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Mixing and moments properties of a non-stationary copula-based Markov process

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Abstract

We provide conditions under which a non-stationary copula-based Markov process is geometric β-mixing and geometric ρ-mixing. Our results generalize some results of Beare (2010) who considers the stationary case. As a particular case we introduce a stochastic process, that we call convolution-based Markov process, whose construction is obtained by using the C-convolution operator which allows the increments to be dependent. Within this subclass of processes we characterize a modified version of the standard random walk where copulas and marginal distributions involved are in the same elliptical family. We study mixing and moments properties to identify the differences compared to the standard case.

JEL classification: C22,C10
Mathematics Subject Classification (2010): 62M10, 62H20

Keywords: Markov process, copula, β-mixing, ρ-mixing, gaussian process.

1 Introduction

In this paper we analyze the temporal dependence properties satisfied by a discrete time *non-stationary* Markov process. Temporal dependence in time series analysis is relevant since it permits to verify how well theoretical models explain temporal persistency observed in financial data. Moreover, it is also a useful tool to establish large sample properties of estimators for dynamic models where the standard iid assumption is no longer allowed. In particular, in this paper we analyze geometric β-mixing and geometric ρ-mixing properties and we give sufficient conditions that ensure they are satisfied.

In the copula approach to univariate time series modelling, the finite dimensional distributions are generated by copulas and marginal distributions. Darsow et al. (1992) provide necessary and sufficient conditions for a copula-based time series to be a Markov process. Recent literature on this topic has mainly focused on the stationary case. Chen and Fan (2006) introduce a copula-based strictly stationary first order Markov process $(Y_t)_t$ generated by $(F(\cdot), C(\cdot,\cdot,\alpha))$ where $F(\cdot)$ is the invariant distribution of Y_t and $C(\cdot,\cdot,\alpha)$ is the parametric copula for (Y_t, Y_{t+1}). The authors show that the β-mixing temporal dependence measure is purely determined by the properties of copulas and does not depend on the invariant marginal distributions. Within the same class of models, Beare (2010) shows that all stationary Markov models generated via symmetric copulas with positive and square integrable densities are geometric β-mixing. Many commonly used bivariate copulas without tail dependence such as the gaussian, Frank and Farlie-Gumbel-Morgestern copulas satisfy this condition. In the same paper the author also shows that, if the density of

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the absolutely continuous part of a copula is bounded away from 0 on a set of Lebesgue measure 1, then the resulting copula-based Markov process is geometric ρ-mixing. Longla and Peligrad (2012) generalize this result by proving a more restrictive condition, namely ϕ-mixing. Chen et al. (2009) show that some asymmetric copulas with tail dependence like Clayton, Gumbel and t-copula generate Markov models which are geometrically ergodic. In all above mentioned papers the stationarity assumption is crucial.

In this contribution we focus on a more general setting considering a *non-stationary* Markov process where the sequence of joint distributions of (Y_t, Y_{t+1}) is given by a sequence of time-dependent copulas $C_{t,t+1}$ and marginal time dependent distributions F_t. Within this general class of Markov processes, we consider a specific subclass of processes generated by the C-convolution operator introduced in Cherubini et al. (2011). The C-convolution makes it possible to generate a non-stationary Markov process $(Y_t)_t$ with dependent increments in which, however, two ingredients are stationary: the distribution of the increments $\Delta Y_{t+1} = Y_{t+1} - Y_t$, given by F_Δ, and the dependence structure between the level at the time t and the next increment, modelled by a copula function C. We call such processes (already considered in Cherubini et al., 2012 and Cherubini et al, 2016) *convolution-based* Markov processes. A significant example of this last approach is provided by a modified version of the standard random walk, $Y_t = Y_{t-1} + \xi_t$ where both the copula associated to (Y_{t-1}, ξ_t) and the marginal distribution of the increment ξ_t belong to the same elliptical family. We study moments and autocorrelation functions in order to emphasize the different temporal properties with respect to the standard case.

The paper is organized as follows. Section 2 presents a general result on mixing properties satisfied by non-stationary copula-based Markov processes. Section 3 introduces convolution-based Markov processes and discusses their main properties in the elliptical and gaussian cases. Section 4 concludes.

2 Copula-based Markov processes and mixing properties

Throughout the paper $(Y_t)_{t\in\mathbb{Z}} = (Y_t)_t$ is a discrete time Markov process. Thanks to the seminal paper of Darsow et al. (1992), the markovianity of a stochastic process can be characterized through a specific requirement that the copulas, representing the dependence structure of the finite dimensional distributions induced by the stochastic process must satisfy (for a detailed discussion on copulas see Nelsen (2006), Joe (1997), Cherubini et al. (2012) and Durante and Sempi (2015)). In particular, in Darsow et al. (1992) it is proved that the Chapman-Kolmogorov equations for transition probabilities are equivalent to the requirement that, if $C_{i,j}$ is the copula associated to the vector (Y_i, Y_j), then

$$C_{s,t}(u,v) = C_{s,r} * C_{r,t}(u,v) = \int_0^1 \frac{\partial}{\partial w} C_{s,r}(u,w) \frac{\partial}{\partial w} C_{r,t}(w,v)\, dw, \ \forall s < r < t.$$

As a consequence, since $(Y_t)_t$ is a discrete time Markov process, if we assume that the set of bivariate copulas $C_{t,t+1}$ (representing the dependence structure of the stochastic process at two adjacent times) is given for $t \in \mathbb{Z}$, then necessarily for $k > 0$ (we remind that the $*$-operator is associative)

$$C_{t,t+k}(u,v) = C_{t,t+k-1} * C_{t+k-1,t+k}(u,v) = C_{t,t+1} * C_{t+1,t+2} * \cdots * C_{t+k-1,t+k}(u,v). \tag{1}$$

Notice that, in the stationary case considered in Beare (2010), $C_{t,t+1} = C$ for all $t \in \mathbb{Z}$, therefore all bivariate copulas $C_{t,t+k}$ are functions of the copula C and of the lag k and not of the time t. In this paper we extend the study to the more general non-stationary case. In particular we analyze the temporal dependence problem with a special attention to mixing properties.

The notion of β-mixing was introduced by Volkonskii and Rozanov (1959 and 1961) and was attributed there to Kolmogorov whereas the ρ-mixing condition was introduced by Kolmogorov and Rozanov (1960). Given a (not necessarily stationary) sequence of random variables $(Y_t)_t$, let

$\mathcal{F}_t^l$ be the σ-field $\mathcal{F}_t^l = \sigma(Y_s,\, t \le s \le l)$ with $-\infty \le t \le l \le +\infty$ and set

$$\tilde{\beta}(\mathcal{F}_{-\infty}^t, \mathcal{F}_{t+k}^{+\infty}) = \sup_{\{A_i\},\{B_j\}} \frac{1}{2} \sum_{i=1}^{I} \sum_{j=1}^{J} |\mathbb{P}(A_i \cap B_j) - \mathbb{P}(A_i)\mathbb{P}(B_j)|, \tag{2}$$

where the supremum is taken over all finite partitions $\{A_1, ...A_I\}$ and $\{B_1, ...B_J\}$ of Ω such that $A_i \in \mathcal{F}_{-\infty}^t$ for each i and $B_j \in \mathcal{F}_{t+k}^{\infty}$ for each j. Given the dependence coefficient

$$\beta_k = \sup_{t \in \mathbb{Z}} \tilde{\beta}(\mathcal{F}_{-\infty}^t, \mathcal{F}_{t+k}^{+\infty}),$$

we say that the sequence $(Y_t)_t$ is $\beta-$mixing (or absolutely regular) if $\beta_k \to 0$ as $k \to +\infty$. In particular, we say that the sequence is geometric β-mixing when the convergence to zero occurs at a geometric rate, i.e., when there exist $c < \infty$ and $\gamma > 0$ such that $\beta_k \le ce^{-\gamma k}$ for all k. Analogously, let

$$\tilde{\rho}(\mathcal{F}_{-\infty}^t, \mathcal{F}_{t+k}^{+\infty}) = \sup_{f,g} |Corr(f, g)|, \tag{3}$$

where the supremum of the correlation is taken over all square integrable r.vs. f and g measurable with respect to $\mathcal{F}_{-\infty}^t$ and $\mathcal{F}_{t+k}^{+\infty}$ respectively with finite and positive variance. Given the dependence coefficient

$$\rho_k = \sup_{t \in \mathbb{Z}} \tilde{\rho}(\mathcal{F}_{-\infty}^t, \mathcal{F}_{t+k}^{+\infty}),$$

we say that the sequence $(Y_t)_t$ is $\rho-$mixing if $\rho_k \to 0$ as $k \to +\infty$. As in the case of β-mixing, the sequence is geometric ρ-mixing when the sequence ρ_k decays to zero at a geometric rate.

In the next two theorems we give conditions on the set of copulas $C_{t,t+1},\ t \in \mathbb{Z}$ in order to guarantee that the resulting Markov process is geometric β-mixing and geometric ρ-mixing respectively. These conditions are based on specific requirements on the maximal correlation coefficients of the copulas $C_{t,t+1}$. We remind that the maximal correlation η of a copula C is given by

$$\eta = \sup_{f,g} \left| \int_0^1 \int_0^1 f(x)g(y)C(dx, dy) \right|$$

where $f, g \in L^2([0, 1])$, $\int_0^1 f(x)dx = \int_0^1 g(x)dx = 0$ and $\int_0^1 f^2(x)dx = \int_0^1 g^2(y)dy = 1$ and we refer to Beare (2010) and Rényi (1959) for more details. In our case, non-stationarity implies that we have a sequence of maximal correlation coefficients $(\eta_t)_t$ associated to the sequence of copulas $(C_{t,t+1})_t$ given by

$$\eta_t = \sup_{f,g} \left| \int_0^1 \int_0^1 f(x)g(y)C_{t,t+1}(dx, dy) \right|.$$

We denote

$$\hat{\eta} = \sup_{t \in \mathbb{Z}} \eta_t.$$

Theorem 2.1. *Let $(Y_t)_t$ be a Markov process. Let $C_{t,t+1}$ be the copula associated to the vector (Y_t, Y_{t+1}) for $t \in \mathbb{Z}$ that we assume to be absolutely continuous, with symmetric and square-integrable density $c_{t,t+1}$ so that $(c_{t,t+1})_t$ is uniformly bounded in $L^2([0, 1])$. If the sequence of maximal correlation coefficients $(\eta_t)_t$ associated to $(C_{t,t+1})_t$ satisfies*

$$\hat{\eta} < 1, \tag{4}$$

then $(Y_t)_t$ is geometric β-mixing.

Proof. The proof follows that of Theorem 3.1 in Beare (2010) who proves a similar result for stationary copula-based Markov processes. First of all, since the stochastic process is Markovian, (2) can be rewritten in terms of the cumulative distribution functions of (Y_t, Y_{t+k}), Y_t and Y_{t+k}

($F_{t,t+k}$, F_t and F_{t+k}, respectively) and the total variation norm $\| \cdot \|_{TV}$ (see Bradley, 2007) and then, applying Sklar's theorem, we can write

$$\begin{aligned}\tilde{\beta}(\mathcal{F}_{-\infty}^{t}, \mathcal{F}_{t+k}^{+\infty}) &= \frac{1}{2} \parallel F_{t,t+k}(x,y) - F_t(x)F_{t+k}(y) \parallel_{TV} = \\ &= \frac{1}{2} \parallel C_{t,t+k}(F_t(x), F_{t+k}(y)) - F_t(x)F_{t+k}(y) \parallel_{TV} \leq \\ &\leq \frac{1}{2} \parallel C_{t,t+k}(u,v) - uv) \parallel_{TV} .\end{aligned}$$

From (1) it follows that all bivariate copulas of type $C_{t,t+k}$ for $t \in \mathbb{Z}$ and $k \geq 1$ are absolutely continuous: let us denote their density as $c_{t,t+k}$. Then

$$\tilde{\beta}(\mathcal{F}_{-\infty}^{t}, \mathcal{F}_{t+k}^{+\infty}) \leq \frac{1}{2} \parallel c_{t,t+k}(u,v) - 1 \parallel_{\mathbb{L}^1} \leq \frac{1}{2} \parallel c_{t,t+k}(u,v) - 1 \parallel_{\mathbb{L}^2}$$

and

$$\beta_k \leq \frac{1}{2} \sup_{t \in \mathbb{Z}} \parallel c_{t,t+k}(u,v) - 1 \parallel_{\mathbb{L}^2} .$$

Since $c_{t,t+1}$ is a symmetric square-integrable joint density with uniform margins, it admits the following series expansion in terms of a complete orthonormal sequence $(\phi_i)_{i \geq 1}$ in $\mathbb{L}^2[0,1]$,

$$c_{t,t+1}(u,v) = 1 + \sum_{i=1}^{+\infty} \lambda_{i,t} \phi_i(u) \phi_i(v),$$

where the eigenvalues $(\lambda_{i,t})_i$ form a square-summable sequence of nonnegative real numbers: notice that, as proved in Lancaster(1958)

$$\max_{i \geq 1} \lambda_{i,t} = \eta_t. \tag{5}$$

Applying (1), we get

$$c_{t,t+k}(u,v) = 1 + \sum_{i=1}^{+\infty} \left(\prod_{j=0}^{k-1} \lambda_{i,t+j} \right) \phi_i(u) \phi_i(v).$$

Then, using (5) and (4), we get

$$\begin{aligned}\parallel c_{t,t+k}(u,v) - 1 \parallel_{\mathbb{L}^2} &= \left\| \sum_{i=1}^{+\infty} \left(\prod_{j=0}^{k-1} \lambda_{i,t+j} \right) \phi_i(u)\phi_i(v) \right\|_{\mathbb{L}^2} = \left[\sum_{i=1}^{+\infty} \left(\prod_{j=0}^{k-1} \lambda_{i,t+j}^2 \right) \right]^{1/2} = \\ &= \left[\sum_{i=1}^{+\infty} \lambda_{i,t}^2 \left(\prod_{j=1}^{k-1} \lambda_{i,t+j}^2 \right) \right]^{1/2} \leq \left[\sum_{i=1}^{+\infty} \lambda_{i,t}^2 \left(\prod_{j=1}^{k-1} \eta_{t+j}^2 \right) \right]^{1/2} \leq \\ &\leq \hat{\eta}^{k-1} \left[\sum_{i=1}^{+\infty} \lambda_{i,t}^2 \right]^{1/2} = \hat{\eta}^{k-1} \parallel c_{t,t+1}(u,v) - 1 \parallel_{\mathbb{L}^2} .\end{aligned} \tag{6}$$

Therefore

$$\beta_k \leq \frac{1}{2} \hat{\eta}^{k-1} \sup_{t \in \mathbb{Z}} \parallel c_{t,t+1}(u,v) - 1 \parallel_{\mathbb{L}^2}$$

which, since $(c_{t,t+1})_t$ is uniformly bounded in $L^2([0,1])$, tends to zero as $k \to +\infty$. □

Notice that, as argued by Beare (2010), any copula exhibiting upper or lower tail dependence will not admit a square integrable density while gaussian, Frank and Farlie-Gumbel-Morgenstern copulas have a maximal correlation coefficient strictly smaller than one provided that the copula parameter lies in the interior of the parameter space. Clearly, if the parameters of the involved bivariate copulas $C_{t,t+1}$ lie in a compact subset of the interior of the parameter space, then the assumptions of Theorem 2.1 are satisfied.

Next theorem shows ρ-mixing conditions.

Theorem 2.2. *Let $(Y_t)_t$ be a Markov process. If $\hat{\eta} < 1$, then $(Y_t)_t$ is geometric ρ-mixing.*

Proof. The proof is a generalization of the proof of Theorem 4.1 in Beare (2010) to the case of non-stationary Markov processes. In fact, Theorem 3.3 in Bradley (2005) ensures that the mixing coefficients ρ_k decay geometrically fast if $\rho_1 < 1$. Therefore, it is sufficient to prove that $\rho_1 < \hat{\eta}$. Let $(F_t)_t$ be the sequence of distributions associated to $(Y_t)_t$ and let (U_t, U_{t+1}) be a random vector with joint distribution $C_{t,t+1}$. Since (Y_t, Y_{t+1}) and $\left(F_t^{-1}(U_t), F_{t+1}^{-1}(U_{t+1})\right)$ (where F_t^{-1} is the quasi-inverse of F_t) share the same distribution, thanks to Proposition 3.6(I)(c) in Bradley (2007), we have that $\sup_{t\in\mathbb{Z}} \tilde{\rho}(\sigma(Y_t), \sigma(Y_{t+1})) = \sup_{t\in\mathbb{Z}} \tilde{\rho}(\sigma(F_t^{-1}(U_t)), \sigma(F_{t+1}^{-1}(U_{t+1})))$. Since $\sigma(F_t^{-1}(U_t)) \subseteq \sigma(U_t)$ and $\sigma(F_{t+1}^{-1}(U_{t+1})) \subseteq \sigma(U_{t+1})$ it follows that $\rho_1 \leq \sup_{t\in\mathbb{Z}} \tilde{\rho}(\sigma(U_t), \sigma(U_{t+1})) = \hat{\eta}$. □

As observed above, the gaussian, Frank and Farlie-Gumbel-Morgestern $C_{t,t+1}$ copulas satisfy $\hat{\eta} < 1$ for parameters in a compact subset of the interior of the respective parameter spaces.

Remark 2.1. *In Theorem 4.2 in Beare (2010) it is proved that, in the stationary case ($C_{t,t+1} = C$), a sufficient condition for the the maximal correlation coefficient to be strictly smaller than one is that the density of the absolutely continuous part of C is bounded away from zero on a set of full measure. In our more general setting, we have that if $c_{t,t+1}$ is the density of the absolutely continuous part of $C_{t,t+1}$, then*

$$\inf_{t\in\mathbb{Z}} c_{t,t+1}(u, v) \geq \epsilon \text{ almost everywhere on } [0,1]^2 \text{ with } \epsilon > 0$$

implies that

$$\hat{\eta} < 1.$$

In fact, by definition of maximal correlation of a copula function we have for all t

$$\int_0^1 \int_0^1 f(x)g(y)C_{t,t+1}(dx, dy) = 1 - \frac{1}{2}\int_0^1 \int_0^1 (f(x) - g(y))^2 C_{t,t+1}(dx, dy) \leq$$
$$\leq 1 - \frac{1}{2}\int_0^1 \int_0^1 (f(x) - g(y))^2 c_{t,t+1}(x, y)\, dxdy \leq 1 - \epsilon,$$

where the last inequality derives from the assumption $\inf_{t\in\mathbb{Z}} c_{t,t+1}(u, v) \geq \epsilon$ almost everywhere on $[0,1]^2$ and implies that the sequence of maximal correlations (η_t) is bounded by $1 - \epsilon$. Therefore, $\hat{\eta}$ cannot exceed $1 - \epsilon$ as required.

Notice that the above condition is satisfied by the Marshall-Olkin copulas and the t-copulas $C_{t,t+1}$ provided that the set of corresponding parameters lies in a compact subset of the parameter space.

3 Convolution-based Markov processes

In this section we propose a construction of a non stationary copula-based Markov process using the C-convolution operator (denoted by $\overset{C}{*}$), introduced in Cherubini et al. (2011) as a tool to recover the distribution of the sum of two dependent random variables. As shown in Cherubini et al. (2011), in Cherubini et al. (2012) and in Cherubini et al. (2016) the C-convolution technique may be used in the construction of (non-stationary) dependent increments stochastic processes. More precisely, if F_t is the cumulative distribution function of Y_t and H_{t+1} that of the increment $\Delta Y_{t+1} = Y_{t+1} - Y_t$ and, moreover, the dependence structure of the couple $(Y_t, \Delta Y_{t+1})$ is modelled by a copula C_t, we may recover the cumulative distribution function of Y_{t+1} iterating the C-convolution across t

$$F_{t+1}(y_{t+1}) = (F_t \overset{C_t}{*} H_{t+1})(y_{t+1}) = \int_0^1 D_1 C_t(w, H_{t+1}(y_{t+1} - F_t^{-1}(w)))dw, \quad t \geq 2 \tag{7}$$

while, the copula associated to (Y_t, Y_{t+1}) is

$$C_{t,t+1}(u, v) = \int_0^u D_1 C_t(w, H_{t+1}(F_{t+1}^{-1}(v) - F_t^{-1}(w)))dw, \quad t \geq 2 \tag{8}$$

where $D_1 C_t(u, v) = \frac{\partial}{\partial u} C_t(u, v)$. Equations (7) and (8) provide the ingredients to construct discrete time Markov processes according to Darsow et al. (1992). We can say that $Y_t \sim (F_t, C_{t,t+1})$ is a *convolution-based Markov process*.

Now suppose that the increments ΔY_{t+1} are identically distributed with absolutely continuous distribution F_Δ and that the dependence between the level Y_t and the next increment ΔY_{t+1} is given by a time-invariant copula C. We can say that the couple (F_Δ, C) generates a convolution-based Markov process $Y_t \sim (F_t, C_{t,t+1})$. Notice that we assume stationarity in the dependence structure "level-increment", $(Y_t, \Delta Y_{t+1})$, and in the marginal distribution of the increment F_Δ but the resulting Markov process is non-stationary. As shown in Cherubini et al. (2011) and in Cherubini et al. (2012) there are closed solutions of the equations (7) and (8) in the sense that the sequence $(F_t, C_{t,t+1})$ belongs to the same family of distributions of (F_Δ, C). In particular, in the case of copulas and marginal distributions belonging to the same *elliptical* family, the C-convolution has a closed form. The most significant cases are provided by the gaussian copula and the t-copula. As a matter of fact, if $F_\Delta \sim N(\mu, \sigma)$ and C is gaussian with a parameter ρ lying in $(-1, 1)$, then the sequence of distributions $(F_t)_t$ recovered by applying relation (7) is gaussian with parameters that will be functions of t, μ, σ and ρ whereas the sequence of copulas $(C_{t,t+1})_t$, determined by applying relation (8), is gaussian characterized by a sequence of associated correlation coefficients that will be functions of t, μ, σ and ρ.

3.1 An elliptical random walk with dependent increments

A particular application of this construction is a modified version of a standard random walk where we allow increments to be dependent. More precisely, we assume that the Markov process $(Y_t)_t$ is obtained through

$$Y_t = Y_{t-1} + \xi_t, \qquad Y_0 = 0, \tag{9}$$

where $(\xi_t)_t$ is a sequence of identically distributed random variables with common distribution F_Δ. Furthermore, the dependence structure of (Y_{t-1}, ξ_t) is given by a time-invariant copula function C.

The process defined in (9) is a generalization of a random walk since increments are not independent. However, we can determine the distribution of Y_t for each t thanks to the C-convolution (7) where F_{t-1} is the distribution function of Y_{t-1} and H is the stationary distribution of ξ_t. We may recover the sequence of distribution functions of Y_t iterating the C-convolution across t. Our model (9) is a sort of a modified version of a random walk process where the independence assumption for the innovations $(\xi_t)_t$ is no longer required: however, its weakness is that in most cases the distribution function cannot be expressed in closed form and it may be evaluated only numerically.

From now on we assume a given bivariate absolutely continuous elliptical distribution family: it is well known that this is specified through a given density generator g (which is a nonnegative function of a scalar argument satisfying $\int_0^{+\infty} g(y)dy < +\infty$), a correlation matrix Σ and a vector of means μ. We denote this bivariate distribution family as $Ell_2(\mu, \Sigma, g)$. It is well known that the bivariate copula associated to the bivariate family $Ell_2(\mu, \Sigma, g)$ depends on the density generator g and the correlation coefficient θ (we will denote it as $C_{g,\theta}$) and that the marginal distributions belong to the family $Ell_1(\mu_i, \sigma_{ii}, g_1)$, $i = 1, 2$, where $g_1(u) = \frac{\pi^{1/2}}{\Gamma(1/2)} \int_u^{+\infty} \frac{g(y)}{\sqrt{y-u}} dy$ (see Fang et al.,1990, for more details on elliptical distributions).

More specifically suppose that innovations $(\xi_t)_t$ are identically distributed according to $Ell_1(0, \sigma_\xi^2, g_1)$ and that the copula between Y_{t-1} and ξ_t is the (stationary) elliptical copula $C_{g,\theta}$ with constant correlation parameter $\theta \in (-1, 1)$. As anticipated in section 3 both sequences $(F_t)_t$, obtained by iterating relation (7), and $(C_{t,t+1})_t$, obtained by iterating relation (8), belong to the same original elliptical families of ξ_t and $C_{g,\theta}$, respectively. More precisely $Y_t \sim Ell_1(0, V_t^2, g_1)$, where, as obtained in the gaussian sub-case in section 4.3.1 of Cherubini et al. (2016), V_t^2 is given by

$$V_t^2 = Var(Y_t) = V_1^2 + (t-1)\sigma_\xi^2 + 2\theta\sigma_\xi \sum_{i=1}^{t-1} V_{t-i}, \quad t \geq 2, \tag{10}$$

where $V_1^2 = \sigma_\xi^2$ since by assumption $Y_1 = \xi_1$ and θ is the correlation coefficient associated to the invariant copula C. Obviously, the copula between Y_t and Y_{t+1}, $C_{g,\tau_{t,t+1}}$, exactly again as in the gaussian case, is characterized by a correlation given by

$$\tau_{t,t+1} = \frac{V_t + \theta\sigma_\xi}{V_{t+1}}, \quad t \geq 2 \tag{11}$$

since $\mathbb{E}[Y_t Y_{t+1}] = V_t^2 + \theta V_t \sigma_\xi$. Hence, from now on we will consider the convolution-based Markov process $Y_t \sim \big(Ell_1(0, V_t^2, g_1), C_{g,\tau_{t,t+1}}\big)$.

The limiting behavior of the standard deviation V_t has also been analyzed in Section 4.3.1 Cherubini et al. (2016) where it is proved that

$$\lim_{t\to+\infty} V_t = \begin{cases} -\frac{\sigma_\xi}{2\theta}, & \text{if } \theta \in (-1, 0) \\ +\infty, & \text{otherwise.} \end{cases} \tag{12}$$

Notice that only in case of negative correlation between Y_{t-1} and ξ_t, the standard deviation of the levels does not explode: in the following we will restrict the analysis to the case $\theta \in (-1, 0)$.

The following lemma establishes a general relation

Lemma 3.1. *Let $(Y_t)_t$ be the convolution-based Markov process $Y_t \sim \big(Ell_1(0, V_t^2, g_1), C_{g,\tau_{t,t+1}}\big)$ where V_t satisfies (10) and $\tau_{t,t+1}$ is given by (11). If $\theta \in (-1, 0)$, then the sequence of correlation coefficients $\tau_{t,t+1}$ associated to (Y_t, Y_{t+1}) satisfy, for every t,*

$$|\tau_{t,t+1}| \leq K < 1,$$

for some constant $0 < K < 1$.

Proof. First notice that $|\tau_{t,t+1}| < 1$, for every t. In fact this is equivalent to $(V_t + \theta\sigma_\xi)^2 < V_{t+1}^2$ which is always verified since $\theta^2 < 1$ by assumption. Moreover

$$\frac{V_t + \theta\sigma_\xi}{V_{t+1}} \to \frac{-\frac{\sigma_\xi}{2\theta} + \theta\sigma_\xi}{-\frac{\sigma_\xi}{2\theta}} = 1 - 2\theta^2.$$

Since $|1 - 2\theta^2| < 1$, we have that the sequence $|\tau_{t,t+1}|$ is bounded by a constant smaller than 1. $\square$

Since the sequence of copulas associated to (Y_t, Y_{t+1}), $C_{t,t+1} = C_{g,\tau_{t,t+1}}$ only depend on the parameter $\tau_{t,t+1}$, Lemma 3.1 implies that the parameters of the copulas $C_{t,t+1}$ belong to a compact subset of the parameter space. An immediate consequence is that, according to Remark 2.1, the convolution-based Student's t Markov process is geometric ρ-mixing.

3.2 A gaussian random walk with dependent increments

From now we will focus on the specific case in which innovations $(\xi_t)_t$ are gaussian identically distributed with zero mean and standard deviation σ_ξ and that the copula between Y_{t-1} and ξ_t is a (stationary) gaussian copula C with constant parameter $\theta \in (-1, 0)$ for all t. In other words we assume $(F_\Delta, C) \equiv (N(0, \sigma_\xi), C(\cdot, \cdot, \theta))$.

In Section 4.3.2 in Cherubini et al. (2016), an alternative representation of the gaussian convolution-based process is introduced and its limiting behavior considered. In fact, more precisely, setting

$$\phi_t = 1 + \frac{\theta\sigma_\xi}{V_{t-1}},$$

it can be easily checked that the stochastic process $(Z_t)_{t\geq 1}$

$$Z_t = \phi_t Z_{t-1} + u_t, \qquad Z_1 \sim N(0, \sigma_\xi^2)$$

with, for $t \geq 2$, $(u_t) \sim i.i.d.\, N(0, \sigma_\xi^2(1-\theta^2))$ and u_t independent of Z_{t-1}, is distributed as the Markov process with dependent increments in (9), being $(Y_{t-1}, Y_t) \sim (Z_{t-1}, Z_t)$ for all $t \geq 2$. Since $\theta < 0$, by (12), we have

$$\phi_t \to 1 - 2\theta^2, \quad t \to +\infty.$$

It is immediate to conclude that the distribution of the Markov process with dependent increments in (9) is asymptotically close to that of an $AR(1)$ process.

Both Theorems 2.1 and 2.2 require the maximal correlation coefficient of the copula $C_{t,t+1}$ to be far from 1 uniformly in t. It is well known that for the gaussian copula the maximal correlation coefficient is equal to the absolute value of the simple correlation coefficient (see Lancaster, 1957). Therefore, according to the notation of Section 2, for each t, $\eta_t = |\tau_{t,t+1}|$.

Proposition 3.1. *The convolution-based gaussian Markov process is both geometric ρ-mixing and β-mixing.*

Proof. Thanks to Lemma 3.1 and Theorem 2.2, the process is geometric ρ-mixing. Furthermore, it is not hard to prove that, if $c_{t,t+1}$ is the density of the gaussian copula $C_{t,t+1}$, for any t

$$\| c_{t,t+1}(u,v) - 1 \|_{\mathbb{L}^2} = \frac{\tau_{t,t+1}^2}{1-\tau_{t,t+1}^2} \leq \frac{\hat{\eta}^2}{1-\hat{\eta}^2}.$$

Thus also Theorem 2.1 applies and geometric β-mixing is ensured. □

In order to understand the differences between the introduced gaussian process and the standard random walk, we study the behavior of moments and autocorrelation functions of the process $(Y_t)_t$ when $t \to +\infty$. It is just the case to recall that in the standard random walk model the k-th order autocorrelation function of $(Y_t)_t$ tends to 1 as $t \to +\infty$, for each lag k. In our more general setting, this is no longer true. The limit of the k-th order autocorrelation function of $(Y_t)_t$ is a function of k and θ as the following proposition shows.

Proposition 3.2. *Let $\theta \in (-1, 0)$. The k-th order autocorrelation function of $(Y_t)_t$ tends to $(1-2\theta^2)^k$ for any $k \geq 1$ as $t \to +\infty$.*

Proof. As proved in section 4.3.1 in Cherubini et al. (2016), using the fact that the $*$-product of two gaussian copulas has a parameter given by the product of the parameters of the copulas involved in the $*$-product, we have that the copula between Y_t and Y_{t+k} is gaussian with parameter

$$\tau_{t,t+k} = \prod_{s=0}^{k-1} \frac{V_{t+s} + \theta\sigma_\xi}{V_{t+s+1}}.$$

Therefore, since as $t \to +\infty$ and for any $s \geq 1$

$$\frac{V_{t+s} + \theta\sigma_\xi}{V_{t+s+1}} \to \frac{-\frac{\sigma_\xi}{2\theta} + \theta\sigma_\xi}{-\frac{\sigma_\xi}{2\theta}} = 1 - 2\theta^2, \tag{13}$$

we easily get the result. □

On the other hand, the innovations $(\xi_t)_t$ are no longer serially independent as in the random walk case and the k-th order autocorrelation function approaches to a limit which again depends on θ and k.

Proposition 3.3. *Let $\theta \in (-1, 0)$. The k-th order autocorrelation function of $(\xi_t)_t$ tends to $-\theta^2(1-2\theta^2)^{k-1}$ for any $k \geq 1$ as $t \to +\infty$.*

Proof. We compute first the autocovariance of order k, with $k \geq 1$, $\mathbb{E}[\xi_t\xi_{t+k}]$. We have

$$\begin{aligned}\mathbb{E}[\xi_t\xi_{t+k}] &= \mathbb{E}[(Y_t - Y_{t-1})(Y_{t+k} - Y_{t+k-1})] = \\ &= \mathbb{E}[Y_tY_{t+k}] - \mathbb{E}[Y_tY_{t+k-1}] - \mathbb{E}[Y_{t-1}Y_{t+k}] + \mathbb{E}[Y_{t-1}Y_{t+k-1}] = \\ &= \tau_{t,t+k}V_tV_{t+k} - \tau_{t,t+k-1}V_tV_{t+k-1} - \tau_{t-1,t+k}V_{t-1}V_{t+k} + \tau_{t-1,t+k-1}V_{t-1}V_{t+k-1}.\end{aligned}$$

Since for any fixed $k \geq 1$, $\tau_{t,t+k} \to (1-2\theta^2)^k$ and $V_t \to -\frac{\sigma_\xi}{2\theta}$ as $t \to +\infty$ we get

$$\begin{aligned}\mathbb{E}[\xi_t\xi_{t+k}] \to &\frac{\sigma_\xi^2}{4\theta^2}\left[(1-2\theta^2)^k - (1-2\theta^2)^{k-1} - (1-2\theta^2)^{k+1} + (1-2\theta^2)^k\right] = \\ &= -\theta^2\sigma_\xi^2(1-2\theta^2)^{k-1}, \quad \textit{as } t \to +\infty.\end{aligned}$$

Moreover, it is immediate to find the statement of the proposition since as $t \to +\infty$

$$corr(\xi_t\xi_{t+k}) \to -\theta^2(1-2\theta^2)^{k-1}.$$

□

4 Concluding remarks

In this paper we provide conditions under which a non-stationary copula-based Markov process is geometric β-mixing and geometric ρ-mixing. Our results represent a generalization of those in Beare (2010), where the author considers the stationary case. The analysis is mainly focused on the particular case of a new Markov process obtained by using the C-convolution operator which generates dependent increments. Among C-convolution Markov processes, the most interesting case is given by a generalization of the standard random walk in which both the copula linking the level of the process with the next increment and the marginal distribution of the increment itself belong to the same elliptical family. When we restrict our attention to the gaussian non-stationary setting we prove that the k-th order autocorrelation function of the process does not converge to 1, as in the random walk case, but to a quantity that depends on the lag and the correlation between the state variable and the innovation, which is assumed to be time-invariant. Additionally, it is proved that the process satisfies the conditions required to be geometric β-mixing and geometric ρ-mixing.

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