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Addendum: Nonlinear integral equations for the sausage model (2017 J.Phys. A50 314005)

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Abstract

We complete the derivation of the sausage model NLIE by giving a proof of the crucial relation (3.24) of the original paper based on the analytic properties of Q and $\bar{Q}$.

1 Introduction

In ref. [1], here below referred as I, we have written the set of Non-Linear Integral Equations (NLIEs) governing the finite size effects of the vacuum as well as the thermodynamics for the integrable deformation of $O(3)$ non-linear sigma model (NLSM), getting it from a manipulation, inspired by those introduced years ago by J. Suzuki [3, 4], of the larger set of Thermodynamic Bethe Ansatz (TBA) equations of the model, known since the original paper by Fateev, Onofri and Zamolodchikov [2]. However, one can realize that (I3.24)[1], a crucial relation in our derivation of

[1] Here we refer to the equations of I as (Ix.xx), for example eq. (3.24) of I is referred as (I3.24). Definitions, notation and symbols are as defined in I.

the sausage model NLIE, is not well-defined because neither Q nor $\bar{Q}$ are analytic on the real axis. Hence $\tilde{Q}$ and $\tilde{\bar{Q}}$ cannot be interpreted as Fourier transforms along the real line[2].

In this Addendum we examine this problem carefully and show that the derivation of the sausage model NLIE remains valid in spite of this potential difficulty .

2 Analyticity strips

Our starting point is that the sausage model Y-system for the ground state has constant solution in the infinite volume limit $\ell = mr \to \infty$:

$$y_k = k(k+2), \quad k = 1, \ldots, N-2; \qquad y_N = y_{N-1} = N-1; \qquad y_0 = 0. \tag{1}$$

The corresponding T-system solution is

$$T_k = k+1, \quad k = 1, \ldots, N-1 \tag{2}$$

and

$$A = \bar{A} = 2. \tag{3}$$

For (I3.13-14) we choose the bounded solutions

$$Q = \bar{Q} = 1. \tag{4}$$

The other linearly independent solutions of the second order difference equations (I3.13) and (I3.14) are $Q = \bar{Q} = \theta$, but these are not bounded.

We assume that we have solved the TBA equations for finite (but large) volume

$$y_a(\theta) = \exp\left\{ \sum_b \frac{I_{ab}}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}\theta'}{\cosh(\theta-\theta')} L_b(\theta') \right\}, \quad a = 1, \ldots, N; \quad y_0 = \mathrm{e}^{-\ell\cosh\theta} y_1(\theta), \tag{5}$$

where I_{ab} is the incidence matrix of the sausage model TBA diagram (including the massive node) and $L_a = \log Y_a$. All y_a functions are defined originally along the real line, where they are real and positive.

The shifts of the left-hand side of the Y-system equations (I3.1-3) along $\mathrm{Im}\,\theta$, often referred to as TBA steps, are $\pm i\pi/2$, so it is convenient to use the notation (α, β) indicating the strip

$$\frac{\pi}{2}\alpha < \mathrm{Im}\,\theta < \frac{\pi}{2}\beta. \tag{6}$$

The above TBA equations themselves allow us to analytically continue the Y-functions to the strip $(-1, 1)$ and we can see that all y_a functions ($a = 1, \ldots, N$) are analytic and non-zero (ANZ) in this strip for large volume and they must be close to the constant solution. y_0 is also ANZ in this strip and it is uniformly small in

[2]We thank Prof. J. Suzuki for pointing this out.

the strip $(-1+\epsilon, 1-\epsilon)$, where ϵ is some fixed, small, but not infinitesimal number. We will abbreviate this property by ANZC, meaning that it is ANZ and close to a constant solution. Then,

$$y_a \text{ is ANZC in } (-1,1) \quad \text{for} \quad a=1,\dots,N; \qquad y_0 \text{ is ANZC in } (-1+\epsilon, 1-\epsilon). \quad (7)$$

We can further extend these "good" strips for the Y-functions and also for the corresponding T-system using the Y-system equations. In the appendix we show that

$$T_k \text{ is ANZC in } (-k-1+\epsilon, k+1-\epsilon), \qquad k=1,\dots,N-1. \quad (8)$$

Now from the definition of A in (I3.13) we find that the ANZC strip for A is $(2+\epsilon, 2k-\epsilon)$, but since A is independent of k, we can take the maximal allowed k value, which gives the strip $(2+\epsilon, 2N-2-\epsilon)$. Similarly for $\bar{A}$ we have $(-2N+2+\epsilon, -2-\epsilon)$.

The defining relation for Q, (I3.13), provides an ANZC strip for Q which is 2 units wider in both directions:

$$Q \text{ is ANZC in } (\epsilon, 2N-\epsilon), \quad (9)$$

and analogously

$$\bar{Q} \text{ is ANZC in } (-2N+\epsilon, -\epsilon). \quad (10)$$

These strips are consistent with both the fact that Q and $\bar{Q}$ are complex conjugates of each other and the crucial relation

$$Q^{[2N]} = \bar{Q}. \quad (11)$$

Therefore, Eq.(I3.16) is still valid if we exclude the real axis from the domain of definition.

3 Fourier transformation

Now the problem with defining the Fourier transform of (the log-derivative of) Q is that the real line is not in the analyticity strip. But the $\mathrm{Im}\,\theta = \pi/2$ line is and there is no problem of defining the Fourier transform of (the log-derivative of) Q^+:

$$\widetilde{Q^+} = q_1. \quad (12)$$

Similarly

$$\widetilde{\bar{Q}^-} = \bar{q}_1. \quad (13)$$

Since

$$Q^{[\alpha]} = (Q^+)^{[\alpha-1]}, \quad (14)$$

in Fourier space we have

$$\widetilde{Q^{[\alpha]}} = p^{\alpha-1} q_1 \quad (15)$$

and analogously

$$\widetilde{\bar{Q}^{[-\beta]}} = p^{1-\beta}\bar{q}_1. \tag{16}$$

Let us now define

$$\widetilde{Q} = \frac{1}{p}q_1, \qquad \text{and} \qquad \widetilde{\bar{Q}} = p\bar{q}_1. \tag{17}$$

Note that although $\widetilde{Q}$, $\widetilde{\bar{Q}}$ are not Fourier transforms of anything, nevertheless we can write the relations

$$\widetilde{Q^{[\alpha]}} = p^{\alpha}\widetilde{Q}, \qquad \text{and} \qquad \widetilde{\bar{Q}^{[-\beta]}} = p^{-\beta}\widetilde{\bar{Q}}. \tag{18}$$

Similarly, instead of the relation $Q^{[2N]} = \bar{Q}$, one can take the Fourier transform of its equivalent form

$$Q^{[2N-1]} = \bar{Q}^{-} \tag{19}$$

since both sides are in their respective analyticity strips to get

$$p^{2N-1}\widetilde{Q} = \frac{1}{p}\widetilde{\bar{Q}}, \tag{20}$$

which is of course equivalent to the relation

$$\widetilde{\bar{Q}} = p^{2N}\widetilde{Q}. \tag{21}$$

This relation was used in the derivation of the sausage model NLIE equations in Fourier space.

We can still apply a procedure of constructing NLIE in Fourier space, initiated by [3] since (I3.20-21) remain valid if we interpret them as Fourier space relations only. However, after eliminating $\tilde{Q}$ and $\tilde{\bar{Q}}$, we arrive at (I3.25-26), where all building blocks are again genuine Fourier transforms. The results for the sausage model NLIE are thus unchanged[3].

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[3] The resulting NLIE, eq. (I3.32-3.34), turns out to be in agreement with the one conjectured by Clare Dunning in [5].

A Derivation of analyticity strips

Using the Y-system equations we look for the maximal analyticity strips. For example y_1 can be written as

$$y_1^+ = \frac{Y_2}{y_1^-} \tag{22}$$

and for $\theta \in (0,1)$ the LHS defines y_1 in the strip $(1,2)$. The numerator on the RHS lives in $(0,1)$ and the denominator in $(-1,0)$. We already know that this RHS is ANZC so we can conclude that y_1 is ANZC also in $(1,2)$. Similar conclusions can be drawn from the equations

$$y_k^+ = \frac{Y_{k-1}Y_{k+1}}{y_k^-} \tag{23}$$

for $k = 3, \dots$. However, we can only conclude that y_2 is ANZC in $(1, 2-\epsilon)$ from

$$y_2^+ = \frac{Y_1 Y_3 Y_0}{y_2^-} \tag{24}$$

because of Y_0 in the numerator. Of course, analogous considerations apply in the negative imaginary direction.

Let us summarize:

$$y_a \text{ is ANZC in } (-2,2) \quad \text{for} \quad a = 1,\dots,N \quad a \neq 2; \qquad y_2 \text{ is ANZC in } (-2+\epsilon, 2-\epsilon). \tag{25}$$

Now continuing this procedure we can convince ourselves that

$$y_3 \text{ is ANZC in } (-3+\epsilon, 3-\epsilon), \qquad y_4 \text{ is ANZC in } (-4+\epsilon, 4-\epsilon), \tag{26}$$

and so on. In the language of the variables Z_k we have

$$Z_k \text{ is ANZC in } (-k+\epsilon, k-\epsilon), \qquad k = 1,\dots,N-1. \tag{27}$$

Finally since the T-system functions are defined as the solution of the basic TBA-like equation

$$T_k^+ T_k^- = Z_k, \tag{28}$$

they have 1 unit wider strips:

$$T_k \text{ is ANZC in } (-k-1+\epsilon, k+1-\epsilon), \qquad k = 1,\dots,N-1. \tag{29}$$

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