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# The exponential of nilpotent supergroups in the theory of Harish-Chandra representations *

C. Carmeli[1][0000−0003−1660−3677], R. Fioresi[2][0000−0003−4075−7641], and V. S. Varadarajan[3][]

[1] DIME, Università di Genova, Genova, Italy carmeli@dime.unige.it
[2] Dipartimento di Matematica, piazza Porta San Donato 5, University of Bologna, 40126 Bologna, Italy
rita.fioresi@unibo.it
[3] Department of Mathematics, UCLA, Los Angeles, 90095-1555 CA, USA

**Abstract.** In this paper we discuss the exponential map in the case of nilpotent superalgebras. This provides global coordinates for nilpotent analytic supergroups, which are useful in the applications.

**Keywords:** Supergeometry · Lie Theory · Representation Theory.

## 1 Introduction

In supersymmetry, originally introduced by Berezin (see [3] and also [4], [21]), the concept of a Lie supergroup is central and it is indeed the search for extra symmetries of physical systems that let to the discovery of supersymmetry and later on of supergeometry (see [25], [26] and the comprehensive treatments [5], [24], [16] and the references within).

First, the notion of Lie superalgebra was introduced and only later on, there was a formalization of the notion of Lie and algebraic supergroup. The exponential map plays an eminent role and was originally introduced and studied by Koszul in [22] and later on the theory was further developed in [18].

In the algebraic setting, the recipe presented in [13], [14], [15] to construct an algebraic supergroup starting from a Lie superalgebra, uses in a implicit way the notion of exponential (see also [22], [6]).

In this paper we want to restrict our attention to a special case, which is however important in the applications, namely the case of a nilpotent analytic subsupergroup of an analytic complex matrix supergroup. We shall employ freely the language of Super Harish-Chandra pair (SCHP) introduced in [21] and developed by Koszul in his fundamental work [22].

In Sec. 2, we present the construction of the exponential, while in the subsequent sections we give applications, important in the study of the Harish-Chandra representations of supergroups (see [19], [7], [8]).

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## 2 The exponential map

Let $\mathfrak{g}$ be a complex contragredient Lie superalgebra, $\mathfrak{g} \neq A(n,n)$, $\mathfrak{g}_1 \neq 0$; hence $\mathfrak{g}$ will be one in the following list of Lie superalgebras (see [20] Prop. 1.1):

$$A(m,n) \text{ with } m \neq n\,,\ B(m,n)\,,\ C(n)\,,\ D(m,n)\,,\ D(2,1;\alpha)\,,\ F(4)\,,\ G(3) \qquad (1)$$

$\mathfrak{g}_0$ is either semisimple or with a one-dimensional center. Hence, by the ordinary theory, we know that the simply connected Lie group $\widetilde{G}$, $\mathfrak{g}_0 = \mathrm{Lie}(\widetilde{G})$ is a matrix complex analytic and algebraic group. Then, the super Harish-Chandra pair SHCP $G = (\widetilde{G}, \mathfrak{g})$ (see [5] Ch. 11 and [12]) can be viewed either as a complex analytic or algebraic supergroup, via the theory of SHCP that establishes an equivalence of categories between the categories of analytic supergroups and SHCP (see [6]). We shall take the first point of view and regard $G$ as a complex analytic matrix supergroup, but later on we will also view $G$ as a pair $G = (\widetilde{G}, \mathcal{O}_G)$, $\mathcal{O}_G$ a sheaf of superalgebras, (see [6]).

Fix $\mathfrak{h}$ a CSA of $\mathfrak{g}$ and fix $P$ a positive system. Let us define $\mathfrak{b}^\pm$ and $\mathfrak{n}^\pm$ the *Borel and nilpotent subsuperalgebras*:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha, \qquad \mathfrak{b}^\pm := \mathfrak{h} \oplus \sum_{\alpha \in \pm P} \mathfrak{g}_\alpha, \qquad \mathfrak{n}^\pm := \sum_{\alpha \in \pm P} \mathfrak{g}_\alpha. \qquad (2)$$

We will call $B^\pm$ *Borel subsupergroup* and $N^\pm$ *unipotent subsupergroup*, their corresponding analytic Lie supergroups in $G$. In particular, $B^\pm$and $N^\pm$ are connected and are algebraic subsupergroups of $G$. Let $A$ be the torus with $\mathrm{Lie}(A) = \mathfrak{h}$.

We want to define the exponential diffeomorphism: $exp : \mathfrak{n}^- \longrightarrow N^-$ for the analytic supergroup $N^-$. To ease the notation we shall drop the index "$-$".

Our purpose in the construction of the exponential diffeomorphism is to obtain *global coordinates* on the nilpotent supergroup $N$; such coordinates are going to be essential for some important applications, (see [7], [8]).

We start with some general remarks on the functor of the $\Lambda$-points, we invite the reader to consult [1] and [2] for the complete details.

Let $M$ be a supermanifold. Instead of looking at the whole functor of points $M(\cdot)$: (smflds) $\to$ (sets), it is sometimes convenient to restrict the functor of points from the category (smflds) to the subcategory (spts) consisting of just the *superpoints*: $k^{0|n}$. These are the supermanifolds $(\{*\}, \Lambda^n)$, where $\Lambda^n$ denotes the Grassmann algebra in $n$ generators over $k$. In this approach the set $M(k^{0|n})$ can be endowed with the structure of an ordinary manifold, but with some peculiarities. The tangent space at a point is a $\Lambda^n_0$-module and the change of coordinates induced by a change of coordinates in $M$ must have $\Lambda^n_0$-linear differential. These are called $\Lambda_0$-manifolds and we denote with ($\Lambda_0$mflds) the corresponding category. The functor

$$(\text{spts}) \to (\Lambda_0\text{mflds}) \qquad k^{0|n} \mapsto M(k^{0|n}) \tag{3}$$

is a full and faithful embedding (see [1] Sec. 4, Theorem 4.5). We notice that, if $V$ is a vector superspace, we have the identification $V(k^{0|n}) \simeq (V \otimes \Lambda^n)_0$ and the previous result is known as the *even rules principle* (see also [10]).

**Proposition 1.** *1. If $G$ is a complex matrix supergroup as above, the $\Lambda_0$-manifold $G(k^{0|n})$ is a group object in the category* ($\Lambda_0$mflds) *and in particular it is an analytic Lie group. Similarly $\mathfrak{g}(k^{0|n})$ is an ordinary Lie algebra.*
*2. The ordinary exponential $\exp_{k^{0|n}} : \mathfrak{g}(k^{0|n}) \to G(k^{0|n})$ is a morphism of $\Lambda_0$ manifolds.*

*Proof.* (1) is a simple check. As for (2), one can readily see that the differential of this map is $\Lambda_0$-linear and the correspondence $\mathfrak{g}(k^{0|n}) \to G(k^{0|n})$ is functorial.

Since the functoriality property of exp in Prop. 1 (refer also to [1, 2] for a thorough treatment of $\Lambda$-points), we can immediately define the exponential morphism for an analytic supergroup $G$.

**Definition 1.** *Let $G$ and $\mathfrak{g}$ as above. We define the exponential map as the morphism of analytic supermanifolds given on the $\Lambda$-points as the ordinary exponential as in Prop 1 (2).*

**Proposition 2.** *Let $N$ be a nilpotent supergroup as above. Then the exponential morphism* $\exp\colon \mathfrak{n} \to N$ *is a global superdiffeomorphism.*

*Proof.* In case $N$ is a unipotent Lie supergroup as in (2), each $G(\Lambda)$ is also a unipotent Lie group and, by a classical result, each $\exp_{k^{0|n}}$ is a diffeomorphism. Hence exp is a superdiffeomorphism.

## 3 The nilpotent subsupergroup $N^-$

In this section we give some applications of the global coordinates we have built in the previous section. Let $\Gamma = N^-AN^+$ denote the *big cell*; it is the open analytic subsupermanifold $\Gamma = (\widetilde{N^-AN^+}, \mathcal{O}_G|_{\widetilde{N^-AN^+}})$ of the analytic supergroup $G = (\widetilde{G}, \mathcal{O}_G)$. We need some preliminary propositions.

**Proposition 3.** *$N^-$ is a section for $\Gamma \to \Gamma/B^+$, the left action of $A$ reads:*

$$A \times \Gamma/B^+ \longrightarrow \Gamma/B^+, \quad (h, nB^+(A)) \mapsto hnh^{-1}B^+(A),$$

*where $n \in N^{\pm}(T)$, $h \in A(T)$, $T \in (\text{smflds})_{\mathbb{C}}$ ($(\text{smflds})_{\mathbb{C}}$ denoting the category of analytic supermanifolds).*

*Proof.* Since the big cell $\Gamma \subset G$ is right $B^+$-invariant and open, and the canonical projection $p\colon G \to G/B^+$ is a submersion, we can define the open subsupermanifold of $G/B^+$:

$$\Gamma/B^+ \coloneqq (\widetilde{\Gamma/B^+}, \mathcal{O}_{G/B^+}|_{\widetilde{\Gamma/B^+}})$$

We have a $N^-$ equivariant diffeomorphism $N^- \longrightarrow \Gamma/B^+$, $n^- \mapsto n^- B^+(T)$, $n^- \in N^-(T)$, $T \in (\mathrm{smflds})_{\mathbb{C}}$. In fact, by the ordinary theory we have a diffeomorphisms of the underlying differentiable manifolds and the differential at the identity is an isomorphism: $\mathfrak{n}^- \cong \mathfrak{g}/\mathfrak{b}^+$.

Clearly $p^{-1}(\Gamma/B^+) = \Gamma$. We are going to construct a section $s\colon \Gamma/B^+ \to \Gamma$. The local splitting $\gamma\colon N^- \times B^+ \to \Gamma$ is an holomorphic morphism such that $\gamma^* \mathcal{O}_{\Gamma/B^+} = \mathcal{O}_{N^-} \otimes 1$. Hence we have an isomorphism $N^- \to \Gamma/B^+$ given by the composition of the "canonical" embedding $i\colon N^- \hookrightarrow N^- \times B^+$ with $\gamma$ and $p$ (which is essentially the same as considering $p \circ \gamma|_{N^- \times \{e\}}$). Its inverse is the required section.

**Proposition 4.** *The (ordinary) torus $A$ normalizes $N^{\pm}$.*

*Proof.* We give the proof for $N^+ = N$. We want to prove that the conjugation

$$\mathrm{conj}(a) : G \to G \qquad \mathrm{conj}(a) = \ell_{a^{-1}} \circ r_a, \qquad a \in \widetilde{A}$$

stabilizes $N$. Since $N$ is connected and the exponential map $\exp\colon \mathfrak{n} \to N$ is surjective it is enough to prove that $(d\mathrm{conj}(a))_1(\mathfrak{n}) \subseteq \mathfrak{n}$ We know from the infinitesimal theory that $\mathrm{ad}(\mathfrak{h})(\mathfrak{n}) = \mathfrak{n}$. Hence, we have

$$\mathrm{Ad}(e^{tX})Y = e^{t\mathrm{ad}X}(Y) \quad \forall X \in \mathfrak{h},\, Y \in \mathfrak{n}$$

so that $\mathrm{Ad}(e^{tX})\mathfrak{n} = \mathfrak{n}$. Since the exponential map of an abelian connected Lie group is surjective we have that $\mathrm{Ad}(A)\mathfrak{n} = \mathfrak{n}$.

By the simply connectedness of $\widetilde{N}$, we get a map $\widetilde{\mathrm{conj}(a)} : \widetilde{N} \to \widetilde{N}$. It is easy to check that the pair

$$\mathrm{Ad}(a) : \mathfrak{n} \to \mathfrak{n} \qquad \widetilde{\mathrm{conj}(a)} : \widetilde{N} \to \widetilde{N}$$

is a SHCP morphism: $(\widetilde{\mathrm{conj}(a)}, \mathrm{Ad}(a)) : (\widetilde{N}, \mathfrak{n}) \longrightarrow (\widetilde{G}, \mathfrak{g})$, so that, by the equivalence of categories between analytic SHCP and analytic supergroups, we have a morphism of super Lie groups $N \to G$. Since its differential coincides with the differential of $\mathrm{conj}(a) : N \longrightarrow G$ and the reduced maps are the same, the two morphisms coincide, hence $\mathrm{conj}(a)N = N$.

Let us fix a character $\chi : A \longrightarrow \mathbb{C}^\times$ of the ordinary torus, that we can trivially extend to a character (still denoted by $\chi$) of the supergroup $B^+$. Define:

$$L^\chi(\Gamma) := \{f \in \mathcal{O}_G(\Gamma) \mid f(gb) = \chi(b)^{-1} f(g)\}$$

We can geometrically view this superspace as the superspace of sections of the line bundle uniquely associated with $\chi$. This superspace is the key for the construction of the infinite dimensional representations of real forms of the analytic supergroup $G$ (see [7], [8] for more details). The actions that we are going to describe are absolutely essential for the realization of such representations.

Since $A$ acts on $N^-$ by conjugation (see Prop. 4), we have a global action of $A$ on $\Gamma$ defined as:

$$a \cdot (n^- b^+) = (a n^- a^{-1}) b^+, \quad a \in \widetilde{A},\ n^- \in N^-(T), b^+ \in B^+(T).$$

Since $A$ also acts on $B^+$ by left translation, we can define the left action of $A$ on $\Gamma$ as:

$$a \cdot (n^- b^+) = (a n^- a^{-1}) a \cdot b^+.$$

Both actions commute with right translations by $B^+$ and hence define representations of $A$ on $L^\chi(\Gamma)$

$$i,\, \ell \colon A \times L^\chi(\Gamma) \to L^\chi(\Gamma)$$

where:

$$i_a(f)(n^- b^+) = f((a^{-1} n^- a) b^+), \qquad \ell_a(f)(n^- b^+) = f((a^{-1} n^- a) a^{-1} b^+) \tag{4}$$

and $a \in \widetilde{A}$, $n^- \in N^-(T)$, $b^+ \in B^+(T)$, $f \in L^\chi(\Gamma)$.

Let $t_\alpha$ denote the global homogeneous exponential coordinates on $N^-$ obtained by Prop. 2.

**Lemma 1.** *Let the notation be as above. Then*

1. $\ell_a f = \chi(a)(i_a f)$
2. $i_a t_\alpha = \chi_\alpha(a) t_\alpha \qquad \forall a \in \widetilde{A}$

*where $\chi_\alpha$ is the character of the maximal torus $A$ obtained by exponentiating the root $\alpha \in \mathfrak{h}^*$.*

*Proof.* (1) follows immediately from the definitions. For (2) let $n = \exp(\sum_{\beta \in P} y_\beta X_{-\beta})$ be an element in $N^-$, then the result comes from the following formal calculation in the exponential global coordinates:

$$t_\alpha(a^{-1} n a) = t_\alpha\big(\exp(\sum_{\beta \in P} y_\beta Ad(a) X_{-\beta})\big) = t_\alpha(\exp(\sum_{\beta \in P} y_\beta \chi_\beta(a) X_{-\beta})$$
$$= \chi_\alpha(a) t_\alpha(n), \quad a \in \widetilde{A},\ y_\beta \in \mathbb{C}$$

## 4 The action of $\mathcal{U}(\mathfrak{g})$ and $G$ on $L^\chi(\Gamma)$

Now we want to use the theory developed so far and extend the action of the maximal torus $A \subset G$ to an action of the whole group on $L^\chi(\Gamma)$. We start by defining the natural action of $\mathcal{U}(\mathfrak{g})$ on the holomorphic functions on any neighbourhood $W$ of the identity of the supergroup $G$.

**Definition 2.** *Let $W \subset G$ be an open neighbourhood of the identity $1_G$ in $G$. There are two well defined actions of $\mathfrak{g}$, hence of $\mathcal{U}(\mathfrak{g})$, on $\mathcal{O}_G(W)$, that read as follows:*

$$\ell(X)f = (-X \otimes 1)\mu^*(f), \qquad \partial(X)f = (1 \otimes X)\mu^*(f), \qquad X \in \mathfrak{g}$$

The actions $\ell$ and $\partial$ commute with each other. Moreover, if $\widetilde{U}$ is open in $\widetilde{G/B^+}$, then $\ell$ is a well defined action on $L^\chi(\widetilde{U})$.

We now want to show that the natural action $\ell$ of $\mathcal{U}(\mathfrak{g})$ on $L^\chi(N^-B^+)$ preserves the polynomial sections on $\widetilde{N^-}$. For this we need some preliminary notation. Since $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{b}^+$, if we fix bases of $\mathfrak{n}^-$ and $\mathfrak{b}^+$, by the PBW (Poincaré Birkhoff Witt) theorem any $X \in \mathcal{U}(\mathfrak{g})$ can be written as

$$X = \sum_{I,J} c_{IJ}(X)\mathrm{B}_I\mathrm{N}_I, \qquad \mathrm{B}_I \in \mathcal{U}(\mathfrak{b}^+), \mathrm{N}_I \in \mathcal{U}(\mathfrak{n}^-) \tag{5}$$

**Lemma 2.** *Let $\phi \in \mathcal{O}_G(N^-B^+)$. In the SHCP notation, $\phi$ is in $L^\chi(N^-B^+)$ if and only if*

$$\phi(X)(nb) = \widetilde{\chi}(b)^{-1}\sum_{IJ} c_{IJ}(b.X)\lambda(\overline{\mathrm{B}}_I)\phi(\mathrm{N}_J)(n), \quad X \in \mathcal{U}(\mathfrak{g}), \quad \lambda = d\chi$$

*where $b.X$ is the adjoint action of $b \in \widetilde{B^+}$ on $\mathcal{U}(\mathfrak{g})$ and as usual $\overline{\mathrm{B}}_I$ denotes the antipode of $\mathrm{B}_I$ in the Hopf superalgebra $\mathcal{U}(\mathfrak{g})$.*

*Proof.* By the very definition we have $\phi \in L^\chi(N^-B^+)$ if

1. $r_b^*\phi = \widetilde{\chi}(b)^{-1}\phi$, $b \in \widetilde{B^+}$
2. $D_Y^L(\phi) = \lambda(\overline{Y})\phi$, $\lambda|_{\mathfrak{g}_0} = d\widetilde{\chi}$.

where as usual $\widetilde{\chi}$ denotes the reduced morphism. The result comes with a calculation.

Notice that once the lemma is established, if $p$ is a polynomial in the global coordinates of $N^-$, we can define $p^\sim \in L^\chi(N^-B^+)$ as:

$$p^\sim(X)(nb) = \widetilde{\chi}(b)^{-1}\sum_{IJ} c_{IJ}(b.X)\lambda(\overline{\mathrm{B}^+}_I)p(\mathrm{N}_J)(n)$$

Vice-versa we can recover $p$ from $p^\sim$ restricting to $N^-$. In the language of SHCP this amounts to two restrictions: we impose $b = 1$ and $X \in \mathcal{U}(\mathfrak{n}^-)$. We shall denote the set of such $p^\sim$ with $\mathcal{P}^\sim$.

**Proposition 5.** *The actions $\ell$ of $\mathcal{U}(\mathfrak{g})$ on $L^\chi(\widetilde{U})$, $p^{-1}(\widetilde{U}) \subset \Gamma$ leave $\mathcal{P}^\sim$ invariant.*

*Proof.* We need to show that, given $Z \in \mathcal{U}(\mathfrak{g})$ and $X \in \mathcal{U}(\mathfrak{n}^-)$, $(D^R_Z p^\sim|_{N^-})(X)$ is a polynomial section. We have (see [5] Sec. 7.4):

$$(D^R_Z p^\sim)(X)(g) = (-1)^{|Z||p|}[p^\sim((g^{-1}.Z)X)](g)$$

Hence if $n \in N^-$, we have:

$$\begin{aligned}(D^R_Z p^\sim)(X)(n) &= (-1)^{|Z||p|}[p^\sim((n^{-1}.Z)X)](n)\\ &= (-1)^{|Z||p|}\sum_{IJ} c_{IJ}((n^{-1}.Z)X)[\lambda(\overline{\mathrm{B}}_I)p^\sim(\mathrm{N}_J)](n)\end{aligned}$$

where $\mathrm{B}_I$ and $\mathrm{N}_J$ are obtained as in (5) applied to $(n^{-1}.Z)X$. The last equality is true by Lemma 2.

Once this is established, we have the following result.

**Theorem 1.** *There is a non-singular $\mathcal{U}(\mathfrak{g})$ invariant pairing between $\mathcal{P}^\sim$ and the Verma module $V_\lambda$:*

$$\langle , \rangle : \mathcal{P}^\sim \times \mathcal{U}(\mathfrak{g}) \longrightarrow \mathbb{C}, \qquad \langle f, u\rangle := (-1)^{|u||f|}(\partial(u)f)(1_G)$$

*Proof.* In order for $\langle , \rangle$ to be a $\mathcal{U}(\mathfrak{g})$ invariant pairing, we need to verify:

$$\langle \ell(c)f, u\rangle \; = \; \langle f, (-1)^{|f||c|}c^T u\rangle, \qquad c, u \in \mathcal{U}(\mathfrak{g}), f \in \mathcal{P}^\sim$$

where $(\cdot)^T$ denotes the antiautomorphism of $\mathcal{U}(\mathfrak{g})$ induced by $X \mapsto -X$ with $X \in \mathfrak{g}$. This is just a check.

Now let $\mathfrak{g}_r$ be a real form of $\mathfrak{g}$ and define the real supergroup $G_r = (\widetilde{G_r}, \mathfrak{g}_r)$, where $\widetilde{G_r}$ is a real form of $\widetilde{G}$, $\mathrm{Lie}(\widetilde{G_r}) = \mathfrak{g}_{r,0}$. Since $\mathfrak{g}_r + \mathfrak{b}^+ = \mathfrak{g}$ as real superalgebras (see [9], Iwasawa decomposition), we have that $S := G_r B^+$ is an open subsupermanifold of $G$.

**Theorem 2.** *Assume $L^\chi(S) \neq 0$ modulo $J$ the submodule generated by the odd part. Then $L^\chi(S)$ contains an element $\psi$ which is an analytic continuation of $1^\sim$ and*

$$\ell(\mathcal{U}(\mathfrak{g}))\psi = \mathcal{P}^\sim \cong \pi_{-\lambda}$$

*where $\pi_{-\lambda}$ the irreducible representation with lowest weight $-\lambda$. Furthermore $L^\chi(S)$ carries a $G_r$ representation defined as:*

$$\begin{cases}(g \cdot f) = l^*_{g^{-1}} f & g \in \widetilde{G_r}\\ X.f = D^R_{\overline{X}} f & X \in \mathfrak{g}_{\mathbb{C}}\end{cases}$$

*where, as usual, $\overline{X}$ is the antipode of $X \in \mathcal{U}(\mathfrak{g})$.*

*Proof.* Direct check.

The closure of $\ell(\mathcal{U}(\mathfrak{g}))\psi$ in $L^\chi(S)$, with a Fréchet superspace structure is a Harish-Chandra representation, with $\ell(\mathcal{U}(\mathfrak{g}))\psi$ as its $K_r$ finite part, where $K_r$ is the supergroup corresponding to the subalgebra $\mathfrak{k}_r$ in the Cartan decomposition of $\mathfrak{g}_r$ (see [9]). The proof of these facts is non trivial, we invite the reader to see [7], [8].

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