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Published Version:
New global bifurcation diagrams for piecewise smooth systems: Transversality of homoclinic points does not imply chaos / Franca, M.; Pospíšil, M.. - In: JOURNAL OF DIFFERENTIAL EQUATIONS. - ISSN 0022-0396. - STAMPA. - 266:2-3(2019), pp. 1429-1461. [10.1016/j.jde.2018.07.078]

Availability:
This version is available at: https://hdl.handle.net/11585/721080 since: 2020-02-03

Published:
DOI: http://doi.org/10.1016/j.jde.2018.07.078

JID:YJDEQ AID:9474 /FLA [m1+; v1.287; Prn:2/08/2018; 12:30] P.1 (1-33)

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New global bifurcation diagrams for piecewise smooth systems: Transversality of homoclinic points does not imply chaos

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Received 11 April 2018; revised 28 July 2018

Abstract

In this paper we consider some piecewise smooth 2-dimensional systems having a possibly non-smooth homoclinic $\vec{\gamma}(t)$. We assume that the critical point $\vec{0}$ lies on the discontinuity surface Ω^0. We consider 4 scenarios which differ for the presence or not of sliding close to $\vec{0}$ and for the possible presence of a transversal crossing between $\vec{\gamma}(t)$ and Ω^0. We assume that the systems are subject to a small non-autonomous perturbation, and we obtain 4 new bifurcation diagrams. In particular we show that, in one of these scenarios, the existence of a transversal homoclinic point guarantees the persistence of the homoclinic trajectory but chaos cannot occur. Further we illustrate the presence of new phenomena involving an uncountable number of sliding homoclinics.

MSC: 34C23; 34C37; 34A36; 37G20

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[1] Partially supported by the GNAMPA project "*Sistemi dinamici, metodi topologici e applicazioni all'analisi nonlineare*".

[2] Partially supported by the Grant VEGA 1/0078/17, VEGA-SAV 2/0153/16 and by the Slovak Research and Development Agency under the contract no. APVV-14-0378.

https://doi.org/10.1016/j.jde.2018.07.078
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Keywords: Homoclinic orbit; Melnikov theory; Piecewise smooth systems; Sliding; Chaos; Transversal homoclinic points

1. Introduction

The appearance of a chaotic pattern for smooth non-autonomous dynamical systems is nowadays a widely investigated topic. A well-established fact is that, if we perturb a smooth dynamical system which admits a transversal homoclinic point, a chaotic pattern arises (see e.g. [22,38,41]).

Consider an autonomous differential equation having a non-degenerate family of trajectories homoclinic to a critical point, say the origin. Mel'nikov theory gives a generic integral condition which is sufficient to ensure the persistence of a homoclinic trajectory to a small time-dependent forcing; the same requirement together with some further weak recurrence properties (e.g. periodicity, almost periodicity) is sufficient to prove the insurgence of chaos.

This fact was first noticed in a two-dimensional example by Mel'nikov in [31]. His work was later refined and generalized in many ways, see e.g. [2,7,12,21,22,30,32–34,37,39,41], and using different approaches. Nowadays, the problem for smooth systems, is well understood in the general $n \geq 2$-dimensional case.

Recently the theory was extended to embrace the case of piecewise smooth systems, see [3–6]. In particular Battelli and Fečkan considered a piecewise smooth setting, but assuming that the critical point does not lie on the discontinuity surface Ω^0. As in the smooth case the condition found by Battelli and Fečkan ensures both persistence of the homoclinic [3] and the insurgence of chaos if the system is recurrent (e.g. almost periodic), either in the case where the unperturbed homoclinic undergoes to sliding, see [4], or in the case where there is no sliding, see [5].

In this paper we consider the piecewise smooth system

$$\dot{\vec{x}} = \vec{f}^{\pm}(\vec{x}) + \varepsilon \vec{g}(t, \vec{x}, \varepsilon), \quad \vec{x} \in \Omega^{\pm}, \tag{PS}$$

where $\Omega^{\pm} = \{\vec{x} \in \Omega \mid \pm G(\vec{x}) > 0\}$, $\Omega^0 := \{\vec{x} \in \Omega \mid G(\vec{x}) = 0\}$, $\Omega \subset \mathbb{R}^2$ is an open set, G is a C^r-function on $\overline{\Omega}$ with $r \geq 2$ and 0 is a regular value of G. Next, $\varepsilon \in \mathbb{R}$ is a parameter, and $\vec{f}^{\pm} \in C_b^r(\Omega^{\pm} \cup \Omega^0, \mathbb{R}^2)$, $\vec{g} \in C_b^r(\mathbb{R} \times \Omega \times \mathbb{R}, \mathbb{R}^2)$ and $G \in C_b^r(\Omega, \mathbb{R})$, i.e., the derivatives of $\vec{f}^{\pm}$, $\vec{g}$ and G are uniformly continuous and bounded up to the r-th order, respectively. Here and in the sequel we use the shorthand notation $\pm$ to represent both the $+$ and $-$ equations and functions.

We assume that the critical point $\vec{0}$ lies on Ω^0. In [10] under this assumptions it was shown that the Mel'nikov condition found by Battelli and Fečkan, together with a further (generic) transversality requirement (always satisfied in 2 dimensions), is enough to prove the persistence of the homoclinic.

The purpose of the present paper is to illustrate some new bifurcation diagrams which arise in this discontinuous setting. In particular we want to illustrate a wide class of piecewise smooth examples in 2 dimensions, which fits the assumption of [10], so that persistence of the homoclinic is ensured, but Mel'nikov chaos is not present. We emphasize that this discrepancy is not present in the smooth setting or even in the piecewise smooth setting but assuming that $\vec{0} \notin \Omega^0$.

More precisely let $\vec{\gamma}(t)$ denote the unperturbed homoclinic: we can split piecewise smooth systems in 2 dimensions into two main open classes (plus some border cases). The first, say **N** where *there is no sliding close to the origin*, containing smooth systems, and the second, say **S** where *there is sliding close to the origin* (see Fig. 1 and Section 3 for details).

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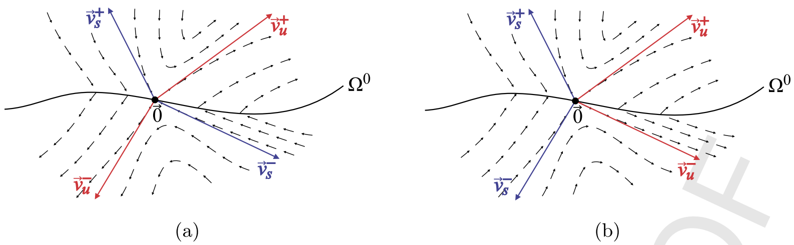

Fig. 1. Different mutual positions of eigenvectors $\vec{v}_s^{\pm}$ and $\vec{v}_u^{\pm}$ in discontinuous systems. Notice that in Fig. 1a trajectories which touch Ω^0 close to the origin cross it transversally (class **N**), while in Fig. 1b they remain on Ω^0 (class **S**).

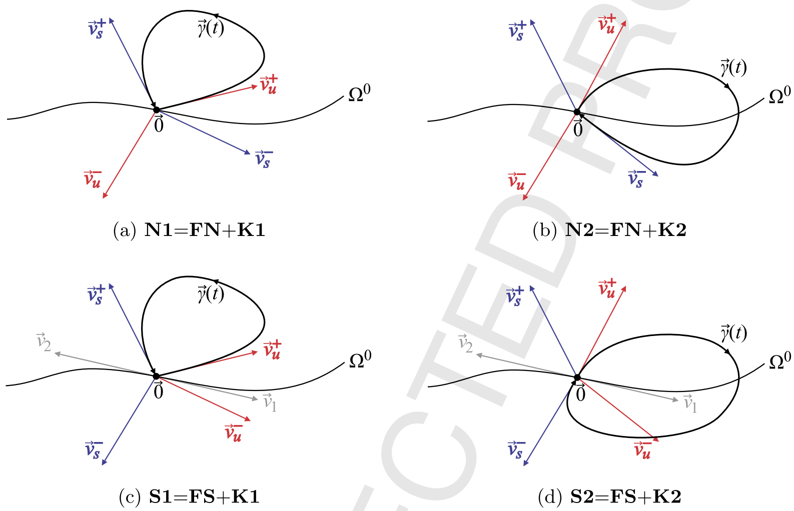

Fig. 2. Subclasses of **N** and **S** as various combinations of assumptions **FN**, **FS**, **K1** and **K2**.

Each of the classes can be split further (see Fig. 2 and Section 3 for details) to **N1** (**S1**) and **N2** (**S2**) depending on whether $\vec{\gamma}(t) \notin \Omega^0$ for any $t \in \mathbb{R}$ or there is some finite $t_0 \in \mathbb{R}$, e.g. $t_0 = 0$, such that $\vec{\gamma}(t)$ crosses transversally Ω^0 at t_0. In both the cases, **N** and **S**, we have some further degenerate cases, which will not be discussed in this article, where $\vec{\gamma}(t)$ is tangent to Ω^0 for some $t \in \mathbb{R}$. In this paper we describe the bifurcation diagrams arising in these four cases as a perturbation is introduced.

The Mel'nikov condition given by [10] (which reduces to the classical one in the smooth cases) ensures the persistence of homoclinic trajectories which do not exhibit sliding phenomena in all the classes **Ni**, **Si** for $i = 1, 2$. Further if we have some weak recurrence properties, e.g. almost periodicity, the insurgence of a chaotic pattern (made up by trajectories which do not slide) is ensured by this Mel'nikov condition for systems **N1** and **S1**, using the methods of [11]. One of the main contribution of this article is to show that chaos does not exist for systems of type **S2** even if we have a transversal homoclinic point, see Theorem 5.7 and Remark 5.9. We conjecture that chaos will be again present for systems **N2**, but this will be the object of future investigation.

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Quite surprisingly this obstruction to chaos (i.e. the fact that the presence of transversal homoclinic points does not imply chaos) seems to be typical of the two-dimensional case (apart from completely decoupled systems).

Further if we perturb systems **S1** or **S2** we prove the appearance of a new phenomenon: the insurgence of an uncountable number of homoclinic trajectories sliding near the homoclinic point. In the **S1** setting these sliding homoclinics might make several loops, following a prescribed sequence of 0s and 1s similarly to the chaotic case (and a "classical" non-sliding chaotic behavior is present), see Theorem 5.5 and property ***Cs***; in the **S2** setting the sliding homoclinics make at most one loop and cross Ω^0 at most once, see Theorem 5.2.

The study of piecewise smooth systems has received a great impulse recently due to its relevance in applications. These equations are commonly used to describe mechanical systems with dry friction or impacts, see e.g. [9]. In particular in the former case it has to be expected that critical points lie on the discontinuity surface: this is the case, e.g., of the inverted pendulum with dry friction. Piecewise smooth systems are also of use in the study of power electronics when we have state dependent switches [1], walking machines [20], relay feedback systems [8], biological systems [36] see also [15,27] and the references therein.

The paper is organized as follows: In Section 2 we define some basic notions, we state some basic assumptions and some classic results concerning chaos in smooth systems. In Section 3 we set the assumptions **FN**, **FS**, **K1**, **K2** that allow to define rigorously systems **N1**, **N2**, **S1**, **S2** and we recall some known results concerning persistence of homoclinic solution and insurgence of chaos in a piecewise-smooth setting. In Section 4 we derive results on the position of trajectories close to the homoclinic in the smooth system. Section 5 is devoted to the main results of this paper – investigation of the four above-mentioned cases. In Section 6 we give the proofs of some technical results: this technical part can be regarded as a sort of appendix, and even if it is needed for the proof, it is not of help for the comprehension of the main argument.

2. Preliminaries

Throughout the paper we use bold letters for matrices, e.g. $\boldsymbol{M}$, the arrows for vectors, e.g. $\vec{m}$, and normal letters for scalars, e.g. m. We shall use the notation $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ for the inner product in $\mathbb{R}^2$ and the norm generated by it, respectively. The lower index shall denote a partial derivative with respect to that variable unless this makes confusion, $\wedge$ stands for the cross product in $\mathbb{R}^2$, i.e. if $\vec{a} = (a_1, a_2)^*$ and $\vec{b} = (b_1, b_2)^*$, then $\vec{a} \wedge \vec{b} = a_1 b_2 - a_2 b_1$.

In Section 2.1 we introduce the definition of solutions for discontinuous systems and in particular of sliding, crossing-sliding, sliding-crossing solutions; then in Section 2.2 we recall some standard results of Mel'nikov theory for smooth systems.

2.1. Definition of solutions in the discontinuous setting

Let us give a definition of what we mean by a solution of the piecewise smooth system (PS). For the simplicity we use the following notation:

$$\begin{aligned} \vec{F}^{\pm}(\vec{x}, t, \varepsilon) &= \vec{f}^{\pm}(\vec{x}) + \varepsilon \vec{g}(t, \vec{x}, \varepsilon), \quad \vec{x} \in \Omega^{\pm} \cup \Omega^0, \\ F_{\perp}^{\pm}(\vec{x}, t, \varepsilon) &= \left(\vec{\nabla} G(\vec{x})\right)^* \vec{F}^{\pm}(\vec{x}, t, \varepsilon), \quad \vec{x} \in \Omega^0. \end{aligned}$$

We say that a function $\vec{x}$ is a solution of (PS) if it is continuous, piecewise C^r, satisfies equation (PS) in $\Omega^{\pm}$, and if $\vec{x}(t_0) \in \Omega^0$ for some $t_0 \in \mathbb{R}$ we have one of the following situations.

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In the first setting we assume that both the flows of (PS+), and of (PS−) at $\vec{x}(t_0)$ point towards Ω^+ (or they both point towards Ω^-), i.e. $F_\perp^+(\vec{x}(t_0), t_0, \varepsilon)F_\perp^-(\vec{x}(t_0), t_0, \varepsilon) > 0$. In this case there is no sliding and the definition is almost obvious. In the second setting we assume $F_\perp^+(\vec{x}(t_0), t_0, \varepsilon)F_\perp^-(\vec{x}(t_0), t_0, \varepsilon) < 0$, i.e., the flows of (PS+) and of (PS−) at $\vec{x}(t_0)$ point in opposite directions: in this case we suppose that there is sliding and we follow Filippov definition.

More precisely let $\vec{x}(t_0) \in \Omega^0$, and ρ be a sufficiently small number.

• If $F_\perp^+(\vec{x}(t_0), t_0, \varepsilon)$ and $F_\perp^-(\vec{x}(t_0), t_0, \varepsilon)$ are both positive for a solution we mean a continuous piecewise C^r function $\vec{x}(t)$ defined for $|t - t_0| < \rho$, such that $\vec{x}(t) \in \Omega^-$ and solves (PS−) for any $t \in (t_0 - \rho, t_0)$, $\vec{x}(t) \in \Omega^+$ and solves (PS+) for any $t \in (t_0, t_0 + \rho)$; the opposite if $F_\perp^\pm(\vec{x}(t_0), t, \varepsilon) < 0$. In this case we say that $\vec{x}(t)$ is a *crossing solution* at $t = t_0$.

• If $F_\perp^+(\vec{x}(t_0), t_0, \varepsilon)F_\perp^-(\vec{x}(t_0), t_0, \varepsilon) < 0$ for solution we mean a continuous piecewise C^r function $\vec{x}(t)$ defined for $|t - t_0| < \rho$, such that one of the following holds

– $\vec{x}(t) \in \Omega^0$ and $F_\perp^+(\vec{x}(t), t, \varepsilon)F_\perp^-(\vec{x}(t), t, \varepsilon) < 0$ for any $t \in (t_0 - \rho, t_0 + \rho)$, and in this interval $\vec{x}(t)$ solves the equation

$$\dot{\vec{x}} = \vec{F}^0(\vec{x}, t, \varepsilon), \quad \text{whenever } \vec{x} \in \Omega^0, \tag{2.1}$$

where

$$\vec{F}^0(\vec{x}, t, \varepsilon) = \big(1 - \beta(\vec{x}, t, \varepsilon)\big)\vec{F}^-(\vec{x}, t, \varepsilon) + \beta(\vec{x}, t, \varepsilon)\vec{F}^+(\vec{x}, t, \varepsilon),$$

$$\beta(\vec{x}, t, \varepsilon) = \frac{F_\perp^-(\vec{x}, t, \varepsilon)}{F_\perp^-(\vec{x}, t, \varepsilon) - F_\perp^+(\vec{x}, t, \varepsilon)}$$

(see [19]). In this case we say that $\vec{x}(t)$ is a *pure-sliding* solution at t_0. We note that $\vec{F}^0$ is a convex combination of $\vec{F}^+$, $\vec{F}^-$ such that $\vec{F}^0$ is tangent to Ω^0; hence the solution of (2.1) will remain on Ω^0 for any $t \in (t_0 - \rho, t_0 + \rho)$.

– $\vec{x}(t) \in \Omega^0$, solves (2.1), and $F_\perp^+(\vec{x}(t), t, \varepsilon)F_\perp^-(\vec{x}(t), t, \varepsilon) < 0$ for any $t \in (t_0 - \rho, t_0]$. Further $\vec{x}(t) \in \Omega^\pm$ and solves (PS±) for any $t \in (t_0, t_0 + \rho)$. In this case we say that $\vec{x}(t)$ is a *sliding-crossing* solution at t_0.

– $\vec{x}(t) \in \Omega^0$, solves (2.1), and $F_\perp^+(\vec{x}(t), t, \varepsilon)F_\perp^-(\vec{x}(t), t, \varepsilon) < 0$ for any $t \in [t_0, t_0 + \rho)$. Further $\vec{x}(t) \in \Omega^\pm$ and solves (PS±) for any $t \in (t_0 - \rho, t_0)$. In this case we say that $\vec{x}(t)$ is a *crossing-sliding* solution at t_0.

In fact the stability of sliding motion of (PS) is inherited by that of any regularized version, and Filippov definition appears to be the most appropriate choice, see [40, Chapter 2], and the introduction of [16] for a discussion of this point in a higher dimensional context.

We say that $\vec{x}(t)$ is a pure crossing or non-sliding solution if it is crossing for any $t_0 \in \mathbb{R}$ such that $\vec{x}(t_0) \in \Omega^0$, otherwise we say that it is a sliding solution. Notice that *local uniqueness of the solution is lost for sliding solutions*. I.e. let $\vec{P} \in \Omega^0$ and $\vec{y}^+(t_0) = \vec{P}$ be a crossing-sliding solution at t_0, so that $\vec{y}^+(t) \in \Omega^+$ in a left neighborhood of t_0. Then there is a solution $\vec{y}^-(t)$ such that $\vec{y}^-(t) \in \Omega^-$ in a left neighborhood of t_0 and $\vec{y}^-(t_0) = \vec{P}$ is a crossing-sliding solution at $t = t_0$; further there will be a solution $\vec{x}(t)$ such that $\vec{x}(t_0) = \vec{P}$ which is pure-sliding at $t = t_0$, i.e. $\vec{x}(t) \in \Omega^0$ in a left and right neighborhood of t_0. A similar non-uniqueness argument holds for sliding-crossing solutions.

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We say that $\vec{\gamma}$ is a **non-sliding** (or sliding) homoclinic trajectory of the system (PS) if it is a non-sliding (or sliding) homoclinic solution in the above sense. Further we say that a homoclinic trajectory $\vec{\psi}$ is **forward sliding** (or **backward sliding**) if there is t_0 such that $\vec{\psi}(t_0)$ is a crossing-sliding point and $\vec{\psi}(t) \in \Omega^0$ for any $t \geq t_0$ ($\vec{\psi}(t_0)$ is a sliding-crossing point and $\vec{\psi}(t) \in \Omega^0$ for any $t \leq t_0$).

Throughout the paper we shall consider the following assumptions:

F0 $\vec{f}(\vec{0}) = \vec{0}$ and the eigenvalues $\lambda_s^\pm$, $\lambda_u^\pm$ of $\boldsymbol{f_x^\pm}(\vec{0})$ are such that $\lambda_s^\pm < 0 < \lambda_u^\pm$.

Denote by $\vec{v}_s^\pm$, $\vec{v}_u^\pm$ the normalized eigenvectors of $\boldsymbol{f_x^\pm}(\vec{0})$ corresponding to $\lambda_s^\pm$, $\lambda_u^\pm$, respectively, and set

$$c_u^{\perp,\pm} = \left[\vec{\nabla} G(\vec{0})\right]^* \vec{v}_u^\pm, \quad c_s^{\perp,\pm} = \left[\vec{\nabla} G(\vec{0})\right]^* \vec{v}_s^\pm. \tag{2.2}$$

We can assume w.l.o.g. that $\pm c_u^{\perp,\pm} \geq 0$, $\pm c_s^{\perp,\pm} \geq 0$, but we need a stronger condition:

F1 $\vec{v}_s^\pm$, $\vec{v}_u^\pm$ are not orthogonal to $\vec{\nabla} G(\vec{0})$, i.e., $c_u^{\perp,-} < 0 < c_u^{\perp,+}$, $c_s^{\perp,-} < 0 < c_s^{\perp,+}$.

Furthermore, we take the assumption on the function $\vec{g}$:

G $\vec{g}(t, \vec{0}, \varepsilon) = \vec{0}$ for any $t, \varepsilon \in \mathbb{R}$.

Some further assumptions on the mutual positions of $\vec{v}_u^\pm$ and $\vec{v}_s^\pm$ will be required later on. These assumptions allow to distinguish between systems **Ni** and **Si**, while the difference between **N1** and **N2** (and **S1** and **S2**) depends on whether or not $\vec{\gamma}(t)$ crosses transversally Ω^0, see Fig. 2.

2.2. *Some remarks on Mel'nikov theory and chaos in smooth systems*

In this section we briefly recall some facts concerning Mel'nikov theory for smooth systems. Therefore we assume that $\vec{f}(\vec{x}) = \vec{f}^\pm(\vec{x})$ for $\vec{x} \in \Omega$, and $\vec{f}$ is C^r in the whole Ω. We also assume **G** and **F0**, which takes a simpler form since $\lambda_s^+ = \lambda_s^-$, $\lambda_u^+ = \lambda_u^-$.

Let us consider

$$\dot{\vec{x}} = \vec{f}(\vec{x}) + \varepsilon \vec{g}(t, \vec{x}, \varepsilon), \quad \vec{x} \in \Omega \tag{S}$$

for $\vec{f}, \vec{g} \in C^r$. We assume that for $\varepsilon = 0$ system (S) admits a homoclinic trajectory, i.e., a solution $\vec{\gamma}(t)$ such that $\lim_{|t| \to +\infty} \vec{\gamma}(t) = (0, 0)$.

Mel'nikov theory gives conditions which guarantee persistence of the homoclinic and, if the system is recurrent in t, the existence of a chaotic pattern. To keep the presentation simpler we assume that $\vec{g}$ is p-periodic in t. However the theory is already developed in the almost periodic and recurrent case, even for discontinuous systems, see [5,11] for more details. To be more precise, let $\mathcal{E} := \{0, 1\}^{\mathbb{Z}}$ be the space of doubly infinite sequences $E : \mathbb{Z} \to \{0, 1\}$, i.e. $E = (e_j)$ where $e_j \in \{0, 1\}$, $j \in \mathbb{Z}$, and

$$\mathcal{E}_0 := \left\{ e \in \mathcal{E} \mid \inf\{m \in \mathbb{Z} \mid e_m = 1\} > -\infty,\ \sup\{m \in \mathbb{Z} \mid e_m = 1\} < \infty \right\},$$
$$\mathcal{E}_+ := \left\{ e \in \mathcal{E} \mid \inf\{m \in \mathbb{Z} \mid e_m = 1\} > -\infty,\ \sup\{m \in \mathbb{Z} \mid e_m = 1\} = \infty \right\},$$
$$\mathcal{E}_- := \left\{ e \in \mathcal{E} \mid \inf\{m \in \mathbb{Z} \mid e_m = 1\} = -\infty,\ \sup\{m \in \mathbb{Z} \mid e_m = 1\} < \infty \right\}.$$

The purpose of the theory is to find sufficient conditions to have the following phenomena for $0 < |\varepsilon| < \varepsilon_0$.

H *Persistence of the homoclinic*. There is $\varepsilon_0 > 0$ such that for any $0 < |\varepsilon| < \varepsilon_0$ there exists a unique C^{r-1} (non-sliding) solution $\vec{x}_b(t, \varepsilon)$, bounded on $\mathbb{R}$ and homoclinic to the origin, and a unique C^{r-1} function $\alpha(\varepsilon)$ satisfying $\alpha(0) = \alpha_0$ such that

$$\sup_{t\in\mathbb{R}} \|\vec{x}_b\big(t + \alpha(\varepsilon), \varepsilon\big) - \vec{\gamma}(t)\| \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

C *Existence of a chaotic pattern*. There is $\varepsilon_0 > 0$ such that, for any $0 < |\varepsilon| < \varepsilon_0$, there is $M \in \mathbb{N}$ large enough, such that for any sequence $E = (e_j) \in \mathcal{E}$ there is a (doubly infinite) sequence of real numbers $(\hat{\alpha}_j)$, $|\hat{\alpha}_j| < 1$ and a (non-sliding) solution $\vec{x}_E(t, \varepsilon)$ such that, if $t \in [(2j-1)pM - 1, (2j+1)pM + 1]$, then

$$\begin{array}{rl} \|\vec{x}_E(t,\varepsilon) - \vec{\gamma}(t - 2pM - \hat{\alpha}_j)\| \le C\varepsilon & \text{if} \ \ e_j = 1, \\ \|\vec{x}_E(t,\varepsilon)\| \le C\varepsilon & \text{if} \ \ e_j = 0. \end{array} \tag{2.3}$$

Further let $\mathcal{S}$ be the set of (non-sliding) solutions as in (2.3), and let F_N, be the time shift map: $F_N[\vec{x}_E(t, \varepsilon)] := \vec{x}_E(t + pN, \varepsilon)$. Then there is $N \in \mathbb{N}$ large enough such that $F_N : \mathcal{S} \to \mathcal{S}$ is topologically conjugated to the Bernoulli-shift on two symbols $\sigma : \mathcal{E} \to \mathcal{E}$, $\sigma((e_j)) = (e_{j+1})$ for $j \in \mathbb{Z}$.

We stress that in property ***H*** we required periodicity just to keep the presentation simpler. In fact all the results hold in the almost periodic case or even with weaker recurrence assumptions, see [5] for more details about this point, see also [11]. Since the system is smooth, all the solutions are obviously non-sliding. We have added this requirement to emphasize that, even in the discontinuous setting, when we have pattern ***H*** or ***C*** we mean that all the solutions involved are non-sliding.

In one of the bifurcation scenarios described in the article we prove the existence of an additional property which is not present in smooth systems. To explain it we need to define the computable constant $L_{\vec{f}0}$, see Remark 3.6 and formulas (6.4), (6.5). However the constant $L_{\vec{f}0}$ has a simple dynamical interpretation: if $L_{\vec{f}0} < 0$ the origin is stable for (2.1), while if $L_{\vec{f}0} > 0$ it is unstable.

Cs *Existence of chaotic-like forward sliding or backward sliding homoclinic trajectories*. Assume $L_{\vec{f}0} < 0$ (or $L_{\vec{f}0} > 0$); there is $\varepsilon_0 > 0$ such that, for any $0 < |\varepsilon| < \varepsilon_0$, there is $M \in \mathbb{N}$ large enough, such that for any sequence $E = (e_j) \in \mathcal{E}_0$ there is an uncountable number of forward sliding homoclinic trajectories (or backward sliding homoclinic trajectories) $\vec{x}_E(t, \varepsilon)$ with the following property: For each $\vec{x}_E(t, \varepsilon)$ there is a (finite) sequence of real numbers $(\hat{\alpha}_j)$, $|\hat{\alpha}_j| < 1$ such that, if $t \in [(2j-1)pM - 1, (2j+1)pM + 1]$ then (2.3) holds.

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Let us define the Mel'nikov function in the simple case $n = 2$.

$$\begin{aligned} \mathcal{M}(\alpha) &= \int_{-\infty}^{+\infty} e^{-\int_0^t \operatorname{tr} \boldsymbol{f_x}(\vec{\gamma}(s))ds} \vec{f}(\vec{\gamma}(t)) \wedge \vec{g}(t+\alpha, \vec{\gamma}(t), 0)dt\,, \\ \mathcal{M}'(\alpha) &= \int_{-\infty}^{+\infty} e^{-\int_0^t \operatorname{tr} \boldsymbol{f_x}(\vec{\gamma}(s))ds} \vec{f}(\vec{\gamma}(t)) \wedge \frac{\partial \vec{g}}{\partial t}(t+\alpha, \vec{\gamma}(t), 0)dt\,. \end{aligned} \tag{2.4}$$

We are ready to state the classical result in the **smooth** case.

Theorem 2.1. *[32] Assume* **F0** *and* **G***, and that there is α^0 such that $\mathcal{M}(\alpha^0) = 0$ and $\mathcal{M}'(\alpha^0) \neq 0$. Then system* (S) *has a persisting homoclinic orbit (property **H**). If further $\vec{g}$ is p-periodic in t, then also the chaos occurs (property **C**).*

When $n \geq 3$ some further transversality conditions are required, see e.g. [32]; and, even to define the function $\mathcal{M}$, we need to employ a solution of the adjoint variational system and the concept of exponential dichotomy, see e.g. [32].

In the whole paper we denote by $\vec{x}(t, \tau, \vec{P})$ a solution of (PS) or of (S) which leaves from $\vec{P}$ at $t = \tau$ (which is locally unique if $\vec{P} \notin \Omega^0$). Let $B = B(\vec{0}, \delta)$ be the ball of radius $\delta > 0$ centered in the origin. We can construct the following sets:

$$\begin{aligned} W^u_{loc}(\tau) &:= \{\vec{P} \in B \mid \vec{x}(t, \tau, \vec{P}) \in B \text{ for } t \leq 0,\ \lim_{t\to-\infty} \vec{x}(t, \tau, \vec{P}) = \vec{0}\}, \\ W^s_{loc}(\tau) &:= \{\vec{P} \in B \mid \vec{x}(t, \tau, \vec{P}) \in B \text{ for } t \geq 0,\ \lim_{t\to+\infty} \vec{x}(t, \tau, \vec{P}) = \vec{0}\}, \\ W^u(\tau) &:= \{\vec{P} \in \mathbb{R}^2 \mid \lim_{t\to-\infty} \vec{x}(t, \tau, \vec{P}) = \vec{0}\}, \\ W^s(\tau) &:= \{\vec{P} \in \mathbb{R}^2 \mid \lim_{t\to+\infty} \vec{x}(t, \tau, \vec{P}) = \vec{0}\}. \end{aligned} \tag{2.5}$$

We state a result proved in [13, §13] or [24, Theorem 2.16].

Lemma 2.2. *Assume* **G** *and consider* (S)*; then $W^u(\tau)$ and $W^s(\tau)$ are C^r immersed manifolds of dimension* 1 *(i.e. each of them is the graph of a C^r curve), varying C^r smoothly with respect to τ and ε. Further if $\delta > 0$ is small enough, then $W^u_{loc}(\tau)$ (or $W^s_{loc}(\tau)$) is a graph on its tangent, say $T^u(\tau)$ (or $T^s(\tau)$), which is ε-close to the line spanned by $\vec{v}_u$ (or by $\vec{v}_s$). Moreover, $W^u_{loc}(\tau) \subset W^u(\tau)$, $W^s_{loc}(\tau) \subset W^s(\tau)$.*

The manifolds $W^u(\tau)$ and $W^s(\tau)$ are the sets of all the initial conditions of the trajectories converging to the origin in the past and in the future, respectively, and they are not invariant for the flow of (S). However, if $\vec{P} \in W^u(\tau)$ then $\vec{x}(t, \tau, \vec{P}) \in W^u(t)$ for any $t, \tau \in \mathbb{R}$. Analogously for $W^s(\tau)$.

In the whole paper we use the following notation: we denote by $W^{u,+}(\tau)$ and by $W^{s,+}(\tau)$ the branches of $W^u(\tau)$ and $W^s(\tau)$ leaving from the origin towards Ω^+, while $W^{u,-}(\tau)$ and $W^{s,-}(\tau)$ denote the branches leaving from the origin towards Ω^-. It follows that $W^u(\tau) = W^{u,-}(\tau) \cup W^{u,+}(\tau)$ and $W^{u,+}(\tau) \cap W^{u,-}(\tau) = (0, 0)$.

3. Homoclinic orbits in discontinuous systems

Here we consider system (PS) and the notation of Section 2. We assume that this system admits a piecewise smooth homoclinic solution $\vec{\gamma}(t)$ for $\varepsilon = 0$. Battelli and Fečkan managed to reprove Theorem 2.1 in this context too, assuming that $\vec{0} \notin \Omega^0$, and $\vec{\gamma}(t) \in \Omega^-$ for $|t| > T$ and $\vec{0} \in \Omega^-$. They considered both the cases where $\vec{\gamma}(t)$ crosses transversally Ω^0 at $t = \pm T$ and

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$\vec{\gamma}(t) \in \Omega^+$ for $|t| < T$, and the case where $\vec{\gamma}(t)$ is a sliding solution so that $\vec{\gamma}(t) \in \Omega^0$ for $|t| \leq T$. In either case they showed that (PS) exhibits a persisting homoclinic as well as the existence of a chaotic pattern (properties ***H*** and ***C***) for $\varepsilon > 0$ small [5]. All the results are carried on in the more general (and difficult) $n \geq 3$ case, and with weak recurrence properties including the almost periodic setting. One of the main difficulties in this higher-dimensional discontinuous context is to redefine the appropriate Mel'nikov function.

Recently the case where $\vec{0} \in \Omega^0$ has been considered, profiting of the approach used by Battelli–Fečkan. In [10] it was considered the case where $\vec{\gamma}(t) \in \Omega^-$ for $t < 0$, $\vec{\gamma}(t) \in \Omega^+$ for $t > 0$ and it crosses transversally the Ω^0 surface at $t = 0$. So no sliding was allowed for $\vec{\gamma}(t)$, and property ***H*** was shown.

As we said in the Introduction we consider four different bifurcation scenarios. In fact we can split piecewise smooth systems exhibiting a homoclinic trajectory into two classes with respect to the position of $\vec{v}_s^{\pm}$ and $\vec{v}_u^{\pm}$. Roughly speaking, in the first class, **N**, the indices s and u are alternating (Fig. 1a). This prevents the existence of sliding solutions in the neighborhood of the origin, while in the other class, **S**, $\vec{v}_s^+$, $\vec{v}_s^-$ and $\vec{v}_u^+$, $\vec{v}_u^-$ lie next to each other (Fig. 1b), which results in the existence of solutions sliding along Ω^0. More precisely, set $\mathcal{T}_u^{\pm} := \{c\vec{v}_u^{\pm} \mid c \geq 0\}$, and denote by Π_u^1 and Π_u^2 the disjoint open sets in which $\mathbb{R}^2$ is divided by the polyline $\mathcal{T}^u := \mathcal{T}_u^+ \cup \mathcal{T}_u^-$. The following assumptions are meant to determine if (PS) is of class **N** or **S**, respectively:

FN $\vec{v}_s^+$ and $\vec{v}_s^-$ lie on the opposite sides with respect to $\mathcal{T}^u$, i.e., $\vec{v}_s^+ \in \Pi_u^1$ and $\vec{v}_s^- \in \Pi_u^2$.
FS $\vec{v}_s^+$ and $\vec{v}_s^-$ lie on the same side with respect to $\mathcal{T}^u$, i.e., $\vec{v}_s^{\pm} \in \Pi_u^2$.

Clearly if $\vec{f}^+ = \vec{f}^-$ then $\mathcal{T}^u$ is a line and Π_u^i is a halfplane for $i = 1, 2$. In fact all smooth systems satisfy **FN**. This way we have partitioned piecewise smooth systems, satisfying **F0** and **F1**, in two classes **N** and **S**, "morally of the same size", the former where no sliding phenomena can take place close to the origin, the latter where sliding phenomena close to the origin are present. Notice that smooth systems are in fact contained in class **N**.

We further divide these systems in two other groups: the one in which $\vec{\gamma}(t)$ lies entirely in Ω^+ and the other one in which $\vec{\gamma}(t)$ crosses transversally the discontinuity surface Ω^0. These are described by the following assumptions:

K1 for $\varepsilon = 0$ there is a unique solution $\vec{\gamma}(t)$ of (PS) homoclinic to the origin such that $\vec{\gamma}(t) \in \Omega^+$ for all $t \in \mathbb{R}$.

K2 for $\varepsilon = 0$ there is a unique solution $\vec{\gamma}(t)$ of (PS) homoclinic to the origin such that

$$\vec{\gamma}(t) \in \begin{cases} \Omega^+, & t < 0, \\ \Omega^0, & t = 0, \\ \Omega^-, & t > 0. \end{cases}$$

Furthermore, $(\vec{\nabla} G(\vec{\gamma}(0)))^* \vec{f}^{\pm}(\vec{\gamma}(0)) < 0$.

Recalling the orientation of $\vec{v}_s^{\pm}$, $\vec{v}_u^{\pm}$ chosen in **F1** we assume w.l.o.g that $\lim_{t\to-\infty} \frac{\dot{\vec{\gamma}}(t)}{\|\dot{\vec{\gamma}}(t)\|} = \vec{v}_u^+$. Further $\lim_{t\to+\infty} \frac{\dot{\vec{\gamma}}(t)}{\|\dot{\vec{\gamma}}(t)\|} = -\vec{v}_s^+$ if **K1** is assumed, while $\lim_{t\to+\infty} \frac{\dot{\vec{\gamma}}(t)}{\|\dot{\vec{\gamma}}(t)\|} = -\vec{v}_s^-$ if **K2** holds.

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In the rest of this section we discuss all the couples of $\{\mathbf{FN}, \mathbf{FS}\} \times \{\mathbf{K1}, \mathbf{K2}\}$ (see Fig. 2). Each of the next four examples represents one class of discontinuous systems. One can compare their phase portraits to Fig. 2. In all the cases $\Omega^0 = \mathbb{R} \times \{0\}$, and $\Omega^\pm = \{(x, y) \in \mathbb{R}^2 \mid \pm y > 0\}$.

Example 3.1. The following system satisfies assumptions **FN** and **K1**

$$\begin{aligned} \dot{x} &= y(1-y) \\ \dot{y} &= x \end{aligned} \quad y > 0, \qquad \begin{aligned} \dot{x} &= y \\ \dot{y} &= x \end{aligned} \quad y < 0.$$

Notice that the system is hamiltonian and for $y > 0$ it admits the first integral $V^+(x, y) = \frac{x^2}{2} - \frac{y^2}{2} + \frac{y^3}{3}$, and the graph of the homoclinic trajectory $\vec{\gamma}$ is contained in the level set $V^+(x, y) = 0$, see Fig. 3a.

Example 3.2. The following system satisfies assumptions **FN** and **K2**

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x(1-x) \end{aligned} \quad y > 0, \qquad \begin{aligned} \dot{x} &= y \\ \dot{y} &= x\left(\frac{9}{8} - x^2\right) \end{aligned} \quad y < 0.$$

The system is hamiltonian and for $y > 0$ we have the first integral $V^+(x, y) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^3}{3}$, while for $y < 0$ we have the first integral $V^-(x, y) = \frac{y^2}{2} - \frac{9x^2}{16} + \frac{x^4}{4}$. The graph of the homoclinic trajectory $\vec{\gamma}$ is contained in the level sets

$$\left\{(x, y) \mid V^+(x, y) = 0,\ y > 0\right\} \cup \left\{(x, y) \mid V^-(x, y) = 0,\ y < 0\right\} \cup \left\{\left(\frac{3}{2}, 0\right)\right\} \tag{3.1}$$

see Fig. 3b.

Example 3.3. The following system satisfies assumptions **FS** and **K1**

$$\begin{aligned} \dot{x} &= y(1-y) \\ \dot{y} &= x \end{aligned} \quad y > 0, \qquad \begin{aligned} \dot{x} &= -y \\ \dot{y} &= -x \end{aligned} \quad y < 0.$$

The system is hamiltonian and for $y > 0$ we have the first integral $V^+(x, y) = \frac{x^2}{2} - \frac{y^2}{2} + \frac{y^3}{3}$. The graph of the homoclinic trajectory $\vec{\gamma}$ is contained in the level set $V^+(x, y) = 0$, see Fig. 3c.

Example 3.4. The following system satisfies assumptions **FS** and **K2**

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x(1-x) \end{aligned} \qquad y > 0,$$

$$\begin{aligned} \dot{x} &= -y - 8xy + 8x^2y + 8y^3 \\ \dot{y} &= -3x + 12x^2 + 4y^2 - 8x^3 - 8xy^2 \end{aligned} \qquad y < 0.$$

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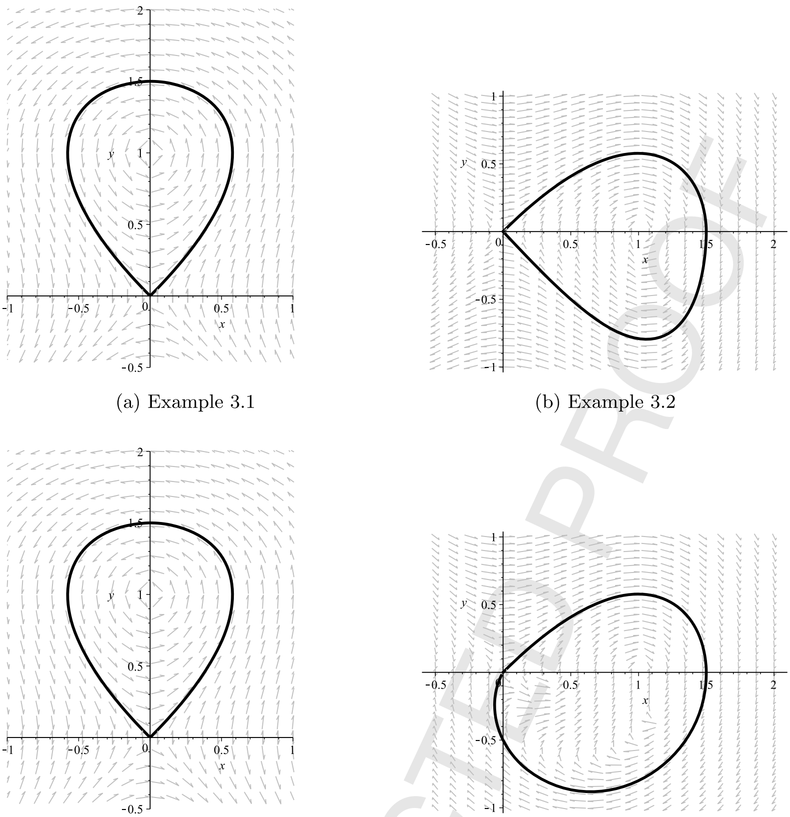

(a) Example 3.1 (b) Example 3.2

(c) Example 3.3 (d) Example 3.4

Fig. 3. Phase portraits of illustrative examples with highlighted homoclinic trajectories.

Again, the system is hamiltonian, in Ω^+ the first integral is $V^+(x,y) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^3}{3}$, and in Ω^- the first integral is $V^-(x,y) = (x^2+y^2-x)^2 - \frac{x^2+y^2}{4}$. The graph of the homoclinic trajectory $\vec{\gamma}$ is contained in (3.1), where the level set $V^-(x,y) = 0$ is a part of limaçon, see Fig. 3d.

Here we note that in general when one tries to find numerically a bounded solution and draw its phase portrait, one may get off the precise trajectory after a single step, although the initial condition/-s was/were taken from the real solution. This is due to the curvature of the solution (see [17] for details).

Now we focus on systems of type **S**. Hence we have the assumption **FS** in mind, so that Π_u^1 does not contain any stable eigenvector. We want to discuss what happens to trajectories leaving from points in Ω^0 close to the origin, both in the perturbed and in the unperturbed case.

Let $\vec{v}_i$ be such that $\|\vec{v}_i\| = 1$, $(\vec{\nabla} G(0))^* \vec{v}_i = 0$ and $\vec{v}_i \in \Pi_u^i$, for $i = 1, 2$ ($\vec{v}_1$ aims towards right in Figs. 2c, 2d).

Let $\vec{y}_i \in \Omega^0$, $\|\vec{y}_i\| \le \rho_0/2$, so that $\vec{y}_i = \vec{v}_i \rho + O(\rho^2)$ for some $0 < \rho < \rho_0$ small. We give the following remarks describing the dynamics close to the origin: their proofs are rather technical and they are postponed to Section 6.1. We recall that the flow on Ω^0 close to the origin is ruled by (2.1), and that Ω^0 is invariant for that equation.

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Remark 3.5. For any $\tau \in \mathbb{R}$, there is a solution $y(t, \tau, \vec{y}_i)$ of (PS) that slides across $\Omega^0 \cap \Pi_u^i$, following equation (2.1). Further this motion is unstable for $i = 1$ and stable for $i = 2$, i.e., the former repels all the close solutions from Π_u^1, while the latter attracts all the close solutions from Π_u^2.

We recall that the solutions leaving from $\vec{y}_i$ are not uniquely defined, as it has to be expected in sliding region, cf. the Introduction.

Remark 3.6. The origin is stable for (2.1) if $L_{\vec{f}^0} < 0$ and unstable if $L_{\vec{f}^0} > 0$, where $L_{\vec{f}^0}$ is the computable constant given in (6.5).

We conclude this part of the section by recalling some known results concerning persistence of a homoclinic, see [10], and insurgence of chaos, see [11].

Theorem 3.7 *([10,11]). Consider* (PS)*; assume that* **F0***,* **F1***,* **K1** *and* **G** *are satisfied, and that there is $\alpha_0 \in \mathbb{R}$ such that $\mathcal{M}(\alpha_0) = 0$ and $\mathcal{M}'(\alpha_0) \neq 0$ where $\mathcal{M}$ is as in* (2.4)*. Then the homoclinic orbit persists, i.e., property* ***H*** *holds; and if g is p-periodic then also a chaotic pattern is present, i.e., property* ***C*** *holds too.*

Theorem 3.7 holds in the general n-dimensional case: property ***H*** follows from [10, Theorem 2.9] while ***C*** follows from [11]. In fact it was proved that all the solutions of the perturbed system close to the original homoclinic lie in the same half space, i.e. the chaos occurs in Ω^+. We emphasize that in [10] assumption **FN** was required, but an inspection of the proof shows that it is not actually needed.

When hypothesis **K2** is considered we need to redefine properly the function $\mathcal{M}$, and we obtain only persistence of the homoclinic.

Theorem 3.8 *([10]). Consider* (PS)*; assume that* **F0***,* **F1***,* **K2** *and* **G** *are satisfied, and that there is $\alpha_0 \in \mathbb{R}$ such that $\mathcal{M}(\alpha_0) = 0$ and $\mathcal{M}'(\alpha_0) \neq 0$ where*

$$\begin{aligned}\mathcal{M}(\alpha) = &\int_{-\infty}^{0} e^{-\int_0^t \operatorname{tr} \vec{f}_x^+(\vec{\gamma}(s))ds}\, \vec{f}^+\big(\vec{\gamma}(t)\big) \wedge \vec{g}\big(t+\alpha, \vec{\gamma}(t), 0\big)dt \\ &+ \int_{0}^{+\infty} e^{-\int_0^t \operatorname{tr} \vec{f}_x^-(\vec{\gamma}(s))ds}\, \vec{f}^-\big(\vec{\gamma}(t)\big) \wedge \vec{g}\big(t+\alpha, \vec{\gamma}(t), 0\big)dt.\end{aligned} \tag{3.2}$$

Then the homoclinic orbit persists, i.e., property ***H*** *holds.*

One of the main purposes of this paper is to show that, if the assumptions of Theorem 3.8 are satisfied and **FS** holds, then chaos, i.e. property ***C***, is not possible. In fact we believe that if **FN** holds, then the assumption of Theorem 3.8 and periodicity will be sufficient to prove the presence of property ***C***: this will be the object of a forthcoming paper.

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4. Solutions close to the homoclinic in smooth systems

Let $\Gamma := \{\vec{\gamma}(t) \mid t \in \mathbb{R}\}$, and denote by E^{in} the compact set enclosed by Γ. Further let $\Gamma^u := \{\vec{\gamma}(t) \mid t \le 0\}$, $\Gamma^s := \{\vec{\gamma}(t) \mid t \ge 0\}$. In this section we consider the smooth system (S), and we investigate what happens in a neighborhood of Γ: this information will be useful also in the context of piecewise smooth system (PS). For this purpose we need to introduce some notation which in fact is used to explain Figs. 4, 5, 6, 7. We invite the reader to refer to the pictures for a better and easier comprehension of the argument. We begin by giving several definitions. We denote by $B(\vec{P}, \delta)$ the open ball of center $\vec{P}$ and of radius $\delta > 0$, and for any set D,

$$B(D, \delta) := \{\vec{Q} \in \mathbb{R}^2 \mid \exists \vec{P} \in D : \|\vec{Q} - \vec{P}\| < \delta\} = \cup\{B(\vec{P}, \delta) \mid \vec{P} \in D\}.$$

We denote by $\vec{\psi}(t)$ the unique vector such that $\langle \vec{\psi}(t), \vec{f}(\vec{\gamma}(t))\rangle = 0$, $\|\vec{\psi}(t)\| = 1$, and $\vec{\gamma}(t) + c\vec{\psi}(t) \in E^{in}$ for any $c > 0$ small enough (it points towards the interior of Γ). Let $\vec{w}$ be a vector transversal to $\dot{\vec{\gamma}}(0)$; for any $\delta > 0$ small we consider the segment

$$L^0(\delta) := \{\vec{Q} = \vec{\gamma}(0) + d\vec{w} \mid |d| \le \delta\}. \tag{4.1}$$

Let $0 < \nu < 1$; follow $W^{u,+}_\varepsilon(\tau)$ and $W^{s,+}_\varepsilon(\tau)$ from the origin towards $L^0(\varepsilon^{1-\nu})$: if ε is small enough $W^{u,+}_\varepsilon(\tau)$ and $W^{s,+}_\varepsilon(\tau)$ both intersect transversally $L^0(\varepsilon^{1-\nu})$ a first time in points denoted respectively by $\vec{\zeta}^u(\tau)$ and $\vec{\zeta}^s(\tau)$. Notice that $\vec{\zeta}^u(\tau)$ and $\vec{\zeta}^s(\tau)$ are C^r both in ε and τ, and that $\vec{\zeta}^u(\tau) - \vec{\zeta}^s(\tau) = O(\varepsilon)$. In fact the classical Mel'nikov theory shows that $\vec{\zeta}^u(\tau) - \vec{\zeta}^s(\tau) = \varepsilon\mathcal{M}(\tau) + o(\varepsilon)$, where $\mathcal{M}(\tau)$ is defined in (2.4). We denote by $\bar{W}^u_\varepsilon(\tau)$ the branch of $W^{u,+}_\varepsilon(\tau)$ between the origin and $\vec{\zeta}^u(\tau)$, similarly we denote by $\bar{W}^s_\varepsilon(\tau)$ the branch of $W^{s,+}_\varepsilon(\tau)$ between the origin and $\vec{\zeta}^s(\tau)$.

Notice that $W^{u,+}_\varepsilon(\tau)$ and $W^{s,+}_\varepsilon(\tau)$ are not invariant for the flow. However, if $\vec{P} \in W^{u,+}_\varepsilon(\tau)$ (or $W^{s,+}_\varepsilon(\tau)$) then $\vec{x}(t, \tau, \vec{P}) \in W^{u,+}_\varepsilon(t)$ (or $W^{s,+}_\varepsilon(t)$) for any $t \in \mathbb{R}$.

Remark 4.1. Let $\nu > 0$ small; if $\vec{P} \in \bar{W}^u_\varepsilon(\tau)$ then $\vec{x}(t, \tau, \vec{P}) \in \bar{W}^u_\varepsilon(t)$ for any $t \le \tau$. Similarly if $\vec{Q} \in \bar{W}^s_\varepsilon(\tau)$ then $\vec{x}(t, \tau, \vec{Q}) \in \bar{W}^s_\varepsilon(t)$ for any $t \ge \tau$. Further $\bar{W}^u_\varepsilon(\tau) \in B(\Gamma^u, \varepsilon^{1-\nu})$ and $\bar{W}^s_\varepsilon(\tau) \in B(\Gamma^s, \varepsilon^{1-\nu})$ for any $\tau \in \mathbb{R}$; in particular $\{\vec{\zeta}^s(\tau)\} = L^0(\varepsilon^{1-\nu}) \cap \bar{W}^s_\varepsilon(\tau)$ and $\{\vec{\zeta}^u(\tau)\} = L^0(\varepsilon^{1-\nu}) \cap \bar{W}^u_\varepsilon(\tau)$.

Let $\vec{C}$ be a point in E^{in} at a finite distance from Γ: $\vec{\gamma}(t)$ makes exactly a complete rotation around $\vec{C}$ as t winds from $-\infty$ to $+\infty$, while $\bar{W}^u_\varepsilon(\tau)$ performs an angle smaller than 2π by construction, say $\theta > 0$ to fix the ideas. However, a priori, $\vec{x}(t, \tau, \vec{\zeta}^u(\tau))$ may perform a complete rotation or more around $\vec{C}$ for $t \le \tau$, e.g. the angle performed may be $2\pi + \theta$, since (S) is non-autonomous. Such a possibility is excluded by Remark 4.1 and a topological argument, see [14, Lemma 3.3].

Let $k > 0$ be a fixed constant; we introduce some further sets

$$\begin{aligned} L^+_\pm(\delta) &:= \{\vec{Q} = d(\vec{v}^u \pm k\vec{v}^s) \mid 0 \le d \le \delta\}, \\ L^-_\pm(\delta) &:= \{\vec{Q} = d(-\vec{v}^u \pm k\vec{v}^s) \mid 0 \le d \le \delta\}. \end{aligned} \tag{4.2}$$

Let $\nu > 0$ and $\delta > 0$ be small constants; set $\sigma^s(\delta, \varepsilon) := (\delta + \varepsilon)^{\frac{|\lambda_s|}{|\lambda_s| + \lambda_u} - \nu}$.

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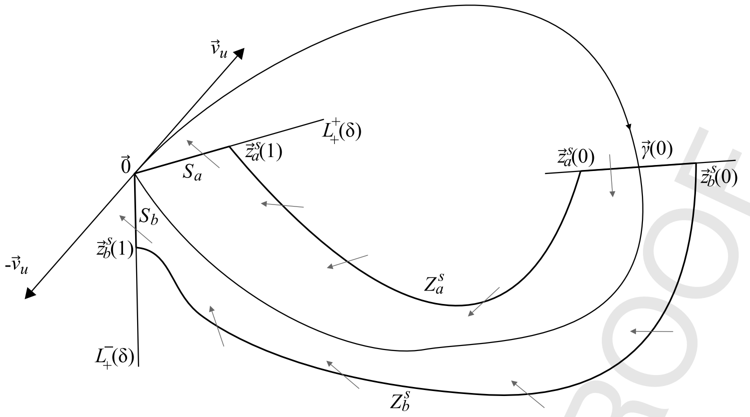

Fig. 4. Curves $\vec{z}^s_a$, $\vec{z}^s_b$. Arrows denote the flow of system (S).

In the next subsection we construct two curves $\vec{z}^s_a : [0,1] \to B(\Gamma^s, \sigma^s(\delta,\varepsilon))$ and $\vec{z}^s_b : [0,1] \to B(\Gamma^s, \sigma^s(\delta,\varepsilon))$, with the following properties, see Fig. 4: $\vec{z}^s_a(0) \in L_0(\varepsilon^{1-\nu}+\delta)$, $\vec{z}^s_b(0) \in L_0(\varepsilon^{1-\nu}+\delta)$, $\vec{z}^s_a(1) \in L^+_+(\sigma^s(\delta,\varepsilon))$, $\vec{z}^s_b(1) \in L^-_+(\sigma^s(\delta,\varepsilon))$. Let $Z^s_a = \{\vec{z}^s_a(s) \mid 0 < s < 1\}$ and $Z^s_b = \{\vec{z}^s_b(s) \mid 0 < s < 1\}$, then Z^s_a, Z^s_b and Γ do not have self-intersections and do not intersect each other. Moreover Z^s_a and Z^s_b lie on the opposite sides with respect to Γ. We denote by S_a, S_b, the segments from the origin respectively to $\vec{z}^s_a(1)$, $\vec{z}^s_b(1)$, and by $\bar{L}^0_s \subset L^0(\varepsilon^{1-\nu}+\delta)$ the segment between $\vec{z}^s_a(0)$ and $\vec{z}^s_b(0)$. Then we denote by $K^s(\delta,\varepsilon)$ the compact set enclosed by S_b, S_a, Z^s_a, $\bar{L}^0_s$ and Z^s_b.

Lemma 4.2. *Let $\kappa > 0$, $\delta = \varepsilon^\kappa$ and $\nu > 0$, then there is $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ we can construct the curves $\vec{z}^s_a(s)$ and $\vec{z}^s_b(s)$ in such a way that $\vec{\zeta}^u(\tau), \vec{\zeta}^s(\tau) \in \bar{L}^0_s$ and the flow of* (S) *on $Z^s_a \cup Z^s_b$ points towards the interior of $K^s(\delta,\varepsilon)$ for any $\tau \in \mathbb{R}$. That means that for any $\vec{P} \in (Z^s_a \cup Z^s_b)$, we have $\vec{x}(t,\tau,\vec{P}) \in K^s(\delta,\varepsilon)$ for $t-\tau > 0$ small enough. Further the flow of* (S) *on $\bar{L}^0$ points towards the interior of $K^s(\delta,\varepsilon)$, while on $S_a \cup S_b \setminus \{(0,0)\}$ it points towards the exterior of $K^s(\delta,\varepsilon)$.*

The construction relies on some simple geometrical facts whose analytic computation is non-trivial and rather cumbersome. In our opinion it is not so relevant for understanding the core of the article, so it is postponed to Section 6 for the interested reader.

Fix $\tau \in \mathbb{R}$; Lemma 4.2 allows us to say that if $\vec{P} \in \bar{L}^0_s$, it stays close to Γ^s for $t > \tau$ until it arrives close to the origin and either crosses (transversally) $S_a \cup S_b \setminus \{(0,0)\}$ in a finite time or it converges to the origin as $t \to +\infty$.

In fact we can say more. Notice that $\vec{\zeta}^u(\tau)$ and $\vec{\zeta}^s(\tau)$ lie in $L^0(\varepsilon^{1-\nu}+\delta)$ for any τ, see Remark 4.1. Let us denote by $A^s(\tau)$ the open segment between $\vec{z}^s_a(0)$ and $\vec{\zeta}^s(\tau)$ and by $B^s(\tau)$ the open segment between $\vec{z}^s_b(0)$ and $\vec{\zeta}^s(\tau)$, so that $\bar{L}^0_s$ is partitioned in $A^s(\tau)$, $B^s(\tau)$ and $\vec{\zeta}^s(\tau)$. Notice that $\bar{W}^s_\varepsilon(\tau)$ splits $K^s(\delta,\varepsilon)$ in two relatively open connected components: let $K^s_A(\tau)$ and $K^s_B(\tau)$ denote respectively the components containing $A^s(\tau)$ and $B^s(\tau)$ (see Fig. 5).

Then we have the following.

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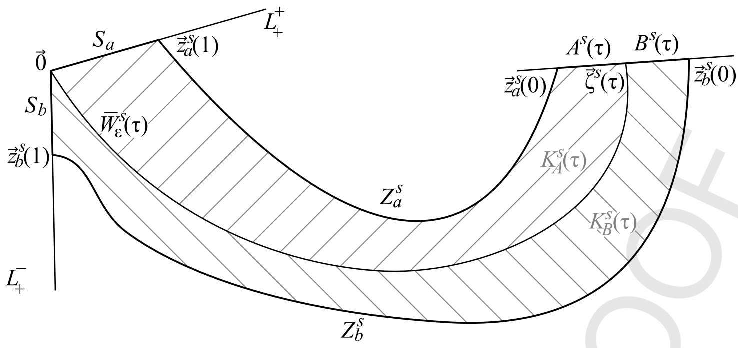

Fig. 5. Notation around $\bar{W}_\varepsilon^s(\tau)$.

Lemma 4.3. *Let $\kappa > 0$, $\delta = \varepsilon^\kappa$ and $\nu > 0$, $\vec{P}_a \in A^s(\tau)$, $\vec{P}_b \in B^s(\tau)$. Then there is $T^s(\vec{P}_a) > \tau$ such that $\vec{x}(t, \tau, \vec{P}_a) \in K_A^s(t) \subset K^s(\delta, \varepsilon)$ for any $t \in [\tau, T^s(\vec{P}_a)]$ and it intersects transversally $S_a \subset L_+^+(\sigma^s(\delta, \varepsilon))$ at $t = T^s(\vec{P}_a)$. Analogously there is $T^s(\vec{P}_b) > \tau$ such that $\vec{x}(t, \tau, \vec{P}_b) \in K_B^s(t) \subset K^s(\delta, \varepsilon)$ for any $t \in [\tau, T^s(\vec{P}_b)]$ and it intersects transversally $S_b \subset L_+^-(\sigma^s(\delta, \varepsilon))$ at $t = T^s(\vec{P}_b)$.*

Proof. Let $\vec{P}_a \in A^s(\tau)$. From elementary phase plane arguments we see that there are three possibilities:

i) $\vec{x}(t, \tau, \vec{P}_a) \in K_A^s(t)$ for any $t \geq \tau$.
ii) There is $T^s(\vec{P}_a) > \tau$ such that $\vec{x}(t, \tau, \vec{P}_a) \in K_A^s(t)$ for any $t \in [\tau, T^s(\vec{P}_a))$ and $\vec{x}(T^s(\vec{P}_a), \tau, \vec{P}_a) \in \bar{W}_\varepsilon^s(T^s(\vec{P}_a))$.
iii) $\vec{x}(t, \tau, \vec{P}_a) \in K_A^s(t)$ for any $t \in [\tau, T^s(\vec{P}_a))$ and it crosses transversally $S_a \cup S_b \setminus \{(0, 0)\}$ at $t = T^s(\vec{P}_a)$.

In case i) then $\vec{x}(t, \tau, \vec{P}_a)$ converges to the origin as $t \to +\infty$: therefore $\vec{P}_a = \vec{\zeta}^s(\tau)$ and this is a contradiction. In case ii) $\vec{x}(t, \tau, \vec{P}_a) \in \bar{W}_\varepsilon^s(t)$ for any $t \geq \tau$, hence again $\vec{P}_a = \vec{\zeta}^s(\tau)$, see Remark 4.1: a contradiction. So only the case iii) may take place and the statement follows.

The case $\vec{P}_b \in B^s(\tau)$ is obtained with a specular argument. □

When we follow trajectories in the past we get the symmetric result.

Set $\sigma^u(\delta, \varepsilon) := (\delta + \varepsilon)^{\frac{\lambda_u}{|\lambda_s|+\lambda_u} - \nu}$. Reasoning as above (see Section 6) we construct two curves $\vec{z}_a^u : [0, 1] \to B(\Gamma^u, \sigma^u(\delta, \varepsilon))$ and $\vec{z}_b^u : [0, 1] \to B(\Gamma^u, \sigma^u(\delta, \varepsilon))$, with the following properties, see Fig. 6: $\vec{z}_a^u(0) \in L_+^+(\sigma^u(\delta, \varepsilon))$, $\vec{z}_b^u(0) \in L_-^+(\sigma^u(\delta, \varepsilon))$, while $\vec{z}_a^u(1) \in L^0(\delta + \varepsilon^{1-\nu})$, $\vec{z}_b^u(1) \in L^0(\delta + \varepsilon^{1-\nu})$. Let $Z_a^u = \{\vec{z}_a^u(s) \mid 0 < s < 1\}$ and $Z_b^u = \{\vec{z}_b^u(s) \mid 0 < s < 1\}$. Then Z_a^u, Z_b^u and Γ do not have self-intersections and do not intersect each other. Moreover Z_a^u and Z_b^u lie on the opposite sides with respect to Γ. We denote by U_a, U_b, the segments from the origin respectively to $\vec{z}_a^u(0)$, $\vec{z}_b^u(0)$, and by $\bar{L}_u^0 \subset L^0(\sigma^u(\delta, \varepsilon))$ the segment between $\vec{z}_a^u(1)$ and $\vec{z}_b^u(1)$. Then we denote by $K^u(\delta, \varepsilon)$ the compact set enclosed by U_b, U_a, Z_a^u, $\bar{L}_u^0$ and Z_b^u.

Lemma 4.4. *Let $\kappa > 0$, $\delta = \varepsilon^\kappa$ and $\nu > 0$, then there is $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ we can construct the curves $\vec{z}_a^u(s)$ and $\vec{z}_b^u(s)$ in such a way that $\vec{\zeta}^u(\tau), \vec{\zeta}^s(\tau) \in \bar{L}_u^0$ and the flow of*

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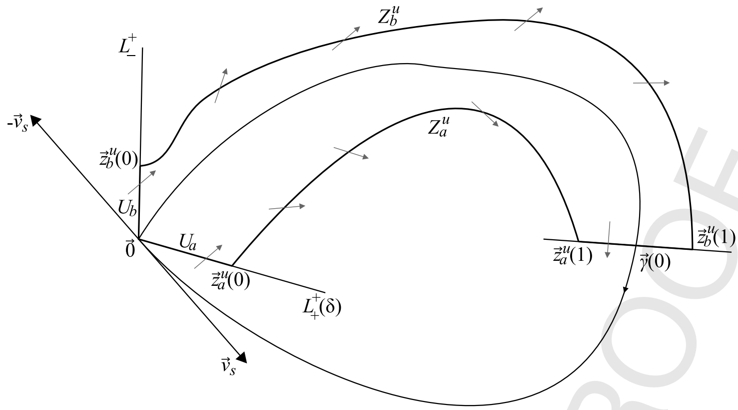

Fig. 6. Curves $\vec{z}_a^u$, $\vec{z}_b^u$. Arrows denote the flow of system (S).

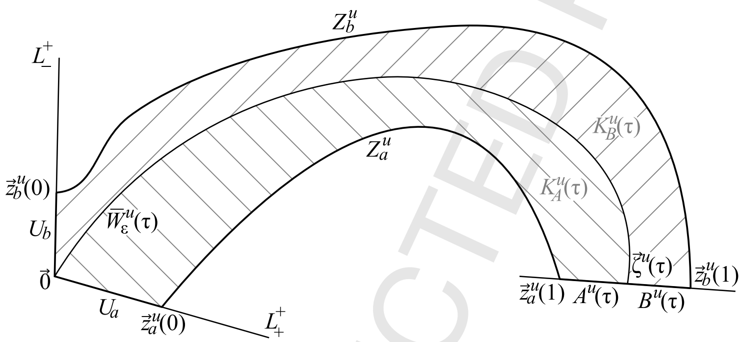

Fig. 7. Notation around $\bar{W}_\varepsilon^u(\tau)$.

(S) *on* $Z_a^u \cup Z_b^u$ *points towards the exterior of* $K^u(\delta, \varepsilon)$ *for any* $\tau \in \mathbb{R}$. *That means that for any* $\vec{P} \in (Z_a^u \cup Z_b^u)$, *we have* $\vec{x}(t, \tau, \vec{P}) \in K^u(\delta, \varepsilon)$ *for* $t - \tau < 0$ *and* $|t - \tau|$ *small enough. Further the flow of* (S) *on* U_a *and* U_b *points towards the interior of* $K^u(\delta, \varepsilon)$, *while on* $\bar{L}_u^0$ *points towards the exterior of* $K^u(\delta, \varepsilon)$.

Then, again, we notice that $\vec{\zeta}^u(\tau)$ splits $\bar{L}_u^0$ in two open and connected segments, $A^u(\tau)$ and $B^u(\tau)$, having as endpoints $\vec{\zeta}^u(\tau)$ and respectively $\vec{z}_a^u(0)$ and $\vec{z}_b^u(0)$ (see Fig. 7). Again $\bar{W}_\varepsilon^u(\tau)$ splits $K^u(\delta, \varepsilon)$ in two relatively open connected components: let $K_A^u(\tau)$ and $K_B^u(\tau)$ denote respectively the components containing $A^u(\tau)$ and $B^u(\tau)$ (see Fig. 5).

Lemma 4.5. *Let* $\kappa > 0$, $\delta = \varepsilon^\kappa$ *and* $\nu > 0$, $\vec{P}_a \in A^u(\tau)$, $\vec{P}_b \in B^u(\tau)$. *Then there is* $T^u(\vec{P}_a) < \tau$ *such that* $\vec{x}(t, \tau, \vec{P}_a) \in K_A^u(t) \subset K^u(\delta, \varepsilon)$ *for any* $t \in [T^u(\vec{P}_a), \tau]$ *and it intersects transversally* $U_a \subset L_+^+(\sigma^u(\delta, \varepsilon))$ *at* $t = T^u(\vec{P}_a)$. *Analogously there is* $T^u(\vec{P}_b) < \tau$ *such that* $\vec{x}(t, \tau, \vec{P}_b) \in K_B^u(t) \subset K^u(\delta, \varepsilon)$ *for any* $t \in [T^u(\vec{P}_b), \tau]$ *and it intersects transversally* $U_b \subset L_-^+(\sigma^u(\delta, \varepsilon))$ *at* $t = T^u(\vec{P}_b)$.

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In fact, for any $\vec{P} \in (A^s(\tau) \cup B^s(\tau))$, and any $\vec{Q} \in (A^u(\tau) \cup B^u(\tau))$ we can give estimates of $T^s(\vec{P})$ and $T^u(\vec{Q})$, see Remark 6.8.

Remark 4.6. Let $E^{out} = \mathbb{R}^2 \setminus (\Gamma \cup E^{in})$. In order to help the reader to visualize the picture we recall that we have two different scenarios. Either $-c\vec{v}^u \in E^{out}$ and $-c\vec{v}^s \in E^{out}$ for any $c > 0$, or E^{in} contains both $-c\vec{v}^u$ and $-c\vec{v}^s$ for $c > 0$ small enough. Perhaps the former situation is the more familiar one. In the former case, for $\varepsilon = 0$, Γ may enjoy asymptotic stability just from inside while in the latter one just from outside, see [38, §13.1] and also the beginning of Section 6.2.

Notice that in the former case L_+^+, U_a, S_a, Z_a^u, Z_a^s lie in E^{in}, while U_b, S_b, Z_b^u, Z_b^s lie in E^{out}. In the latter case we have the reversed situation.

5. The four bifurcation scenarios

In this section we apply the results developed in the previous section for smooth system (S) to the piecewise smooth system (PS). In this case, we consider appropriate parts of Ω^0 (in general curvilinear) instead of line segments L^0, $L_\pm^+$, $L_\pm^-$. The analogous statements are obtained with trivial changes.

5.1. FN+K1

In this case Theorem 3.7 applies, so we have the both – a persisting homoclinic (property ***H***) and the existence of chaos if $\vec{g}$ is periodic (property ***C***). Further notice that sliding is not allowed, so there are no sliding homoclinic solutions.

5.2. FN+K2

In this case Theorem 3.8 applies, so we have a persisting homoclinic as in property ***H*** and we conjecture that property ***C*** on the existence of chaos is possible if $\vec{g}$ is periodic: this will be the object of future investigation. Further notice that sliding is not allowed, so there are no sliding homoclinic solutions.

5.3. FS+K1

In this subsection we always assume **F0**, **F1**, **FS**, **K1** and **G**.

We first observe that, for $\varepsilon = 0$, Γ is surrounded by sliding solutions; if $L_{\vec{f}^0} < 0$ they are sliding towards the origin in the future and they get away from a neighborhood of the origin in the past, while if $L_{\vec{f}^0} > 0$ we have the reversed situation. We recall that $L_{\vec{f}^0}$ is a computable constant, whose value is given in (6.5).

Proposition 5.1. *Consider* (PS) *for $\varepsilon = 0$, and assume* **F0**, **F1**, **FS**, **K1**. *For any $\sigma > 0$ there is $\delta > 0$ such that if $\vec{P} \in \Omega^+ \cap E^{out}$, $\vec{P} \in B(\Gamma, \delta)$, there are $t_1 < 0 < t_2$ such that $\vec{x}(t, 0, \vec{P}) \in \Omega^+$ for $t \in (t_1, t_2)$ and it is a sliding solution at $t = t_1$ and at $t = t_2$.*

Further $\vec{x}(t, 0, \vec{P}) \in E^{out}$ for any $t \in \mathbb{R}$, and if $L_{\vec{f}^0} < 0$ then $\vec{x}(t, 0, \vec{P}) \in B(\Gamma, \sigma)$ for any $t \geq t_1$, $\vec{x}(t, 0, \vec{P}) \in \Omega^0$ whenever $t \geq t_2$ and it converges to the origin as $t \to +\infty$, while if $L_{\vec{f}^0} > 0$ then $\vec{x}(t, 0, \vec{P}) \in B(\Gamma, \sigma)$ for any $t \leq t_2$, $\vec{x}(t, 0, \vec{P}) \in \Omega^0$ whenever $t \leq t_1$ and it converges to the origin as $t \to -\infty$.

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Proof. The proof follows from a straightforward application of Lemmas 4.3 and 4.5. □

We emphasize that no sliding homoclinic trajectories are possible for $\varepsilon = 0$. Now we turn to consider the $\varepsilon > 0$ case.

We recall that Theorem 3.7 holds in this setting too, so for $\varepsilon > 0$ we have the persistence of a non-sliding homoclinic and the insurgence of chaos made up by non-sliding solutions.

Let $\vec{\zeta}^u(\tau) = \vec{\gamma}(0) + d^u(\tau)\vec{\psi}(0)$ and $\vec{\zeta}^s(\tau) = \vec{\gamma}(0) + d^s(\tau)\vec{\psi}(0)$ for some $d^u(\tau), d^s(\tau) \in \mathbb{R}$, see (4.1). We recall that the Mel'nikov function $\mathcal{M}(\tau)$ measures the first order term $D(\tau)$ of the distance with sign between $\vec{\zeta}^u(\tau)$ and $\vec{\zeta}^s(\tau)$. Namely there is $c \neq 0$ independent of ε and τ such that

$$D(\tau) := d^u(\tau) - d^s(\tau) = c\varepsilon\mathcal{M}(\tau) + o(\varepsilon) \tag{5.1}$$

where $o(\varepsilon)$ is uniform with respect to $\tau \in \mathbb{R}$. Assume to fix the ideas that $c > 0$ so that $\mathcal{M}(\tau) > 0$ implies $D(\tau) > 0$. Then either $\vec{\zeta}^u(\tau) \in B^s(\tau)$ and $\vec{\zeta}^s(\tau) \in A^u(\tau)$, or $\vec{\zeta}^u(\tau) \in A^s(\tau)$ and $\vec{\zeta}^s(\tau) \in B^u(\tau)$. To fix the ideas assume the former takes place when $D(\tau) < 0$, and the latter for $D(\tau) > 0$. This corresponds to the case $d\vec{v}_u^-, d\vec{v}_s^- \in E^{out}$ for $d > 0$ sufficiently small, i.e. Fig. 2c.

So we have the following results completing the picture of Theorem 3.7 when the additional assumption **FS** holds.

Theorem 5.2. *Consider* (PS) *and assume* **F0**, **F1**, **K1**, **G** *and* **FS**, *and that there is $\alpha_0 \in \mathbb{R}$ such that $\mathcal{M}(\alpha_0) = 0$ and $\mathcal{M}'(\alpha_0) \neq 0$ where $\mathcal{M}$ is as in* (2.4)*. Then if $L_{\vec{f}^0} < 0$ there is $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ there is an uncountable number of **forward sliding** homoclinic trajectories. Analogously if $L_{\vec{f}^0} > 0$, then there is $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ there is an uncountable number of **backward sliding** homoclinic trajectories.*

Proof. Assume that $\mathcal{M}(\alpha_1) < 0$, so that $\vec{\zeta}^u(\alpha_1) \in B^s(\alpha_1)$. So using Lemma 4.3 we see that there is $\mathfrak{T}^u(\alpha_1) := T^s(\vec{\zeta}^u(\alpha_1)) > \alpha_1$ such that $\vec{x}(t, \alpha_1, \vec{\zeta}^u(\alpha_1))$ is in Ω^+ for any $\alpha_1 < t < \mathfrak{T}^u(\alpha_1)$, and it is in Ω^0 in a right neighborhood of $t = \mathfrak{T}^u(\alpha_1)$, i.e., $\vec{x}(t, \alpha_1, \vec{\zeta}^u(\alpha_1))$ is a crossing-sliding solution at $t = \mathfrak{T}^u(\alpha_1)$. So if $L_{\vec{f}^0} < 0$ then $\vec{x}(t, \alpha_1, \vec{\zeta}^u(\alpha_1))$ is a forward sliding homoclinic.

Analogously assume that $\mathcal{M}(\alpha_2) > 0$ so that $\vec{\zeta}^s(\alpha_2) \in B^u(\alpha_2)$. Using Lemma 4.5 we see that there is $\mathfrak{T}^s(\alpha_2) := T^u(\vec{\zeta}^s(\alpha_2)) < \alpha_2$ such that $\vec{x}(t, \alpha_2, \vec{\zeta}^s(\alpha_2))$ is in Ω^+ for any $\alpha_2 > t > \mathfrak{T}^s(\alpha_2)$, and it is in Ω^0 in a left neighborhood of $t = \mathfrak{T}^s(\alpha_2)$, so that $\vec{x}(t, \alpha_2, \vec{\zeta}^s(\alpha_2))$ is a sliding-crossing solution at $t = \mathfrak{T}^s(\alpha_2)$. So if $L_{\vec{f}^0} > 0$ then $\vec{x}(t, \alpha_2, \vec{\zeta}^s(\alpha_2))$ is a backward sliding homoclinic.

The proof of Theorem 5.2 now easily follows observing that there are uncountably many α_1, α_2 such that $\mathcal{M}(\alpha_1) < 0 < \mathcal{M}(\alpha_2)$. □

Proposition 5.3. *Let $\vec{C} \in E^{in}$, $\vec{C} \notin B(\Gamma, \sqrt{\varepsilon})$. In the assumption of Theorem 5.2, if $\vec{g}$ is p-periodic in t, for any $0 < \varepsilon < \varepsilon_0$, for any $k \in \mathbb{N}$ there is an uncountable number of sliding homoclinic trajectories performing exactly k loops around $\vec{C}$ if $L_{\vec{f}^0} \neq 0$. Such homoclinic trajectories are forward-sliding if $L_{\vec{f}^0} < 0$ and backward-sliding if $L_{\vec{f}^0} > 0$.*

To prove Proposition 5.3 we need the following preliminary result. Let $\vec{P} \in W^u_\varepsilon(\tau)$ and denote by $\tilde{W}^u(\tau)$ the branch of $W^u_\varepsilon(\tau)$ between the origin and $\vec{P}$. Let $\vec{C}$ be as in Proposition 5.3. Let us consider a parametrization $\Psi(s)$ of $\tilde{W}^u(\tau)$ such that $\Psi(0) = (0, 0)$ and $\Psi(1) = \vec{P}$. Let $\alpha \in \mathbb{R}$ and consider the trajectory $\vec{x}(t,\alpha,\vec{\zeta}^u(\alpha))$. We consider polar coordinates with respect to $\vec{C}$ for $\Psi(s)$ and $\vec{x}(t,\alpha,\vec{\zeta}^u(\alpha))$, i.e. we set

$$\begin{aligned}\Psi(s)-\vec{C} &= R(s)\big(\cos\big(\Theta(s)\big),\sin\big(\Theta(s)\big)\big),\\ \vec{x}\big(t,\alpha,\vec{\zeta}^u(\alpha)\big)-\vec{C} &= \rho(t)\big(\cos\big(\theta(t)\big),\sin\big(\theta(t)\big)\big).\end{aligned}$$

Assume $L_{\vec{f}^0}<0$ and let $\theta(t):=\vec{x}(t,\alpha_1,\vec{\zeta}^u(\alpha_1))$; observe that $\lim_{t\to-\infty}\theta(t)=:\theta(-\infty)$ exists. We can assume w.l.o.g. that $\theta(-\infty)=\Theta(0)\in(-\pi;\pi]$ (they are equal modulo 2π). The proof of Proposition 5.3 is based on the following Lemma borrowed from [18, Lemma 4.3].

Lemma 5.4. *Assume* **F0** *and* **G**. *Then* $\theta(\tau)=\Theta(1)$.

We recall that the above lemma is trivial if the system is autonomous, but it is not if the system is nonautonomous, since the graphs of $\vec{x}(t,\tau,\vec{P})$ and of $\tilde{W}^u(\tau)$ are distinct.

Proof of Proposition 5.3. Assume for definiteness $L_f<0$, the other case being analogous.

From Theorem 3.7 we know that for any $k\in\mathbb{N}$ there is a non-sliding homoclinic trajectory $\vec{x}(t)$ of (PS) such that $\vec{x}(t)\in\Omega^+$ for any $t\in\mathbb{R}$ and it performs exactly k rotations around $\vec{C}$, say counterclockwise to fix the ideas. Namely let L^1 be a segment of size $\varepsilon^{1-\nu}$, centered in $\vec{\gamma}(1)$ and transversal to $\dot{\vec{\gamma}}(1)$; we can choose $\nu>0$ small enough so that $L^1\subset K^s(\delta,\varepsilon)$: then $\vec{x}(t)$ intersects transversally L^1 exactly k times where $\nu>0$ is small enough. Let $\theta(t)$ be the angular coordinate of $\vec{x}(t)-\vec{C}$; then we can choose $\tau\in\mathbb{R}$ such that $\vec{x}(\tau)\in L^1$ and $\vec{x}(t)\notin L^1$ for any $t>\tau$, so that $2(k-1)\pi<\Delta=\theta(\tau)-\theta(-\infty)<2k\pi$, and $\theta(+\infty)-\theta(-\infty)=2k\pi$.

Let $\tilde{W}^u(\tau)$ be the branch of $W^u_\varepsilon(\tau)$ between the origin and $\vec{P}=\vec{x}(\tau)$; let $\Theta(s)$ be the angular coordinate of its parametrization $\Psi(s)$. From Lemma 5.4, we see that $\Theta(1)-\Theta(0)=\Delta\in(2(k-1)\pi;2k\pi)$.

Notice that $\vec{x}(\tau)\in W^s_\varepsilon(\tau)$ and $\vec{x}(t)\in K^s(\delta,\varepsilon)$ for any $t\geq\tau$. Since the crossing between $W^u_\varepsilon(\tau)$ and $W^s_\varepsilon(\tau)$ in $\vec{x}(\tau)$ is topologically transversal, following $\Psi(s)$ we find $s_1<1<s_2$ such that $\Psi(s_1)=\vec{Q}\in K^s_A(\tau)$ and $\Psi(s_2)=\vec{R}\in K^s_B(\tau)$, while $\Psi(1)=\vec{P}\in W^s_\varepsilon(\tau)$. I.e. $\vec{Q},\vec{R}\in W^u_\varepsilon(\tau)$ lie in the opposite sides with respect to $W^s_\varepsilon(\tau)$. Further, using a continuity argument, we can assume w.l.o.g. that

$$2(k-1)\pi<\Theta(s_i)-\Theta(0)<2k\pi\,,\qquad i=1,2.$$

Using Lemma 4.3 we see that $\vec{x}(t,\tau,\vec{R})$ is forced to stay in $K^s_B(\tau)$ for $t\geq\tau$ until it crosses Ω^0, and then it slides to the origin. Further, using again Lemma 5.4 we see that $\vec{x}(t,\tau,\vec{R})$ makes exactly $k-1$ complete rotations around $\vec{C}$ for $t\leq\tau$. So it follows that $\vec{x}(t,\tau,\vec{R})$ is a forward-sliding homoclinic which performs exactly k rotations around $\vec{C}$.

If we assume $L_{\vec{f}^0}>0$ we need to start from trajectories in $W^s_\varepsilon(\tau)$ which perform exactly $k-1$ complete rotations around $\vec{C}$ for $t\geq\tau$, and intersect L^1 transversally at $t=\tau$. Then we proceed as above and we find backward-sliding homoclinic making k rotations around $\vec{C}$. □

In fact, with trivial adaption in the proof of Proposition 5.3, we can prove the following.

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Theorem 5.5. *In the assumption of Theorem 5.2, if $\vec{g}$ is p-periodic in t, then* (PS) *has properties **C** and **Cs**, i.e., after a perturbation the system exhibits chaotic behavior and chaotic-like forward or backward sliding homoclinics.*

We conjecture that property ***Cs*** might hold even for any sequence $E \in \mathcal{E}_+$ if $L_{\vec{f}^0} > 0$ and for any $E \in \mathcal{E}_-$ if $L_{\vec{f}^0} < 0$: this could be the object of further investigation.

5.4. FS+K2

Now we turn to consider the case where **F0**, **F1**, **FS**, **K2** and **G** holds. Observe first that there are no sliding homoclinics for $\varepsilon = 0$.

As in Section 5.2, Theorem 3.8 ensures the persistence of a unique non-sliding homoclinic orbit transversally crossing Ω^0 if the function $\mathcal{M}$ defined in (3.2) has a unique non-degenerate zero. Our aim is to show that, even if g is periodic, for $\varepsilon > 0$ neither the existence of chaos (property ***C***) nor an infinite number of sliding homoclinics (property ***Cs***) is possible, while we have the appearance of uncountably many sliding homoclinic trajectories making just one "loop".

Recall that the chaotic trajectories, if any, lie in $B(\Gamma, c\varepsilon)$ for some $c > 0$.

Lemma 5.6. *Consider* (PS) *and assume* **F0**, **F1**, **FS**, **K2** *and* **G**.

Let $\vec{C} \in E^{in}$ be a point at a finite distance from Γ, so that $\vec{C} \notin B(\Gamma, \sqrt{\varepsilon})$. Then there is no solution $\vec{\phi}(t)$ of (PS) *with the following property:*

• There are $-\infty \le T_1 < T_2 \le \infty$ such that $\vec{\phi}(t) \in B(\Gamma, \sqrt{\varepsilon})$ for any $t \in [T_1, T_2]$, and in this interval $\vec{\phi}(t)$ performs at least two complete rotations around $\vec{C}$.

Proof. To help the reader to follow the proof we provide Fig. 8. Notice that in this context L_+^+ is a broken line segment partly located in Ω^+ and partly in Ω^-, while L_-^+ and L_+^- are curvilinear segment contained in Ω^0.

When **FS** is assumed, $\vec{\zeta}^s(\tau) \in W^{s,-}(\tau) \cap L^0(c\varepsilon)$. From Lemma 4.3 (with a trivial adaption to take account of the fact that L_+^- is not a straight line), we see that if $\vec{Q}_a \in A^s(\tau)$ then there is $T^s(\vec{Q}_a) > 0$ such that $\vec{x}(t, \tau, \vec{Q}_a) \in \Omega^-$ for any $\tau < t < T^s(\vec{Q}_a)$ and it intersects $L_+^+(\sigma^s(c\varepsilon, \varepsilon))$ at $t = T^s(\vec{Q}_a)$, while if $\vec{Q}_b \in B^s(\tau)$ then $\vec{x}(t, \tau, \vec{Q}_b) \in \Omega^-$ for any $\tau < t < T^s(\vec{Q}_b)$ and it intersects $L_+^-(\sigma^s(c\varepsilon, \varepsilon)) \subset \Omega^0$ at $t = T^s(\vec{Q}_b)$.

Hence $\vec{x}(t, \tau, \vec{Q}_b)$ is crossing-sliding at $t = T^s(\vec{Q}_b)$. So either it stays in Ω^0 for any $t > T^s(\vec{Q}_b)$, and it converges to the origin as $t \to +\infty$ (so it does not make a complete loop around $\vec{C}$ for $t > \tau$), or it stays in Ω^0 until it gets out from a $\sqrt{\varepsilon}$-neighborhood of the origin (so it does not anymore belong to $B(\Gamma, \sqrt{\varepsilon})$). Further $\vec{x}(t, \tau, \vec{Q}_a)$ will stay close to $W^{u,-}(t)$ for t in some right neighborhood of $T^s(\vec{Q}_a)$, and eventually it will get out from $B(\Gamma, \sqrt{\varepsilon})$, and the statement follows.

Now we make an analogous reasoning backwards in time. Hence if $\vec{Q} = \vec{\zeta}^u(\tau)$ then $\vec{x}(t, \tau, \vec{Q})$ performs less than one loop around $\vec{C}$ for $t \le \tau$ and the lemma is proved. If $\vec{Q}_a \in A^u(\tau)$ then from Lemma 4.5 we see that, there is $T^u(\vec{Q}_a) < 0$ such that $\vec{x}(t, \tau, \vec{Q}_a) \in \Omega^+$ for any $T^u(\vec{Q}_a) < t < \tau$ and it intersects $L_+^+(\sigma^u(c\varepsilon, \varepsilon))$ at $t = T^u(\vec{Q}_a)$. So $\vec{x}(t, \tau, \vec{Q}_a)$ will stay close to $W^{s,+}(t)$ for t in some left neighborhood of $T^u(\vec{Q}_a)$, and eventually it will get out from $B(\Gamma, \sqrt{\varepsilon})$. If $\vec{Q}_b \in B^u(\tau)$ then from Lemma 4.5 we see that, $\vec{x}(t, \tau, \vec{Q}_b) \in \Omega^+$ for any $T^u(\vec{Q}) < t < \tau$ and it intersects $L_-^+(\sigma^u(c\varepsilon, \varepsilon)) \subset \Omega^0$ at $t = T^u(\vec{Q}_b)$. So reasoning as above, we conclude that it cannot make more than a loop around $\vec{C}$ for $t \le \tau$, staying in $B(\Gamma, \sqrt{\varepsilon})$, and the lemma follows. □

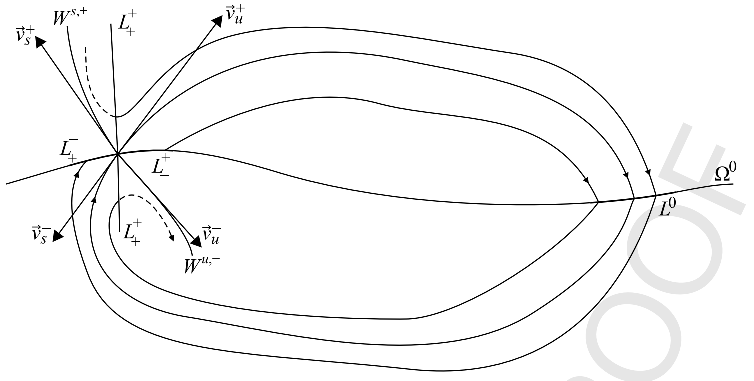

Fig. 8. Behavior of solutions close to Γ. Note that curvilinear segments L_-^+, L_+^-, L^0 lie on Ω^0, while L_+^+ are linear and different for each one of $\Omega^\pm$.

As a consequence of the above lemma and of Theorem 3.8 we obtain the next result.

Theorem 5.7. *Consider* (PS) *and assume* **F0***,* **F1***,* **FS***,* **K2** *and* **G**. *Assume that there is α_0 such that $\mathcal{M}(\alpha^0) = 0$ and $\mathcal{M}'(\alpha^0) \neq 0$; then the homoclinic persists (property **H**), but neither chaos (property **C**) nor an infinite number of sliding homoclinics (property **Cs**) are present even if $\vec{g}$ is periodic.*

Moreover, we have the existence of sliding homoclinics:

Theorem 5.8. *Consider* (PS) *and assume* **F0***,* **F1***,* **FS***,* **K2** *and* **G**. *Then we get the same conclusion as in Theorem 5.2.*

Notice that backward and forward sliding homoclinics make exactly one loop.

Proof. The proof of the existence of sliding-homoclinic is analogous to the one of Theorem 5.2.

For briefness we just consider the $L_{\vec{f}^0} < 0$ case. Assume to fix the ideas that $\vec{\zeta}^u(\tau) \in B^s(\tau)$ iff $D(\tau) < 0$ and that the constant c in (5.1) is positive. Assume further that $\mathcal{M}'(\alpha^0) < 0 = \mathcal{M}(\alpha^0)$. Then again $\vec{\zeta}^u(\tau) \in B^s(\tau)$ and $D(\tau) < 0$ if $\alpha^0(\varepsilon) < \tau < \alpha^0(\varepsilon) + \sigma$, for some $\sigma > 0$. So, reasoning as above, from Lemma 4.3 we see that for any $\tau \in (\alpha^0(\varepsilon); \alpha^0(\varepsilon) + \sigma)$ there is $\mathfrak{T}^u(\tau) := T^s(\vec{\zeta}^u(\tau))$ such that $\vec{x}(t, \tau, \vec{\zeta}^u(\tau)) \in \Omega^-$ for any $t \in (\tau, \mathfrak{T}^u(\tau))$, it is in Ω^0 for any $t > \mathfrak{T}^u(\tau)$ and it converges to the origin as $t \to +\infty$. Hence $\vec{x}(t, \tau, \vec{\zeta}^u(\tau))$ is a forward-sliding homoclinic. □

Now we make a brief digression concerning the transversality of the crossing between $W^u(\tau)$ and $W^s(\tau)$.

Remark 5.9. Consider the smooth system (S). The existence of $\alpha^0 \in \mathbb{R}$ such that $\mathcal{M}(\alpha^0) = 0 \neq \mathcal{M}'(\alpha^0)$ is sufficient to have a transversal crossing between $W^u(\alpha^0)$ and $W^s(\alpha^0)$. Then, using the flow, we see that there is a transversal crossing between $W^u(\tau)$ and $W^s(\tau)$ for any $\tau \in \mathbb{R}$ and any $\varepsilon > 0$.

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Now we consider (PS) and we assume that we are in the assumptions of Theorem 5.7 so that there is $\alpha(\varepsilon)$ such that $\vec{\zeta}^u(\alpha(\varepsilon)) = \vec{\zeta}^s(\alpha(\varepsilon)) \in [\Omega^0 \cap W^u(\alpha(\varepsilon)) \cap W^s(\alpha(\varepsilon))]$. It follows that $\vec{x}_b(t) = \vec{x}(t, \alpha(\varepsilon), \vec{\zeta}^u(\alpha(\varepsilon)))$ is a homoclinic trajectory and that $W^u(\tau)$ intersects $W^s(\tau)$ in $\vec{x}_b(\tau)$ for any $\tau \in \mathbb{R}$. We denote by $\bar{W}^u(\tau)$ and $\bar{W}^s(\tau)$ the branch of $W^u(\tau)$ and $W^s(\tau)$ between $\vec{0}$ and $\vec{x}_b(\tau)$. A priori $W^u(\tau)$ and $W^s(\tau)$ are just locally Lipschitz and they do not have a definite tangent. However, since for $\varepsilon = 0$ they are piecewise C^1, with a standard continuity argument we see that $\bar{W}^u(\alpha(\varepsilon))$ and $\bar{W}^s(\alpha(\varepsilon))$ are C^1, and that if $\bar{\tau} > \alpha(\varepsilon)$, then $\bar{W}^s(\bar{\tau})$ is C^1 while $\bar{W}^u(\bar{\tau})$ is piecewise C^1 and it has a corner in its unique transversal crossing with Ω^0, see [3, §3] for a further discussion on the transversality and a precise estimate of the jump discontinuity of the tangent of $\bar{W}^u(\bar{\tau})$ (in the general $n \geq 2$ dimensional case). However in $\vec{x}_b(\bar{\tau})$ the tangents of $W^u(\bar{\tau})$ and $W^s(\bar{\tau})$ are well defined and they cross each other transversally.

So, even if we have a transversal homoclinic point, the periodicity of $\vec{g}$ does not give chaos as we said in the Introduction.

To conclude this section we stress that all the results of this article are easily generalized to the case where $\vec{\gamma}(t)$ intersects the discontinuity surface Ω^0 transversally more than once, by combining the methods of [10] (which are based on [3,5]).

6. Proofs of the technical results

6.1. *Proofs of Remarks 3.5 and 3.6*

In this subsection we give a full fledged proofs of Remarks 3.5 and 3.6, and we evaluate explicitly the constant $L_{\vec{f}^0}$ whose value rules the motion along Ω^0 close to the origin.

Proof of Remark 3.5. Recall that $\|\vec{v}_i\| = 1$, $(\vec{\nabla}G(0))^*\vec{v}_i = 0$ and $\vec{v}_i \in \Pi_u^i$, for $i = 1, 2$. Notice that $\vec{v}_i$ is uniquely determined and that there are positive constants $c^\pm$, $d^\pm$ such that:

$$\vec{v}_1 := c^\pm \vec{v}_u^\pm - d^\pm \vec{v}_s^\pm \in T_{\vec{0}}\Omega^0 = \left(\vec{\nabla}G(0)\right)^\perp, \quad \vec{v}_2 = -\vec{v}_1. \tag{6.1}$$

We emphasize that for each $\vec{v}_i$ there are two possible expressions using either $+$ or $-$ eigenvectors, and that $c^\pm$, $d^\pm$ just depend on the angles between $\vec{\nabla}G(\vec{0})$, $\vec{v}_u^\pm$, $\vec{v}_s^\pm$.

We consider some $\vec{y}_i \in \Omega^0 \cap \Pi_u^i$, $i = 1, 2$ such that $\|\vec{y}_i\| \leq \rho_0/2$. Then $\vec{y}_i = \vec{v}_i\rho + O(\rho^2)$ for some $0 < \rho < \rho_0$.

Let us recall that $\pm c_u^{\perp,\pm} > 0$ and $\pm c_s^{\perp,\pm} > 0$, see (2.2). Consequently, $\vec{f}^\pm(\vec{y}_i) = \boldsymbol{f}_{\boldsymbol{x}}^{\pm}(\vec{0})\vec{v}_i\rho + O(\rho^2)$ and

$$\begin{aligned}
\left(\vec{\nabla}G(\vec{y}_1)\right)^* \vec{f}^\pm(\vec{y}_1) =& \left(\vec{\nabla}G(\vec{0})\right)^* \boldsymbol{f}_{\boldsymbol{x}}^{\pm}(\vec{0})\left(c^\pm \vec{v}_u^\pm - d^\pm \vec{v}_s^\pm\right)\rho + O\left(\rho^2\right) \\
=& \left(\lambda_u^\pm c^\pm c_u^{\perp,\pm} - \lambda_s^\pm d^\pm c_s^{\perp,\pm}\right)\rho + O\left(\rho^2\right), \\
\left(\vec{\nabla}G(\vec{y}_2)\right)^* \vec{f}^\pm(\vec{y}_2) =& -\left(\lambda_u^\pm c^\pm c_u^{\perp,\pm} - \lambda_s^\pm d^\pm c_s^{\perp,\pm}\right)\rho + O\left(\rho^2\right).
\end{aligned} \tag{6.2}$$

Therefore for $\rho > 0$ small enough we obtain

$$\begin{aligned}
(\vec{\nabla}G(\vec{y}_1))^* \vec{f}^+(\vec{y}_1) > 0, \qquad & (\vec{\nabla}G(\vec{y}_2))^* \vec{f}^+(\vec{y}_2) < 0, \\
(\vec{\nabla}G(\vec{y}_1))^* \vec{f}^-(\vec{y}_1) < 0, \qquad & (\vec{\nabla}G(\vec{y}_2))^* \vec{f}^-(\vec{y}_2) > 0.
\end{aligned}$$

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Then from [19] we get the stability properties of the motion of $x(t,\tau;\vec{y}_i)$ for any $\tau \in \mathbb{R}$ and the proof is concluded. □

Proof of Remark 3.6. Let us define the following non-zero constant

$$\beta(\vec{0}) := \frac{(\vec{\nabla}G(\vec{0}))^* \boldsymbol{f}_{\boldsymbol{x}}^-(\vec{0})\vec{\nu}_1}{(\vec{\nabla}G(\vec{0}))^*(\boldsymbol{f}_{\boldsymbol{x}}^-(\vec{0}) - \boldsymbol{f}_{\boldsymbol{x}}^+(\vec{0}))\vec{\nu}_1} \tag{6.3}$$

and notice that the value of $\beta(\vec{0})$ remains the same if we replace $\vec{\nu}_1$ by $\vec{\nu}_2 = -\vec{\nu}_1$ in (6.3). We stress that $\beta(\vec{y}^i,t,\varepsilon) = \beta(\vec{0}) + O(\rho+\varepsilon)$ where $O(\rho+\varepsilon)$ is uniform with respect to $t \in \mathbb{R}$, see (2.1).

It is easy to check that $0 < \beta(\vec{0}) < 1$, so, using a continuity argument, we see that $0 < \beta(\vec{x},t,\varepsilon) < 1$ whenever $\|\vec{x}\| \le \rho$, $0 < \rho < \rho_0$ and $0 < \varepsilon < \varepsilon_0$.

Notice that by construction $\vec{F}^0(\vec{x},t,\varepsilon) \in T_{\vec{x}}\Omega^0$ for any $\vec{x} \in \Omega^0$, $\|\vec{x}\| \le \rho_0$. Hence differentiating (2.1) we get

$$\left(\left[1-\beta(\vec{0})\right]\boldsymbol{f}_{\boldsymbol{x}}^-(\vec{0}) + \beta(\vec{0})\boldsymbol{f}_{\boldsymbol{x}}^+(\vec{0})\right)\vec{\nu}_i = L_{\vec{f}^0}\vec{\nu}_i. \tag{6.4}$$

Let $\cos(\theta^\pm) = \langle \vec{v}_u^\pm, \vec{v}_s^\pm \rangle$, then we get

$$L_{\vec{f}^0}\vec{\nu}_1 = \left[1-\beta(\vec{0})\right]\left\{\lambda_u^- c^- \vec{v}_u^- - \lambda_s^- d^- \vec{v}_s^-\right\} + \beta(\vec{0})\left\{\lambda_u^+ c^+ \vec{v}_u^+ - \lambda_s^+ d^+ \vec{v}_s^+\right\}.$$

Hence $L_{\vec{f}^0}$ is as follows:

$$\begin{aligned} L_{\vec{f}^0} :=& \left[1-\beta(\vec{0})\right]\left\{\lambda_u^- c^-\left[c^- - d^-\cos\left(\theta^-\right)\right] - \lambda_s^- d^-\left[-d^- + c^-\cos\left(\theta^-\right)\right]\right\} \\ &+\beta(\vec{0})\left\{\lambda_u^+ c^+\left[c^+ - d^+\cos\left(\theta^+\right)\right] - \lambda_s^+ d^+\left[-d^+ + c^+\cos\left(\theta^+\right)\right]\right\}. \\ \beta(\vec{0}) :=& \frac{c^-\lambda_u^- c_u^{\perp,-} - d^-\lambda_s^- c_s^{\perp,-}}{c^-\lambda_u^- c_u^{\perp,-} - d^-\lambda_s^- c_s^{\perp,-} - (c^+\lambda_u^+ c_u^{\perp,+} - d^+\lambda_s^+ c_s^{\perp,+})}. \end{aligned} \tag{6.5}$$

Then we see that

$$\vec{F}^0\left(\vec{y}^i,t,\varepsilon\right) = \rho L_{\vec{f}^0}\vec{\nu}_i + o(\rho)$$

where $o(\rho)$ is uniform with respect to ε and t. Hence the origin is an unstable fixed point of (2.1) if $L_{\vec{f}^0} > 0$, and it is a stable fixed point if $L_{\vec{f}^0} < 0$, both in the perturbed and in the unperturbed case (in this paper we do not consider the non-generic case $L_{\vec{f}^0} = 0$). □

6.2. *Proofs of Lemmas 4.2 and 4.4*

In this subsection we always assume that $\vec{f} \in C^r$ with $r \ge 2$. The purpose of this section is to borrow some ideas from [38, §13.1], to construct the curves $\vec{z}_a^s$, $\vec{z}_b^s$, $\vec{z}_a^u$, $\vec{z}_b^u$ and to prove Lemmas 4.2, 4.4.

The proofs of this subsection are very technical and introduce a lot of notations, but in fact mainly rely on some linearizations and some simple geometrical interpretations of the pictures.

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We suggest the reader to rely on the figures of this section to follow the arguments and the notation.

We begin by recalling some well-established facts concerning stability of the homoclinic trajectories of autonomous systems (in two dimensions), i.e., for system (S) when $\varepsilon = 0$: we denote by W^s and W^u the stable and unstable manifold of such a system, respectively. Then we construct two auxiliary autonomous smooth systems which are ε-close to the original one, and which enable us to construct the curves $\vec{z}^{u,s}_{a,b}$ of Lemmas 4.2, 4.4.

Let $\vec{P} \in \mathbb{R}^2$. In the whole section we denote by $\vec{y}(t, \vec{P})$ and by $\vec{y}^{\ell}(t, \vec{P})$, respectively, the solution of system (S) for $\varepsilon = 0$ and the solution of the linear system $\dot{\vec{x}} = \boldsymbol{f_x}(\vec{0})\vec{x}$ leaving from $\vec{P}$ at $t = 0$.

As we said in Remark 4.6, although there are two possible scenarios, we just consider the "more usual" one, $-c\vec{v}_u \in E^{out}$, $-c\vec{v}_s \in E^{out}$ for $c > 0$ small, the other being analogous. Following [38, §13.1], we say that Γ is one sided stable (unstable), if there is $\delta > 0$ such that for any $\vec{P} \in B(\Gamma, \delta) \cap E^{in}$ the trajectory $\vec{y}(t, \vec{P})$ has Γ as ω-limit set ($\vec{y}(t, \vec{P})$ has Γ as α-limit set), i.e. $\vec{y}(t, \vec{P})$ approaches Γ as $t \to +\infty$ ($t \to -\infty$). It turns out that Γ is one sided stable if $\operatorname{tr}\boldsymbol{f_x}(\vec{0}) = \lambda_1 + \lambda_2 =: D_1 < 0$, or if $D_1 = 0$ and $D_2 := \int_{-\infty}^{+\infty} \operatorname{tr}\boldsymbol{f_x}(\vec{\gamma}(t))dt < 0$; analogously Γ is one sided unstable if $D_1 > 0$, or if $D_1 = 0$ and $D_2 > 0$, see [38, Lemma 13.1, Theorem 13.2]. In fact D_1, D_2 are the first two terms of the so-called Dulac sequence which gives a complete answer to the problem of establishing one sided stability for Γ, see e.g. [38, §13.1].

6.2.1. Supporting results

We begin by defining some further segments, see Fig. 9. Let $\delta < \delta_0 \ll 1$ so that $\delta|\ln(\delta)| \ll 1$, and set

$$\mathcal{S}_1^{\ell,\pm}(\delta) := \left\{ \pm\frac{\vec{v}_u}{|\ln(\delta)|} + d\vec{v}_s \mid |d| \le \delta \right\}, \quad \mathcal{S}_2^{\ell}(\delta) := \left\{ d\vec{v}_u + \frac{\vec{v}_s}{|\ln(\delta)|} \mid |d| \le \delta \right\}.$$

Through the whole Section 6.2.1, we estimate explicitly the crossing times τ_i and T_u^+ just for completeness, but they are not really used in this article.

Let $0 < d < \delta \min\{1, |\delta \ln(\delta)|^{\frac{\lambda_u}{|\lambda_s|}-1}\}$ and $\vec{P}_+^{\ell} = d\vec{v}_u + \frac{\vec{v}_s}{|\ln(\delta)|} \in \mathcal{S}_2^{\ell}(\delta)$, $\vec{P}_-^{\ell} = -d\vec{v}_u + \frac{\vec{v}_s}{|\ln(\delta)|} \in \mathcal{S}_2^{\ell}(\delta)$. Let us focus on the solution of $\dot{\vec{x}} = \boldsymbol{f_x}(\vec{0})\vec{x}$ leaving from $\vec{P}_{\pm}^{\ell}$, i.e.

$$\vec{y}^{\ell}\big(t, \vec{P}_{\pm}^{\ell}\big) = \pm d\, \mathrm{e}^{\lambda_u t}\vec{v}_u + \frac{\mathrm{e}^{\lambda_s t}}{|\ln(\delta)|}\vec{v}_s. \tag{6.6}$$

Notice that $\vec{y}^{\ell}(t, \vec{P}_+^{\ell})$ crosses transversally $\mathcal{S}_1^{\ell,+}(\delta)$ in $\vec{Q}_+^{\ell}$, while $\vec{y}^{\ell}(t, \vec{P}_-^{\ell})$ crosses transversally $\mathcal{S}_1^{\ell,-}(\delta)$ in $\vec{Q}_-^{\ell}$ where

$$\vec{Q}_{\pm}^{\ell} = \frac{1}{|\ln(\delta)|}\big(\pm\vec{v}_u + |d\ln(\delta)|^{\frac{|\lambda_s|}{\lambda_u}}\vec{v}_s\big) \tag{6.7}$$

at $t = T_1^{\ell} = \frac{|\ln|d\ln(\delta)||}{\lambda_u}$ (recall that $\delta|\ln(\delta)| \ll 1$), since $\frac{|d\ln(\delta)|^{\frac{|\lambda_s|}{\lambda_u}}}{|\ln(\delta)|} < \delta$.

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Recall that there are $c_u, c_s > 0$ such that $\vec{\gamma}(t) \sim c_u\, e^{\lambda_u t}\vec{v}_u$ as $t \to -\infty$ and $\vec{\gamma}(t) \sim c_s\, e^{\lambda_s t}\vec{v}_s$ as $t \to +\infty$, see e.g. [11, Appendix]. Therefore there are $\tau_1 < 0 < \tau_2$ such that $\vec{\gamma}(t)$ intersects transversally $\mathcal{S}_1^{\ell,+}(\delta)$ and $\mathcal{S}_2^{\ell}(\delta)$, respectively at $t = \tau_1$ and at $t = \tau_2$. Further notice that

$$\tau_1 \sim -\frac{\ln|\ln(\delta)|}{\lambda_u}, \qquad \tau_2 \sim \frac{\ln|\ln(\delta)|}{|\lambda_s|}. \tag{6.8}$$

Let $\boldsymbol{P}$ be the matrix whose range and kernel are spanned by $\vec{v}_s$ and $\vec{v}_u$, respectively.

Follow W^u from the origin in the direction of $-\vec{v}_u$, and denote by $\vec{\xi}_-^u$ the first intersection with $\mathcal{S}_1^{\ell,-}(\delta)$. Let $\bar{d}_1^-$, $\bar{d}_1^+$, $\bar{d}_2$ be such that $\boldsymbol{P}\vec{\xi}_-^u = \bar{d}_1^-\vec{v}_s$, $\boldsymbol{P}\vec{\gamma}(\tau_1) = \bar{d}_1^+\vec{v}_s$ and $(\mathbb{I} - \boldsymbol{P})\vec{\gamma}(\tau_2) = \bar{d}_2\vec{v}_u$, then we set

$$\mathcal{S}_1^{\pm}(\delta) := \left\{\pm\frac{\vec{v}_u}{|\ln(\delta)|} + \left(d + \bar{d}_1^{\pm}\right)\vec{v}_s \mid |d| \le \delta\right\},$$
$$\mathcal{S}_2(\delta) := \left\{(d + \bar{d}_2)\vec{v}_u + \frac{\vec{v}_s}{|\ln(\delta)|} \mid |d| \le \delta\right\}.$$

Let $0 < d < \delta$ for δ sufficiently small and $\vec{P}_+ = (d + \bar{d}_2)\vec{v}_u + \frac{\vec{v}_s}{|\ln(\delta)|} \in \mathcal{S}_2(\delta)$. Following [38, Lemma 13.1] we approximate the trajectory $\vec{y}(t, \vec{P}_+)$ by $\vec{y}^{\ell}(t, \vec{P}_+^{\ell})$ defined in (6.6). Notice that $\vec{y}^{\ell}(t, \vec{P}_+^{\ell})$ crosses transversally $L_+^+(\delta)$ at $t = T_+^{\ell,s}$ in $\vec{Q}_{L,+}^{\ell}(d)$, and then $\mathcal{S}_1^{\ell,+}(\delta)$ at $t = T_1^{\ell}$ in $\vec{Q}_+^{\ell}$, see (6.7). Further, for any $\vec{Q}_+(d) = d(\vec{v}_u + k\vec{v}_s) \in L_+^+(\delta)$, the trajectory $\vec{y}^{\ell}(t, \vec{Q}_+(d))$ intersects transversally $\mathcal{S}_1^{\ell,+}(\tilde{\delta})$ for some $\tilde{\delta}$ at $t = T_+^{\ell,u}(d)$ in $\vec{Q}_u^{\ell}(d)$, where

$$\begin{aligned}
\vec{Q}_{L,+}^{\ell}(d) &= \frac{d^{\frac{|\lambda_s|}{|\lambda_s|+\lambda_u}}}{|k\ln(\delta)|^{\frac{\lambda_u}{|\lambda_s|+\lambda_u}}}(\vec{v}_u + k\vec{v}_s),\\
\vec{Q}_u^{\ell}(d) &= \frac{1}{|\ln(\delta)|}\left(\vec{v}_u + k|d\ln(\delta)|^{1+\frac{|\lambda_s|}{\lambda_u}}\vec{v}_s\right),\\
T_+^{\ell,s}(d) &= \frac{|\ln|kd\ln(\delta)||}{\lambda_u + |\lambda_s|}, \quad T_+^{\ell,u}(d) = \frac{|\ln|d\ln(\delta)||}{\lambda_u}.
\end{aligned} \tag{6.9}$$

Using a contraction principle we see that $\vec{y}(t, \vec{P}_+)$ and $\vec{y}(t, \vec{Q}_+)$ are well approximated by the explicitly known $\vec{y}^{\ell}(t, \vec{P}_+^{\ell})$ and $\vec{y}^{\ell}(t, \vec{Q}_+)$, i.e. we have the following.

Lemma 6.1 *([38, Lemma 13.1]). Let $0 < d < \delta$, where $\delta < \delta_0$ is small enough, and set $\bar{\delta} = \delta^{\frac{|\lambda_s|}{|\lambda_s|+\lambda_u} - \bar{\nu}}$, $\tilde{\delta} = \delta^{1+\frac{|\lambda_s|}{\lambda_u} - \tilde{\nu}}$, where $\tilde{\nu} > \bar{\nu} > 0$ are arbitrary small constants. Let*

$$\vec{P}_+ = \vec{P}_+(d) = (d + \bar{d}_2)\vec{v}_u + \frac{\vec{v}_s}{|\ln(\delta)|} \in \mathcal{S}_2(\delta).$$

Then the trajectory $\vec{y}(t, \vec{P}_+)$ intersects transversally $L_+^+(\bar{\delta})$ at $t = T_+^s(d)$ in $\Pi_+^s(d)(\vec{v}_u + k\vec{v}_s)$. Further, for any $\vec{Q}_+(d) = d(\vec{v}_u + k\vec{v}_s) \in L_+^+(\delta)$, the trajectory $\vec{y}(t, \vec{Q}_+(d))$ intersects transversally $S_1^+(\tilde{\delta})$ at $t = T_+^u(d)$ in $\vec{Q}_u = \vec{Q}_u(d) = \vec{\gamma}(\tau_1) + \Pi_+^u(d)\vec{v}_s$. Moreover, as $d \to 0$, we have

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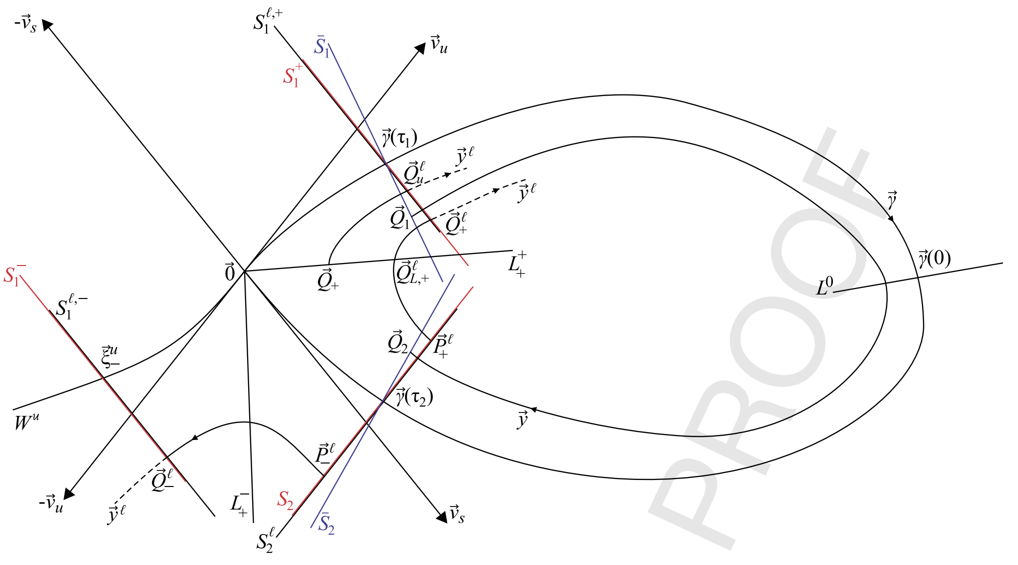

Fig. 9. Notation of intersections in a neighborhood of the origin.

$$
\begin{aligned}
\Pi_+^s(d) &\sim d^{\frac{|\lambda_s|}{|\lambda_s|+\lambda_u}}\,|k\ln(\delta)|^{-\frac{\lambda_u}{|\lambda_s|+\lambda_u}}, &\quad \Pi_+^u(d) &\sim kd^{1+\frac{|\lambda_s|}{\lambda_u}}\,|\ln(\delta)|^{\frac{|\lambda_s|}{\lambda_u}},\\
T_+^s(d) &\sim \frac{|\ln|kd\ln(\delta)||}{\lambda_u+|\lambda_s|}, & T_+^u(d) &\sim \frac{|\ln|d\ln(\delta)||}{\lambda_u}.
\end{aligned}
\tag{6.10}
$$

Finally $\vec{y}(t,\vec{P}_+) \in B(\Gamma,\bar{\delta})$ for $0 \le t \le T_+^s$ and $\vec{y}(t,\vec{Q}_+) \in B(\Gamma,\tilde{\delta})$ for $0 \le t \le T_+^u$.

Proof. The proof is obtained by applying a change of variable that reduces W^u to the y axis and then by applying a fixed point argument, see [38, Lemma 13.1]. The fact that $\vec{y}(t,\vec{P}_+) \in B(\Gamma,\bar{\delta})$ and $\vec{y}(t,\vec{Q}_+) \in B(\Gamma,\tilde{\delta})$ is not explicitly stated in [38] but easily follows from an inspection of the proof. □

Then, for any $\vec{Q} \in \mathcal{S}_1^+(\delta)$ we see that $\vec{y}(t,\vec{Q})$ follows Γ and intersects transversally $\mathcal{S}_2(\delta)$ at some T_2 close to $\tau_2 - \tau_1$.

We recall that $\vec{\psi}(t)$ denotes the unique vector such that $\langle\vec{\psi}(t), \vec{f}(\vec{\gamma}(t))\rangle = 0$, $\|\vec{\psi}(t)\| = 1$, and $\vec{\gamma}(t) + c\vec{\psi}(t) \in E^{in}$ for any $c > 0$ small enough (it points towards the interior of Γ).

We need to introduce two further segments

$$
\bar{\mathcal{S}}_i(\delta) = \left\{\vec{Q} = \vec{\gamma}(\tau_i) + d\vec{\psi}(\tau_i) \mid |d| \le \delta\right\}, \qquad i = 1,2.
$$

Lemma 6.2. *Let $|d| \le \delta$, $\vec{Q}_1 = \vec{Q}_1(d) = \vec{\gamma}(\tau_1) + d\vec{\psi}(\tau_1) \in \bar{\mathcal{S}}_1(\delta)$. Then the trajectory $\vec{y}(t,\vec{Q}_1)$ intersects transversally $\bar{\mathcal{S}}_2(\delta^{1-\nu})$ in some $\vec{Q}_2$ at $t = \bar{T}_2$, where $\nu > 0$ is an arbitrary small constant. Further $\vec{y}(t,\vec{Q}_1) \in B(\Gamma,\delta^{1-\nu})$ for any $t \in [0,\bar{T}_2]$, and*

$$
\begin{aligned}
\vec{Q}_2 &= \vec{\gamma}(\tau_2) + \Pi_2(d)\vec{\psi}(\tau_2),\\
\bar{T}_2 &\sim |\tau_1| + \tau_2 \sim \ln|\ln(\delta)|\left[\frac{1}{|\lambda_s|} + \frac{1}{\lambda_u}\right], \qquad \text{as } \delta \to 0.
\end{aligned}
$$

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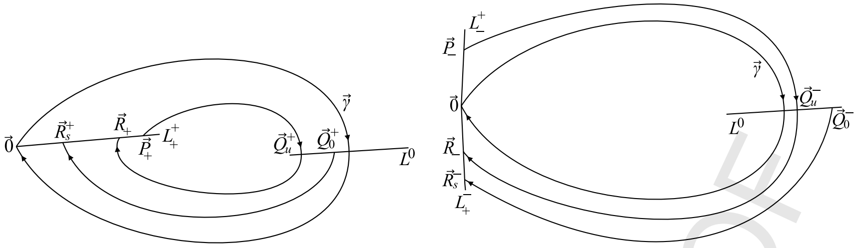

(a) (b)

Fig. 10. Solutions of (S) for $\varepsilon = 0$ in (10a) E^{in}, (10b) E^{out}.

Finally $\Pi_2(d)$ *is* C^r*, monotone increasing,*

$$\Pi_2(0) = 0 \quad \textit{and} \quad \Pi_2'(0) = \exp\left[\int_0^{\bar{T}_2} \operatorname{div}\big(\vec{f}\big(\vec{\gamma}(s)\big)\big)ds\right].$$

Proof. $\bar{T}_2$ is obtained from (6.8). The rest follows from [23, Theorem 12.15] with trivial changes: in fact in [23, Theorem 12.15] the statement is developed assuming that $\vec{\gamma}$ is periodic, so there is a unique section, say $\bar{\mathcal{S}}_1$, and Π_2 is a Poincaré map. □

Now, using the fact that $\bar{\mathcal{S}}_1(\delta)$ is not orthogonal to $\mathcal{S}_1^+(\delta)$ and $\bar{\mathcal{S}}_2(\delta)$ is not orthogonal to $\mathcal{S}_2(\delta)$, independently of δ, we get the following.

Lemma 6.3. *Let* $|d| \le \delta$*, and let* $\vec{Q} = \vec{Q}(d) = \vec{\gamma}(\tau_1) + d\vec{v}_s \in \mathcal{S}_1^+(\delta)$*. Then* $\vec{y}(t, \vec{Q})$ *intersects transversally* $\mathcal{S}_2(\delta^{1-\nu})$ *in some* $\vec{R}$ *at* $t = T_2$*.*

$$\begin{aligned}\vec{R} =& \vec{\gamma}(\tau_2) + c(d)\Pi_2(d)\vec{v}_u,\\ T_2 \sim& |\tau_1| + \tau_2 \sim \ln|\ln(\delta)|\left[\frac{1}{|\lambda_s|} + \frac{1}{\lambda_u}\right], \qquad \textit{as } \delta \to 0\end{aligned}$$

where c *is a smooth function depending just on the angles between* $\bar{\mathcal{S}}_1(\delta)$*,* $\mathcal{S}_1^+(\delta)$*, and* $\bar{\mathcal{S}}_2(\delta)$*,* $\mathcal{S}_2(\delta)$*, and* $c(0) > 0$*.*

From Lemmas 6.1 and 6.3 we easily get the following (see Fig. 10a).

Lemma 6.4. *Fix* $k > 0$*,* $0 < \delta \le \delta_0$ *small enough,* $0 < \bar{\nu} < \nu$ *arbitrarily small* $\sigma = \min\{\frac{|\lambda_s|}{\lambda_u}, 1\}$*. Let* $\vec{w}$ *aim towards* E^{in}*,* $d, D \in (0, \delta)$*, and set*

$$\vec{P}_+(d) = d(\vec{v}_u + k\vec{v}_s) \in L_+^+(\delta), \qquad \vec{Q}_+^0(D) = \vec{\gamma}(0) + D\vec{w} \in L^0(\delta).$$

Then there are $0 < \mathcal{T}_+^u(d) < \mathcal{T}_+(d)$*,* $\mathcal{T}_+^s(D) > 0$ *such that the trajectory* $\vec{y}(t, \vec{P}_+(d))$ *intersects transversally a first time* $L^0(\delta^{1+\frac{|\lambda_s|}{\lambda_u}-\bar{\nu}})$ *at* $t = \mathcal{T}_+^u(d)$ *in* $\vec{Q}_+^u(d)$ *and then* $L_+^+(\delta^{\frac{|\lambda_s|}{\lambda_u}-\nu})$ *at* $t = \mathcal{T}_+(d)$ *in* $\vec{R}_+(d)$*, and*

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$$\begin{aligned} \vec{Q}^u_+(d) &= \vec{\gamma}(0) + \Psi^u_+(d)\vec{w} \quad \textit{where} \quad d^{1+\frac{|\lambda_s|}{\lambda_u}+\bar{\nu}} \le \Psi^u_+(d) \le d^{1+\frac{|\lambda_s|}{\lambda_u}-\bar{\nu}}, \\ \vec{R}_+(d) &= \Psi_+(d)(\vec{v}_u + k\vec{v}_s) \quad \textit{where} \quad d^{\frac{|\lambda_s|}{\lambda_u}+\nu} \le \Psi_+(d) \le d^{\frac{|\lambda_s|}{\lambda_u}-\nu}. \end{aligned} \tag{6.11}$$

Further $\vec{y}(t, \vec{Q}^0_+(D))$ *intersects* $L^+_+(\delta^{\frac{|\lambda_s|}{|\lambda_s|+\lambda_u}-\bar{\nu}})$ *at* $t = \mathcal{T}^s_+(D)$ *in* $\vec{R}^s_+(D)$, *and*

$$\vec{R}^s_+(D) = \Psi^s_+(D)(\vec{v}_u + k\vec{v}_s) \ \textit{where} \ D^{\frac{|\lambda_s|}{|\lambda_s|+\lambda_u}+\bar{\nu}} \le \Psi^s_+(D) \le D^{\frac{|\lambda_s|}{|\lambda_s|+\lambda_u}-\bar{\nu}}. \tag{6.12}$$

Moreover $\vec{y}(t, \vec{P}_+(d)) \in B(\Gamma, \delta^{\sigma-\nu}) \cap E^{in}$ *for any* $0 \le t \le \mathcal{T}_+(d)$, *and* $\vec{y}(t, \vec{Q}^0_+(D)) \in B(\Gamma, \delta^{\frac{|\lambda_s|}{|\lambda_s|+\lambda_u}-\nu}) \cap E^{in}$ *for any* $0 \le t \le \mathcal{T}^s_+(D)$, *and*

$$\mathcal{T}^u_+(d) \sim \frac{|\ln(d)|}{\lambda_u}, \quad \mathcal{T}^s_+(D) \sim \frac{|\ln(D)|}{\lambda_u + |\lambda_s|}, \quad \mathcal{T}_+(d) \sim \mathcal{T}^u_+(d) + \mathcal{T}^s_+\big(\Psi^u_+(d)\big). \tag{6.13}$$

Remark 6.5. Following [38, Theorem 13.2] we can improve the estimates for $\Psi_+(d)$: we can obtain a complete expansion to any order via [38, §13.1]. In particular $\Psi_+(d) = Ad^{\frac{|\lambda_s|}{\lambda_u}}$ where $A > 0$, and if $\lambda_s + \lambda_u = 0$ then $A = \exp[\int_{-\infty}^{+\infty} \operatorname{div}(\vec{f}(\vec{\gamma}(s)))ds]$. Notice that the integral defining A is convergent iff $\lambda_s + \lambda_u = 0$. However the estimates in Lemma 6.4 are enough for our purpose.

Repeating the argument of Lemma 6.1 we obtain the following.

Lemma 6.6. *Let* $0 < d < \delta$, *where* $\delta < \delta_0$ *is small enough, and set* $\bar{\delta} = \delta^{\frac{|\lambda_s|}{|\lambda_s|+\lambda_u}-\bar{\nu}}$, $\tilde{\delta} = \delta^{1+\frac{|\lambda_s|}{\lambda_u}-\tilde{\nu}}$, *where* $\tilde{\nu} > \bar{\nu} > 0$ *are arbitrary small constants. Let*

$$\vec{P}_- = \vec{P}_-(d) = (-d + \bar{d}_2)\vec{v}_u + \frac{\vec{v}_s}{|\ln(\delta)|} \in \mathcal{S}_2(\delta).$$

Then the trajectory $\vec{y}(t, \vec{P}_-)$ *intersects transversally* $L^-_+(\bar{\delta})$ *at* $t = T^s_-$ *in* $\Pi^s_-(d)(-\vec{v}_u + k\vec{v}_s)$. *Further, for any* $\vec{Q}_-(d) = d(-\vec{v}_u + k\vec{v}_s) \in L^-_+(\delta)$, *the trajectory* $\vec{y}(t, \vec{Q}_-(d))$ *intersects transversally* $\mathcal{S}^-_1(\tilde{\delta})$ *at* $t = T^u_-$ *in* $\vec{Q}_u = \vec{Q}_u(d) = \vec{\xi}^u_- + \Pi^u_-(d)\vec{v}_s$. *Moreover, as* $d \to 0$, *we have*

$$\begin{aligned} \Pi^s_-(d) &\sim d^{\frac{|\lambda_s|}{|\lambda_s|+\lambda_u}} |k\ln(\delta)|^{-\frac{\lambda_u}{|\lambda_s|+\lambda_u}}, \quad & \Pi^u_-(d) &\sim kd^{1+\frac{|\lambda_s|}{\lambda_u}} |\ln(\delta)|^{\frac{|\lambda_s|}{\lambda_u}}, \\ T^s_-(d) &\sim \frac{|\ln|kd\ln(\delta)||}{\lambda_u + |\lambda_s|}, & T^u_-(d) &\sim \frac{|\ln|d\ln(\delta)||}{\lambda_u}. \end{aligned} \tag{6.14}$$

Finally $\vec{y}(t, \vec{P}_-) \in B(\Gamma, \bar{\delta})$ *for* $0 \le t \le T^s_-$ *and* $\vec{y}(t, \vec{Q}_-) \in B(\Gamma, \tilde{\delta})$ *for* $0 \le t \le T^u_-$.

Then, from Lemmas 6.6 and 6.3, we get the following, see Fig. 10b.

Lemma 6.7. *Fix* $k > 0$, $0 < \delta \le \delta_0$ *small enough,* $0 < \bar{\nu} < \nu$ *arbitrarily small,* $\sigma = \min\{\frac{|\lambda_s|}{\lambda_u}, 1\}$. *Let* $\vec{w}$ *aim towards* E^{in}, $d, D \in (0, \delta)$, *and set*

$$\vec{P}_-(d) = d(\vec{v}_u - k\vec{v}_s) \in L^+_-(\delta), \qquad \vec{Q}^-_0(D) = \vec{\gamma}(0) - D\vec{w} \in L^0(\delta).$$

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Then there are $0 < \mathcal{T}_u^-(d) < \mathcal{T}_-(d)$, $\mathcal{T}_s^-(D) > 0$ *such that the trajectory* $\vec{y}(t, \vec{P}_-(d))$ *intersects transversally a first time* $L^0(\delta^{1+\frac{|\lambda_s|}{\lambda_u}-\bar{\nu}})$ *at* $t = \mathcal{T}_u^-(d)$ *in* $\vec{Q}_u^-(d)$ *and then* $L_+^-(\delta^{\frac{|\lambda_s|}{\lambda_u}-\nu})$ *at* $t = \mathcal{T}_-(d)$ *in* $\vec{R}_-(d)$*; further* $\vec{y}(t, \vec{Q}_0^-(D))$ *intersects* $L_+^-(\delta^{\frac{|\lambda_s|}{|\lambda_s|+\lambda_u}-\bar{\nu}})$ *at* $t = \mathcal{T}_s^-(D)$ *in* $\vec{R}_s^-(D)$ *with*

$$\begin{aligned} \vec{Q}_u^-(d) &= \vec{\gamma}(0) - \Psi_u^-(d)\vec{w} && \text{where} \quad d^{1+\frac{|\lambda_s|}{\lambda_u}+\bar{\nu}} \le \Psi_u^-(d) \le d^{1+\frac{|\lambda_s|}{\lambda_u}-\bar{\nu}}, \\ \vec{R}_-(d) &= \Psi_-(d)(-\vec{v}_u + k\vec{v}_s) && \text{where} \quad d^{\frac{|\lambda_s|}{\lambda_u}+\nu} \le \Psi_-(d) \le d^{\frac{|\lambda_s|}{\lambda_u}-\nu}, \\ \vec{R}_s^-(D) &= \Psi_s^-(D)(-\vec{v}_u + k\vec{v}_s) && \text{where} \quad D^{\frac{|\lambda_s|}{|\lambda_s|+\lambda_u}+\bar{\nu}} \le \Psi_s^-(D) \le D^{\frac{|\lambda_s|}{|\lambda_s|+\lambda_u}-\bar{\nu}}. \end{aligned} \tag{6.15}$$

Moreover $\vec{y}(t, \vec{P}_-(d)) \in B(\Gamma, \delta^{\sigma-\nu}) \cap E^{out}$ *for any* $0 \le t \le \mathcal{T}_-(d)$*, and* $\vec{y}(t, \vec{Q}_0^-(D)) \in B(\Gamma, \delta^{\frac{|\lambda_s|}{|\lambda_s|+\lambda_u}-\nu}) \cap E^{out}$ *for any* $0 \le t \le \mathcal{T}_s^-(D)$*, and*

$$\mathcal{T}_u^-(d) \sim \frac{|\ln(d)|}{\lambda_u}, \quad \mathcal{T}_s^-(D) \sim \frac{|\ln(D)|}{\lambda_u + |\lambda_s|}, \quad \mathcal{T}_-(d) \sim \mathcal{T}_u^-(d) + \mathcal{T}_s^-\big(\Psi_u^-(d)\big).$$

Remark 6.8. Let $\vec{P} \in (A^s(\tau) \cup B^s(\tau))$, $\vec{Q} \in (A^u(\tau) \cup B^u(\tau))$, and assume $\|\vec{P} - \vec{\zeta}^s(\tau)\| \le \delta$ and $\|\vec{Q} - \vec{\zeta}^u(\tau)\| \le \delta$. In the assumptions of Lemmas 4.3, and 4.5 we see that there is $k_0 > 0$ such that, we have

$$T^s(\vec{P}) - \tau \ge \frac{|\ln(\delta|\ln(\delta)|)|}{\lambda_u + |\lambda_s| - k_0\varepsilon}, \qquad \tau - T^u(\vec{P}) \ge \frac{|\ln(\delta|\ln(\delta)|)|}{\lambda_u + |\lambda_s| - k_0\varepsilon}.$$

The proof follows easily from Lemmas 6.6 and 6.3.

6.2.2. Proofs of Lemmas 4.2, 4.4

Let $\vec{f}^\perp(\vec{x})$ be the vector field such that $\langle \vec{f}^\perp(\vec{x}), \vec{f}(\vec{x})\rangle = 0$, $\|\vec{f}^\perp(\vec{x})\| = \|\vec{f}(\vec{x})\|$, i.e. $\vec{f}^\perp(\vec{x})$ is the C^r function obtained rotating by $\vec{f}(\vec{x})$ by $\pi/2$: we choose the orientation in such a way that $\vec{f}^\perp(\vec{\gamma}(t)) = \|\vec{f}^\perp(\vec{\gamma}(t))\|\vec{\psi}(t)$.

Let λ_1^g, λ_2^g be the eigenvalues of $\vec{g}_x(0)$. Then if $\|\vec{x}\|$ is small enough we have $\frac{\|\vec{g}(\vec{x})\|}{\|\vec{f}(\vec{x})\|} \le \frac{2\max\{|\lambda_1^g|, |\lambda_2^g|\}}{\min\{\lambda_u, |\lambda_s|\}}$. Then let

$$K := 2\sup\big\{\|\vec{g}(\vec{x})\|/\|\vec{f}(\vec{x})\| \mid \vec{x} \in B(\Gamma, 1) \cap \Omega\big\}$$

and notice that K is bounded. We introduce the auxiliary functions $\vec{f}_{\text{in}}$ and $\vec{f}_{\text{out}}$ defined as follows:

$$\begin{aligned} \vec{f}_{\text{in}}(\vec{x}) &= \vec{f}(\vec{x}) + \varepsilon K \vec{f}^\perp(\vec{x}), \\ \vec{f}_{\text{out}}(\vec{x}) &= \vec{f}(\vec{x}) - \varepsilon K \vec{f}^\perp(\vec{x}). \end{aligned} \tag{6.16}$$

We have chosen the functions $\vec{f}_{\text{in}}(\vec{x})$ and $\vec{f}_{\text{out}}(\vec{x})$ so that the flow of $\dot{\vec{x}} = \vec{f}_{\text{in}}(\vec{x})$ and of $\dot{\vec{x}} = \vec{f}_{\text{out}}(\vec{x})$ on Γ aims towards E^{in} or towards E^{out} (always assuming that $-c\vec{v}_u \in E^{out}$ for any $c > 0$ small).

Then we denote by $\vec{y}_{in}(t, \vec{P})$ and $\vec{y}_{out}(t, \vec{P})$, respectively, the solutions of $\dot{x} = \vec{f}_{\text{in}}(\vec{x})$ and of $\dot{x} = \vec{f}_{\text{out}}(\vec{x})$ leaving from $\vec{P}$ at $t = 0$. Notice that the origin is still a saddle critical point for both the systems, and that the eigenvalues and the eigenvectors of $\boldsymbol{\partial_x f_{in}}(\vec{0})$ and $\boldsymbol{\partial_x f_{out}}(\vec{0})$ are

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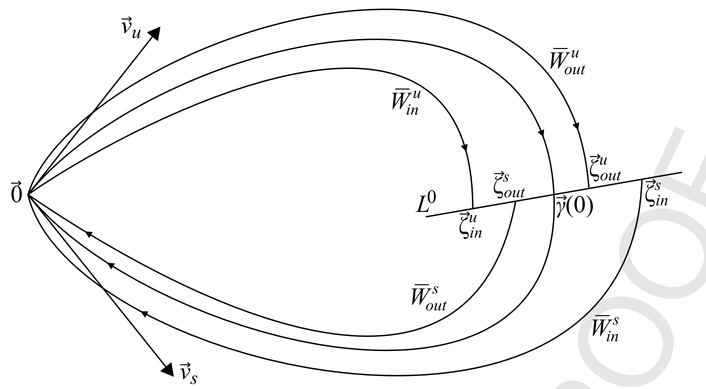

Fig. 11. Illustration of Lemma 6.9.

$O(\varepsilon)$ close to the ones of $\boldsymbol{f_x}(\vec{0})$. Let $W^{u,+}_{in}$, $W^{s,+}_{in}$ be the branches of the unstable and the stable manifold of $\dot{x} = \vec{f}_{\text{in}}(\vec{x})$, which are close to $W^{u,+}$, $W^{s,+}$, respectively; and similarly we define $W^{u,+}_{out}$, $W^{s,+}_{out}$. Let $\vec{\zeta}^u_{in}$, $\vec{\zeta}^s_{in}$, $\vec{\zeta}^u_{out}$, $\vec{\zeta}^s_{out}$ be the first intersections of $W^{u,+}_{in}$, $W^{s,+}_{in}$, $W^{u,+}_{out}$, $W^{s,+}_{out}$ with $L^0(\varepsilon^{1-\nu})$. We denote by $\bar{W}^u_{in}$ the branch of $W^{u,+}_{in}$ between the origin and $\vec{\zeta}^u_{in}$. Analogously we define $\bar{W}^s_{in}$, $\bar{W}^u_{out}$, $\bar{W}^s_{out}$.

Let $\tilde{L}^0_u$ be the segment between $\vec{\zeta}^u_{in}$ and $\vec{\zeta}^u_{out}$, $\tilde{L}^0_s$ the segment between $\vec{\zeta}^s_{in}$ and $\vec{\zeta}^s_{out}$. Now we state a simple result illustrated by Fig. 11.

Lemma 6.9. *There is $c > 0$ such that $\bar{W}^u_{in} \in B(\Gamma^u, c\varepsilon) \cap E^{in}$, $\bar{W}^u_{out} \in B(\Gamma^u, c\varepsilon) \cap E^{out}$, $\bar{W}^s_{out} \in B(\Gamma^s, c\varepsilon) \cap E^{in}$, $\bar{W}^s_{in} \in B(\Gamma^s, c\varepsilon) \cap E^{out}$. Further there are positive constants $c^u_{in}, c^u_{out}, c^s_{in}, c^s_{out}$ such that*

$$\begin{aligned} \vec{\zeta}^u_{in} &= \vec{\gamma}(0) + (c^u_{in}\varepsilon + o(\varepsilon))\vec{w}, \quad & \vec{\zeta}^u_{out} &= \vec{\gamma}(0) - (c^u_{out}\varepsilon + o(\varepsilon))\vec{w}, \\ \vec{\zeta}^s_{in} &= \vec{\gamma}(0) - (c^s_{in}\varepsilon + o(\varepsilon))\vec{w}, \quad & \vec{\zeta}^s_{out} &= \vec{\gamma}(0) + (c^s_{out}\varepsilon + o(\varepsilon))\vec{w} \end{aligned}$$

for $\vec{w}$ aiming towards E^{in}. Finally $\vec{\zeta}^u(\tau) \in \tilde{L}^0_u$ and $\vec{\zeta}^s(\tau) \in \tilde{L}^0_s$.

Proof. Assume for simplicity that $\vec{w} = \vec{\psi}(0)$ in the definition of L^0, cf. (4.1). Then, from classical arguments of Mel'nikov theory, we see that

$$\vec{\zeta}^u_{in} = \vec{y}_{in}\big(0, \vec{\zeta}^u_{in}\big) = \vec{\gamma}(0) + K\varepsilon \int_{-\infty}^{0} \vec{f}^{\perp}\big(\vec{\gamma}(s)\big)ds + o(\varepsilon).$$

So the claim concerning $\bar{W}^u_{in}$ and $\vec{\zeta}^u_{in}$ easily follows. The others are analogous.

If $\vec{w} \neq \vec{\psi}(0)$ but it aims towards E^{in}, the argument still goes through but we have different values of the constants $c^{u,s}_{in,out}$ (depending on the angle between $\vec{w}$ and $\vec{\psi}(0)$).

The fact that $\vec{\zeta}^u(\tau) \in \tilde{L}^0_u$, $\vec{\zeta}^s(\tau) \in \tilde{L}^0_s$ follows by construction. □

Let $\nu > 0$ be small and set $\bar{c} = \max\{c^u_{in}, c^u_{out}, c^s_{in}, c^s_{out}\}$, $0 < d < \delta$, and

$$\begin{aligned}\vec{Q}_a(d) &:= \vec{\gamma}(0) + (d + \bar{c}\varepsilon)\vec{w} \in [E^{in} \cap L^0(\delta + \bar{c}\varepsilon)], \\ \vec{Q}_b(d) &:= \vec{\gamma}(0) - (d + \bar{c}\varepsilon)\vec{w} \in [E^{out} \cap L^0(\delta + \bar{c}\varepsilon)]\end{aligned} \tag{6.17}$$

see Fig. 12. Then consider the trajectory $\vec{y}_{in}(t, \vec{Q}_a(d))$ for $t > 0$. Notice that

$$\vec{Q}_a(d) - \vec{\zeta}^s_{in} = \big[d + \big(c^s_{in} + \bar{c}\big)\varepsilon + o(\varepsilon)\big]\vec{w},$$

therefore, with a trivial adaption of Lemma 6.4 (cf. (6.12)) we see that there is $T_a = \mathcal{T}^+_{s,in}(D_{in})$ with $D_{in} = d + (c^s_{in} + \bar{c})\varepsilon + o(\varepsilon)$ such that $\vec{y}_{in}(t, \vec{Q}_a(d))$ intersects L^+_+ transversally at $t = T_a$ in $\vec{R}^+_{s,in}(D_{in}) = \Psi^+_{s,in}(D_{in})(\vec{v}_u + k\vec{v}_s)$ where

$$(d + \varepsilon)^{\frac{|\lambda_s|}{|\lambda_s| + \lambda_u} + \nu} \le \Psi^+_{s,in}(D_{in}) \le (d + \varepsilon)^{\frac{|\lambda_s|}{|\lambda_s| + \lambda_u} - \nu}. \tag{6.18}$$

Further $\vec{y}_{in}(t, \vec{Q}_a(d)) \in B(\Gamma, (d + \varepsilon)^{\frac{|\lambda_s|}{|\lambda_s| + \lambda_u} - \nu})$ for any $0 < t < T_a$. Then the curve $\vec{z}^s_a : [0, 1] \to \mathbb{R}^2$ of Lemma 4.2 is a reparametrization of $\vec{y}_{in}(t, \vec{Q}_a(d))$ such that $\vec{z}^s_a(0) = \vec{y}_{in}(0, \vec{Q}_a(d)) = \vec{Q}_a(d) \in L^0$ and $\vec{z}^s_a(1) = \vec{y}_{in}(T_a, \vec{Q}_a(d)) = \vec{R}^+_{s,in}(D_{in}) \in L^+_+$.

Analogously consider the trajectory $\vec{y}_{out}(t, \vec{Q}_b(d))$ for $t > 0$. Notice that

$$\vec{Q}_b(d) - \vec{\zeta}^s_{out} = -\big[d + \big(c^s_{out} + \bar{c}\big)\varepsilon + o(\varepsilon)\big]\vec{w}.$$

Therefore, with a trivial adaption of Lemma 6.7 (cf. line 3 in (6.15)) we see that there is $T_b = \mathcal{T}^-_{s,out}(D_{out})$ with $D_{out} = d + (c^s_{out} + \bar{c})\varepsilon + o(\varepsilon)$ such that $\vec{y}_{out}(t, \vec{Q}_b(d))$ intersects L^-_+ transversally at $t = T_b$ in

$$\vec{R}^-_{s,out}(D_{out}) = \Psi^-_{s,out}(D_{out})(-\vec{v}_u + k\vec{v}_s)$$

where

$$(d + \varepsilon)^{\frac{|\lambda_s|}{|\lambda_s| + \lambda_u} + \nu} \le \Psi^-_{s,out}(D_{out}) \le (d + \varepsilon)^{\frac{|\lambda_s|}{|\lambda_s| + \lambda_u} - \nu}. \tag{6.19}$$

Further $\vec{y}_{out}(t, \vec{Q}_b(d)) \in B(\Gamma, (d + \varepsilon)^{\frac{|\lambda_s|}{|\lambda_s| + \lambda_u} - \nu})$ for any $0 < t < T_b$. Then the curve $\vec{z}^s_b : [0, 1] \to \mathbb{R}^2$ of Lemma 4.2 is a reparametrization of $\vec{y}_{out}(t, \vec{Q}_b(d))$ such that $\vec{z}^s_b(0) = \vec{y}_{out}(0, \vec{Q}_b(d)) = \vec{Q}_b(d) \in L^0$ and $\vec{z}^s_b(1) = \vec{y}_{out}(T_b, \vec{Q}_b(d)) = \vec{R}^-_{s,out}(D_{out}) \in L^-_+$. In the notation preceding Lemma 4.2, by construction, the flow of the non-autonomous system (S) on Z^s_a and Z^s_b points towards the interior of $K^s(\delta, \varepsilon)$; further recall that $\bar{L}^0_s$ is the segment between $\vec{z}^s_a(0)$ and $\vec{z}^s_b(0)$, so that $\vec{\zeta}^s(\tau) \in \tilde{L}^0_s \subset \bar{L}^0_s$, hence Lemma 4.2 is proved.

Analogously to prove Lemma 4.4 we follow for $t \le 0$ the trajectories $\vec{y}_{out}(t, \vec{Q}_a(d))$ and $\vec{y}_{in}(t, \vec{Q}_b(d))$, where $\vec{Q}_a(d)$ and $\vec{Q}_b(d)$ are given by (6.17). Then, reasoning as above, we see that there are $T^a < 0$, $T^b < 0$ such that $\vec{y}_{out}(t, \vec{Q}_a(d))$ intersects transversally L^+_+ at $t = T^a$ and $\vec{y}_{in}(t, \vec{Q}_b(d))$ intersects transversally L^+_- at $t = T^b$. Then $\vec{z}^u_a$ is a reparametrization of $\vec{y}_{out}(t, \vec{Q}_a(d))$ such that $\vec{z}^u_a(1) = \vec{Q}_a(d)$ and $\vec{z}^u_a(0) = \vec{y}_{out}(T^a, \vec{Q}_a(d))$, while $\vec{z}^u_b$ is a

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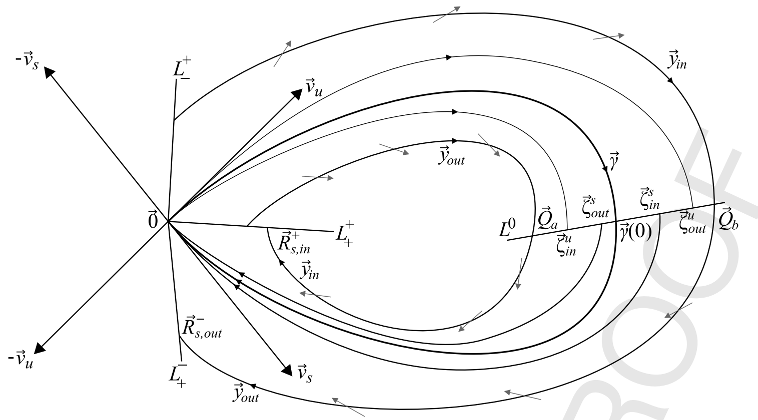

Fig. 12. Proof of Lemma 4.2. Arrows denote the flow of system (S).

reparametrization of $\vec{y}_{in}(t, \vec{Q}_b(d))$ such that $\vec{z}_b^u(1) = \vec{Q}_b(d)$ and $\vec{z}_b^u(0) = \vec{y}_{in}(T^b, \vec{Q}_b(d))$. Then, using the notation preceding Lemma 4.2, we see that the flow of the non-autonomous system (S) on Z_a^s and Z_b^s points towards the exterior of $K^u(\delta, \varepsilon)$. Further $K^u(\delta, \varepsilon) \subset B(\Gamma, (\delta+\varepsilon)^{\frac{\lambda_u}{|\lambda_s|+\lambda_u}-\nu})$, and $\vec{\zeta}^u(\tau) \in \tilde{L}_u^0 \subset \bar{L}_u^0$, hence Lemma 4.4 is proved.

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[25] [26] [28] [29] [35]

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