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# Second Order Approximation of Extended Thermodynamics of a Monatomic Gas and Hyperbolicity Region.

Francesca Brini* Tommaso Ruggeri†

**Abstract**

The Rational Extended Thermodynamics theory describes non-equilibrium phenomena for rarefied gases and it is usually approximated in the neighborhood of an equilibrium state. Consequently, the hyperbolicity of its differential system holds only in some domain of the state-variables (called *hyperbolicity region*). In this paper, we present a second order approximation with respect to non-equilibrium variables, in the case of a monatomic gas theory with 13 fields. We verify that, in the case of one-dimensional space, the radius of the hyperbolicity region is larger than the corresponding radius of the first order approximation. Moreover, when the model involves three-dimensional field variables, we prove that the equilibrium state for differential systems with quadratic approximation is inside the hyperbolicity region. This fact is in contrast with the first order models that, in some cases of three-dimensional field variables, present the equilibrium point at the boundary of the hyperbolicity region.

## 1 Introduction

Rational Extended Thermodynamics (RET) is a well-known phenomenological field theory, that is able to describe non-equilibrium phenomena of rarefied gases [1, 2, 3]. The macroscopic RET theory is closed through the requirement of universal principles for constitutive equations, such as the objectivity principle, the entropy principle and the convexity of entropy (thermodynamical stability). This gives to the theory a particularly elegant and robust structure, both from the mathematical and the physical points of view. In fact, the RET models are expected to be hyperbolic PDE systems with a *convex extension* and can be put in a symmetric form using the *main field* as set of variables [4, 5], so

*Department of Mathematics and AM$^2$, University of Bologna, via Saragozza,8, 40123 Bologna, Italy. francesca.brini@unibo.it

†Department of Mathematics and AM$^2$, University of Bologna, via Saragozza,8, 40123 Bologna, Italy. tommaso.ruggeri@unibo.it

that the well-posedness of the Cauchy problem is guaranteed. The relation between Hamiltonian structure and hyperbolic structure of the equation systems is investigated in [6]. The hyperbolicity property is very important for a realistic physical description of non-stationary phenomena, since it is associated to finite speeds of disturbances, in contrast to the infinite speed predicted by the parabolic models typical of which is the Thermodynamics of Irreversible Processes (TIP), like Navier-Stokes-Fourier ones.

Usually, the RET systems are linearized with respect to the non-equilibrium variables and therefore the validity of the theory is restricted only to a neighborhood of an equilibrium state [1, 2, 3]. The closure obtained via RET in the case of the 13-field theory of monatomic gases is the same as the one obtained via kinetic considerations by Grad [7]. Also the hyperbolicity condition remains valid only in a neighborhood of the equilibrium state, which is called *region of hyperbolicity*.

The analysis on the determination of the region of hyperbolicity started more than 25 years ago by Müller and Ruggeri [1] in the case of 13 moments. The studies were carried on for a one-dimensional system with linear expansion and extended to the corresponding one-dimensional 14-moment monatomic theory by Brini [8] and to the 13-moment model for monatomic degenerate gases by Ruggeri and Trovato [9], both for linear and quadratic expansions.

A new relevant question arises from the work by Cai, Fan and Li [10], who showed that the thermodynamic equilibrium belongs to the boundary of the hyperbolicity region when the heat flux and velocity vectors are no longer parallel to a fixed direction. Under such an assumption, the authors concluded that the equilibrium state is unstable, since an arbitrary small perturbation of the equilibrium can bring out of the hyperbolicity region. The same authors suggested in a very ingenious mathematical way different tools to overcome the problem [10, 11], modifying the system ad hoc in order to 'make' the equilibrium point interior to the hyperbolicity region or introducing a new globally hyperbolic system. Unfortunately, as the only system of balance laws compatible with the entropy principle in the first order approximation is the Grad one, we conclude that the new system cannot be put in a balance form and there cannot be any entropy principle for the modified model. This means that the system is not usable in the case of weak solutions (in particular for shock waves) and, furthermore, it loses physical meaning, since an entropy principle is mandatory in all continuum theories!

Moreover, the global hyperbolicity is an excessively stringent requirement and only ideal cases (e.g. Euler fluid with ideal gas assumption) fulfill this requirement for all possible values of the field variables. The lack of hyperbolicity could also be associated to physical effects (phase transition, instability, etc..) as, for example, in van der Waals fluids (see e.g. [12]), in Born-Infield nonlinear electrodynamics [13], in non-linear elasticity (see e.g. [14]) and in the Boltzmann-Vlasov Equation [15].

The structure of the differential RET system is justified by the kinetic theory. Nevertheless in the phenomenological RET the field components are not seen as moments of a distribution function and, therefore, no integrability conditions are necessary. Unfortunately, the procedure for the construction of the phenomenological RET becomes complicated when we want to consider a theory far from equilibrium, taking into account successive approximations or an increasing number of fields (in this framework some formal attempts were done by Pennisi and co-workers see e.g [16, 17]).

On the other hand, in the case of a rarefied gas another method to close the differential system is to recall that the field variables are moments of a distribution function, so that the closure of the moment balance equations, truncated at some level, can be obtained using the so-called *Maximum Entropy Principle* (MEP).

The theory derived through this approach was called by Müller and Ruggeri *Molecular Extended Thermodynamics* [1]. The principle of maximum entropy has its roots in statistical mechanics. It was developed by Jaynes [18] in the context of the theory of information starting from the Shannon entropy. *MEP states that the probability distribution that represents the current state of knowledge in the best way is the one with the largest entropy.*

Concerning the applicability of MEP to non-equilibrium thermodynamics, the idea was originally introduced by Kogan [19], who demonstrated that the Grad's distribution function maximizes the entropy. Afterwards, in 1987, Dreyer showed for the first time in [20] that the MEP closure is equivalent to closure of the phenomenological RET.

In 1993 the MEP procedure was generalized to the case of any number of moments by Müller and Ruggeri in the first edition of their book [1], proving that the PDE system is symmetric hyperbolic if the Lagrange multipliers are chosen as field variables. The complete equivalence between the entropy principle closure and the MEP ones in Molecular Extended Thermodynamics was finally proved in 1997 by Boillat and Ruggeri [21]. In fact, they demonstrated that the Lagrange multipliers coincide with the so-called *main field* that symmetrize any hyperbolic system compatible with an entropy principle and a convex entropy law [4, 5].

All the results recalled above are valid also for phenomena far from equilibrium, provided that the integrals are convergent. Actually, the problem of the convergence of the moments remains still one of the main questions in a far-from-equilibrium theory. In particular, the index of truncation $N$ must be even [19, 21, 22]. This implies, for example, that a theory with 13 moments corresponding to $N = 3$ is not allowed far from equilibrium!

The question about convergence is not the only one in the full non-linear case. Indeed, Junk and co-workers have shown [23, 24] that even the complete RET theory, obtained through the maximum entropy principle (MEP), without any approximation, presents problems, since there are moment states, physically admissible, that correspond to singularities.

The integrability conditions and the Junk problem do not arise if the distribution function obtained as solution of the variational problem is considered only in the neighborhood of a local equilibrium state, and is formally expanded as the perturbation of an equilibrium distribution $f^{(E)}$. In the case of 13 moments for a monatomic gas, if the approximation is stopped at the first order, the closure gives again the same results as the phenomenological RET and the Grad approach [20]. The same holds also in the recent theory of 14 fields for a polyatomic gas [3, 25]. Thus, the closure of the moment system at first order expansion can be obtained in three different ways: RET, Grad, and MEP. A remarkable point is that all closures are equivalent to each others!

Brini and Ruggeri [26], considered for the first time the expansion of the distribution function that maximizes the entropy until a generic order $\alpha$, introducing the notation $\mathrm{ET}_M^\alpha$ that represents a RET theory with $M$ moments and an $\alpha$ expansion order for the distribution function. An alternative approximation was proposed in [27].

We recall that in the non-linear case the hyperbolicity is guaranteed for any value of field variables, while the problem of the hyperbolicity region arises when we introduce the approximation. On the other hand, using the approximation we can invert the map between the main field and the densities that are physical variables, since, in general, such an inversion cannot be done analytically. Therefore, we expect that the domain of hyperbolicity increases with the approximation order more than increasing the number of moments.

The aim of this paper is to construct a RET model with quadratic expansion with respect to the non-equilibrium variables, using the technique indicated in [26]. In such a way, we can verify that in the one-dimensional variables the radius of the domain of hyperbolicity increases. Moreover, in the case of three-dimensional variables, we avoid the troubles related to the instability of the equilibrium state, by dealing with a local-hyperbolic model that can be written in the balance form and is compatible with the entropy principle.

To better contextualize the problem in the next sections we will start from the well-known case of a monatomic gas described by the 13-moment Grad system. Then, we will pass to the quadratic expansion of the 13-moment theory both for one-dimensional and three-dimensional field variables. To our knowledge, the $\mathrm{ET}_{13}^2$ differential system and the corresponding hyperbolicity region are presented here for the first time.

The paper is organized as follows. Section 2 contains a review of the Extended Thermodynamics theories for monatomic gases and their approximations; in Section 3 we present the $\mathrm{ET}_{13}^2$ system. A new approach to identify the hyperbolicity region is introduced in Section 4. The study of the domain of hyperbolicity in the one-dimensional case is available in Section 5, while the case of three-dimensional field variables is analyzed in Section 6. Finally, in Section 7 one can find some conclusions and final remarks.

## 2 $\mathrm{ET}_M^\alpha$ of rarefied monatomic gases

According to the kinetic theory, the state of a rarefied gas is described by the phase density $f(\mathbf{x}, t, \mathbf{c})$, where $f(\mathbf{x}, t, \mathbf{c})d\mathbf{c}$ is the number density of the monatomic molecules at point $\mathbf{x}$ and time $t$ that present velocities between $\mathbf{c}$ and $\mathbf{c} + d\mathbf{c}$. The time evolution of the phase density is governed by the well-known Boltzmann equation

$$\partial_t f + c_i \partial_i f = Q, \tag{1}$$

where $\partial_t$ denotes the partial derivative with respect to the time $t$, $\partial_i = \partial/\partial x_i$ ($i = 1, 2, 3$) is the partial derivative with respect to $x_i$, $\mathbf{c} \equiv (c_1, c_2, c_3)$ denotes the microscopic velocity, $Q$ is the term related to the particle collisions and, as usual, we will always omit the symbol of sum for repeated indices. A common procedure provides an infinite hierarchy of balance laws, called *moment system*, starting from (1). We define the moments as ($m$ is the atomic mass)

$$\begin{aligned} &F = m \int_{\mathbb{R}^3} f(\mathbf{x}, t, \mathbf{c})\, d\mathbf{c}, \\ &F_{k_1 k_2 k_3 \dots k_j} = m \int_{\mathbb{R}^3} f(\mathbf{x}, t, \mathbf{c})\, c_{k_1} c_{k_2} c_{k_3} \dots c_{k_j}\, d\mathbf{c}, \quad j = 1, 2, \dots,\ k_j = 1, 2, 3 \end{aligned} \tag{2}$$

and the production terms as

$$P_{k_1 k_2 \dots k_j} = m \int_{\mathbb{R}^3} Q\, c_{k_1} c_{k_2} c_{k_3} \dots c_{k_j}\, d\mathbf{c}, \qquad j = 2, 3, \dots$$

where $P_{k_1 k_2}$ is a deviatoric tensor (traceless). So, the infinite hierarchy presents a peculiarly elegant form, since the flux component of one equations becomes the density component of the following one:

$$\begin{array}{l} \partial_t F + \partial_i F_i = 0 \\ \qquad \swarrow \\ \partial_t F_{k_1} + \partial_i F_{i k_1} = 0 \\ \qquad \quad \swarrow \\ \partial_t F_{k_1 k_2} + \partial_i F_{i k_1 k_2} = P_{<k_1 k_2>} \\ \qquad \quad \ \swarrow \\ \partial_t F_{k_1 k_2 k_3} + \partial_i F_{i k_1 k_2 k_3} = P_{k_1 k_2 k_3} \\ \vdots \\ \partial_t F_{k_1 k_2 \dots k_j} + \partial_i F_{i k_1 k_2 \dots k_j} = P_{k_1 k_2 \dots k_j} \\ \vdots \end{array} \tag{3}$$

The first 5 scalar equations $(3)_{1,2}$ and the trace of $(3)_3$ are conservation laws and represents respectively the conservation of mass, momentum and energy respectively. In RET, this set of balance laws is truncated at some finite tensorial order $N$. Denoting with $\mathbf{F}$, $\mathbf{F}^i$ and $\mathbf{P}$ respectively the density, flux and production vectors:

$$\mathbf{F} \equiv (F, F_{k_1}, F_{k_1 k_2} \dots . F_{k_1 k_2 \dots k_N})^T, \qquad \mathbf{F}^i \equiv (F_i, F_{i k_1}, F_{i k_1 k_2} \dots F_{i k_1 k_2 \dots k_N})^T,$$

$$\mathbf{P} \equiv (0, 0_{k_1}, P_{k_1 k_2}, \dots P_{k_1 k_2 \dots k_N})^T,$$

(the $0_{k_1}$ denotes the zero vector), the system of truncated balance laws can be written as

$$\partial_t \mathbf{F} + \partial_i \mathbf{F}^i = \mathbf{P}. \tag{4}$$

The truncation at tensorial order $N$ entails a problem of closure of the system, as already described in the introduction. The first procedure of RET to close the moment system, [1, 2] considers the remaining quantities (last flux and production terms) as local constitutive functions of the the densities field $\mathbf{F}$. The restriction on the constitutive equations are obtained through the validity requirement of the universal principles such as the Galilean invariance of the balance laws, the entropy principle and the requirement of convexity of the entropy density [1, 2, 3]. A different approach (already described in the introduction) is provided by MEP and it is possible to show [2] that the phase density that maximizes the entropy $h$

$$h = -k_{\mathcal{B}} \int_{\mathbb{R}^3} f \ln f d\mathbf{c}$$

is given by

$$f_N = \exp(-1 - m\chi_N / k_{\mathcal{B}}) \qquad \text{with} \quad \chi_N = u'_A(\mathbf{x}, t) \cdot c_A, \tag{5}$$

where $k_{\mathcal{B}}$ denotes the Boltzmann constant, the symbol sum $\sum_{A=0}^{N}$ in $(5)_2$ is omitted and will be omitted also in the following formulas. Moreover, the main field components $u'_A$ (Lagrange multipliers) and the quantity $c_A$ are respectively:

$$u'_A = \begin{cases} u' & \text{if } A = 0 \\ u'_{k_1 \dots k_A} & \text{if } 1 \leq A \leq N \end{cases}, \quad c_A = \begin{cases} 1 & \text{if A} = 0 \\ c_{k_1} \dots c_{k_A} & \text{if } 1 \leq A \leq N. \end{cases}$$

At equilibrium all the main field components vanish, except the first five [2] and the phase density (5) reduces to the equilibrium Maxwellian one, $f_{\mathcal{M}}$:

$$f^{(E)} = f_{\mathcal{M}} = \frac{\rho}{m} \left( \frac{m}{2\pi k_{\mathcal{B}} T} \right)^{3/2} \exp\left[ -\frac{mC^2}{2k_{\mathcal{B}} T} \right],$$

where $\rho$ denotes the gas density, $T$ the temperature, $\mathbf{v}$ the velocity and $\mathbf{C} = \mathbf{c} - \mathbf{v}$ is the peculiar velocity ($C^2 = C_i C_i$). For processes sufficiently close to an equilibrium state the phase density $f_N$ could be approximated by $f_N^{(\beta)}$ through a Taylor's expansion of order $\beta > 0$ ($\beta \in \mathbb{N}$)

$$f_N^{(\beta)} = f_{\mathcal{M}} \Bigg( 1 + \Lambda_{A_1} c_{A_1} + \frac{1}{2} \Lambda_{A_1} \Lambda_{A_2} c_{A_1} c_{A_2} + \dots + \\ + \frac{1}{\beta!} \Lambda_{A_1} \Lambda_{A_2} \dots \Lambda_{A_\beta} c_{A_1} c_{A_2} \dots c_{A_\beta} \Bigg),$$

where $\Lambda_A$ is proportional to the non equilibrium part of the main field components:

$$\Lambda_A = -\frac{m}{k_{\mathcal{B}}} (u'_A - u'_A|_E).$$

In this way also the components $F_{ik_1k_2k_3\dots k_N}$ that are not in the list of the densities are approximated as

$$F^{(\beta)}_{ik_1k_2k_3\dots k_N} = m\int_{\mathbb{R}^3} f^{(\beta)}_N(\mathbf{x},t,\mathbf{c})c_ic_{k_1}c_{k_2}c_{k_3}\dots c_{k_N}d\mathbf{c}, \tag{6}$$

and in general the procedure gives rise to a RET theory with $M$ moments and an expansion order $\beta$, $\mathrm{ET}^{\beta}_{M}$. In particular, if we focus on the 13-moment models, $\mathrm{ET}^{1}_{13}$ corresponds to the usual Grad system [7].

Unfortunately, the approximation of the phase density entails a local validity of the convexity of the entropy and of the hyperbolicity property, that turn out to hold only in a neighborhood of the equilibrium point.

The previous calculations are often done at zero velocity ($\mathbf{v}=0$), since the corresponding expression for a generic velocity is easily determined thanks to the Galilean invariance. Usually, one solves the previous relations substituting the microscopic velocity $\mathbf{c}$ with the corresponding peculiar velocity $\mathbf{C}$, and deriving the so called *non-convective* part of densities, fluxes, productions and main field, evaluated at zero velocity, that we will denote with a hat ($\hat{\mathbf{F}}$,$\hat{\mathbf{F}}^i$, $\hat{\mathbf{P}}$, $\hat{\mathbf{u}}'$). Then, the dependence on $\mathbf{v}$ is obtained through the theorem proved in [28], for which there exists a matrix $\mathbf{X}(\mathbf{v})$ such that

$$\mathbf{F} = \mathbf{X}(\mathbf{v})\hat{\mathbf{F}}, \qquad \mathbf{F}^i = \mathbf{X}(\mathbf{v})\left(\hat{\mathbf{F}}^i + v^i\hat{\mathbf{F}}\right), \quad \mathbf{P} = \mathbf{X}(\mathbf{v})\hat{\mathbf{P}}.$$

while for the main field components we have

$$\mathbf{u}' = \hat{\mathbf{u}}'\mathbf{X}^{-1}(\mathbf{v}).$$

The matrix $\mathbf{X}(\mathbf{v})$ is an exponential matrix that becomes polynomial in the velocity components in the case of moments structure (see [28] for details).

In order to determine the non-convective expression of main field, density and flux vectors for a certain RET theory obtained truncating at the $N$ tensorial order and with an expansion order $\beta$, we have to proceed as follows (see also the Appendix of [26], where the problem was already considered). We first rewrite the truncated system (4) using the multi-index:

$$\partial_t F_A + \partial_i F_{iA} = P_A, \quad A = 0,1,\dots,N$$

with

$$F_A = \begin{cases} F & \text{if } A=0 \\ F_{k_1\dots k_A} & \text{if } 1\le A\le N \end{cases}, \quad F_{iA} = \begin{cases} F_i & \text{if } A=0 \\ F_{ik_1\dots k_A} & \text{if } 1\le A\le N \end{cases}$$

and

$$P_A = P_{k_1\dots k_A} \qquad 2\le A\le N, \quad \text{with } P_{kk}=0.$$

We introduce the notation $\Delta \hat{F}_A = \hat{F}_A - \hat{F}|_E$ and we have

$$\begin{aligned}\Delta \hat{F}_A = &\int_{\mathbb{R}^3} C_A (f_N^{(\beta)} - f_{\mathcal{M}}) d\mathbf{C} = \\ &= \int_{\mathbb{R}^3} f_{\mathcal{M}} C_A \left[ \Lambda_{B_1} C_{B_1} + \frac{1}{2} \Lambda_{B_1} \Lambda_{B_2} C_{B_1} C_{B_2} + \cdots \right. \\ &\left. + \frac{1}{\beta !} \Lambda_{B_1} \Lambda_{B_2} \cdots \Lambda_{B_\beta} C_{B_1} C_{B_2} \dots C_{B_\beta} \right] d\mathbf{C}.\end{aligned} \tag{7}$$

The previous relations constitute a set of $M$ equations for the $M$ components of $\Lambda_A$. To solve them we start with expansion of the main field components where the up index indicates the order of approximation of $\Lambda_A$:

$$\Lambda_A = \Lambda_A^{(1)} + \Lambda_A^{(2)} + \Lambda_A^{(3)} + \dots \Lambda_A^{(\beta)}. \tag{8}$$

In this way the values of $\Lambda_A^{(a)}$ are determined through a recursive procedure. Let define

$$J_{AB_1 B_2 \dots B_a} = \int_{\mathbb{R}^3} f_{\mathcal{M}} C_A C_{B_1} C_{B_2} \dots C_{B_a} \, d\mathbf{C}, \qquad (a = 1, \dots, \beta).$$

At the first order we have the linear system

$$J_{AB} \Lambda_B^{(1)} = \Delta \hat{F}_A. \tag{9}$$

Taking into account that $J_{AB}$ is symmetric and positive definite, the values of $\Lambda_A^{(1)}$ are determined in terms of the densities $F_A$. Then, the second order terms are the solution of the equations for the second order expansion:

$$J_{AB} \Lambda_B^{(2)} = -\frac{1}{2} \Lambda_{B_1}^{(1)} \Lambda_{B_2}^{(1)} J_{AB_1 B_2}, \tag{10}$$

where $\Lambda_A^{(2)}$ are the roots of a linear system containing the same matrix $J_{AB}$ as for (9). The third order of expansion is treated in a similar way:

$$J_{AB} \Lambda_B^{(3)} = -\frac{1}{2} \left( \Lambda_{B_1}^{(1)} \Lambda_{B_2}^{(2)} + \Lambda_{B_1}^{(2)} \Lambda_{B_2}^{(1)} \right) J_{AB_1 B_2} - \frac{1}{6} \Lambda_{B_1}^{(1)} \Lambda_{B_2}^{(1)} \Lambda_{B_3}^{(1)} J_{AB_1 B_2 B_3}. \tag{11}$$

The procedure has to be iterated up to the $\beta$ order and, since the matrix $J_{AB}$ is not singular, we have a unique solution. If one employs an algebraic manipulator or a numerical algorithm in order to automate it, the multi-index notation turns out to be complicated, although very elegant.

For this reason we report here also the three-index notation introduced as first by Boillat and Ruggeri in [21]. The idea is that we can rewrite moments and Lagrange multipliers using only a 3-index tensor taking into account that the components of $\mathbf{c}$ (or $\mathbf{C}$) can be only $c_1, c_2$ or $c_3$ and therefore there exists a correspondence one to one between for example the moments given by (2) and

$$F_{pqr} = \int f c_1^p c_2^q c_3^r d\mathbf{c},$$

where the power integer index $p, q, r$ satisfy $p+q+r=0,1,2,\dots N$; so, for example, $F=F_{000}$, $F_1=F_{100}$, $F_2=F_{010}$, $F_3=F_{001}$.

Referring to this representation, the expression of (7), becomes

$$\Delta\hat{F}_{pqr}=\int_{\mathbb{R}^3} f_{\mathcal{M}}\,[\Lambda_{n_1m_1s_1}C_1^{n_1}C_2^{m_1}C_3^{s_1}+ \\ +\frac{1}{2}\Lambda_{n_1m_1s_1}\Lambda_{n_2m_2s_2}C_1^{n_1+n_2}C_2^{m_1+m_2}C_3^{s_1+s_2}+\dots\Big]\,d\mathbf{C}.$$

If we consider the integrals (no sum on the index $k$)

$$J_k^j=\int_{\mathbb{R}}\left(\frac{1}{2\pi\theta}\right)^{1/2}\exp\left[-\frac{C_k^2}{2\theta}\right]C_k^j dC_k=(2\theta)^{n/2}\frac{(1+(-1)^n)}{\pi}\Gamma\left(\frac{j+1}{2}\right)$$

where $\Gamma$ is the gamma function and $\theta=k_{\mathcal{B}}T/m$, and we introduce the quantities

$$Y_{p,q,r}=\frac{\rho}{m}J_1^pJ_2^qJ_3^r$$

assuming that

$$p+q+r=0,\dots,N,$$

the linear systems (9), (10), (11) can be rewritten respectively as

$$\sum_{n1+m1+s1=0}^{N} Y_{n_1+p,m_1+q,s_1+r}\,\Lambda^{(1)}_{n_1m_1s_1}=\Delta\hat{F}_{pqr},$$

$$\sum_{n1+m1+s1=0}^{N} Y_{n_1+p,m_1+q,s_1+r}\,\Lambda^{(2)}_{n_1m_1s_1}= \\ =-\frac{1}{2}\sum_{n1+m1+s1=0}^{N}\ \sum_{n_2+m_2+s_2=0}^{N} Y_{p+n_1+n_2,q+m_1+m_2,r+s_1+s_2}\,\Lambda^{(1)}{}_{n_1m_1s_1}\Lambda^{(1)}_{n_2m_2s_2},$$

$$\sum_{n1+m1+s1=0}^{N} Y_{n_1+p,m_1+q,s_1+r}\,\Lambda^{(3)}{}_{n_1m_1s_1}= \\ =-\frac{1}{2}\sum_{n1+m1+s1=0}^{N}\ \sum_{n_2+m_2+s_2=0}^{N} Y_{p+n_1+n_2,q+m_1+m_2,r+s_1+s_2} \\ \left(\Lambda^{(1)}{}_{n_1m_1s_1}\Lambda^{(2)}{}_{n_2m_2s_2}+\Lambda^{(2)}_{n_1m_1s_1}\Lambda^{(1)}{}_{n_2m_2s_2}\right)- \\ -\frac{1}{6}\sum_{n1+m1+s1=0}^{N}\ \sum_{n_2+m_2+s_2=0}^{N}\ \sum_{n_3+m_3+s_3=0}^{N} Y_{p+n_1+n_2+n_2,q+m_1+m_2+m_3,r+s_1+s_2+s_3}, \\ \Lambda^{(1)}_{n_1m_1s_1}\Lambda^{(1)}_{n_2m_2s_2}\Lambda^{(1)}_{n_3m_3s_3}$$

and so on.

Referring to the definition introduced in [3], we call an $N$–system a RET system obtained imposing that the truncation index is equal to $N$ and keeping all tensors $F_A = F_{k_1 k_2 \dots k_A}$ with $1 \le A \le N$. The $N^-$–system is a RET system with the same truncation order $N$, but where for the last balance equation we consider only the trace with respect to the last two indexes ($F_{k_1 k_2 \dots k_{N-2} ll}$).

# 3 The $\mathrm{ET}^2_{13}$ system of monatomic gas

In this paper we will consider the 13-moment theory, that is to say the $3^-$–system. The first 13 components of the non-convective density vector read

$$\hat{\mathbf{F}} \equiv \left(\hat{F}, \hat{F}_k, \hat{F}_{jk}, \hat{F}_{kll}\right) = (\rho, 0, p\delta_{jk} - \sigma_{jk}, 2q_k) \quad j,k = 1,2,3,$$

where $p$ is the pressure, $\sigma_{jk}$ denotes the $jk$-component of the deviatoric part of the viscous tensor (that for monatomic gases coincide with the viscous tensor itself), while $q_k$ is the $k$-component of the heat flux. In this case the triangular matrix necessary to determine the convective parts reads [28]

$$\mathbf{X}(\mathbf{v}) = \begin{pmatrix} 1 & & & \\ v_k & \delta_{ik} & & \\ v_j v_k & \delta_{ik} v_j + \delta i j v_k & \delta_{ij}\delta_{hk} & \\ v^2 v_k & v^2 \delta_{ik} + 2 v_i v_k & 2\delta_{ik} v_h + \delta_{ih} v_k & \delta_{ik} \end{pmatrix}. \tag{12}$$

Following the previous procedures, after some cumbersome calculations, the form of the $\mathrm{ET}^2_{13}$ is determined as:

$$\begin{aligned}
&\partial_t \rho + \partial_k \rho v_k = 0,\\
&\partial_t \rho v_i + \partial_k (\rho v_i v_k + p\delta_{ik} - \sigma_{ik}) = 0\\
&\partial_t (\rho v_i v_j + p\delta_{ij} - \sigma_{ij}) + \partial_k \left\{ \rho v_i v_j v_k + p(v_i \delta_{jk} + v_j \delta_{ik} + v_k \delta_{ij}) - \sigma_{ij} v_k - \right.\\
&- \sigma_{jk} v_i - \sigma_{ik} v_j + \frac{2}{5}(q_i \delta_{jk} + q_j \delta_{ik} + q_k \delta_{ij}) + \underline{\Big[\frac{8}{25p}(\sigma_{il} q_l \delta_{jk} +}\\
&\left. \underline{+\sigma_{jl} q_l \delta_{ik} + \sigma_{kl} q_l \delta_{ij}) - \frac{4}{5p}(q_i \sigma_{jk} + q_j \sigma_{ik} + q_k \sigma_{ij})\Big]} \right\} = P_{<ij>},\\
&\partial_t (\rho v^2 v_i + 2(\rho\varepsilon + p) v_i - 2\sigma_{il} v_l + 2q_i) + \partial_k \left\{ \rho v^2 v_i v_k + 2\rho\varepsilon v_i v_k + \right.\\
&+ p(v^2 \delta_{ik} + 4 v_i v_k) - \sigma_{ik} v^2 - 2\sigma_{il} v_l v_k - 2\sigma_{kl} v_i v_l +\\
&+ \frac{4}{5} q_l v_l \delta_{ik} + \frac{14}{5} q_i v_k + \frac{14}{5} q_k v_i + 5\frac{p^2}{\rho}\delta_{ik} - 7\frac{p}{\rho}\sigma_{ik} + \underline{\Big[\frac{2}{\rho}\sigma_{il}\sigma_{kl} +}\\
&\underline{+ \frac{1}{25p}(112 q_i q_k + 36 q^2 \delta_{ik} + 16(\sigma_{il} q_l v_k + \sigma_{kl} q_l v_i + \sigma_{lm} q_l v_m \delta_{ik}) -}\\
&\left. \underline{-40(\sigma_{il} q_k v_l + \sigma_{kl} q_i v_l + \sigma_{ik} q_l v_l))\Big]} \right\} = P_{ill},
\end{aligned} \tag{13}$$

where the underlined terms are the quadratic ones and the conservation law of energy is given by the trace of equation $(13)_3$. We consider here a rarefied monatomic gas, so that $p = k_{\mathcal{B}}\rho T$ and $\varepsilon = (3/2)k_{\mathcal{B}}T$.

### 3.1 Dimensionless variables

For convenience, in the next sections we will use dimensionless quantities referring to the velocity $c_0 = \sqrt{k_{\mathcal{B}}T/m} = \sqrt{\theta}$. In particular, we will introduce the dimensionless[1] viscous tensor and heat flux as

$$\tilde{\sigma}_{ij} = \frac{\sigma_{ij}}{\rho c_0^2}, \qquad \tilde{q}_i = \frac{q_i}{\rho c_0^3}, \tag{14}$$

for $i, j = 1, 2, 3$, while the dimensionless non-convective characteristic velocity, is denoted with $\tilde{\lambda}$:

$$\tilde{\lambda} = \frac{\lambda - v_1}{c_0}.$$

## 4 The hyperbolicity region

In this section we suggest a general way to determine the hyperbolicity region and its boundaries, starting from the knowledge of the characteristic polynomial associated to the PDE system.

We denote the generic vector of the field variables as $\mathbf{u}$. Moreover, in the next sections we will deal with one-dimensional or with three-dimensional field variables but we assume anyway that such variables depend only on $t$ and on $x_1$:

$$\partial_t \mathbf{F} + \partial_1 \mathbf{F}^1 = \mathbf{P} \Longrightarrow \mathbf{A}(\mathbf{u})\partial_t \mathbf{u} + \mathbf{B}(\mathbf{u})\partial_1 \mathbf{u} = \mathbf{P}, \tag{15}$$

where $\mathbf{A}(\mathbf{u}) = \partial_{\mathbf{u}} \mathbf{F}$ and $\mathbf{B}(\mathbf{u}) = \partial_{\mathbf{u}} \mathbf{F}^1$ are square matrix. The hyperbolicity of system (15) is guaranteed if $\mathbf{A}$ is a non-singular matrix and the generalized eigenvalue problem $(\mathbf{B} - \lambda \mathbf{A})\,\mathbf{r} = 0$ presents all real characteristic velocities $\lambda$ and a complete set of eigenvectors. The characteristic polynomial associated to the system (15) has the form

$$\mathcal{P}(\lambda) = \sum_{k=0}^{n} a_k \lambda^k, \quad \text{with } a_k = a_k(\mathbf{u}), \tag{16}$$

and the hyperbolicity region corresponds to the values of $\mathbf{u}$ for which equation $\mathcal{P}(\lambda) = 0$ presents all real roots and the right eigenvectors form a basis.

For many models the grade of the characteristic polynomial does not allow an analytical study of the roots and different approaches can be employed to overcome these difficulties.

The simplest method is the numerical procedure that identifies the hyperbolicity region starting from a grid of points in the $\mathbf{u}$-space, checking if the

[1] We remark that here the dimensionless quantities are different with respect to those in [1, 2], since in the past $c = \sqrt{(5/3)}c_0$ was used in place of $c_0$.

hyperbolicity property is satisfied for each point. This method can be used also a posteriori to validate the results obtained in a different manner.

Here, we suggest the following procedure, that can be seen as a generalization of that proposed in [1] and is capable of identifying the boundaries of the hyperbolicity region. Such boundaries are the set of the values of $\mathbf{u} = \mathbf{u}_B$ that correspond to at least a pair of real coincident eigenvalues. In fact, by continuity arguments, the transition from real to complex conjugate roots takes place when at least two real eigenvalues coincide. In order to determine $\mathbf{u}_B$ we require that when $\mathbf{u} = \mathbf{u}_B$ the characteristic polynomial with $n \geq 2$, can be factorized as

$$\mathcal{P}(\lambda) = (\lambda - \lambda_*)^2 \sum_{j=0}^{n-2} b_j \lambda^j, \tag{17}$$

if $\lambda_*$ denotes the double root and $b_j$ are suitable coefficients that can be determined imposing the equivalence between (16) and (17). The requirement that there exists a double root implies two constraints for the coefficients $a_k$ in terms of the parameter $\lambda_*$ (see (17)):

$$\begin{aligned} &\mathcal{P}(\lambda_*) = 0, \quad \Leftrightarrow \quad \sum_{k=0}^{n} a_k \, \lambda_*^k = 0, \\ &\mathcal{P}'(\lambda_*) = 0, \quad \Leftrightarrow \quad \sum_{k=0}^{n} k \, a_k \, \lambda_*^{k-1} = 0. \end{aligned} \tag{18}$$

It is possible to verify that the coefficients $b_j$, $(0 \leq j \leq n-2)$ can be expressed in terms of $\lambda_*$ and $a_k$ $(0 \leq k \leq n)$:

$$b_j = \sum_{l=j+2}^{n} (l - j - 1) \lambda_*^{l-j-2} a_l, \qquad 0 \leq j \leq n-2. \tag{19}$$

As the $a_k$ are functions of the field $\mathbf{u}$, the conditions (18) constitute the parametric equations for the boundaries of the hyperbolicity region in terms of the parameter $\lambda_*$. Just to better clarify the procedure, we consider here the case of $n = 5$. In that case, the following compatibility conditions are deduced (18):

$$\begin{cases} \lambda_*^5 a_5 + \lambda_*^4 a_4 + \lambda_*^3 a_3 + \lambda_*^2 a_2 + \lambda_* a_1 + a_0 = 0, \\ 5\lambda_*^4 a_5 + 4\lambda_*^3 a_4 + 3\lambda_*^2 a_3 + 2\lambda_* a_2 + a_1 = 0, \end{cases} \tag{20}$$

with the following $b_j$ (see (19)) :

$$\begin{aligned} &b_3 = a_5, \quad b_2 = 2\lambda_* a_5 + a_4, \quad b_1 = 3\lambda_*^2 a_5 + 2\lambda_* a_4 + a_3, \\ &b_0 = 4\lambda_*^3 a_5 + 3\lambda_*^2 a_4 + 2\lambda_* a_3 + a_2. \end{aligned}$$

# 5 The hyperbolicity region of $\mathrm{ET}_{13}$ in the one-dimensional field variables

In this section we will focus on the model that describes one-dimensional physical phenomena dependent only on one spatial variable (that we will call $x_1$), under the assumption that the velocity and the heat flux vectors are both parallel to the $x_1$-direction, i.e. $\mathbf{v} = (v_1, 0, 0)$, $\mathbf{q} = (q_1, 0, 0)$. We assume also that $\sigma_{12} = \sigma_{13} = \sigma_{23} = 0$ and $\sigma_{22} = \sigma_{33} = -\sigma_{11}/2$ (the viscous tensor is traceless). Thus, we will deal with five field variables $\mathbf{u} = (\rho, v_1, T, \sigma_{11}, q_1)$ and the set of the corresponding five PDE's of the form (15), that can be deduced from (13) neglecting the quadratic terms and supposing $\partial_k = \delta_{k1}\partial_{x_1}$.

The structure of the hyperbolicity region for a monatomic gas described by the one-dimensional $\mathrm{ET}^1_{13}$ (i.e. the well-known Grad system) was studied in the literature [1, 2]. We reconstruct here the model and the results to create a comparison benchmark for the next cases.

First of all, the explicit expressions of matrices $\mathbf{A}$ and $\mathbf{B}$ are easily deduced; $\mathbf{A}$ turns out to be non-singular ($\det(\mathbf{A}) = -6k_{\mathcal{B}}\rho^2/m$) and referring to the dimensionless "non-convective" characteristic velocities $\tilde{\lambda}$ and to the dimensionless quantities introduced in (14), one gets the characteristic polynomial (already determined in [1]):

$$\mathcal{P}^{(1)}_{1D}(\tilde{\lambda}) = \frac{k_{\mathcal{B}}}{m}\theta^{1/2}p^2\tilde{\lambda}\Big[a_4^{(1)}\tilde{\lambda}^4 + a_2^{(1)}\tilde{\lambda}^2 + a_1^{(1)}\tilde{\lambda} + a_0^{(1)}\Big],$$

with

$$\begin{aligned} a_4^{(1)} &= 6, \quad a_2^{(1)} = -\tfrac{4}{5}(39 - 31\tilde{\sigma}_{11}), \\ a_1^{(1)} &= -\tfrac{576\tilde{q}_1}{25}, \quad a_0^{(1)} = \tfrac{18}{5}(5 - 10\tilde{\sigma}_{11} + 7\tilde{\sigma}_{11}^2), \end{aligned} \tag{21}$$

and where the subscript $1D$ stands for the case of one-dimensional field variables and the index (1) is associated to the first order expansion of the theory. At equilibrium the values of $\tilde{\lambda}$ are all real and distinct [1, 2]

$$\tilde{\lambda}_1\Big|_E = 0, \qquad \tilde{\lambda}_{2,3}\Big|_E = \pm\sqrt{\frac{13-\sqrt{94}}{5}}, \qquad \tilde{\lambda}_{4,5}\Big|_E = \pm\sqrt{\frac{13+\sqrt{94}}{5}}, \tag{22}$$

and this fact guarantees the existence of a basis of eigenvectors and the hyperbolicity property at the equilibrium state. By continuity arguments, such a property has to be valid in a neighborhood of the equilibrium. The analytic equations for the boundaries of the hyperbolicity region could be deduced ([1, 2]) through the procedure described in Section 3 for $n = 4$, obtaining from (21) and (18), the parametric equations for the hyperbolic region:

$$\begin{cases} 9\lambda_*^4 + \frac{2}{5}(-39 + 31\tilde{\sigma}_{11})\lambda_*^2 - \frac{9}{5}(5 - 10\sigma_{11} + 7\tilde{\sigma}_{11}^2) = 0, \\[2mm] 3\lambda_*^3 + \dfrac{1}{5}(-39 + 31\tilde{\sigma}_{11})\lambda_* - \dfrac{72\tilde{q}_1}{25} = 0, \end{cases}$$

that can be made explicit as

$$\tilde{q}_1 = \pm\frac{\sqrt{5}}{648}\sqrt{\gamma_1 + \sqrt{\gamma_2}}\,(-2\gamma_1 + \sqrt{\gamma_2}),$$

$$\text{where} \qquad \gamma_1 = 39 - 31\tilde{\sigma}_{11},\ \ \gamma_2 = 3546 + 4\tilde{\sigma}_{11}(-1617 + 949\tilde{\sigma}_{11}).$$

The corresponding hyperbolicity region is the domain enclosed in the two curves for $\sigma_{11} \leq 3(89 - \sqrt{7545})/8$, as shown in Figure 1 and coincide with the one deduced in [2], except for different dimensionless variables.

Figure 1: The boundaries of the hyperbolicity region for the one-dimensional $\mathrm{ET}^1_{13}$ model in the $(\tilde{q}_1, \tilde{\sigma}_{11})$ space. The hyperbolicity region is denoted by 'I', while the zones that do not satisfy hyperbolicity property is labeled with 'II'. The Grey circle represents the circle with the maximum radius, centered in the equilibrium point and inscribed within the hyperbolicity region.

We move now to the second order of expansion $\mathrm{ET}^2_{13}$, still in the case of one-dimensional field variables. In that case, the characteristic polynomial associated with the system (13) with $\partial_k = \delta_{k1}\partial_{x_1}$, becomes

$$\mathcal{P}^{(2)}_{1D}(\tilde{\lambda}) = \frac{k_{\mathcal{B}}}{m}\theta^{1/2}p^2\left[a^{(2)}_5\tilde{\lambda}^5 + a^{(2)}_4\tilde{\lambda}^4 + +a^{(2)}_3\tilde{\lambda}^3 + a^{(2)}_2\tilde{\lambda}^2 + a^{(2)}_1\tilde{\lambda} + a^{(2)}_0\right], \tag{23}$$

where the index (2) corresponds to the second order approximation theory, and

$$\begin{aligned}
a^{(2)}_5 &= 6, \qquad a^{(2)}_4 = -\frac{1104\tilde{q}_1}{25},\\
a^{(2)}_3 &= -\frac{2}{625}(150(\tilde{\sigma}_{11} - 1)(18\tilde{\sigma}_{11} - 65) - 19684\tilde{q}_1^2),\\
a^{(2)}_2 &= -\frac{72\tilde{q}_1}{625}(148\tilde{q}_1^2 + 8\tilde{\sigma}_{11}^2 + 945\tilde{\sigma}_{11} - 975),\\
a^{(2)}_1 &= \frac{2}{625}(5625(-1 + \tilde{\sigma}_{11})^2 - 36\tilde{q}_1^2(76\tilde{\sigma}_{11}^2 - 619\tilde{\sigma}_{11} + 580)),\\
a^{(2)}_0 &= \frac{72}{25}\tilde{q}_1(\tilde{\sigma}_{11} - 1)^2(2\tilde{\sigma}_{11} - 5).
\end{aligned} \tag{24}$$

Obviously, at equilibrium the roots of $\mathcal{P}^{(2)}_{13,1D} = 0$ coincide with those of the first order expansion and the same considerations about hyperbolicity remain valid. Beyond the equilibrium point, we deal with the fifth degree polynomial (23), that cannot be factorized analytically. However, one can refer to Section 3 in order to determine the parametric equations of the boundaries, that turn out to be exactly the ones presented as an example in (20). In Figure 2 such boundaries are plotted in the $(\tilde{q}_1, \tilde{\sigma}_{11})$-plane. The hyperbolicity region is composed by

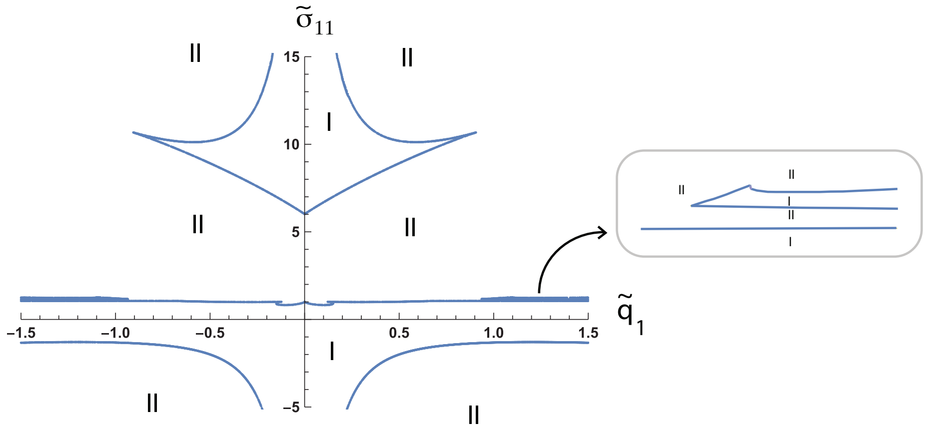


Figure 2: The boundaries of the hyperbolicity region in the $(\tilde{q}_1, \tilde{\sigma}_{11})$ for the $\mathrm{ET}^2_{13}$ model,when one-dimensional variables are taken into account. The hyperbolicity region is denoted by 'I', while the zones that do not satisfy hyperbolicity property is labeled with 'II'. A zoom is also presented to show the details of the structure of the hyperbolicity region.

different zones that we have denoted with the symbol 'I', bounded by the line described by the parametric equations (20) with the coefficients (24).

We stress that the results presented in Figure 2 are compatible with those obtained with a different approach in [9], if one considers the limit of non-degeneracy of the gas.

The hyperbolicity regions corresponding to $\mathrm{ET}^1_{13}$ and $\mathrm{ET}^2_{13}$ present completely different structures, so that it is hard to compare them quantitatively. For this reason, we construct the maximum circles centered in the equilibrium state and inscribed in the regions and compare their radius. In Figure 1 and in Figure 3 one can find the maximum circle for $\mathrm{ET}^1_{13}$ and $\mathrm{ET}^2_{13}$. The region corresponding to the quadratic expansion turns out to be larger in the neighborhood of the equilibrium state, that is to say where the expansion is valid. In fact, for the Grad system the maximum radius[2] is $\tilde{r}^{(1)} \simeq 0.5432$, while for the second order expansion the radius is $\tilde{r}^{(2)} \simeq 0.8220$.

Figure 3: The maximum radius circle with the center in the equilibrium point, inscribed in the hyperbolicity region for $\mathrm{ET}^2_{13}$ in the $(\tilde{q}_1, \tilde{\sigma}_{11})$-plane.

[2] The values of the dimensionless radius $\tilde{r}^{(1)}$ calculated in [1, 2] is different with respect to the present one, since different dimensionless variables are used, as already remarked.

# 6 The hyperbolicity region of $\mathrm{ET}_{13}$ for three-dimensional field variables

In this section following [10], we will consider all the 13 field components $\mathbf{u} = (\rho, v_j, T, \sigma_{jk}, q_j)$ for $j, k = 1, 2, 3$ (with $\sum_{l=1}^{3} \sigma_{ll} = 0$) assuming that the variables still depend on $t$ and only on one space variable $x_1$. The corresponding equations system has the structure of (15) and is formed by 13 equations.

The hyperbolicity region is now a manifold in a multi-dimensional space, difficult to visualize. Just to get an idea of the structure of the region, in what follows we will refer to its two-dimensional sections obtained under the assumptions that all the non-equilibrium variables vanish except two of them. Unfortunately, for three-dimensional field variables the results cannot be obtained in a completely analytical way, even for the linearized model: all the results presented in this section were achieved through some numerical steps.

Referring to $\mathrm{ET}_{13}^{1}$, calculating the corresponding matrices $\mathbf{A}$ and $\mathbf{B}$ and taking into account the dimensionless "non-convective" characteristic velocities $\tilde{\lambda}$ and the dimensionless quantities (14), it is possible to determine the characteristic polynomial $\mathcal{P}(\tilde{\lambda}) = \left(\det\left[\mathbf{B} - (\tilde{\lambda} c_0 + v_1)\mathbf{A}\right]\right)$ and, in particular, to verify that matrix $\mathbf{A}$ is non-singular $(\det(\mathbf{A}) = -24 k_{\mathcal{B}} \rho^4/m)$. The analytic expression of the complete characteristic polynomial is very cumbersome and we will not report it here in its general form. At equilibrium the eigenvalues are all real:

$$\tilde{\lambda}_{1,2,3,4,5}\Big|_E = 0, \qquad \tilde{\lambda}_{6,7}\Big|_E = \sqrt{\frac{7}{5}}, \qquad \tilde{\lambda}_{8,9}\Big|_E = -\sqrt{\frac{7}{5}},$$

$$\tilde{\lambda}_{10,11}\Big|_E = \pm\sqrt{\frac{13-\sqrt{94}}{5}}, \qquad \tilde{\lambda}_{12,13}\Big|_E = \pm\sqrt{\frac{13+\sqrt{94}}{5}},$$

and there exists a basis of independent eigenvectors, so that the hyperbolicity property is guaranteed at equilibrium. The hyperbolicity region of the three-dimensional $\mathrm{ET}_{13}$ would be a manifold in a space in 8 dimensions, since 8 is the number of the non-equilibrium field components. To get an idea of its structure we have focused on its 28 two-dimensional sections (obtained under the assumption that only two non-equilibrium variables are different from zero). Here we will present only two of them. We remark, also, that if all the non-equilibrium variables vanish except $q_1$ and $\sigma_{11}$, the solutions of the 1D case (Figure 1) is obviously achieved.

For the $\mathrm{ET}_{13}^{1}$ theory, the section of the hyperbolicity region in the plane $(\tilde{q}_1, \tilde{\sigma}_{12})$ presents a singularity due to the fact that the equilibrium point belongs to the boundary of the region, as already predicted by Cai, Fan and Li [10, 11].

In fact, the corresponding characteristic polynomial is factorized as

$$\mathcal{P}^{(1)}_{3D}(\tilde{\lambda})\,|_{q_1,\sigma_{12}} = \frac{24k_{\mathcal{B}}p^4\theta^{7/2}}{15625m}\tilde{\lambda}^2\big(25\tilde{\lambda}^3 - 35\tilde{\lambda} - 14\tilde{q}_1\big)\Big(c_8^{(1)}\tilde{\lambda}^8 + c_6^{(1)}\tilde{\lambda}^6 + c_5^{(1)}\tilde{\lambda}^5 + \\ + c_4^{(1)}\tilde{\lambda}^4 + c_3^{(1)}\tilde{\lambda}^3 + c_2^{(1)}\tilde{\lambda}^2 + c_1^{(1)}\tilde{\lambda}^1 + c_0^{(1)}\Big)\,,$$

and referring to the dimensionless variables (14), we have

$$\begin{aligned} &c_8^{(1)} = 625, \quad c_6^{(1)} = -4125, \quad c_5^{(1)} = -2750\tilde{q}_1, \quad c_4^{(1)} = 1200\tilde{\sigma}_{12}^2 + 6425, \\ &c_3^{(1)} = 5180\tilde{q}_1, \quad c_2^{(1)} = 200\tilde{\sigma}_{12}^2 + 1344\tilde{q}_1^2 - 2625, \\ &c_1^{(1)} = 2240\tilde{q}_1\tilde{\sigma}_{12} - 1050\tilde{q}_1, \quad c_0^{(1)} = -700\tilde{\sigma}_{12}^2. \end{aligned} \tag{25}$$

From (25) it is evident that when $q_1$ vanishes, the last eighth grade polynomial assumes a quadratic form with at least a couple of imaginary roots, inasmuch as $c_8^{(1)} = 0$ and $c_0^{(1)} < 0$. The complete structure of the hyperbolicity region presented in Figure 4 was obtained applying the method introduced in Section 3 to the eighth grade polynomial, since it is easily shown that the third grade polynomial presents always real roots for $|\tilde{q}_1| \leq \frac{5}{3}\sqrt{\frac{7}{15}}$.

To consider a simple case that can be studied in a completely analytic way we focus, then, on the section in $(\tilde{\sigma}_{12}, \tilde{\sigma}_{13})$–plane. If all the non-equilibrium variables vanish except $\tilde{\sigma}_{12}$ and $\tilde{\sigma}_{13}$ and we pose $\tilde{\sigma}^2 = \tilde{\sigma}_{12}^2 + \tilde{\sigma}_{13}^2$, the characteristic polynomial reduces to

$$\mathcal{P}^{(1)}_{3D}\Big|_{\sigma_{12},\sigma_{13}}(\tilde{\lambda}) = \frac{24k_{\mathcal{B}}\theta^{5/2}p^4}{(125m)}\tilde{\lambda}^3(-7 + 5\tilde{\lambda}^2)\Big[d_8^I\tilde{\lambda}^8 + \\ + d_6^{(1)}\tilde{\lambda}^6 + d_4^{(1)}\tilde{\lambda}^4 + d_4^{(1)}\tilde{\lambda}^4 + d_2^{(1)}\tilde{\lambda}^2 + d_0^{(1)}\Big]\,,$$

with

$$d_8^{(1)} = 25, \qquad d_6^{(1)} = -165, \qquad d_4^{(1)} = 257 + 48\tilde{\sigma}^2,$$

$$d_2^{(1)} = -105 + 8\tilde{\sigma}^2, \qquad d_0^{(1)} = -28\tilde{\sigma}^2.$$

The square brackets contain a quadratic eighth grade polynomial with a positive coefficient $d_8^{(1)}$ and a non-positive term $d_0^{(1)}$ that is proportional to $\tilde{\sigma}^2$. Thus, the roots of the eighth polynomial are all real if and only if $\sigma = \tilde{\sigma}_{12} = \tilde{\sigma}_{13} = 0$. Thus, the section of the hyperbolicity region in the plane $(\tilde{\sigma}_{12}, \tilde{\sigma}_{13})$ reduces only to the equilibrium point, implying the instability of the equation system [10, 11].

We pass now to the second order expansion model $\mathrm{ET}^2_{13}$, that is studied here for the first time under the assumption of three-dimensional field variables. The determinant of matrix $\mathbf{A}$ remains the same as for $\mathrm{ET}^1_{13}$ and the hyperbolicity of the PDE system is guaranteed if the generalized eigenvalues problem presents all real eigenvalues and a basis of eigenvectors. Also for the quadratic theory the thirteenth grade characteristic polynomial is not easy to 'handle'! We have

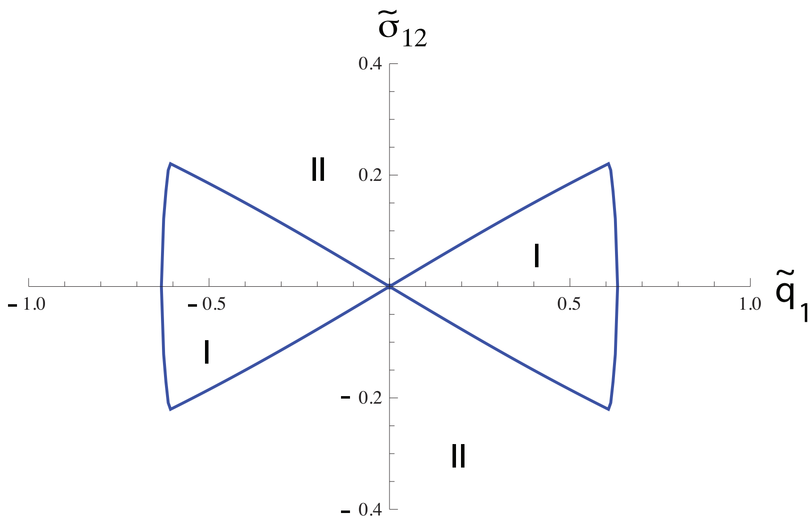


Figure 4: Section of the hyperbolicity region in the $(\tilde{q}_1, \tilde{\sigma}_{12})$-plane for the $\mathrm{ET}^1_{13}$ theory. The equilibrium state is on the boundary of the region [11]. The hyperbolicity region is denoted by 'I', while the zones that do not satisfy hyperbolicity property is labeled with 'II'.

studied the 28 two-dimensional sections in the 8 dimension space discovering that there exists a neighborhood of the equilibrium point completely contained in the hyperbolicity region. Here we analyse the same sections as for the linear theory. If we assume that the only non-vanishing non-equilibrium variables are $\tilde{q}_1$ and $\tilde{\sigma}_{12}$, the characteristic polynomial reduces to

$$\mathcal{P}^{(2)}_{3D}\Big|_{\sigma_{12},\sigma_{13}}(\tilde{\lambda}) = \frac{8k_{\mathcal{B}}\theta^{5/2}p^4}{(6103515625m)}\mathcal{P}_4(\tilde{\lambda})\mathcal{P}_9(\tilde{\lambda}),$$

where

$$\begin{aligned}\mathcal{P}_4(\tilde{\lambda}) =& 3125\tilde{\lambda}^4 - 13500\tilde{q}_1\tilde{\lambda}^3 - (4375 - 17760\tilde{q}_1^2 + 2500\tilde{\sigma}_{12}^2)\tilde{\lambda}^2 + \\ &+ (8750\tilde{q}_1 - 7168\tilde{q}_1^3 + 3200\tilde{q}_1\tilde{\sigma}_{12}^2)\tilde{\lambda} - 4200\tilde{q}_1^2 + 1250\tilde{\sigma}_{12}^2\end{aligned}$$

and

$$\mathcal{P}_9(\tilde{\lambda}) = \sum_{k=0}^{9} c_k^{(2)}\tilde{\lambda}^k \tag{26}$$

with

$$
\begin{aligned}
c_9^{(2)} =& 5859375, \quad c_8^{(2)} = -68437500\tilde{q}_1, \\
c_7^{(2)} =& -38671875 + 281112500\tilde{q}_1^2 - 15000000\tilde{\sigma}_{12}^2 \\
c_6^{(2)} =& 318093750\tilde{q}_1 - 540912000\tilde{q}_1^3 + 104200000\tilde{q}_1\tilde{\sigma}_{12}^2, \\
c_5^{(2)} =& 60234375 - 927002500\tilde{q}_1^2 + 520434240\tilde{q}_1^4 + 70481250\tilde{\sigma}_{12}^2 - \\
& - 250496000\tilde{q}_1^2\tilde{\sigma}_{12}^2 + 9600000\tilde{\sigma}_{12}^4, \\
c_4^{(2)} =& -328875000\tilde{q}_1 + 1228649000\tilde{q}_1^3 - 235720192\tilde{q}_1^5 - \\
& - 268515000\tilde{q}_1\tilde{\sigma}_{12}^2 + 274594560\tilde{q}_1^3\tilde{\sigma}_{12}^2 - 21248000\tilde{q}_1\tilde{\sigma}_{12}^4 \\
c_3^{(2)} =& -24609375 + 600075000\tilde{q}_1^2 - 751718400\tilde{q}_1^4 + 38191104\tilde{q}_1^6 - \\
& - 38906250\tilde{\sigma}_{12}^2 + 246656000\tilde{q}_1^2\tilde{\sigma}_{12}^2 - 148789248\tilde{q}_1^4\tilde{\sigma}_{12}^2 + \\
& + 45825000\tilde{\sigma}_{12}^4 + 19230720\tilde{q}_1^2\tilde{\sigma}_{12}^4),
\end{aligned}
$$

$$
\begin{aligned}
c_2^{(2)} =& 68906250\tilde{q}_1 - 450360000\tilde{q}_1^3 + 172045440\tilde{q}_1^5 + 37237500\tilde{q}_1\tilde{\sigma}_{12}^2 + \\
& + 13531200\tilde{q}_1^3\tilde{\sigma}_{12}^2 + 36225024\tilde{q}_1^5\tilde{\sigma}_{12}^2 - 192540000\tilde{q}_1\tilde{\sigma}_{12}^4 - \\
& - 23617536\tilde{q}_1^3\tilde{\sigma}_{12}^4 \\
c_1^{(2)} =& (-63000000\tilde{q}_1^2 + 119952000\tilde{q}_1^4 + 11250000\tilde{\sigma}_{12}^2 + 66345000\tilde{q}_1^2\tilde{\sigma}_{12}^2 - \\
& - 566605440\tilde{q}_1^4\tilde{\sigma}_{12}^2 - 20362500\tilde{\sigma}_{12}^4 + 159264000\tilde{q}_1^2\tilde{\sigma}_{12}^4 + \\
& + 8257536\tilde{q}_1^4\tilde{\sigma}_{12}^4 - 21600000\tilde{\sigma}_{12}^6) \\
c_0^{(2)} =& 18900000\tilde{q}_1^3 - 9000000\tilde{q}_1\tilde{\sigma}_{12}^2 - 56028000\tilde{q}_1^3\tilde{\sigma}_{12}^2 + 16290000\tilde{q}_1\tilde{\sigma}_{12}^4 - \\
& - 30105600\tilde{q}_1^3\tilde{\sigma}_{12}^4 + 17280000\tilde{q}_1\tilde{\sigma}_{12}^6.
\end{aligned}
$$

First of all we remark that there exists a complete symmetry if the signs of $\tilde{q}_1$ and or $\tilde{\sigma}_{12}$ are changed, as also evident in Figure 5. Moreover, it is possible to verify that $\mathcal{P}_4(\tilde{\lambda})$ presents always real roots, so that the boundaries of the hyperbolicity region section are determined studying the properties of polynomial (26). In this regard, we have applied the method introduced in Section 3, analyzing the compatibility equations in (18) of the ninth grade polynomial. The two-dimensional section of the hyperbolicity region is shown in Figure 5, where it is clear that there exists a neighborhood of the equilibrium state completely enclosed in the region.

The differences between the hyperbolicity region for the first and the second order expansion are unequivocal also when $\tilde{\sigma}_{12}$ and $\tilde{\sigma}_{13}$ are the only non-vanishing non-equilibrium variables. In that case, the characteristic polynomial reduces to:

$$
\mathcal{P}_{3D}^{(2)}\Big|_{\sigma_{12},\sigma_{13}}(\tilde{\lambda}) = \frac{24k_{\mathcal{B}}p^4\theta^{5/2}}{3125m}\tilde{\lambda}(5\tilde{\lambda}^4 - (7+4\tilde{\sigma}^2)\tilde{\lambda}^2 + 2\tilde{\sigma}^2)\mathcal{P}_8(\tilde{\lambda}),
$$

with

$$
\mathcal{P}_8(\tilde{\lambda}) = \left[d_8^{(2)}\tilde{\lambda}^8 + d_6^{(2)}\tilde{\lambda}^6 + d_4^{(2)}\tilde{\lambda}^4 + d_2^{(2)}\tilde{\lambda}^2 + d_0^{(2)}\right], \tag{27}
$$

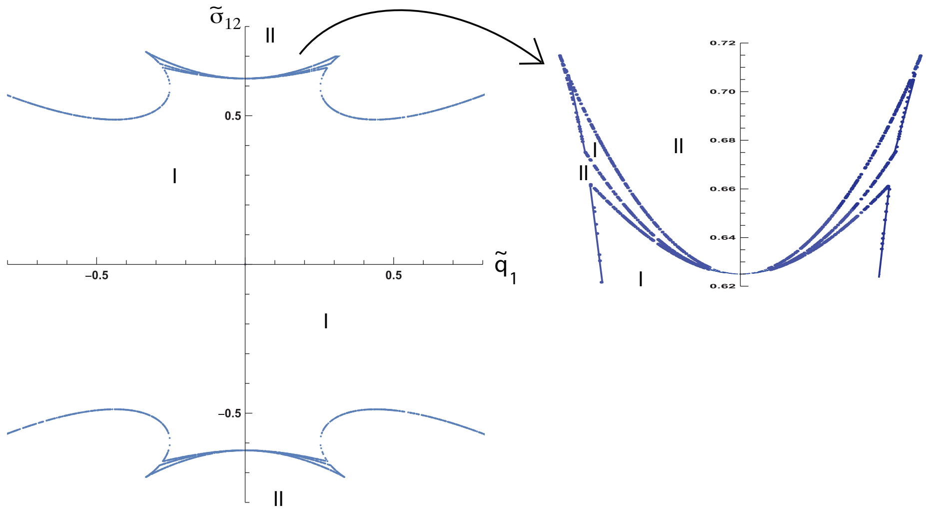


Figure 5: The section of the hyperbolicity region in the $(\tilde{q}_1, \tilde{\sigma}_{12})$–plane for the $\mathrm{ET}^2_{13}$ three-dimensional theory, on the left the zoom of some details. The hyperbolicity region is denoted by 'I', while the zones that do not satisfy hyperbolicity property is labeled with 'II'.

$\tilde{\sigma}^2 = \tilde{\sigma}^2_{12} + \tilde{\sigma}^2_{13}$ and, moreover,

$$d_8^{(2)} = 625, \quad d_6^{(2)} = (-4125 - 1600\tilde{\sigma}^2), \quad d_4^{(2)} = (6425 + 7518\tilde{\sigma}^2 + 1024\tilde{\sigma}^4),$$

$$d_2^{(2)} = (-2625 - 4150\tilde{\sigma}^2 + 4888\tilde{\sigma}^4) \quad d_0^{(2)} = 1200\tilde{\sigma}^2 - 2172\tilde{\sigma}^4 - 2304\sigma^6.$$

We notice that $\mathcal{P}^{(2)}_{3D}|_{\sigma_{12},\sigma_{13}}(\tilde{\lambda})$ contains the product of a fourth grade quadratic polynomial (with real roots for any value of $\tilde{\sigma}^2$) and the eighth grade quadratic polynomial $\mathcal{P}_8(\tilde{\lambda})$. It is proven that such last polynomial (27) presents real roots if $\tilde{\sigma}^2 \leq 25/64$ (i.e. $d_0^{(2)} \geq 0$). In fact if we write the polynomial in $\mu = \tilde{\lambda}^2$, (27) becomes a fourth grade polynomial that at equilibrium presents 3 positive and a null roots, Moreover, since $d_0^{(2)} > 0$, there exists a neighborhood of the equilibrium state for which all the roots $\mu$ are real and positive. In this way the reality of the eigenvalues $\tilde{\lambda} = \pm\sqrt{\mu}$ is guaranteed at least in a neighborhood of the equilibrium. Hence, we conclude that in the quadratic approximation there exists a neighborhood of the equilibrium point within the hyperbolicity region, overcoming the instability problem.

# 7 Conclusions

In this paper, we have presented the second order RET model for a monatomic gas described by 13 moments and we have studied its hyperbolicity properties, comparing the results with those for the corresponding linearized model, already known in the literature. We have verified that the radius of the hyperbolicity regions for $\mathrm{ET}^2_{13}$ is in general, larger than those for $\mathrm{ET}^1_{13}$. This fact suggests the idea that the hyperbolicity region grows together with the approximation order. Moreover, there exists a neighborhood of the equilibrium state enclosed in the region, in contrast to the three-dimensional first-order theory. This fact suggests a way to overcome the problems already studied by Cai, Fan Li [10, 11], without loss of validity of the entropy principle and keeping the balance law structure of the PDE system. The extension of these results to Extended Thermodynamics of polyatomic gases will be soon submitted and also comparisons with experimental data will be carried on to test the second order theory.

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