bracket-generating condition*, playing a remarkable role in Control Theory. Hörmander v.f.s are also fundamental in the study of hypoellipticity for linear PDEs, due to the celebrated result by [Hörmander (1967)] (see the survey [Bramanti (2014)]).

The notion of X-subunit path sets the basis for the definition of the *control distance* d_X associated with the family X: broadly speaking, the d_X-distance between x and y is obtained by minimizing the life-time of the X-subunit curves connecting x and y. The idea of modeling a more intrinsic geometry attached to X by means of the X-subunit paths proved to be one of the most fruitful ideas in the theory of sub-elliptic PDEs, as well as in Control Theory and in Geometric Measure Theory.

P.5. Applications: the CBHD Theorem for Lie algebras, and for ODEs

As anticipated at the end of Sec. P.2, a momentous problem for our investigation is the study of the convergence of the CBHD series in the realm of a *finite-dimensional Lie algebra* $\mathfrak{g}$. Due to its finite-dimensionality, $\mathfrak{g}$ supports a natural differentiable structure, where the Poincaré ODE-PDEs (P.8)-(P.9) are meaningful (besides being consequences of their abstract counterparts).

Hence, if x and y are close to 0 in $\mathfrak{g}$, we shall prove that

$$t \mapsto Z(t) := \sum_{n=1}^{\infty} Z_n(x,y)\, t^n$$

solves in $\mathfrak{g}$ (for $|t| \leq 1$) the Poincaré ODE (P.8). This fact has remarkable applications in Lie group theory: indeed, the *CBHD local operation*

$$(x,y) \mapsto x \diamond y := \sum_{n=1}^{\infty} Z_n(x,y) \tag{P.12}$$

has a "germ of associativity" on a small neighborhood Ω of 0 in $\mathfrak{g}$:

$$x \diamond (y \diamond z) = (x \diamond y) \diamond z, \quad \text{for every } x,y,z \in \Omega. \tag{P.13}$$

This result is a crucial point in solving Lie's Third Theorem in its local form (see Sec. P.7). Last but not least, the Poincaré PDEs (P.9) can be used to determine an enlarged domain of convergence for (P.12), even in infinite-dimensional contexts (see [Biagi and Bonfiglioli (2014); Blanes and Casas (2004); Mérigot (1974)]).

Deeply related to the convergence problem is the answer to our previous question, concerning the legitimacy of the limit "$\lim_{n\to\infty}$" in (P.7). A positive answer produces a non-trivial analog, *in the realm of ODEs*, of the CBHD Theorem. Roughly put, since the flow of a v.f. X is of "exponential-type", see (P.2), it is natural to presume that the composition of two flows $\exp(tY) \circ \exp(tX)$ may be ruled by a closed CBHD-type formula.

At the *finite* level of Taylor expansions, we have already stated that this is indeed the case, see (P.7), modulo suitable remainders. But much more is true if we deal with a finite-dimensional Lie-algebra V of smooth vector fields on a domain Ω. We shall derive *the CBHD Theorem for ODEs*:

$$\exp(Y)\big(\exp(X)(x)\big) = \exp(X \diamond Y)(x), \tag{P.14}$$

when $X, Y \in V$ are close to the zero vector field (and $x \in \Omega$ is fixed). In accordance with the spirit of the book, we prove (P.14) via an ODE argument: we show that

$$F(t) = \exp(tY)\big(\exp(X)(x)\big) \quad \text{and} \quad G(t) = \exp\big(X \diamond (tY)\big)(x)$$

solve (up to $t = 1$) the same Cauchy problem! In order to obtain $G'(t)$, we need many of our previous ingredients: Hadamard's Formula (P.10), Poincaré's ODE (P.8) in V, and the integral equation of variation (P.4) in the autonomous case.

As anticipated at the end of Sec. P.1, one can apply the CBHD Theorem for ODEs (P.14) when V is the Lie algebra of a real Lie group, but it is also possible to use this result in the absence of any background Lie group structure: this will be our approach for *the very construction of Lie groups* starting from Lie algebras V of smooth v.f.s satisfying minimal assumptions.

P.6. Applications to Maximum Principles

In Chap. 8 of the book we introduce an important class of linear PDOs, the *semielliptic operators* L of second order: we have chosen this name in place of the more usual (and longer) 'Picone elliptic-parabolic PDO' to mean that the second-order matrix of L is everywhere positive *semidefinite*. In this section, we denote by c the zero-order term of L.

We establish fundamental tools used for the study of these operators, of an undisputed independent interest in the PDE literature:

(a) *the Weak Maximum Principle* (WMP);
(b) *the Strong Maximum Principle* (SMP);
(c) *the Maximum Propagation Principle* (MPP) *for principal v.f.s*;
(d) *the MPP for the drift v.f.*

(a). Broadly put, we say that L satisfies the WMP on the set Ω if every $u \in C^2(\Omega)$ satisfying $Lu \geq 0$ must be non-positive on Ω whenever this is true on $\partial\Omega$ (in a suitable weak sense). Applications of the WMP are also provided, such as:

- comparison principles and *a priori* estimates;
- the uniqueness of the classical solution of the Dirichlet problem;
- the Green and Poisson operators related to L;
- the Maximum-Modulus Principle and the Maximum Principle.

The latter is important and it establishes that, when $c \equiv 0$, any L-subharmonic function (i.e., $Lu \geq 0$) takes its maximum on $\partial\Omega$, if L satisfies the WMP on Ω.

The semielliptic assumption on L is the natural hypothesis for the validity of the WMP: indeed, we show that (when $c \equiv 0$) the validity of the WMP on every bounded open set implies that L must be semielliptic. However, the semielliptic hypothesis is not sufficient as it stands, and some assumptions on the sign of c are needed. For example, $c < 0$ is a sufficient condition, as is $c \leq 0$ together with the existence of a so-called L-barrier for L.

Whereas our study of the WMP makes no use of any earlier result of the book, it serves as a bridge for the investigation of the SMP and of the MPP; the latter are indeed deeply connected with earlier topics and techniques of the book, such as the Connectivity Theorem and the use of integral curves.

(b). We say that L satisfies the SMP on the connected open set Ω if, whenever $u \in C^2(\Omega)$ satisfies

$$Lu \geq 0 \text{ and } u \leq 0 \text{ on } \Omega, \quad \text{and } F(u) = \{x : u(x) = 0\} \text{ is non-void,}$$

then $u \equiv 0$ on Ω. Actually, we shall prove the SMP for

$$L = \sum_{j=1}^{m} X_j^2 + X_0, \quad \text{where } X_1, \dots, X_m \text{ are Hörmander v.f.s.} \tag{P.15}$$

In studying the SMP, we are interested in how large $F(u)$ is (ideally, we aim at $F(u) = \Omega$); in particular, it is interesting to investigate those v.f.s whose integral curves "propagate" $F(u)$: the richer the class of these v.f.s, the larger $F(u)$; whence the validity of the SMP.

(c). It is clear from the above discussion that, in studying the SMP, the following classical approach (due to [Bony (1969)]) is the right path:

- given a set F, we are interested in those v.f.s X whose integral curves remain in F once they intersect F: in this case, we say that the set F is *X-invariant*;
- a theorem, due to Nagumo and Bony, tells us that F is X-invariant if and only if X is *tangent to F* in a weak sense (generalizing that of differential geometry);
- the so-called Hopf Lemma for L shows that any so-called *principal vector field X* for L is automatically tangent to $F(u)$.

Thus, if X is a principal v.f. for F, then $F(u)$ is X-invariant, so that

$F(u)$ propagates along the integral curves of any principal v.f. X for L.

This is precisely the statement of the MPP for L.

Finally, the SMP for L in (P.15) follows due to the Connectivity Theorem, since the piecewise integral curves of $\pm X_1, \ldots, \pm X_m$ connect any pair of points of Ω, and the latter v.f.s are all principal for L.

(d). In dealing with PDOs of the form (P.15), we said nothing about the MPP along the v.f. X_0. This was not an oversight: the problem is that X_0 may not be a principal v.f. for L, as the Heat operator $H = \sum_{j=1}^{N} \partial_j^2 - \partial_t$ shows; as a matter of fact, H satisfies the WMP (on any bounded set) but it violates the SMP!

Nonetheless, we shall prove a redeeming fact: despite the possible lack of the X_0-invariance, still we have the *positive X_0-invariance* of $F(u)$. This means that, if $\gamma : [0,T] \to \Omega$ is an integral curve of X_0 with $\gamma(0) \in F(u)$, then we have $\gamma(t) \in F(u)$ for every $t \in [0,T]$. The proof of this fact is extremely delicate: we follow the approach by [Amano (1979)], where PDOs of the following form are considered (here $X_0, X_1, \ldots, X_N$ are regular enough v.f.s)

$$Lu = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}(X_i u) + X_0 u; \tag{P.16}$$

also in this case we say that X_0 is the *drift* of L. Later, after we have studied PDOs of Amano form (P.16), we shall return to Hörmander PDOs (P.15).

P.7. Applications to Lie groups

In the realm of Lie groups, we show how the preceding flow theory produces *simple proofs* of the following results:

(i) the CBHD Theorem for Lie groups;

(ii) the local Third Theorem of Lie;
(iii) the global Third Theorem of Lie in the nilpotent case;
(iv) the construction of Carnot groups;
(v) the exponentiation of finite-dimensional Lie algebras of vector fields into Lie groups (under minimal assumptions).

Let us briefly analyze each topic.

(i). The CBHD Theorem for Lie groups $\mathbb{G}$ states that

$$\mathrm{Exp}(X)\mathrm{Exp}(Y) = \mathrm{Exp}(X \diamond Y), \tag{P.17}$$

for every $X, Y \in \mathrm{Lie}(\mathbb{G})$ sufficiently close to the null v.f.; here $\diamond$ is the CBHD operation in (P.12) and Exp is the Exponential Map for $\mathbb{G}$. The cornerstone for the proof of (P.17) is furnished by the following identity, valid for every $X, Y \in \mathrm{Lie}(\mathbb{G})$ and for time t close to 0,

$$\mathrm{Exp}(tX)\mathrm{Exp}(tY) = \Psi_t^Y\big(\Psi_t^X(e)\big),$$

where e is the identity element of $\mathbb{G}$. Thus the lhs of (P.17) is simply a composition of flows: this allows us to reduce the CBHD Theorem on a Lie group to the CBHD Theorem for ODEs, see (P.14).

(ii). The local (real) Third Theorem of Lie states that, for any finite-dimensional real Lie algebra $\mathfrak{g}$, there exists a local Lie group (on a neighborhood U of the origin of $\mathfrak{g}$) such that the smooth vector fields on U which are invariant under the (local) left-translations form a Lie algebra isomorphic to $\mathfrak{g}$. As a meaningful application of the results in Sec. P.5, we prove that this local Lie group is given by the CBHD operation $\diamond$ on U, see (P.12). The germ of associativity (P.13) enjoyed by this local operation is the main ingredient.

(iii). When $\mathfrak{g}$ is nilpotent, the result in (ii) can easily be globalized: the operation $\diamond$ is well defined throughout $\mathfrak{g}$, and $(\mathfrak{g}, \diamond)$ is a Lie group with Lie algebra isomorphic to $\mathfrak{g}$: this is the *global* version of Lie's Third Theorem for real and nilpotent Lie algebras. The global form of the Third Theorem of Lie holds for any finite-dimensional Lie algebra, not necessarily nilpotent. In this book we restrict to considering the latter case for two reasons: firstly, the general case requires a deep knowledge of differential geometry, which is beyond our scope; secondly the local case and the global nilpotent case are sufficiently interesting for our intent, since their proofs can be carried out in a constructive way as they exploit the CBHD series.

(iv). As a by-product of (iii), we shall derive a version of Lie's Third Theorem for finite-dimensional stratified Lie algebras $\mathfrak{s}$: a Lie algebra $\mathfrak{s}$ is stratified when it admits a decomposition of the form

$$\mathfrak{s} = V \oplus [V, V] \oplus [V, [V, V]] \oplus [V, [V, [V, V]]] \oplus \cdots,$$

where V is a subspace of $\mathfrak{s}$. If $\mathfrak{s}$ is finite-dimensional ($N = \dim \mathfrak{s}$, say), then $\mathfrak{s}$ is necessarily nilpotent. We can therefore equip $\mathfrak{s}$ with the Lie group structure $(\mathfrak{s}, \diamond)$

in (iii); it turns out that this Lie group can be further endowed with a homogeneous structure δ_λ, turning it into a so-called *homogeneous Carnot group* (HCG, for short) $\mathbb{G} = (\mathbb{R}^N, \diamond, \delta_\lambda)$, such that $\mathrm{Lie}(\mathbb{G})$ is isomorphic to $\mathfrak{s}$.

HCGs (and their sub-Laplacian operators) are studied in [Bonfiglioli *et al.* (2007)], by Lanconelli, Uguzzoni and one of us. The way HCGs are presented in the present book is more intrinsic than what is done in [Bonfiglioli *et al.* (2007)], in that we here shift the focus to the stratified Lie algebra as a *datum*, and the associated HCG is obtained by the constructive global Third Theorem of Lie. In this way, the many well-behaved properties of HCGs (mostly, the existence of dilations) are simple by-products of the Lie-algebra properties.

(v). Let us consider the following question:

(Q): *given a Lie subalgebra V of the smooth v.f.s on $\mathbb{R}^N$, is it possible to find a Lie group $\mathbb{G}$ whose manifold is $\mathbb{R}^N$ (with its usual differentiable structure) such that* $\mathrm{Lie}(\mathbb{G}) = V$*?*

Notice that we are requiring[3] the equality '$\mathrm{Lie}(\mathbb{G}) = V$', and not an equality 'up to an isomorphism' (as it happens with the global Third Theorem of Lie). We know that the following conditions are necessary for (Q) to have a positive answer:

- every $X \in V$ must be global;
- V must satisfy Hörmander's bracket-generating condition;
- the dimension of V must be equal to N.

We will show that these are also independent and sufficient conditions on V for a positive answer to (Q). Coherently within the spirit of the book, the following ingredients will be used: the CBHD Theorem for ODEs, in order to equip $V \equiv \mathbb{R}^N$ with a local-Lie-group structure; the use of a prolongation argument for ODEs in order to globalize this local Lie group.

How to read the book

About the Appendices. We assume that the reader is already acquainted with the basic notions of manifold theory and Lie groups; however, since we all know that notations and definitions may vary from book to book and from author to author, we equip our monograph with a short Appendix (App. C) containing the needed Lie group theory, where the reader can find all the definitions and the prerequisites clearly stated. The same is done with manifold theory in Chap. 4, whenever it is compulsory to fix the symbols and the nomenclature.

App. A, with some basic results of algebra and linear algebra, is functional to the reading of the rest of the book; once again it collects definitions and results which will avoid notational ambiguities and will spare the reader an endless search in the literature for those prerequisites which may not be standard for everybody.

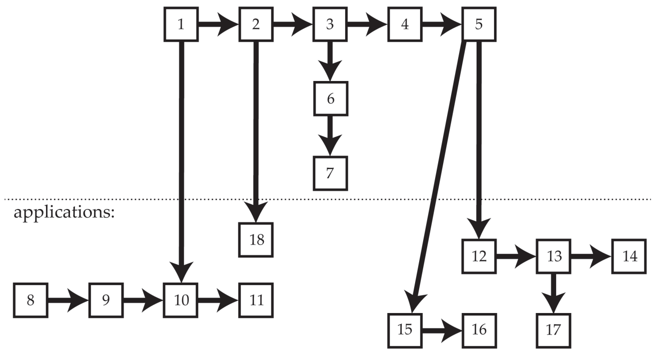
[3]Here $\mathrm{Lie}(\mathbb{G})$ is meant as a Lie algebra of derivations, i.e., of vector fields in the sense of differential operators of order 1.

As regards App. B, devoted to background material on ODEs, this has another purpose: we hope that the reader has solid knowledge of basic ODE theory; but, as we already discussed, it is now increasingly rare to find an exhaustive treatment of parametric dependence in ODE textbooks. Thus, we provide an analysis of this subject (so important in our book), starting from the continuous dependence and ending with the C^ω case (the latter being usually omitted in textbooks); we skip most of the proofs, yet provide a complete presentation of all the relevant material.

How to use this book. Several parts of this monograph have been taught by the second-named author during his classes for the Master Degree and for the PhD in Mathematics at Bologna University. It was then our intention to write a book that could be used by students in their mathematic investigation as well as by teachers in giving their lectures; in this regard, the book contains:

- basic topics for an introductory course on ODEs (Chap. 1, App. B); and advanced topics in ODEs (Chap.s 12, 13);
- applications, suitable to a course on Lie groups (Chap.s 14-to-17 and 5);
- applications, suitable to PDE-oriented courses (Chap.s 8-to-11, 16, 17);
- introductory topics of Control Theory (Chap.s 6, 7);
- introductory topics of Differential Geometry (Chap.s 1-to-4).

Further material. We hope that students may benefit from the 182 exercises, and from the 58 figures. The interested reader can find some bibliographical references, grouped by chapter, at the end of the book, in the Further Readings section. The following figure describes the interdependence of the different chapters, and is therefore a guide through the reading of the book.



Bologna,
July, 2018

Stefano Biagi
Andrea Bonfiglioli

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