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Generalized Samuel Multiplicities of Monomial Ideals and Volumes

This is the submitted version (pre peer-review, preprint) of the following publication:

Published Version:

Generalized Samuel Multiplicities of Monomial Ideals and Volumes / Achilles R.; Manaresi M.. - In: EXPERIMENTAL MATHEMATICS. - ISSN 1058-6458. - STAMPA. - 31:2(2022), pp. 611-620. [10.1080/10586458.2019.1671919]

Availability:

This version is available at: <https://hdl.handle.net/11585/729776> since: 2022-11-07

Published:

DOI: <http://doi.org/10.1080/10586458.2019.1671919>

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GENERALIZED SAMUEL MULTIPLICITIES OF MONOMIAL IDEALS AND VOLUMES

R. ACHILLES AND M. MANARESI

ABSTRACT. We describe conjecturally the generalized Samuel multiplicities c_0, \dots, c_{d-1} of a monomial ideal $I \subset K[x_1, \dots, x_d]$ in terms of its Newton polyhedron $\Gamma(I)$. More precisely, we conjecture that c_i equals the sum of the normalized $(d-i)$ -volumes of pyramids over the projections of the $(d-i-1)$ -dimensional compact faces of $\Gamma(I)$ along the infinite-directions of i -unbounded facets in which they are contained. For c_0 proofs are known (Guibert, Jeffries and Montaña) and for c_{d-1} a proof will be given.

1. INTRODUCTION

In this paper, based on computations with the free softwares `Germe` by A. Montesinos [11] and `REDUCE` [12] by A. C. Hearn [8] and `REDUCE` developers, we give a conjecture that in the case of monomial ideals links the generalized multiplicities defined algebraically in [3] with volumes derived from the Newton polyhedra of the ideals, thus extending a result of B. Teissier [14].

In 1988, B. Teissier [14, p. 131] proved that for an \mathfrak{m} -primary monomial ideal I of a local ring A the Samuel multiplicity is equal to the normalized volume of the complement of the Newton polyhedron of the ideal I . In 1999, G. Guibert [7] generalized Teissier's result. Precisely, Guibert defines the local Segre class of an ideal generated by a set of germs of holomorphic functions and, under a non-degeneracy condition, he describes such a class by Minkowski mixed volumes of polytopes. As a special case he obtains that for a certain class of monomial ideals the local Segre class is a normalized volume of the simplex generated by the origin and the vertices of the Newton polyhedron, see [7, 4.2]. By [4], the local Segre class is the so called j -multiplicity of the ideal. In 2013, J. Jeffries and J. Montaña [9] gave a different proof that the

2010 *Mathematics Subject Classification*. Primary 13H15; Secondary 13F20, 52B20.

Key words and phrases. Monomial ideal, generalized Samuel multiplicity, Newton polyhedron.

j -multiplicity of a monomial ideal is the normalized volume of the pyramid of the ideal.

The j -multiplicity of an ideal is different from zero if and only if its analytic spread is maximal, that is, equal to the Krull-dimension d of A . A result of C. Bivià-Ausina [5] states that the analytic spread diminished by one is the maximum of the dimensions of compact faces of the Newton polyhedron of I .

According to [3] the j -multiplicity is only the first coordinate of the generalized Samuel multiplicity vector $c(I) = (c_0(I), \dots, c_d(I))$. Here we present and illustrate a conjecture which expresses the other components of $c(I)$ in terms of the Newton polyhedron of I . Our conjecture extends the known result of G. Giubert and J. Jeffries and J. Montaña regarding $c_0(I)$, but we shall prove it here only for $c_{d-1}(I)$.

2. GENERALIZED SAMUEL MULTIPLICITIES

This section is a quick review of a generalization of Samuel's multiplicity by a sequence of numbers, the so-called generalized Samuel multiplicity, which we have introduced in [3].

Let A be a d -dimensional Noetherian local ring (A, \mathfrak{m}) with unique maximal ideal \mathfrak{m} or a standard graded algebra $A = \bigoplus_{i \geq 0} A_i$ such that A_0 is a field and $\mathfrak{m} = (A_1)A$ is the unique homogeneous maximal ideal of A . Let $I \subset A$ be an arbitrary ideal (not necessarily \mathfrak{m} -primary).

In order to define the generalized Samuel multiplicity $c(I)$, consider $G_I(A) := \bigoplus_{j \geq 0} I^j/I^{j+1}$, the associated graded ring of A with respect to I and the bigraded ring

$$T = \bigoplus_{i,j \geq 0} T_{ij} = G_{\mathfrak{m}}(G_I(A)) = \bigoplus_{i,j \geq 0} \frac{\mathfrak{m}^i I^j + I^{j+1}}{\mathfrak{m}^{i+1} I^j + I^{j+1}},$$

where $T_{00} = A/\mathfrak{m} = K$ is a field.

Let $H^{(0,0)}(i, j) := \dim_K T_{ij}$ be the Hilbert function of the bigraded ring T and let

$$H^{(1,1)}(i, j) := \sum_{q=0}^j \sum_{p=0}^i H^{(0,0)}(p, q)$$

its twofold sum transform. For both $i, j \gg 1$ this function becomes a polynomial in (i, j) , which can be written in the form

$$\sum_{k+l \leq d} a_{k,l}^{(1,1)} \binom{i+k}{k} \binom{j+l}{l}.$$

Following [3] define the *generalized Samuel multiplicity* to be the vector

$$\begin{aligned} \left(a_{0,d}^{(1,1)}, a_{1,d-1}^{(1,1)}, \dots, a_{d,0}^{(1,1)} \right) &=: (c_0(T), c_1(T), \dots, c_d(T)) =: c(T) \\ &=: (c_0(I), c_1(I), \dots, c_d(I)) =: c(I). \end{aligned}$$

The first coefficient $c_0(I)$ plays an important role as an intersection number and was introduced in [2]. It is called the *j-multiplicity* $j(I) := c_0(I)$.

The generalized Samuel multiplicities depend only on the highest dimensional components of T , see [15] or [3, Proposition 1.2]:

Proposition 1. *With the preceding notation,*

$$c(I) = c(T) = \sum_P \text{length}(T_P) \cdot c(T/P),$$

where P runs through all highest dimensional prime ideals of T .

By analogy with the application of $c(I)$ to intersection theory, we shall call $c_i(T/P) \neq 0$ a *movable contribution* to $c_i(I)$ if there is an integer $k > i$ such that $c_k(T/P) \neq 0$.

3. A CONJECTURE AND SOME RESULTS

Let I be an ideal in $R = K[x_1, \dots, x_d] = K[\mathbf{x}]$ (K a field) minimally generated by the monomials

$$\mathbf{x}^{v_1} := x_1^{v_1(1)} \dots x_d^{v_1(d)}, \dots, \mathbf{x}^{v_r} := x_1^{v_r(1)} \dots x_d^{v_r(d)},$$

that is, $v_1 = (v_1(1), \dots, v_1(d)), \dots, v_r = (v_r(1), \dots, v_r(d))$ are the points of $\mathbb{Z}_{\geq 0}^d$ corresponding to the exponents of the generators of I .

The *Newton polyhedron* $\Gamma(I)$ of I is defined as the convex hull of $\{v \in \mathbb{Z}_{\geq 0}^d \mid \mathbf{x}^v \in I\}$ in \mathbb{R}^d , that is,

$$\begin{aligned} \Gamma(I) &:= \text{conv}(\{v \in \mathbb{Z}_{\geq 0}^d \mid x_1^{v(1)} \dots x_d^{v(d)} \in I\}) \\ &= \text{conv}(\{v_1, \dots, v_r\}) + \mathbb{R}_{\geq 0}^d, \end{aligned}$$

where $+$ denotes the Minkowski sum (for the equality see [13, Lemma 4.3]).

A hyperplane

$$H = \{v \in \mathbb{R}^d \mid \langle v, a \rangle = b\} \quad (\text{with } a \in \mathbb{R}_{\geq 0}^d, b \in \mathbb{R})$$

is called a *supporting hyperplane* of the Newton polyhedron $\Gamma(I)$ if

$$\Gamma(I) \subset H^+ = \{v \in \mathbb{R}^d \mid \langle v, a \rangle \geq b\} \quad \text{and} \quad \Gamma(I) \cap H \neq \emptyset.$$

A subset $F \subset \Gamma(I)$ is called a *proper face* of $\Gamma(I)$ if there exists a supporting hyperplane H of $\Gamma(I)$ such that $F = \Gamma(I) \cap H$. The

boundary of $\Gamma(I)$ is a set of faces of dimension $d - 1$, called *facets* of $\Gamma(I)$, some of them compact.

The zero-dimensional faces are called *vertices* of $\Gamma(I)$. We shall denote the set of vertices by $\text{vert}(I)$. Note that the monomials corresponding to the points in $\text{vert}(I)$ are part of the set of minimal generators of I , so by renumbering we will assume that

$$\text{vert}(I) = \{v_1, \dots, v_s\} \text{ with some } s \leq r,$$

hence

$$\Gamma(I) = \text{conv}(\{v_1, \dots, v_r\}) + \mathbb{R}_{\geq 0}^d = \text{conv}(\{v_1, \dots, v_s\}) + \mathbb{R}_{\geq 0}^d.$$

Any face F can be described using its vertices and infinite-directions. Let e_j denote the unit vector with non-zero j th component, let H be a supporting hyperplane such that $F = \Gamma(I) \cap H$ and let a be a normal vector to H . Then the *infinite-directions* of F are given by those e_j such that the j th component of a is zero. If v_{i_1}, \dots, v_{i_s} are the vertices of F , then

$$F = \text{conv}(\{v_{i_1}, \dots, v_{i_s}\}) + \sum_{j: a(j)=0} \mathbb{R}_{\geq 0} e_j.$$

Of course, the compact faces are precisely those that do not have infinite directions e_j .

By the Minkowski-Weyl Theorem for convex polyhedra, there are uniquely determined finitely many closed half spaces

$$H_i^+ = \{v \in \mathbb{R}^d \mid \langle v, a_i \rangle \geq b_i\} \text{ (with } a_i \in \mathbb{Z}_{\geq 0}^d, b_i \in \mathbb{Z}_{\geq 0}), i = 1, \dots, t,$$

such that

$$\Gamma(I) = H_1^+ \cap \dots \cap H_t^+.$$

Then $F_i := H_i \cap \Gamma(I)$, $i = 1, \dots, t$, are the facets of $\Gamma(I)$. We will assume that H_1, \dots, H_r are the hyperplanes corresponding to the unbounded facets and that H_{r+1}, \dots, H_t are those corresponding to the compact facets.

To each bounded facet $F = \text{conv}(\{v_{i_1}, \dots, v_{i_s}\})$ of $\Gamma(I)$ we associate the polytope (or pyramid)

$$\hat{F} := \text{conv}(0, F) = \text{conv}(\{0, v_{i_1}, \dots, v_{i_s}\})$$

and denote by $\text{vol}_d(\hat{F})$ its d -dimensional volume and by

$$\text{Vol}_d(\hat{F}) := d! \text{vol}_d(\hat{F})$$

its *normalized volume*.

A facet $F \subset \Gamma(I)$ is called an *h-unbounded facet* if the normal vector a to its supporting hyperplane has at least $h > 0$ coordinates $a(j)$ which are zero, that is, the facet has h infinite-directions e_j .

Let $\mathcal{F}(k)$ be the set of all $(d - (k + 1))$ -unbounded facets containing at least one k -dimensional compact face F^k , $0 \leq k \leq d - 2$. We define $\mathcal{F}(d - 1)$ to be the set of all compact or bounded facets of $\Gamma(I)$.

To each couple (F^k, F^{d-1}) , with F^k a k -dimensional compact face and $F^{d-1} \in \mathcal{F}(k)$ containing F^k , we associate a $(k + 1)$ -dimensional normalized volume $\text{Vol}(F^k, F^{d-1})$ in the following way. A normal vector to the facet F^{d-1} lies on at least $d - (k + 1)$ coordinate hyperplanes. We project F^k on all linear subspaces $\mathbb{R}^{k+1} \subseteq \mathbb{R}^d$ obtained by intersecting $d - (k + 1)$ of these coordinate hyperplanes, that is, we project F^k along all possible choices of $d - (k + 1)$ infinite-directions of the facet F^{d-1} . We obtain polytopes of dimension at most k . We consider only the k -dimensional polytopes $\text{pr}_{\mathbb{R}^{k+1}}(F^k) \subset \mathbb{R}^{k+1}$ obtained by the aforementioned projections and set

$$\hat{F}^k := \text{conv}(\{0, \text{pr}_{\mathbb{R}^{k+1}}(F^k)\}),$$

which has dimension k or $k + 1$. The volume associated to the couple (F^k, F^{d-1}) is

$$\text{Vol}(F^k, F^{d-1}) := \min_{\mathbb{R}^{k+1}} \text{Vol}_{k+1}(\hat{F}^k).$$

Conjecture 1. *For each $k = 0, \dots, d - 1$, the generalized Samuel multiplicity of a monomial ideal I is*

$$c_{d-(k+1)}(I) = \sum_{F^{d-1} \in \mathcal{F}(k)} \min_{F^k} \{\text{Vol}(F^k, F^{d-1})\},$$

where the minimum is taken over all compact faces F^k of $\Gamma(I)$ that are contained in the facet F^{d-1} .

Conjecture 2. *Each summand in the formula of Conjecture 1 corresponds, in the sense of Proposition 1, to the contribution of a highest dimensional primary component of $T = G_{\mathfrak{m}}(G_I(R))$ to $c_{d-(k+1)}(I)$.*

In particular, the number of compact facets of $\Gamma(I)$ is equal to the number of d -dimensional associated prime ideals of T that contain $\mathfrak{m} = (x_1, \dots, x_d)R$.

Note that in general the zero-ideal \mathfrak{n} of $T \cong K[x_1, \dots, x_d, y_1, \dots, y_r]/\mathfrak{n}$ is a binomial but not a monomial ideal, see [6].

Our conjectures are confirmed by many examples, but so far we do not have a proof except for Conjecture 1 in the extremal cases $k = 0$ and $k = d - 1$, as it is stated in the following two theorems.

Theorem 1 (Jeffries and Montaña, [9]). *If $I \subset K[x_1, \dots, x_d]$ is a monomial ideal and F_{r+1}, \dots, F_t are the compact facets of the Newton polyhedron $\Gamma(I)$, then*

$$c_0(I) = \sum_{i=r+1}^t d! \operatorname{vol}(\hat{F}_i) = \sum_{i=r+1}^t \operatorname{Vol}(\hat{F}_i).$$

Theorem 2. *Let I be a monomial ideal in $R = K[x_1, \dots, x_d]$ generated by $x_1^{v_1(1)} \dots x_d^{v_1(d)}, \dots, x_1^{v_r(1)} \dots x_d^{v_r(d)}$ and $m_j = \min\{v_1(j), \dots, v_r(j)\}$, $j = 1, \dots, d$. Then*

$$c_{d-1}(I) = m_1 + \dots + m_d.$$

Proof. By [3, Proposition 2.3], $c_{d-1}(I) \neq 0$ if and only if $\dim R/I = d - 1$. If $\dim R/I < d - 1$, then none of the variables x_j appears in all monomials generating I , hence $m_j = 0$ for all j , and the result is true. If $\dim R/I = d - 1$, then again by [3, Proposition 2.3],

$$c_{d-1}(I) = \sum_P e(IR_P) \cdot e(R/P),$$

where P runs through all $(d-1)$ -dimensional associated prime ideals of R/I , that is, prime ideals of the form (x_j) for some j , see [10, Satz 9]. Therefore $IR_P = (x_j^{m_j})R_P$ and $e(IR_P) = m_j$. By [10] the $(d-1)$ -dimensional part of the primary decomposition of I is $(x_1^{m_1}) \cap (x_2^{m_2}) \cap \dots \cap (x_d^{m_d})$, which is of degree $m_1 + \dots + m_d$. \square

The following corollary of Theorem 2 states that the Conjecture 1 is true for $k = 0$.

Corollary 3. *Using the preceding notations, for $j = 1, \dots, d$ set*

$$F_j := \operatorname{conv}(\{v \in \operatorname{vert}(I) \mid v(j) = m_j\}) + \sum_{1 \leq i \leq d, i \neq j} \mathbb{R}_{\geq 0} e_i$$

and $\operatorname{vert}(F_j) := \operatorname{vert}(I) \cap F_j$.

Then $\mathcal{F}(0) = \{F_1, \dots, F_d\}$. If $v \in \operatorname{vert}(F_j)$, then $\operatorname{Vol}(v, F_j) = m_j$ and it holds

$$c_{d-1}(I) = \sum_{j=1}^d \min_{v \in \operatorname{vert}(F_j)} \{\operatorname{Vol}(v, F_j)\}.$$

Proof. Since each $v \in \Gamma(I)$ is the sum of a convex combination of the vertices v_1, \dots, v_s of $\Gamma(I)$ and of some $w \in \mathbb{R}_{\geq 0}^d$, we have

$$v(j) \geq \min\{v_1(j), \dots, v_s(j)\} + w(j) \geq \min\{v_1(j), \dots, v_s(j)\},$$

hence

$$m_j := \min\{v_1(j), \dots, v_r(j)\} = \min\{v_1(j), \dots, v_s(j)\} = \min_{v \in \Gamma(I)} \{v(j)\}.$$

It follows that F_1, \dots, F_d are precisely the $(d-1)$ -unbounded facets of $\Gamma(I)$, that is, $\mathcal{F}(0) = \{F_1, \dots, F_d\}$.

Since the projection of $\mathbb{R}^d \rightarrow \mathbb{R}$ with center $\sum_{1 \leq i \leq d, i \neq j} \mathbb{R}e_i$ sends $v \in \text{vert}(F_j)$ to $v(j) = m_j$, we have $\text{Vol}(v, F_j) = v(j) = m_j$. Then, by Theorem 2, we obtain the desired formula for c_{d-1} , which finishes the proof. \square

4. EXAMPLES

We illustrate the theorems and the conjecture by examples of monomial ideals $I \subset R = K[x_1, \dots, x_d]$, K an arbitrary field. We set $\mathfrak{m} := (x_1, \dots, x_d)R$ and $T := G_{\mathfrak{m}}(G_I(R))$. All the examples will show a closed relation between the summands in the formula of Conjecture 1 and the highest dimensional primary components of T .

We begin with the simplest case of a monomial ideal generated by one monomial in two variables. The first two examples are covered by Theorems 1 and 2.

Example 1 (Figure 1). Let $I = (x^3y^2) \subset R = K[x, y]$. We have

$$c(I) = (c_0(I), c_1(I), c_2(I)) = (0, 5, 0) = 2 \cdot (0, 1, 0) + 3 \cdot (0, 1, 0),$$

where the summands are the contributions of the components of the bigraded ring $G_{\mathfrak{m}}(G_I(R))$, see Proposition 1.

The Newton polyhedron $\Gamma(I)$ has only one vertex $v = (3, 2)$ and two (unbounded) facets $F_1 = v + R_{\geq 0}e_2$ and $F_2 = v + R_{\geq 0}e_1$ (see Figure 1), hence $\mathcal{F}(1) = \emptyset$ and $c_0(I) = 0$. We have $\mathcal{F}(0) = \{F_1, F_2\}$ and $\text{Vol}(v, F_1) = 3$ and $\text{Vol}(v, F_2) = 2$, hence $c_1(I) = 5$.

Example 2 (Figure 2). Let $I = (x^2y^5, x^3y^4, x^4y^2, x^6y) \subset R = K[x, y]$. We have

$$\begin{aligned} c(I) &= (c_0(I), c_1(I), c_2(I)) \\ &= (24, 3, 0) = 16 \cdot (1, 0, 0) + 8 \cdot (1, 0, 0) + 2 \cdot (0, 1, 0) + (0, 1, 0), \end{aligned}$$

where the summands are the contributions of the components of the bigraded ring $G_{\mathfrak{m}}(G_I(R))$, see Proposition 1.

The Newton polyhedron $\Gamma(I)$ has three vertices $v_1 = (2, 5)$, $v_3 = (4, 2)$, $v_4 = (6, 1)$, two (unbounded) facets $F_1 = v_1 + R_{\geq 0}e_2$, $F_2 = v_4 +$

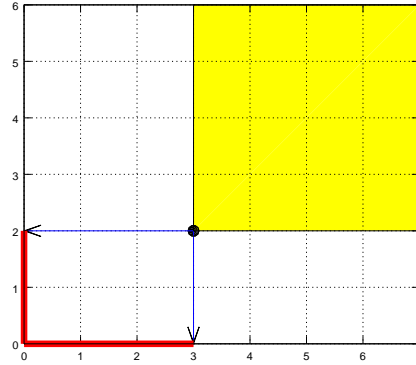


FIGURE 1. Projection along the infinite-directions of the facets gives $c_1(I)$, which is the red distance.

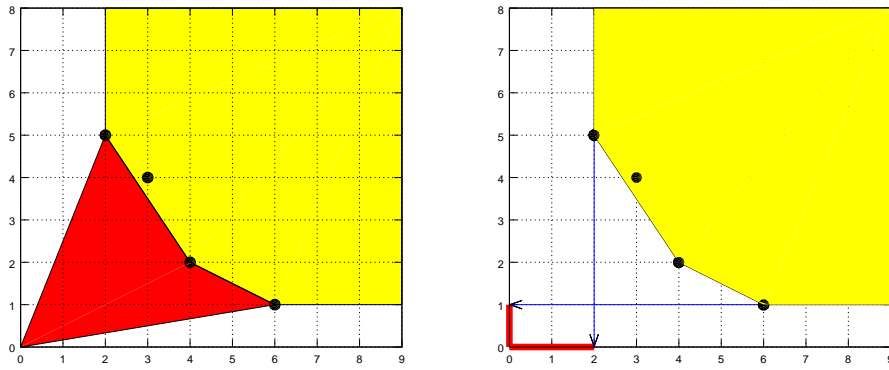


FIGURE 2. The red area is $c_0(I)/2$, the red distance $c_1(I)$.

$R_{\geq 0}e_1$ and two bounded facets: the line segments $F_3 = \text{conv}(v_1, v_3)$, $F_4 = \text{conv}(v_3, v_4)$ (see Figure 2), hence $\mathcal{F}(1) = \{F_3, F_4\}$ and

$$c_0(I) = \text{Vol}(\text{conv}(0, F_3)) + \text{Vol}(\text{conv}(0, F_4)) = \begin{vmatrix} 4 & 2 \\ 2 & 5 \end{vmatrix} + \begin{vmatrix} 6 & 1 \\ 4 & 2 \end{vmatrix} = 16 + 8.$$

We have $\mathcal{F}(0) = \{F_1, F_2\}$ and

$$c_1(I) = \text{Vol}(v_1, F_1) + \text{Vol}(v_4, F_2) = 2 + 1 = 3.$$

In the following example some of the compact faces of $\Gamma(I)$ do not contribute to the generalized Samuel multiplicity $c(I)$. In this example there is also a movable contribution, therefore the number of the highest dimensional components of T is one less than the number of summands in the conjectured formula.

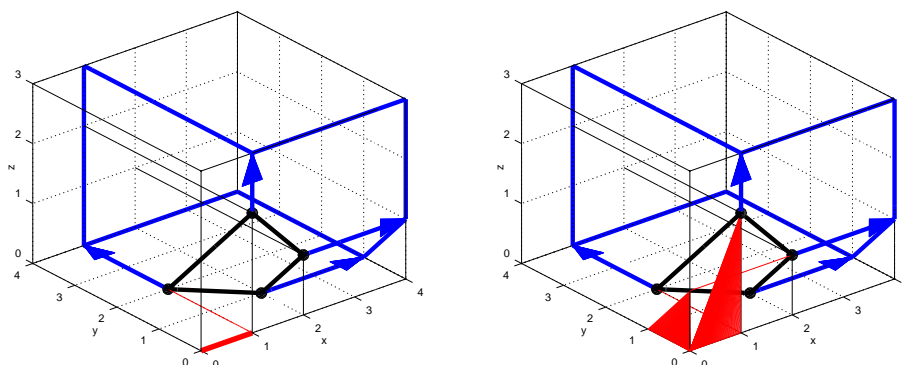


FIGURE 3. Infinite-directions (blue arrows) of the unbounded facets, $c_2(I)$ (red distance) and $c_1(I)/2$ (red area).

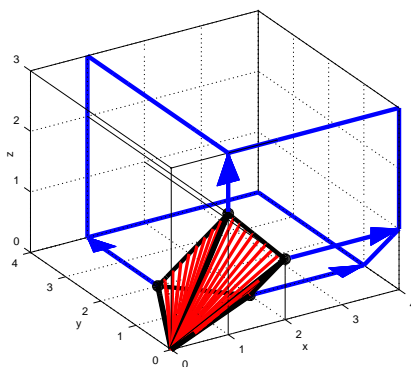


FIGURE 4. The volume of the red pyramid is $c_0/6$.

Example 3 (Figures 3, 4 and 5). Let $I = (x^2y, x^2z, xy^2, xz^2) \subset K[x, y, z]$. By a computer computation (using [1]) we have

$$\begin{aligned} c(I) &= (c_0(I), c_1(I), c_2(I), c_3(I)) \\ &= (9, 3, 1, 0) = 3 \cdot (3, 0, 0, 0) + (0, 1, 0, 0) + (0, 2, 1, 0), \end{aligned}$$

where the summands are the contributions of the highest dimensional components of the bigraded ring T , see Proposition 1. The contribution 2 in the last vector is a movable contribution to $c_1(I)$. This can be read off also from the Newton polyhedron $\Gamma(I)$, see Figure 6.

According to the program Germenes [11], the compact faces of $\Gamma(I)$ are the vertices $v_1 = (2, 1, 0)$, $v_2 = (2, 0, 1)$, $v_3 = (1, 2, 0)$, $v_4 = (1, 0, 2)$, the line segments v_1v_2 , v_1v_3 , v_2v_4 , v_3v_4 and the quadrilateral facet

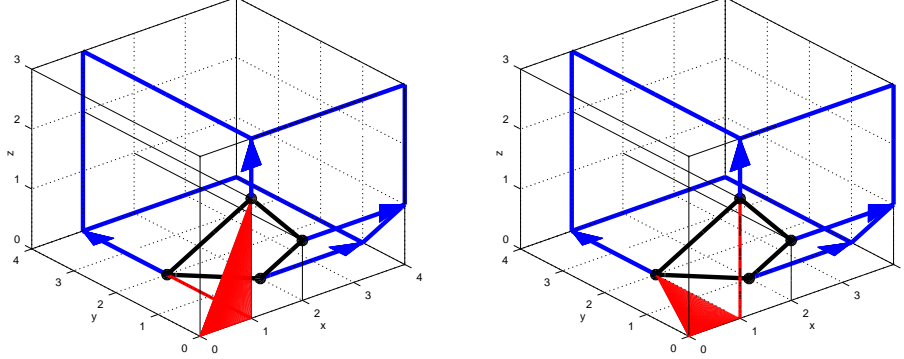


FIGURE 5. A movable contribution (to $c_1(I)/2$, red area) can be realized by at least two projections.

$v_1v_2v_4v_3$. The unbounded facets are

$$\begin{aligned} F_1 &= v_3v_4 + \mathbb{R}_{\geq 0} e_2 + \mathbb{R}_{\geq 0} e_3, & F_2 &= v_2v_4 + \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0} e_3, \\ F_3 &= v_1v_3 + \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0} e_2, & F_4 &= v_1v_2 + \mathbb{R}_{\geq 0} e_1. \end{aligned}$$

We observe that the set of bounded facets is $\mathcal{F}(2) = \{v_1v_2v_4v_3\}$ and

$$c_0(I) = \text{Vol}(\text{conv}(0, v_1, v_2, v_4, v_3)) = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 3 + 6 = 9,$$

see Figure 4.

The set of 1-unbounded facets that contain a compact one-dimensional face is $\mathcal{F}(1) = \{F_1, F_2, F_3, F_4\}$, and we have

$$\text{Vol}(v_1v_2, F_4) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1,$$

$$\text{Vol}(v_1v_3, F_3) = \min \left\{ \begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix}, \begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix} \right\} = 0,$$

$$\text{Vol}(v_2v_4, F_2) = \min \left\{ \begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix}, \begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix} \right\} = 0,$$

$$\text{Vol}(v_3v_4, F_1) = \min \left\{ \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} \right\} = 2$$

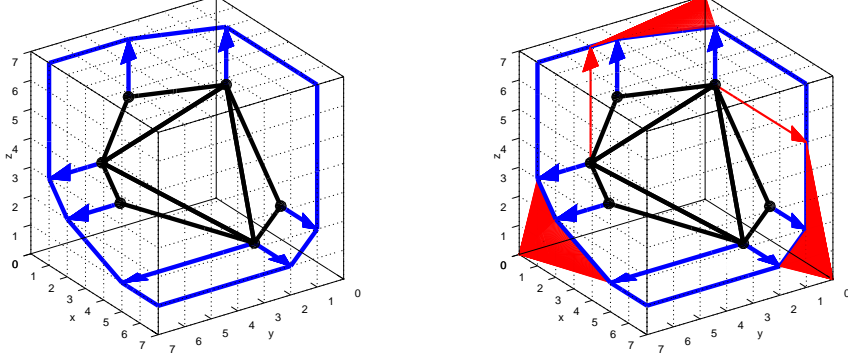


FIGURE 6. $\Gamma(I)$ with compact (black) and unbounded (blue) facets; projections of compact edges along infinite-directions (blue arrows) give $c_1(I)/2$ (red area).

(the last minimum is given by two different projections and is a movable contribution, see Figure 5), hence

$$\begin{aligned} c_1(I) &= \text{Vol}(v_1v_2, F_4) + \text{Vol}(v_1v_3, F_3) + \text{Vol}(v_2v_4, F_2) + \text{Vol}(v_3v_4, F_1) \\ &= 1 + 0 + 0 + 2 = 3. \end{aligned}$$

The set of 2-unbounded facets is $\mathcal{F}(0) = \{F_1, F_2, F_3\}$. We have $\text{Vol}(v_3, F_1) = 1$, $\text{Vol}(v_4, F_1) = 1$, $\text{Vol}(v_4, F_2) = 0$, $\text{Vol}(v_1, F_3) = 0$, $\text{Vol}(v_3, F_3) = 0$, hence

$$\begin{aligned} c_2(I) &= \min\{\text{Vol}(v_3, F_1), \text{Vol}(v_4, F_1)\} + \text{Vol}(v_4, F_2) + \\ &\quad + \min\{\text{Vol}(v_1, F_3), \text{Vol}(v_3, F_3)\} = 1 + 0 + 0 = 1. \end{aligned}$$

Example 4 (Figure 6). Let $I = (xy^4z^5, x^2y^5z^2, xy^5z^3, x^5yz^2, x^2yz^5, x^5y^2z) \subset K[x, y, z]$. By a computer computation we have

$$\begin{aligned} c(I) &= (c_0(I), c_1(I), c_2(I), c_3(I)) = (168, 26, 3, 0) = \\ &= 19 \cdot (1, 0, 0, 0) + 103 \cdot (1, 0, 0, 0) + 22 \cdot (1, 0, 0, 0) + \\ &\quad + 24 \cdot (1, 0, 0, 0) + 7 \cdot (0, 1, 0, 0) + (0, 3, 1, 0) + \\ &\quad + 8 \cdot (0, 1, 0, 0) + (0, 1, 0, 0) + 4 \cdot (0, 1, 0, 0) + \\ &\quad + 3 \cdot (0, 1, 0, 0) + (0, 0, 1, 0) + (0, 0, 1, 0), \end{aligned}$$

where the summands are the contributions of the highest dimensional components of the bigraded ring T , see Proposition 1. The contribution 3 in the sixth vector is a movable contribution to $c_1(I)$.

The software Germenes [11] gives the following description of the Newton polyhedron $\Gamma(I)$. The compact faces of $\Gamma(I)$ are the 6 vertices

$v_1 = (1, 4, 5)$, $v_2 = (2, 5, 2)$, $v_3 = (1, 5, 3)$, $v_4 = (5, 1, 2)$, $v_5 = (2, 1, 5)$, $v_6 = (5, 2, 1)$, the 9 line segments v_4v_6 , v_2v_6 , v_5v_6 , v_4v_5 , v_3v_6 , v_3v_2 , v_3v_5 , v_1v_5 , v_1v_3 and the 4 triangles (bounded facets) $v_4v_5v_6$, $v_2v_3v_6$, $v_3v_5v_6$, $v_1v_3v_5$. There are 7 unbounded facets:

$$\begin{aligned} F_1 &= v_1v_3 + \mathbb{R}_{\geq 0} e_2 + \mathbb{R}_{\geq 0} e_3, & F_2 &= v_4v_5 + \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0} e_3, \\ F_3 &= v_6 + \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0} e_2, & F_4 &= v_1v_5 + \mathbb{R}_{\geq 0} e_3, \\ F_5 &= v_2v_3 + \mathbb{R}_{\geq 0} e_2, & F_6 &= v_2v_6 + \mathbb{R}_{\geq 0} e_2, & F_7 &= v_4v_6 + \mathbb{R}_{\geq 0} e_1. \end{aligned}$$

From the set of bounded facets $\mathcal{F}(2) = \{v_4v_5v_6, v_2v_3v_6, v_3v_5v_6, v_1v_3v_5\}$ we get

$$\begin{aligned} c_0(I) &= \text{Vol}(\text{conv}(0, v_4, v_5, v_6)) + \text{Vol}(\text{conv}(0, v_2, v_3, v_6)) + \\ &\quad + \text{Vol}(\text{conv}(0, v_3, v_5, v_6)) + \text{Vol}(\text{conv}(0, v_1, v_3, v_5)) = \\ &= \begin{vmatrix} 5 & 1 & 2 \\ 5 & 2 & 1 \\ 2 & 1 & 5 \end{vmatrix} + \begin{vmatrix} 2 & 5 & 2 \\ 1 & 5 & 3 \\ 5 & 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 5 & 3 \\ 2 & 1 & 5 \\ 5 & 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 4 & 5 \\ 2 & 1 & 5 \\ 1 & 5 & 3 \end{vmatrix} = \\ &= 24 + 22 + 103 + 19 = 168. \end{aligned}$$

We have $\mathcal{F}(1) = \{F_1, F_2, F_4, F_5, F_6, F_7\}$ and

$$\begin{aligned} \text{Vol}(v_1v_3, F_1) &= \min \left\{ \begin{vmatrix} 1 & 4 \\ 1 & 5 \end{vmatrix}, \begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} \right\} = \min\{1, 2\} = 1, \\ \text{Vol}(v_4v_5, F_2) &= \min \left\{ \begin{vmatrix} 5 & 1 \\ 2 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 1 & 5 \end{vmatrix} \right\} = \min\{3, 3\} = 3 \end{aligned}$$

(here the minimum is attained twice, that is, by two different projections which indicates a movable contribution),

$$\begin{aligned} \text{Vol}(v_1v_5, F_4) &= \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = 7, & \text{Vol}(v_2v_3, F_5) &= \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 4, \\ \text{Vol}(v_2v_6, F_6) &= \begin{vmatrix} 5 & 1 \\ 2 & 2 \end{vmatrix} = 8, & \text{Vol}(v_4v_6, F_7) &= \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3, \end{aligned}$$

hence

$$\begin{aligned} c_1(I) &= \text{Vol}(v_1v_3, F_1) + \text{Vol}(v_4v_5, F_2) + \text{Vol}(v_1v_5, F_4) + \\ &\quad + \text{Vol}(v_2v_3, F_5) + \text{Vol}(v_2v_6, F_6) + \text{Vol}(v_4v_6, F_7) = \\ &= 1 + 3 + 7 + 4 + 8 + 3 = 26. \end{aligned}$$

We observe that the compact 1-dimensional faces v_5v_6 , v_3v_6 , v_3v_5 , that is, the edges of the big triangle $v_3v_5v_6$, do not contribute to $c_1(I)$

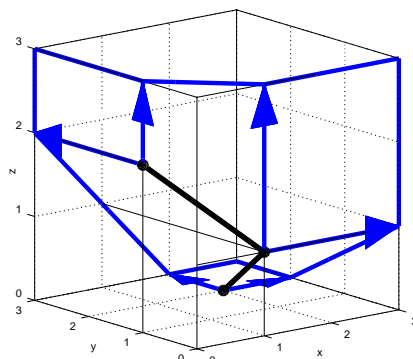


FIGURE 7. The triangle defined by 3 affinely independent vertices is not a compact facet.

since they lie on no 1-unbounded facets. Moreover, as in the previous example, there is a movable contribution, namely $\text{Vol}(v_4v_5, F_2) = 3$.

The set of 2-unbounded facets is $\mathcal{F}(0) = \{F_1, F_2, F_3\}$, and we have $\text{Vol}(v_1, F_1) = 1$, $\text{Vol}(v_3, F_1) = 1$, $\text{Vol}(v_4, F_2) = 1$, $\text{Vol}(v_5, F_2) = 1$, $\text{Vol}(v_6, F_3) = 1$, hence

$$c_2(I) = \min\{\text{Vol}(v_1, F_1), \text{Vol}(v_3, F_1)\} + \min\{\text{Vol}(v_4, F_2), \text{Vol}(v_5, F_2)\} + \text{Vol}(v_6, F_3) = 1 + 1 + 1 = 3.$$

Example 5 (Figure 7). Let $I = (xz, x^2y^2, yz^2) \subset K[x, y, z]$. By a computer computation we have

$$\begin{aligned} c(I) &= (c_0(I), c_1(I), c_2(I), c_3(I)) = (0, 7, 0, 0) = \\ &= 2 \cdot (0, 1, 0, 0) + 2 \cdot (0, 1, 0, 0) + 2 \cdot (0, 1, 0, 0) + (0, 1, 0, 0), \end{aligned}$$

where the summands are the contributions of the highest dimensional components of the bigraded ring T , see Proposition 1.

A computation with the program Germenes [11] shows the compact faces of $\Gamma(I)$ are the vertices $v_1 = (1, 0, 1)$, $v_2 = (2, 2, 0)$, $v_3 = (0, 1, 2)$ and the line segments v_1v_2 , v_1v_3 . There are no compact or bounded facets, but 6 unbounded facets:

$$\begin{aligned} F_1 &= v_3 + \mathbb{R}_{\geq 0} e_2 + \mathbb{R}_{\geq 0} e_3, & F_2 &= v_1 + \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0} e_3, \\ F_3 &= v_2 + \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0} e_2, & F_4 &= v_1v_2 + \mathbb{R}_{\geq 0} e_1, \\ F_5 &= v_1v_3 + \mathbb{R}_{\geq 0} e_3, & F_6 &= v_1v_2v_3 + \mathbb{R}_{\geq 0} e_2. \end{aligned}$$

We observe that $\mathcal{F}(2) = \emptyset$, hence

$$c_0(I) = 0 \neq \text{Vol}_3(\text{conv}(\{0, v_1, v_2, v_3\})) = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 6.$$

This means that v_1, v_2, v_3 are affinely independent, but the local Segre class is zero and not equal to the normalized volume of the simplex generated by the origin and v_1, v_2, v_3 as claimed in [7, 4.2].

The set of 1-unbounded facets which contain a compact 1-dimensional face is $\mathcal{F}(1) = \{F_4, F_5, F_6\}$, and we have

$$\begin{aligned} \text{Vol}(v_1v_2, F_4) &= \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = 2, & \text{Vol}(v_1v_2, F_6) &= \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = 2, \\ \text{Vol}(v_1v_3, F_5) &= \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2, & \text{Vol}(v_1v_3, F_6) &= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, \end{aligned}$$

hence

$$\begin{aligned} c_1(I) &= \text{Vol}(v_1v_2, F_4) + \text{Vol}(v_1v_2, F_6) + \text{Vol}(v_1v_3, F_5) + \text{Vol}(v_1v_3, F_6) \\ &= 2 + 2 + 2 + 1 = 7. \end{aligned}$$

The set of 2-unbounded facets is $\mathcal{F}(0) = \{F_1, F_2, F_3\}$ and we have

$$\text{Vol}(v_3, F_1) = 0, \quad \text{Vol}(v_1, F_2) = 0, \quad \text{Vol}(v_2, F_3) = 0,$$

hence $c_2(I) = 0$.

In the following example there is only one 1-dimensional compact face, but it lies on three 2-unbounded facets.

Example 6. Let $d = 4$, $I = (x_1^3x_2x_3x_4, x_1x_2x_3x_4^2)$. By a computer computation we have

$$\begin{aligned} c(I) &= (c_0(I), c_1(I), c_2(I), c_3(I), c_4(I)) = (0, 0, 7, 4, 0) = \\ &= 5 \cdot (0, 0, 1, 0, 0) + (0, 0, 1, 0, 0) + (0, 0, 1, 0, 0) + (0, 0, 0, 1, 0) + \\ &\quad + (0, 0, 0, 1, 0) + (0, 0, 0, 1, 0) + (0, 0, 0, 1, 0), \end{aligned}$$

where the summands are the contributions of the highest dimensional components of the bigraded ring T , see Proposition 1.

The compact faces of the Newton polyhedron $\Gamma(I)$ are the vertices $v_1 = (3, 1, 1, 1)$, $v_2 = (1, 1, 1, 2)$ and the line segment v_1v_2 . There are

no compact facets, but 5 unbounded facets:

$$\begin{aligned} F_1 &= v_2 + \mathbb{R}_{\geq 0} e_2 + \mathbb{R}_{\geq 0} e_3 + \mathbb{R}_{\geq 0} e_4, \\ F_2 &= v_1 v_2 + \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0} e_3 + \mathbb{R}_{\geq 0} e_4, \\ F_3 &= v_1 v_2 + \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0} e_2 + \mathbb{R}_{\geq 0} e_4, \\ F_4 &= v_1 + \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0} e_2 + \mathbb{R}_{\geq 0} e_3, \\ F_5 &= v_1 v_2 + \mathbb{R}_{\geq 0} e_2 + \mathbb{R}_{\geq 0} e_3. \end{aligned}$$

Obviously $\mathcal{F}(3) = \mathcal{F}(2) = \emptyset$, hence $c_0(I) = c_1(I) = 0$. We have $\mathcal{F}(1) = \{F_2, F_3, F_5\}$ and

$$\begin{aligned} \text{Vol}(v_1 v_2, F_2) &= \min\left\{\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}\right\} = \min\{2, 1\} = 1, \\ \text{Vol}(v_1 v_2, F_3) &= \min\left\{\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}\right\} = \min\{2, 1\} = 1, \end{aligned}$$

$$\text{Vol}(v_1 v_2, F_5) = \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 5,$$

We observe that in the computations of $\text{Vol}(v_1 v_2, F_2)$ and $\text{Vol}(v_1 v_2, F_3)$ the projection of the line segment $v_1 v_2$ on the $\{x_2, x_3\}$ -plane gives the point $(1, 1)$ and must not be considered. We obtain $c_2(I) = \text{Vol}(v_1 v_2, F_2) + \text{Vol}(v_1 v_2, F_3) + \text{Vol}(v_1 v_2, F_5) = 1 + 1 + 5 = 7$.

The set of 3-unbounded facets is $\mathcal{F}(0) = \{F_1, F_2, F_3, F_4\}$, and we have

$$\begin{aligned} \text{Vol}(v_2, F_1) &= 1, & \text{Vol}(v_1, F_2) &= 1, & \text{Vol}(v_2, F_2) &= 1, \\ \text{Vol}(v_1, F_3) &= 1, & \text{Vol}(v_2, F_3) &= 1, & \text{Vol}(v_1, F_4) &= 1, \end{aligned}$$

hence

$$\begin{aligned} c_3(I) &= \text{Vol}(v_2, F_1) + \min\{\text{Vol}(v_1, F_2), \text{Vol}(v_2, F_2)\} + \\ &\quad + \min\{\text{Vol}(v_1, F_3), \text{Vol}(v_2, F_3)\} + \text{Vol}(v_1, F_4) = \\ &= 1 + 1 + 1 + 1 = 4. \end{aligned}$$

We now present an example in dimension 5 in which several compact 1-dimensional faces lie on the same 3-unbounded facet.

Example 7. Let $d = 5$, $I = (x_1^3 x_2 x_3 x_4 x_5, x_1 x_2^2 x_3 x_4 x_5, x_1 x_2 x_3 x_4 x_5^5)$. By a computer computation we have

$$\begin{aligned} c(I) &= (c_0(I), c_1(I), c_2(I), c_3(I), c_4(I), c_5(I)) = (0, 0, 26, 6, 5, 0) = \\ &= 22 \cdot (0, 0, 1, 0, 0, 0) + 2 \cdot (0, 0, 1, 0, 0, 0) + 2 \cdot (0, 0, 1, 0, 0, 0) + \\ &\quad + 2 \cdot (0, 0, 0, 1, 0, 0) + (0, 0, 0, 1, 0, 0) + (0, 0, 0, 1, 0, 0) + \\ &\quad + (0, 0, 0, 1, 0, 0) + (0, 0, 0, 1, 0, 0) + (0, 0, 0, 0, 1, 0) + \\ &\quad + (0, 0, 0, 0, 1, 0) + (0, 0, 0, 0, 1, 0) + (0, 0, 0, 0, 1, 0) + \\ &\quad + (0, 0, 0, 0, 1, 0), \end{aligned}$$

where the summands are the contributions of the highest dimensional components of the bigraded ring T , see Proposition 1.

The software Germenes [11] shows that the compact faces of $\Gamma(I)$ are the vertices $v_1 = (3, 1, 1, 1, 1)$, $v_2 = (1, 2, 1, 1, 1)$, $v_3 = (1, 1, 1, 1, 5)$, the line segments $v_1 v_2$, $v_1 v_3$, $v_2 v_3$ and the triangle $v_1 v_2 v_3$. The facets, all of them unbounded, are:

$$\begin{aligned} F_1 &= v_2 v_3 + \mathbb{R}_{\geq 0} e_2 + \mathbb{R}_{\geq 0} e_3 + \mathbb{R}_{\geq 0} e_4 + \mathbb{R}_{\geq 0} e_5, \\ F_2 &= v_1 v_3 + \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0} e_3 + \mathbb{R}_{\geq 0} e_4 + \mathbb{R}_{\geq 0} e_5, \\ F_3 &= v_1 v_2 v_3 + \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0} e_2 + \mathbb{R}_{\geq 0} e_4 + \mathbb{R}_{\geq 0} e_5, \\ F_4 &= v_1 v_2 v_3 + \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0} e_2 + \mathbb{R}_{\geq 0} e_3 + \mathbb{R}_{\geq 0} e_5, \\ F_5 &= v_1 v_2 + \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0} e_2 + \mathbb{R}_{\geq 0} e_3 + \mathbb{R}_{\geq 0} e_4, \\ F_6 &= v_1 v_2 v_3 + \mathbb{R}_{\geq 0} e_3 + \mathbb{R}_{\geq 0} e_4. \end{aligned}$$

Obviously $\mathcal{F}(4) = \mathcal{F}(3) = \emptyset$, hence $c_0(I) = c_1(I) = 0$.

We have $\mathcal{F}(2) = \{F_3, F_4, F_6\}$ and

$$\begin{aligned} \text{Vol}(v_1 v_2 v_3, F_3) &= \text{Vol}(v_1 v_2 v_3, F_4) = \min \left\{ \left| \begin{array}{ccc} 3 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{array} \right|, \left| \begin{array}{ccc} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 5 \end{array} \right| \right\} = \\ &= \min\{2, 4\} = 2, \end{aligned}$$

$$\text{Vol}(v_1 v_2 v_3, F_6) = \left| \begin{array}{ccc} 3 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 5 \end{array} \right| = 22.$$

Observe that in the computations of $\text{Vol}(v_1 v_2 v_3, F_3)$ and $\text{Vol}(v_1 v_2 v_3, F_4)$ four projections of the compact triangle $v_1 v_2 v_3$ give a line segment and

must not be considered. Summing up we get

$$\begin{aligned} c_2(I) &= \text{Vol}(v_1v_2v_3, F_3) + \text{Vol}(v_1v_2v_3, F_4) + \text{Vol}(v_1v_2v_3, F_6) \\ &= 2 + 2 + 22 = 26. \end{aligned}$$

The set of 3-unbounded facets containing 1-dimensional compact faces is $\mathcal{F}(1) = \{F_1, F_2, F_3, F_4, F_5\}$ and we have

$$\begin{aligned} \text{Vol}(v_2v_3, F_1) &= \min \left\{ \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} \right\} = \min\{1, 4\} = 1, \\ \text{Vol}(v_1v_3, F_2) &= \min \left\{ \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} \right\} = \min\{2, 4\} = 2, \\ \text{Vol}(v_1v_2, F_3) &= \text{Vol}(v_1v_2, F_4) = \min \left\{ \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \right\} = \min\{2, 1\} = 1, \\ \text{Vol}(v_1v_3, F_3) &= \text{Vol}(v_1v_3, F_4) = \min \left\{ \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} \right\} = \min\{2, 4\} = 2, \\ \text{Vol}(v_2v_3, F_3) &= \text{Vol}(v_2v_3, F_4) = \min \left\{ \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} \right\} = \min\{1, 4\} = 1, \\ \text{Vol}(v_1v_2, F_5) &= \min \left\{ \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \right\} = \min\{2, 1\} = 1, \end{aligned}$$

hence

$$\begin{aligned} c_3(I) &= \text{Vol}(v_2v_3, F_1) + \text{Vol}(v_1v_3, F_2) + \\ &\quad + \min\{\text{Vol}(v_1v_2, F_3), \text{Vol}(v_1v_3, F_3), \text{Vol}(v_2v_3, F_3)\} + \\ &\quad + \min\{\text{Vol}(v_1v_2, F_4), \text{Vol}(v_1v_3, F_4), \text{Vol}(v_2v_3, F_4)\} + \\ &\quad + \text{Vol}(v_1v_2, F_5) = 1 + 2 + 1 + 1 + 1 = 6. \end{aligned}$$

From the list of the facets we see that there are five 4-unbounded facets, precisely $\mathcal{F}(0) = \{F_1, F_2, F_3, F_4, F_5\}$ and we have

$$\begin{aligned} \text{Vol}(v_2, F_1) &= 1, & \text{Vol}(v_3, F_1) &= 1, & \text{Vol}(v_1, F_2) &= 1, & \text{Vol}(v_3, F_2) &= 1, \\ \text{Vol}(v_1, F_3) &= 1, & \text{Vol}(v_2, F_3) &= 1, & \text{Vol}(v_3, F_3) &= 1, & \text{Vol}(v_1, F_4) &= 1, \\ \text{Vol}(v_2, F_4) &= 1, & \text{Vol}(v_3, F_4) &= 1, & \text{Vol}(v_1, F_5) &= 1, & \text{Vol}(v_2, F_5) &= 1, \end{aligned}$$

$$\begin{aligned} c_4(I) &= \min\{\text{Vol}(v_2, F_1), \text{Vol}(v_3, F_1)\} + \min\{\text{Vol}(v_1, F_2), \text{Vol}(v_3, F_2)\} + \\ &\quad + \min\{\text{Vol}(v_1, F_3), \text{Vol}(v_2, F_3), \text{Vol}(v_3, F_3)\} + \\ &\quad + \min\{\text{Vol}(v_1, F_4), \text{Vol}(v_2, F_4), \text{Vol}(v_3, F_4)\} + \\ &\quad + \min\{\text{Vol}(v_1, F_5), \text{Vol}(v_2, F_5)\} = 1 + 1 + 1 + 1 + 1 = 5. \end{aligned}$$

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