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The exponential of nilpotent supergroups in the theory of Harish-Chandra representations ^{*}

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Abstract. In this paper we discuss the exponential map in the case of nilpotent superalgebras. This provides global coordinates for nilpotent analytic supergroups, which are useful in the applications.

Keywords: Supergeometry · Lie Theory · Representation Theory.

1 Introduction

In supersymmetry, originally introduced by Berezin (see [3] and also [4], [21]), the concept of a Lie supergroup is central and it is indeed the search for extra symmetries of physical systems that led to the discovery of supersymmetry and later on of supergeometry (see [25], [26] and the comprehensive treatments [5], [24], [16] and the references within).

First, the notion of Lie superalgebra was introduced and only later on, there was a formalization of the notion of Lie and algebraic supergroup. The exponential map plays an eminent role and was originally introduced and studied by Koszul in [22] and later on the theory was further developed in [18].

In the algebraic setting, the recipe presented in [13], [14], [15] to construct an algebraic supergroup starting from a Lie superalgebra, uses in an implicit way the notion of exponential (see also [22], [6]).

In this paper we want to restrict our attention to a special case, which is however important in the applications, namely the case of a nilpotent analytic subsupergroup of an analytic complex matrix supergroup. We shall employ freely the language of Super Harish-Chandra pair (SCHP) introduced in [21] and developed by Koszul in his fundamental work [22].

In Sec. 2, we present the construction of the exponential, while in the subsequent sections we give applications, important in the study of the Harish-Chandra representations of supergroups (see [19], [7], [8]).

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2 The exponential map

Let \mathfrak{g} be a complex contragredient Lie superalgebra, $\mathfrak{g} \neq A(n, n)$, $\mathfrak{g}_1 \neq 0$; hence \mathfrak{g} will be one in the following list of Lie superalgebras (see [20] Prop. 1.1):

$$A(m, n) \text{ with } m \neq n, B(m, n), C(n), D(m, n), D(2, 1; \alpha), F(4), G(3) \quad (1)$$

\mathfrak{g}_0 is either semisimple or with a one-dimensional center. Hence, by the ordinary theory, we know that the simply connected Lie group \tilde{G} , $\mathfrak{g}_0 = \text{Lie}(\tilde{G})$ is a matrix complex analytic and algebraic group. Then, the super Harish-Chandra pair SHCP $G = (\tilde{G}, \mathfrak{g})$ (see [5] Ch. 11 and [12]) can be viewed either as a complex analytic or algebraic supergroup, via the theory of SHCP that establishes an equivalence of categories between the categories of analytic supergroups and SHCP (see [6]). We shall take the first point of view and regard G as a complex analytic matrix supergroup, but later on we will also view G as a pair $G = (\tilde{G}, \mathcal{O}_G)$, \mathcal{O}_G a sheaf of superalgebras, (see [6]).

Fix \mathfrak{h} a CSA of \mathfrak{g} and fix P a positive system. Let us define \mathfrak{b}^\pm and \mathfrak{n}^\pm the *Borel and nilpotent subsuperalgebras*:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha, \quad \mathfrak{b}^\pm := \mathfrak{h} \oplus \sum_{\alpha \in \pm P} \mathfrak{g}_\alpha, \quad \mathfrak{n}^\pm := \sum_{\alpha \in \pm P} \mathfrak{g}_\alpha. \quad (2)$$

We will call B^\pm *Borel subsupergroup* and N^\pm *unipotent subsupergroup*, their corresponding analytic Lie supergroups in G . In particular, B^\pm and N^\pm are connected and are algebraic subsupergroups of G . Let A be the torus with $\text{Lie}(A) = \mathfrak{h}$.

We want to define the exponential diffeomorphism: $\exp: \mathfrak{n}^- \rightarrow N^-$ for the analytic supergroup N^- . To ease the notation we shall drop the index “-”.

Our purpose in the construction of the exponential diffeomorphism is to obtain *global coordinates* on the nilpotent supergroup N ; such coordinates are going to be essential for some important applications, (see [7], [8]).

We start with some general remarks on the functor of the Λ -points, we invite the reader to consult [1] and [2] for the complete details.

Let M be a supermanifold. Instead of looking at the whole functor of points $M(\cdot): (\text{smflds}) \rightarrow (\text{sets})$, it is sometimes convenient to restrict the functor of points from the category (smflds) to the subcategory (spts) consisting of just the *superpoints*: $k^{0|n}$. These are the supermanifolds $(\{*\}, \Lambda^n)$, where Λ^n denotes the Grassmann algebra in n generators over k . In this approach the set $M(k^{0|n})$ can be endowed with the structure of an ordinary manifold, but with some peculiarities. The tangent space at a point is a Λ_0^n -module and the change of coordinates induced by a change of coordinates in M must have Λ_0^n -linear

differential. These are called Λ_0 -manifolds and we denote with $(\Lambda_0\text{mflds})$ the corresponding category. The functor

$$(\text{spts}) \rightarrow (\Lambda_0\text{mflds}) \quad k^{0|n} \mapsto M(k^{0|n}) \quad (3)$$

is a full and faithful embedding (see [1] Sec. 4, Theorem 4.5). We notice that, if V is a vector superspace, we have the identification $V(k^{0|n}) \simeq (V \otimes \Lambda^n)_0$ and the previous result is known as the *even rules principle* (see also [10]).

Proposition 1. *1. If G is a complex matrix supergroup as above, the Λ_0 -manifold $G(k^{0|n})$ is a group object in the category $(\Lambda_0\text{mflds})$ and in particular it is an analytic Lie group. Similarly $\mathfrak{g}(k^{0|n})$ is an ordinary Lie algebra.*

2. The ordinary exponential $\exp_{k^{0|n}} : \mathfrak{g}(k^{0|n}) \rightarrow G(k^{0|n})$ is a morphism of Λ_0 manifolds.

Proof. (1) is a simple check. As for (2), one can readily see that the differential of this map is Λ_0 -linear and the correspondence $\mathfrak{g}(k^{0|n}) \rightarrow G(k^{0|n})$ is functorial.

Since the functoriality property of \exp in Prop. 1 (refer also to [1,2] for a thorough treatment of Λ -points), we can immediately define the exponential morphism for an analytic supergroup G .

Definition 1. *Let G and \mathfrak{g} as above. We define the exponential map as the morphism of analytic supermanifolds given on the Λ -points as the ordinary exponential as in Prop 1 (2).*

Proposition 2. *Let N be a nilpotent supergroup as above. Then the exponential morphism $\exp : \mathfrak{n} \rightarrow N$ is a global superdiffeomorphism.*

Proof. In case N is a unipotent Lie supergroup as in (2), each $G(\Lambda)$ is also a unipotent Lie group and, by a classical result, each $\exp_{k^{0|n}}$ is a diffeomorphism. Hence \exp is a superdiffeomorphism.

3 The nilpotent subsupergroup N^-

In this section we give some applications of the global coordinates we have built in the previous section. Let $\Gamma = N^-AN^+$ denote the *big cell*; it is the open analytic subsupermanifold $\Gamma = (\widetilde{N^-AN^+}, \mathcal{O}_G|_{\widetilde{N^-AN^+}})$ of the analytic supergroup $G = (\widetilde{G}, \mathcal{O}_G)$. We need some preliminary propositions.

Proposition 3. *N^- is a section for $\Gamma \rightarrow \Gamma/B^+$, the left action of A reads:*

$$A \times \Gamma/B^+ \longrightarrow \Gamma/B^+, \quad (h, nB^+(A)) \mapsto hnh^{-1}B^+(A),$$

where $n \in N^\pm(T)$, $h \in A(T)$, $T \in (\text{smflds})_{\mathbb{C}}$ ($(\text{smflds})_{\mathbb{C}}$ denoting the category of analytic supermanifolds).

Proof. Since the big cell $\Gamma \subset G$ is right B^+ -invariant and open, and the canonical projection $p: G \rightarrow G/B^+$ is a submersion, we can define the open subsupermanifold of G/B^+ :

$$\Gamma/B^+ := (\widetilde{\Gamma/B^+}, \mathcal{O}_{G/B^+}|_{\widetilde{\Gamma/B^+}})$$

We have a N^- equivariant diffeomorphism $N^- \rightarrow \Gamma/B^+$, $n^- \mapsto n^-B^+(T)$, $n^- \in N^-(T)$, $T \in (\text{smflds})_{\mathbb{C}}$. In fact, by the ordinary theory we have a diffeomorphisms of the underlying differentiable manifolds and the differential at the identity is an isomorphism: $\mathfrak{n}^- \cong \mathfrak{g}/\mathfrak{b}^+$.

Clearly $p^{-1}(\Gamma/B^+) = \Gamma$. We are going to construct a section $s: \Gamma/B^+ \rightarrow \Gamma$. The local splitting $\gamma: N^- \times B^+ \rightarrow \Gamma$ is an holomorphic morphism such that $\gamma^*\mathcal{O}_{\Gamma/B^+} = \mathcal{O}_{N^-} \otimes 1$. Hence we have an isomorphism $N^- \rightarrow \Gamma/B^+$ given by the composition of the ‘‘canonical’’ embedding $i: N^- \hookrightarrow N^- \times B^+$ with γ and p (which is essentially the same as considering $p \circ \gamma|_{N^- \times \{\epsilon\}}$). Its inverse is the required section.

Proposition 4. *The (ordinary) torus A normalizes N^{\pm} .*

Proof. We give the proof for $N^+ = N$. We want to prove that the conjugation

$$\text{conj}(a) : G \rightarrow G \quad \text{conj}(a) = \ell_{a^{-1}} \circ r_a, \quad a \in \widetilde{A}$$

stabilizes N . Since N is connected and the exponential map $\exp: \mathfrak{n} \rightarrow N$ is surjective it is enough to prove that $(d\text{conj}(a))_1(\mathfrak{n}) \subseteq \mathfrak{n}$. We know from the infinitesimal theory that $\text{ad}(\mathfrak{h})(\mathfrak{n}) = \mathfrak{n}$. Hence, we have

$$\text{Ad}(e^{tX})Y = e^{t\text{ad}X}(Y) \quad \forall X \in \mathfrak{h}, Y \in \mathfrak{n}$$

so that $\text{Ad}(e^{tX})\mathfrak{n} = \mathfrak{n}$. Since the exponential map of an abelian connected Lie group is surjective we have that $\text{Ad}(A)\mathfrak{n} = \mathfrak{n}$.

By the simply connectedness of \widetilde{N} , we get a map $\widetilde{\text{conj}}(a) : \widetilde{N} \rightarrow \widetilde{N}$. It is easy to check that the pair

$$\text{Ad}(a) : \mathfrak{n} \rightarrow \mathfrak{n} \quad \widetilde{\text{conj}}(a) : \widetilde{N} \rightarrow \widetilde{N}$$

is a SHCP morphism: $(\widetilde{\text{conj}}(a), \text{Ad}(a)) : (\widetilde{N}, \mathfrak{n}) \rightarrow (\widetilde{G}, \mathfrak{g})$, so that, by the equivalence of categories between analytic SHCP and analytic supergroups, we have a morphism of super Lie groups $N \rightarrow G$. Since its differential coincides with the differential of $\text{conj}(a) : N \rightarrow G$ and the reduced maps are the same, the two morphisms coincide, hence $\text{conj}(a)N = N$.

Let us fix a character $\chi : A \rightarrow \mathbb{C}^\times$ of the ordinary torus, that we can trivially extend to a character (still denoted by χ) of the supergroup B^+ . Define:

$$L^\chi(\Gamma) := \{f \in \mathcal{O}_G(\Gamma) \mid f(gb) = \chi(b)^{-1}f(g)\}$$

We can geometrically view this superspace as the superspace of sections of the line bundle uniquely associated with χ . This superspace is the key for the construction of the infinite dimensional representations of real forms of the analytic supergroup G (see [7], [8] for more details). The actions that we are going to describe are absolutely essential for the realization of such representations.

Since A acts on N^- by conjugation (see Prop. 4), we have a global action of A on Γ defined as:

$$a \cdot (n^- b^+) = (a n^- a^{-1}) b^+, \quad a \in \tilde{A}, n^- \in N^-(T), b^+ \in B^+(T).$$

Since A also acts on B^+ by left translation, we can define the left action of A on Γ as:

$$a \cdot (n^- b^+) = (a n^- a^{-1}) a \cdot b^+.$$

Both actions commute with right translations by B^+ and hence define representations of A on $L^\chi(\Gamma)$

$$i, \ell: A \times L^\chi(\Gamma) \rightarrow L^\chi(\Gamma)$$

where:

$$i_a(f)(n^- b^+) = f((a^{-1} n^- a) b^+), \quad \ell_a(f)(n^- b^+) = f((a^{-1} n^- a) a^{-1} b^+) \quad (4)$$

and $a \in \tilde{A}$, $n^- \in N^-(T)$, $b^+ \in B^+(T)$, $f \in L^\chi(\Gamma)$.

Let t_α denote the global homogeneous exponential coordinates on N^- obtained by Prop. 2.

Lemma 1. *Let the notation be as above. Then*

1. $\ell_a f = \chi(a)(i_a f)$
2. $i_a t_\alpha = \chi_\alpha(a) t_\alpha \quad \forall a \in \tilde{A}$

where χ_α is the character of the maximal torus A obtained by exponentiating the root $\alpha \in \mathfrak{h}^*$.

Proof. (1) follows immediately from the definitions. For (2) let $n = \exp(\sum_{\beta \in P} y_\beta X_{-\beta})$ be an element in N^- , then the result comes from the following formal calculation in the exponential global coordinates:

$$\begin{aligned} t_\alpha(a^{-1} n a) &= t_\alpha\left(\exp\left(\sum_{\beta \in P} y_\beta \text{Ad}(a) X_{-\beta}\right)\right) = t_\alpha\left(\exp\left(\sum_{\beta \in P} y_\beta \chi_\beta(a) X_{-\beta}\right)\right) \\ &= \chi_\alpha(a) t_\alpha(n), \quad a \in \tilde{A}, y_\beta \in \mathbb{C} \end{aligned}$$

4 The action of $\mathcal{U}(\mathfrak{g})$ and G on $L^\chi(\Gamma)$

Now we want to use the theory developed so far and extend the action of the maximal torus $A \subset G$ to an action of the whole group on $L^\chi(\Gamma)$. We start by defining the natural action of $\mathcal{U}(\mathfrak{g})$ on the holomorphic functions on any neighbourhood W of the identity of the supergroup G .

Definition 2. Let $W \subset G$ be an open neighbourhood of the identity 1_G in G . There are two well defined actions of \mathfrak{g} , hence of $\mathcal{U}(\mathfrak{g})$, on $\mathcal{O}_G(W)$, that read as follows:

$$\ell(X)f = (-X \otimes 1)\mu^*(f), \quad \partial(X)f = (1 \otimes X)\mu^*(f), \quad X \in \mathfrak{g}$$

The actions ℓ and ∂ commute with each other. Moreover, if \widetilde{U} is open in $\widetilde{G/B^+}$, then ℓ is a well defined action on $L^\times(\widetilde{U})$.

We now want to show that the natural action ℓ of $\mathcal{U}(\mathfrak{g})$ on $L^\times(N^-B^+)$ preserves the polynomial sections on $\widetilde{N^-}$. For this we need some preliminary notation. Since $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{b}^+$, if we fix bases of \mathfrak{n}^- and \mathfrak{b}^+ , by the PBW (Poincaré Birkhoff Witt) theorem any $X \in \mathcal{U}(\mathfrak{g})$ can be written as

$$X = \sum_{I,J} c_{IJ}(X)B_I N_J, \quad B_I \in \mathcal{U}(\mathfrak{b}^+), N_J \in \mathcal{U}(\mathfrak{n}^-) \quad (5)$$

Lemma 2. Let $\phi \in \mathcal{O}_G(N^-B^+)$. In the SHCP notation, ϕ is in $L^\times(N^-B^+)$ if and only if

$$\phi(X)(nb) = \widetilde{\chi}(b)^{-1} \sum_{IJ} c_{IJ}(b.X)\lambda(\overline{B}_I)\phi(N_J)(n), \quad X \in \mathcal{U}(\mathfrak{g}), \quad \lambda = d\widetilde{\chi}$$

where $b.X$ is the adjoint action of $b \in \widetilde{B^+}$ on $\mathcal{U}(\mathfrak{g})$ and as usual \overline{B}_I denotes the antipode of B_I in the Hopf superalgebra $\mathcal{U}(\mathfrak{g})$.

Proof. By the very definition we have $\phi \in L^\times(N^-B^+)$ if

1. $r_b^*\phi = \widetilde{\chi}(b)^{-1}\phi$, $b \in \widetilde{B^+}$
2. $D_Y^L(\phi) = \lambda(\overline{Y})\phi$, $\lambda|_{\mathfrak{g}_0} = d\widetilde{\chi}$.

where as usual $\widetilde{\chi}$ denotes the reduced morphism. The result comes with a calculation.

Notice that once the lemma is established, if p is a polynomial in the global coordinates of N^- , we can define $p^\sim \in L^\times(N^-B^+)$ as:

$$p^\sim(X)(nb) = \widetilde{\chi}(b)^{-1} \sum_{IJ} c_{IJ}(b.X)\lambda(\overline{B^+}_I)p(N_J)(n)$$

Vice-versa we can recover p from p^\sim restricting to N^- . In the language of SHCP this amounts to two restrictions: we impose $b = 1$ and $X \in \mathcal{U}(\mathfrak{n}^-)$. We shall denote the set of such p^\sim with \mathcal{P}^\sim .

Proposition 5. The actions ℓ of $\mathcal{U}(\mathfrak{g})$ on $L^\times(\widetilde{U})$, $p^{-1}(\widetilde{U}) \subset \Gamma$ leave \mathcal{P}^\sim invariant.

Proof. We need to show that, given $Z \in \mathcal{U}(\mathfrak{g})$ and $X \in \mathcal{U}(\mathfrak{n}^-)$, $(D_Z^R p^\sim|_{N^-})(X)$ is a polynomial section. We have (see [5] Sec. 7.4):

$$(D_Z^R p^\sim)(X)(g) = (-1)^{|Z||p|} [p^\sim((g^{-1}.Z)X)](g)$$

Hence if $n \in N^-$, we have:

$$\begin{aligned} (D_Z^R p^\sim)(X)(n) &= (-1)^{|Z||p|} [p^\sim((n^{-1}.Z)X)](n) \\ &= (-1)^{|Z||p|} \sum_{IJ} c_{IJ}((n^{-1}.Z)X) [\lambda(\overline{B}_I) p^\sim(N_J)](n) \end{aligned}$$

where B_I and N_J are obtained as in (5) applied to $(n^{-1}.Z)X$. The last equality is true by Lemma 2.

Once this is established, we have the following result.

Theorem 1. *There is a non-singular $\mathcal{U}(\mathfrak{g})$ invariant pairing between \mathcal{P}^\sim and the Verma module V_λ :*

$$\langle \cdot, \cdot \rangle : \mathcal{P}^\sim \times \mathcal{U}(\mathfrak{g}) \longrightarrow \mathbb{C}, \quad \langle f, u \rangle := (-1)^{|u||f|} (\partial(u)f)(1_G)$$

Proof. In order for $\langle \cdot, \cdot \rangle$ to be a $\mathcal{U}(\mathfrak{g})$ invariant pairing, we need to verify:

$$\langle \ell(c)f, u \rangle = \langle f, (-1)^{|f||c|} c^T u \rangle, \quad c, u \in \mathcal{U}(\mathfrak{g}), f \in \mathcal{P}^\sim$$

where $(\cdot)^T$ denotes the antiautomorphism of $\mathcal{U}(\mathfrak{g})$ induced by $X \mapsto -X$ with $X \in \mathfrak{g}$. This is just a check.

Now let \mathfrak{g}_r be a real form of \mathfrak{g} and define the real supergroup $G_r = (\widetilde{G}_r, \mathfrak{g}_r)$, where \widetilde{G}_r is a real form of \widetilde{G} , $\text{Lie}(\widetilde{G}_r) = \mathfrak{g}_{r,0}$. Since $\mathfrak{g}_r + \mathfrak{b}^+ = \mathfrak{g}$ as real superalgebras (see [9], Iwasawa decomposition), we have that $S := G_r B^+$ is an open subsupermanifold of G .

Theorem 2. *Assume $L^\times(S) \neq 0$ modulo J the submodule generated by the odd part. Then $L^\times(S)$ contains an element ψ which is an analytic continuation of 1^\sim and*

$$\ell(\mathcal{U}(\mathfrak{g}))\psi = \mathcal{P}^\sim \cong \pi_{-\lambda}$$

where $\pi_{-\lambda}$ the irreducible representation with lowest weight $-\lambda$. Furthermore $L^\times(S)$ carries a G_r representation defined as:

$$\begin{cases} (g \cdot f) = l_{g^{-1}}^* f & g \in \widetilde{G}_r \\ X.f = D_{\overline{X}}^R f & X \in \mathfrak{g}_\mathbb{C} \end{cases}$$

where, as usual, \overline{X} is the antipode of $X \in \mathcal{U}(\mathfrak{g})$.

Proof. Direct check.

The closure of $\ell(\mathcal{U}(\mathfrak{g}))\psi$ in $L^\times(S)$, with a Fréchet superspace structure is a Harish-Chandra representation, with $\ell(\mathcal{U}(\mathfrak{g}))\psi$ as its K_r finite part, where K_r is the supergroup corresponding to the subalgebra \mathfrak{k}_r in the Cartan decomposition of \mathfrak{g}_r (see [9]). The proof of these facts is non trivial, we invite the reader to see [7], [8].

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