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(Article begins on next page)

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SCHAUDER ESTIMATES AT THE BOUNDARY FOR SUB-LAPLACIANS IN CARNOT GROUPS

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ABSTRACT. In this paper we present a new approach to prove Schauder estimates at the boundary for sub-Laplacian type operators in Carnot groups. While internal Schauder estimates have been deeply studied, up to now subriemannian estimates at the boundary are known only in the Heisenberg groups. The proof of these estimates in the Heisenberg setting, due to Jerison ([34]), is based on the Fourier transform technique and cannot be repeated in general Lie groups. After the result of Jerison no new contribution to the boundary problem has been provided. In this paper we introduce a new method, which allows to build a Poisson kernel starting from the fundamental solution, from which we deduce the Schauder estimates at non characteristic boundary points.

CONTENTS

1

1. Introduction

1.1. **Aim of this work.** The aim of this work is to introduce a new approach to obtain Schauder estimates to the boundary for sub-Laplacian type operators in Carnot groups.

As it is well known, Schauder estimates at the boundary in the Euclidean setting are based on two main ingredients. The first one, which is the core of the Schauder method, is the local reduction of general uniformly elliptic operators to the Laplace operator. The second one, which seems elementary in the Euclidean setting, is a reflection technique which reduces the boundary Schauder estimates to internal ones. Unfortunately, this technique cannot be applied in the strong anisotropic setting of a Carnot group, since a Laplace type operator in this framework is not invariant with respect to reflection, nor can be approximated by any invariant operator. In the special case of the Heisenberg group, Schauder estimates are a classical result due to Jerison (see [34]), but not even this technique, based on the Fourier transform, can be extended to general Lie groups. After that, no new contribution has been provided to the problem, which is still open, while its solution would be necessary for the development of nonlinear PDE's theory in this setting.

In this paper we introduce a completely different approach, which is new even in the Riemannian setting, that allows to build a Poisson kernel starting from the knowledge of

a smooth fundamental solution for the problem on the whole space and eliminates any use of Fourier transform in the full rank case.

1.2. **Carnot groups.** A Carnot group $\mathbb{G}$ can be identified with $\mathbb{R}^n$ with a polynomial group law $(\mathbb{G}, \cdot)$, whose Lie algebra $\mathfrak{g}$ admits a step κ stratification. Precisely there exist linear subspaces $V^1, ..., V^\kappa$ such that

$$\mathfrak{g} = V^1 \oplus ... \oplus V^\kappa, \quad [V^1, V^{i-1}] = V^i, \text{ if } i \leq \kappa, \quad [V^1, V^\kappa] = \{0\}. \tag{1.1}$$

We will call horizontal tangent bundle the subspace V^1, and we will choose a basis for it denoted by $\{X_1, \cdots, X_m\}$. By the assumption on the Lie algebra, this basis can be completed to a basis $\{X_1, \cdots, X_n\}$ of $\mathfrak{g}$ with the list of their commutators. On the vector space V^1 we define a Riemannian metric which makes orthonormal the vector fields $X_1, \cdots, X_m$. Several equivalent left invariant distances d can be introduced on the whole space requiring that their restriction to V^1 is equivalent to the fixed Riemannian metric (see for example Nagel, Stein and Wainger in [43]). The subriemannian gradient of a regular function f will be denoted by $\nabla f = (X_1 f, \cdots, X_m f)$ and f will be called of class C^1 if this gradient is continuous with respect to the distance d. More generally, spaces of Hölder continuous functions $C^{k,\alpha}$ can be defined in terms of this distance and this gradient. We will study here a subelliptic operator expressed as follows:

$$\Delta = \sum_{i=1}^{m} (X_i^2 + b_i X_i), \tag{1.2}$$

with regular coefficients b_i. Operators of this type are hypoelliptic and have been deeply studied after the first works of Folland and Stein [23], Rothschild and Stein [44], Jerison and Sanchez-Calle [36], Fefferman and Sanchez-Calle [20], Kohn and Nirenberg [37], and Jerison [34, 35] (see also [4] for a recent monograph). Their fundamental solution Γ_Δ is of class C^∞ far from the diagonal and it can be estimated in term of the distance as follows

$$\Gamma_\Delta(x, y) \approx \frac{1}{d^{Q-2}(x, y)}, \tag{1.3}$$

for a suitable integer Q, called homogeneous dimension of the space (see (2.7) for a precise definition). A kernel with the behavior of Γ_Δ is called of local type 2. In general we will say that a kernel K is of local type λ with respect to the distance d if for every open bounded set V and for every $p \geq 0$ there exists a positive constant C_p such that, for every $x, y \in V$, with $x \neq y$

$$|X_{i_1} \cdots X_{i_p} K(x, y)| \leq C_p d(x, y)^{\lambda - p - 2} \Gamma_\Delta(x, y). \tag{1.4}$$

A well established theory of singular integrals in Hörmander setting (due to Folland and Stein [23], Rothschild and Stein [44], Greiner and Stein [30]) allows to prove interior Schauder estimates. For more recent results we quote the Hölder estimates by Citti [15], the Schauder estimates of Xu [49] and Capogna and Han [14] for uniformly subelliptic operators, Bramanti and Brandolini [5] for heat-type operators and the results of Lunardi [39], Di Francesco and Polidoro [18], Gutierrez and Lanconelli [32], Bramanti and Zhu [7] and Simon [46] for a large class of operators. The problem at the boundary is completely different and largely unsolved.

1.3. **Schauder estimates at the boundary.** A surface M in a Carnot group, smooth in the Euclidean sense, can be locally expressed as the zero level set of a function $f \in C^\infty$, but it can have points in M where its subriemannian gradient vanishes. At these points, called characteristic, the geometry of the surface is not completely understood. Far from characteristic points, properties of regular surfaces have been largely studied starting from the papers of Kohn and Nirenberg in [37], Jerison in [34] and more recently by Franchi, Serapioni and Serra Cassano, [24, 25] (see also the references therein). The stratification defined in (1.1) induces a stratification on the tangent plane of the manifold M. We will call $\hat{V}^1 = V^1 \cap TM$, $\hat{V}^2 = V^2 \cap TM$, $\cdots$, $\hat{V}^\kappa = V^\kappa \cap TM$. It is not restrictive to assume that $X_1 \in V^1$ is normal to $\hat{V}^1$ with respect to the metric fixed in V^1 so that we can denote by $\{\hat{X}_i\}_{i=2,\cdots,m}$ a basis of $\hat{V}^1$. We also require that the following condition holds:

$$\mathrm{Lie}(\hat{V}^1) = TM. \tag{1.5}$$

Under this assumption the manifold M has a Hörmander structure, and $\hat{V}^1$ inherits a metric from the immersion in V^1. Hence a distance $\hat{d}$ and corresponding classes of Hölder continuous functions $\hat{C}^{k,\alpha}(M)$ are well defined. For every choice of regular coefficients $(b_i)_{i=2,\cdots,m}$, a Laplace-type operator

$$\hat{\Delta} = \sum_{i=2}^m \hat{X}_i^2 + \sum_{i=2}^m b_i \hat{X}_i \tag{1.6}$$

is defined on M, with fundamental solution $\hat{\Gamma}_{\hat{\Delta}}$.

It has been proved by Kohn and Nirenberg in [37] that, if D is a smooth open set with smooth boundary and g a smooth function defined on the boundary of D, the problem

$$\Delta u = 0 \text{ in } D, \quad u = g \text{ on } \partial D \tag{1.7}$$

has a unique solution, of class C^∞ up to the boundary at non characteristic points. At the characteristic points very few results are known (see [35], already quoted, and [13], [27] and [48], where existence of non tangential limits up to the boundary are established). In this paper we prove the exact analogous of the classical Schauder estimates at the boundary, providing estimates of the $\hat{C}^{2,\alpha}$ norm of the solution in terms of the Hölder norm of the data. Precisely our result can be stated as follows.

Theorem 1.1. *Let $D \subset \mathbb{G}$ be a smooth, bounded domain and assume that the vector fields $\{X_i\}_{i=1,\cdots,m}$ satisfy the assumption* (1.5). *Denote u the unique solution to*

$$\Delta u = f \text{ in } D, \quad u = g \text{ on } \partial D,$$

where $f \in C^\alpha(\bar{D})$ and $g \in \hat{C}^{2,\alpha}(\partial D)$ and $0 < \alpha < 1$. If $\bar{x} \in \partial D$ and V is an open neighborhood of $\bar{x}$ without characteristic points, for every $\varphi \in C_0^\infty(V)$ we have $\varphi u \in C^{2,\alpha}(\bar{D} \cap V)$ and

$$\|\varphi u\|_{C^{2,\alpha}(\bar{D}\cap V)} \leq C(\|g\|_{\hat{C}^{2,\alpha}(\partial D)} + \|f\|_{C^\alpha(\bar{D})}). \tag{1.8}$$

We believe that, even if we prove our results under assumption (1.5), the method presented here will open the possibility to establish Schauder estimates for non characteristic points in any Carnot group, since any Carnot group can be lifted to a group satisfying assumption (1.5).

As we mentioned before, up to now subriemannian boundary Schauder estimates are known only for the Heisenberg group (see [34]) and are based on the construction of a Poisson kernel. If D is an open bounded set, and V is a neighborhood of a non characteristic point $\overline{x} \in \partial D$, we say that $P : C^\infty(\partial D \cap V) \to C^\infty(V \cap \overline{D})$ is a local Poisson operator for the problem (1.7) if, for every $g \in C^\infty(\partial D \cap V)$, the function $u = P(g)$ satisfies $\Delta u = 0$ in $D \cap V$ and $u(x) = g(x)$ for all $x \in \partial D \cap V$.

The construction of the Poisson kernel contained in [34] is based on the Fourier transform and cannot be directly repeated in general Lie groups. General measure theory ensures the existence of a Poisson kernel under very weak assumptions on the vector fields (see for example Lanconelli and Uguzzoni [38]), but this theory only allows to establish L^p regularity of the solution at the boundary. A Poisson kernel has been built by Ferrari and Franchi [21] in the very special case of a set D of the form $\mathbb{R}^+ \times \hat{\mathbb{G}}$, that enables to obtain Schauder estimates via a direct symmetrization argument which cannot be applied in general Lie groups satisfying assumption (1.5).

Our construction of the Poisson operator is based on the knowledge of a smooth fundamental solution, its restriction to the boundary, and on the properties of singular integrals. Since our result is local, we can locally express the boundary of D as the graph of a smooth function w, and perform a change of variable to reduce the boundary to a plane. In the new coordinates the vector fields will explicitly depend on the function w defining the boundary, and will not be homogeneous in general. For sub-Laplacian type operators associated to these vector fields we will obtain the following expression of the Poisson kernel.

Theorem 1.2. *Let* $D = \{(x_1, \hat{x}) \in \mathbb{R} \times \mathbb{R}^{n-1} : x_1 > 0\} \subset \mathbb{G}$ *be a non characteristic half space and let* $g \in C^\infty(\partial D)$. *Let* $\bar{x} \in \partial D$, *let* V_0 *be a neighborhood of* $\bar{x}$ *in* $\mathbb{R}^n$ *and let*

$$K_1(g)(\hat{y}) := \int_{\partial D \cap V_0} \Gamma_\Delta((0, \hat{y}), (0, \hat{z})) \hat{\Delta} g(\hat{z}) d\hat{z}. \tag{1.9}$$

There exists a lower order operator R *of type 3/2 with respect to the distance* $\hat{d}$ *defined on* ∂D, *such that for every neighborhood* V *of* $\bar{x}$ *in* $\mathbb{R}^n$, $V \subset\subset V_0$, *the operator*

$$P(g)(x) := \int_{\partial D \cap V_0} \Gamma_\Delta(x, (0, \hat{y}))(K_1 + R)(g)(\hat{y}) d\hat{y} \tag{1.10}$$

is a Poisson kernel in V.

The representation (1.10) and the properties of the fundamental solution immediately ensure that $P(g)$ satisfies the equation in (1.7). In order to show that P is a Poisson operator, we only have to show that $P(g) = g$ on the boundary $\{x_1 = 0\}$. Denoting by $E_{\Gamma_\Delta(0,\cdot)}$ the operator associated to the kernel $\Gamma_\Delta((0, \hat{x}), (0, \hat{z}))$, this is equivalent to say that $K_1 + R$ is the inverse of the operator $E_{\Gamma_\Delta(0,\cdot)}$. Under the assumption (1.5) this is proved using the fundamental solution $\hat{\Gamma}_{\hat{\Delta}}$ of the operator $\hat{\Delta}$ defined in (1.6). Indeed $\hat{\Gamma}_{\hat{\Delta}}$ satisfies the following approximate reproducing formula:

Theorem 1.3. *Let* $D = \{(x_1, \hat{x}) \in \mathbb{R} \times \mathbb{R}^{n-1} : x_1 > 0\} \subset \mathbb{G}$ *be a non characteristic half plane. If* $\bar{x} \in \partial D$, *then there exists a neighborhood* V *of* $\bar{x}$ *in* $\mathbb{G}$ *such that the fundamental*

solution admits the following representation:

$$\hat{\Gamma}_{\hat{\Delta}}(\hat{x}, \hat{y}) = \int_{\partial D \cap V} \Gamma_\Delta((0, \hat{x}), (0, \hat{z})) \Gamma_\Delta((0, \hat{z}), (0, \hat{y})) d\hat{z} + \hat{R}_{\hat{\Delta}}(\hat{x}, \hat{y}), \qquad (1.11)$$

for every $x = (0, \hat{x}), y = (0, \hat{y}) \in \partial D \cap V$, where $\hat{R}_{\hat{\Delta}}$ is a kernel of type $5/2$ with respect to the distance $\hat{d}$.

This theorem ensures that K_1 is the inverse of the operator $E_{\Gamma_\Delta(0,\cdot)}$ up to a remainder. The proof of Theorem 1.2 will be concluded with a standard version of the parametrix method, which allows to carefully handle the remainder and to prove that $K_1 + R$ is indeed the inverse of $E_{\Gamma_\Delta(0,\cdot)}$.

Theorem 1.3 expresses $E_{\Gamma_\Delta(0,\cdot)}$ as the square root of the operator associated to $\hat{\Gamma}_{\hat{\Delta}}$. This result, well known in the Euclidean setting and due to Caffarelli and Silvestre [8], was not known for general Carnot groups, but only in the special case when the group $\mathbb{G}$ is expressed as $\mathbb{G} = \mathbb{R} \times \hat{\mathbb{G}}$ (see Ferrari and Franchi in [21]). These proofs strongly rely on the splitting of the space as direct product, which is not satisfied in general Carnot groups, making impossible to follow their approach. The proof in our setting is inspired by the results of Evans in [19] (in the Euclidean case) and of Capogna, Citti and Senni (in Carnot groups) in [12].

1.4. **Structure of the paper and sketch of the proofs.** The paper starts with Section 2, where we fix notations and recall known properties of Carnot groups and their Riemannian approximation.

In Section 3 we show that a non characteristic plane can always be represented as the plane $\{(x_1, \hat{x}) \in \mathbb{R} \times \mathbb{R}^{n-1} : x_1 = 0\}$ with the canonical exponential change of variables described in (2.2). In these coordinates the vector fields attain an explicit polynomial representation recalled in (2.3). Moreover, Section 3 contains the proof of Theorem 1.3 under the assumption that the boundary of D is a non characteristic plane and the vector fields are homogeneous. The proof of this theorem is the most technical part of the paper and it is based on a Riemannian approximation and a parabolic regularization of the operator Δ. Precisely, the Riemannian approximation of the Laplace type operator Δ is an operator of the form

$$\Delta_\varepsilon = \Delta + \varepsilon^2 \sum_{i=m+1}^{n} X_i^2, \qquad (1.12)$$

and its parabolic regularization leads to the operator

$$L_\varepsilon := \partial_t - \Delta_\varepsilon. \qquad (1.13)$$

In a neighborhood of any non characteristic point z of the plane ∂D we will apply a new version of the freezing and parametrix methods to approximate the fundamental solution Γ_ε of L_ε in terms of the fundamental solution $\hat{\Gamma}_\varepsilon$ of a suitable tangential heat operator $\partial_t - \hat{\Delta}_\varepsilon$. The parametrix method has already been largely used in the subriemannian setting for estimating the fundamental solution in terms of a known one (see for example [44, 45, 36, 15, 3]). Here we are inspired by the papers [12] and [16] where the relation between the fundamental solution on the whole space and its restriction to the boundary was studied in

the framework of a diffusion driven motion by curvature. The main technical difficulty in our setting is due to the fact that neither the geometry of the subriemannian space nor the structure of the subriemannian operators is naturally represented as the direct sum of the tangential and the normal part. This splitting is true in the Riemannian approximation, and this is the reason for using this approximation. However the subriemannian structure and its Riemannian approximation have different homogeneous dimension. Hence we need to introduce a non homogeneous version of the parametrix method, which leads to the existence of a constant C such that

$$\left| \Gamma_\varepsilon((0, \hat{x}, t), (0, \hat{y}, \tau)) - \frac{\hat{\Gamma}_\varepsilon((\hat{x}, t), (\hat{y}, \tau))}{\sqrt{t - \tau}} \right| \leq C \Gamma_\varepsilon((0, \hat{x}, t), (0, \hat{y}, \tau))(t - \tau)^{1/4} \qquad (1.14)$$

for every $\hat{x}$ and $\hat{y} \in \partial D$, (this is done in Proposition 3.13 below). The key point here is to prove that C is independent of ε. The proof is quite delicate, and it is based on an interplay between the Riemannian and subriemannian nature of our operators. Since all constants in (1.14) are independent of ε, we can let ε go to 0 and obtain an analogous estimate for subriemannian operators. Denoting by Γ the fundamental solution of $\partial_t - \Delta$ and by $\hat{\Gamma}$ the fundamental solution of the operator $\partial_t - \hat{\Delta}$, we will prove in Theorem 3.2 that there exists a constant $C = C(T)$ such that for all $z = (0, \hat{z})$, $x = (0, \hat{x})$ in ∂D and for every t and τ, with $0 < t - \tau < T$, we have

$$\left| \Gamma((0, \hat{x}, t), (0, \hat{y}, \tau)) - \frac{\hat{\Gamma}((\hat{x}, t), (\hat{y}, \tau))}{\sqrt{t - \tau}} \right| \leq C \Gamma((0, \hat{x}, t), (0, \hat{y}, \tau))(t - \tau)^{1/4} \qquad (1.15)$$

Now, integrating in the time variable, we deduce the proof of Theorem 1.3 for homogeneous vector fields and on a plane (also called Lemma 3.16).

In Section 4 we provide the full proof of Theorem 1.3 on smooth manifolds. Since this is a local result, we show that, via a suitable change of variables, it is possible to identify the boundary of D with the plane $\{x_1 = 0\}$. With this change of variables the vector fields (X_i) become non homogeneous, but they still define an Hörmander structure. In Section 4.1 we describe this procedure and recall some properties of subriemannian spaces in this generality. Then, in Section 4.2 we apply a new simplified version of the parametrix method of Rothschild and Stein [44] tailored on the present setting, and locally we reduce the vector fields to homogeneous ones. With this instrument we can deduce the proof of Theorem 1.3 for smooth surfaces from the one obtained on planes, previously proved in Section 3.

Finally, Section 5 contains the construction of the Poisson kernel, which allows to prove Theorem 1.2. The main idea of the proof of this theorem has been outlined above. First, we use Theorem 1.3 to build an approximated kernel. After that, a standard version of the parametrix method is applied to obtain the Poisson kernel from the approximating one. The Schauder estimates stated in Theorem 1.1 are a consequence of the boundedness of the operator associated to the Poisson kernel and they will be proved with the same instruments as in [34]. In Section 6, in order to clarify our approach in finding a Poisson kernel, we apply it to the special case of Heisenberg groups $\mathbb{H}^n$, with $n \geq 2$.

2. NOTATIONS AND KNOWN RESULTS

2.1. **The subriemannian structure.** As recalled in Section 1.2, a Carnot group $\mathbb{G}$ is $\mathbb{R}^n$, with the group low induced by the exponential map and the stratification $V^1 \oplus \cdots \oplus V^\kappa$ of the tangent space recalled in (1.1). The stratification induces a natural notion of degree of a vector field:

$$deg(X) = j \quad \text{whenever } X \in V^j. \tag{2.1}$$

If $\{X_i\}_{i=1,\cdots,n}$ is the stratified basis introduced in subsection 1.2, we will write also $deg(i)$ instead of $deg(X_i)$. Via the exponential map, $\mathbb{R}^n$ is endowed with a Lie group structure and the resulting group is denoted by $\mathbb{G}$. Since in this setting the exponential map around a fixed point y is a global diffeomorphism, every other point x can be uniquely represented as $x = \exp(v_1 X_1) \exp(\sum_{i=2}^n v_i X_i)(y)$. Consequently we can define a logarithmic function $\Theta_{X_1,\cdots,X_n,y}$ as the inverse of the exponential map:

$$\Theta_{X_1,\cdots,X_n,y} : \mathbb{G} \to \mathfrak{g}, \quad \Theta_{X_1,\cdots,X_n,y}(x) = (v_1, \cdots, v_n). \tag{2.2}$$

We will simply denote Θ_y instead of $\Theta_{X_1,\cdots,X_n,y}$ when no ambiguity may arise. Note that we are using exponential canonical coordinates of second type around a fixed point $y \in \mathbb{G}$, which will simplify notations while dealing with a boundary problem.

In particular, the fixed point y, around which we choose the axes, has coordinates 0 and the vector field X_1 is represented as $X_1 = \partial_1$ and all the others vector fields $(X_i)_{i\geq 2}$ coincide with the partial derivative ∂_i at the fixed point $y = 0$. In any other point they can be represented in these coordinates as

$$X_1 = \partial_1, \quad X_i = \partial_i + \sum_{deg(j)>deg(i)} a_{ij}(v)\partial_j \quad i = 2, \cdots, n, \tag{2.3}$$

where a_{ij} are homogeneous polynomials of degree $deg(j) - deg(i)$ depending only on variables v_h, with $deg(h) \leq deg(j) - deg(i)$ (see for example [44] for a detailed proof). Note that if $deg(i) = \kappa$ then $X_i = \partial_i$.

By construction the vectors $\{X_i\}_{i=1,\cdots,m}$ and their commutators span $\mathfrak{g}$ at every point, and consequently verify Hörmander's finite rank condition ([33]). Due to the stratification of the algebra, a natural family of dilation $(\delta_\lambda)_{\lambda>0}$ acts on points $v = \sum_{i=1}^n v_i X_i \in \mathfrak{g}$ as follows:

$$\delta_\lambda(v) := \lambda^{deg(i)} v_i. \tag{2.4}$$

On V^1, which is generated by $X_1, \cdots X_m$, we define a Riemannian metric which makes $X_1, \cdots, X_m$ an orthonormal basis. The associated norm will be extended to an homogeneous norm to the whole $\mathfrak{g}$ defined as follows:

$$\|v\| := \sum_{i=1}^n |v_i|^{1/deg(i)}. \tag{2.5}$$

Via the logarithmic function defined in (2.2) the dilation δ_λ induces a one-parameter group of automorphisms on $\mathbb{G}$, again denoted by δ_λ. A function $f : \mathbb{G} \to \mathbb{R}$ is called homogeneous of degree α if $f(\delta_\lambda(x)) = \lambda^\alpha f(x)$ for any $\lambda > 0$ and $x \in \mathbb{G}$. In particular we can define

a gauge distance $d(\cdot, \cdot)$ homogeneous of degree 1, as the image of the norm through the function Θ:

$$d(y, x) := \|\Theta_{X_1, \cdots, X_n, y}(x)\|. \tag{2.6}$$

The gauge function is homogeneous of order

$$Q := \sum_{i=1}^{\kappa} i \, dim(V^i) \tag{2.7}$$

with respect to the dilation. Hence Q is called the homogeneous dimension of the space and there exist constants C_1, C_2 such that

$$C_1 r^Q \leq |B(x, r)| \leq C_2 r^Q \qquad \forall \, r > 0, \; x \in \mathbb{G},$$

where $B(x, r)$ denotes the metric ball centered in x with radius r, and $| \cdot |$ denotes the Lebesgue measure.

Any vector field X will be identified with the first order differential operator with its same coefficients. If φ is a continuous function defined in an open set V of $\mathbb{G}$ and if, for every $i = 1, \cdots, m$, there exists the Lie derivative $X_i \varphi$ then we call horizontal gradient of φ the vector

$$\nabla \varphi = \sum_{i=1}^{m} (X_i \varphi) X_i. \tag{2.8}$$

The associated classes of Hölder continuous functions will be defined as follows:

Definition 2.1. *Let $0 < \alpha < 1$, $V \subset \mathbb{G}$ be an open set, and u be a function defined on V. We say that $u \in C^\alpha(V)$ if there exists a positive constant M such that for every $x, x_0 \in V$*

$$|u(x) - u(x_0)| \leq M d^\alpha(x, x_0). \tag{2.9}$$

We put

$$\|u\|_{C^\alpha(V)} = \sup_{x \neq x_0} \frac{|u(x) - u(x_0)|}{d^\alpha(x, x_0)} + \sup_V |u|.$$

Iterating this definition, if $k \geq 1$ we say that $u \in C^{k,\alpha}(V)$ if $X_i u \in C^{k-1,\alpha}(V)$ for all $i = 1, \cdots, m$.

The Laplace type operator defined in (1.2) is a differential operator of degree 2, in the sense of the following definition.

Definition 2.2. *Let $\{X_{i_j}\}$ be differential operators of order 1 and degree $deg(X_{i_j})$. We say that the differential operator $Y_1 = X_{i_1} \cdots X_{i_p}$ has order p and degree $\sum_{j=1}^{p} deg(X_{i_j})$. Moreover, if Y is a differential operator represented as*

$$Y = a Y_1, \tag{2.10}$$

where a is a homogeneous function of degree α, then we say that Y is homogeneous of degree $deg(Y_1) - \alpha$. A differential operator will be called of degree $k - \alpha$ if it is a sum of operators with maximum degree $k - \alpha$.

Following [22] we recall the definition of kernel of type α:

Definition 2.3. *We say that K is a kernel of type α, if K is of class C^∞ away from 0 and it is homogeneous of degree $\alpha - Q$.*

In a Carnot group, this implies that K satisfies condition (1.4).

2.2. **The Riemannian approximation of the structure.** One of the key technical instruments that we will use is a Riemannian approximation of the subriemannian structure.

For every $\varepsilon > 0$, we extend the Riemannian metric defined on V^1 to a left invariant Riemannian metric defined on $\mathfrak{g}$ by requesting that

$$(X_{1,\varepsilon}, \cdots, X_{n,\varepsilon}) := (X_1, \cdots, X_m, \varepsilon^{deg(m+1)-1}X_{m+1}, \cdots, \varepsilon^{deg(n)-1}X_n) \qquad (2.11)$$

is an orthonormal frame. We say that these vector fields have ε-degree equal to 1, and we write $deg_\varepsilon(X_{i,\varepsilon}) = 1$. Since the Lie algebra generated by these vectors also contains the commutators of these vector fields, we also consider the vector fields

$$X_{i,\varepsilon} := X_{i-n+m} \text{ and } deg_\varepsilon(X_{i,\varepsilon}) := deg(X_{i-n+m}) \text{ for all } i = n+1, \cdots, 2n-m. \quad (2.12)$$

Let d_{cc} and $d_{cc,\varepsilon}$ denote the control distances associated with the vector fields $X_1, \cdots, X_m$ and $X_{1,\varepsilon}, \cdots, X_{n,\varepsilon}$, respectively. It is well known (see for instance [31] and the references therein) that $(\mathbb{G}, d_{cc,\varepsilon})$ converges in the Gromov-Hausdorff sense, as $\varepsilon \to 0$, to the subriemannian space $(\mathbb{G}, d_{cc})$. Although the subriemannian structure is formally recovered in the limit for $\varepsilon \to 0$, we will need to recognize that the structure and all constants appearing in the estimates are stable in the limit. In addition we will need to recognize that the space has a property of ε-homogeneity, with respect to the natural distance.

A classical estimate of the distance $d_{cc,\varepsilon}$ is due to Nagel, Stein and Wainger in [43]. From the whole family $\{X_{i,\varepsilon}\}_{i=1,\cdots,2n-m}$ it is possible to select different bases $\{X_{i_j,\varepsilon}\}_{i_j \in I}$, for different choices of indices $I = (i_1, \cdots, i_n) \subset \{1, ..., 2n-m\}^n$. As a consequence each vector v has different representations $v = \sum_{i_j \in I} v_{i_j,\varepsilon} X_{i_j,\varepsilon}$ in terms of the different bases. The optimal choice of indices, denoted $I_{v,\varepsilon}$, is the one which minimize the ε-homogeneous gauge distance:

$$\|v\|_\varepsilon = \sum_{i_j \in I_{v,\varepsilon}} |v_{i_j,\varepsilon}|^{1/deg_\varepsilon(i_j)} = \min_I \sum_{i_j \in I} |v_{i_j,\varepsilon}|^{1/deg_\varepsilon(i_j)}. \qquad (2.13)$$

This norm can be explicitly written as follows: if $v = \Theta_{X_1,\cdots,X_n,y}(x)$ then

$$\|v\|_\varepsilon = \sum_{i=1}^m |v_i| + \sum_{i=m+1}^n \min\left\{\frac{|v_i|}{\varepsilon^{deg(i)-1}}, |v_i|^{1/deg(i)}\right\}. \qquad (2.14)$$

In [11] it is proved that the associated ball box distance

$$d_\varepsilon(y,x) = \|\Theta_{X_1,\cdots,X_n,y}(x)\|_\varepsilon \qquad (2.15)$$

is locally equivalent to the distance $d_{cc,\varepsilon}$, with equivalence constants independent of ε. Let us explicitly note that this distance has different behavior in 0 and at infinity. Indeed, if v_i are small with respect to ε for every $i \geq m+1$, then the distance d_ε has a Riemannian behavior, while it is purely subriemannian for v_i large.

It is worthwhile to note that for every $\varepsilon > 0$ there exists a constant C_ε such $|B_\varepsilon(x,r)| = C_\varepsilon r^n$, where $B_\varepsilon(x,r)$ denotes the ball $\{y \in \mathbb{G} \,|\, d_\varepsilon(x,y) < r\}$ and $|\cdot|$ the Lebesgue measure. In particular for every $\varepsilon > 0$ the homogeneous dimension of the Riemannian space is n, while by (2.7) for $\varepsilon = 0$ the homogeneous dimension of the space is Q, with $Q > n$. Hence this notion of homogeneity is not stable with respect to ε, and the constant C_ε blows up for $\varepsilon \to 0$. However it has recently proved in [11] the following uniform doubling property:

Proposition 2.4. *There is a constant C independent of ε such that for every $x \in \mathbb{G}$ and $r > 0$,*

$$|B_\varepsilon(x,2r)| \le C|B_\varepsilon(x,r)|. \tag{2.16}$$

The doubling inequality (2.16) can be considered as a weak form of homogeneity, and suggests that it is possible to give a new definition of ε- homogeneity. Following [44] we will give the following definition of local homogeneous functions and operators

Definition 2.5. *A function f is locally homogeneous of ε-degree α in a neighborhood of a point z with respect to the metric (2.15) if $f \circ \Theta_{X_1,\cdots,X_n,z}^{-1}$ is homogeneous of degree α, with respect to the norm $\|\cdot\|_\varepsilon$ defined in (2.14). A differential operator Y is homogeneous of local ε-degree α in a neighborhood of a point z, with respect to the metric (2.15) if $d\Theta_{X_1,\cdots,X_n,z}(Y)$ is homogeneous of degree α.*

In particular this definition implies the following property:

Remark 2.6. *If a is a homogeneous function of ε-degree α and Y_1 is an operator of ε-degree k, then $a(z^{-1}x)Y_1$ is a homogeneous operator of ε-degree $k - \alpha$. This implies that for every other smooth function f of local ε-degree β in a neighborhood of a point z, $a(z^{-1}x)(Y_1f)$ is a smooth function of ε-degree $\beta + \alpha - k$ in a neighborhood of a point z, and $|a(z^{-1}x)(Y_1f)(x)| \le Cd_\varepsilon^{\beta+\alpha-k}(x,z)$.*

If $\varphi \in C^\infty(\mathbb{G})$ we define the ε-gradient of φ as follows

$$\nabla_\varepsilon \varphi := \sum_{i=1}^{n}(X_{i,\varepsilon}\varphi)X_{i,\varepsilon}.$$

In terms of the vector fields with ε-degree 1, defined in (2.11), we consider the associated heat operator

$$L_\varepsilon := \partial_t - \sum_{i=1}^{n} X_{i,\varepsilon}^2 - \sum_{i=1}^{m} b_i X_{i,\varepsilon}, \tag{2.17}$$

(recall that b_i are the smooth coefficients introduced in (1.2)). In analogy with the operator introduced in (2.8), the heat operator associated to the subriemannian structure has the form

$$L := \partial_t - \sum_{i=1}^{m} X_i^2 - \sum_{i=1}^{m} b_i X_i. \tag{2.18}$$

The behavior of these operators in interior points is well known: they admit fundamental solutions respectively $\Gamma_\varepsilon(x,t)$ and $\Gamma(x,t)$ of class C^∞ out of the pole (see [36] for precise estimates of $\Gamma(x,t)$ and [10] for estimates of $\Gamma_\varepsilon(x,t)$ uniform in ε).

In our work we will need estimates which are uniform in the variable ε. We start with the following definition.

Definition 2.7. *We say that a family of kernels $(P_\varepsilon)_{\varepsilon>0}$, defined on $\mathbb{G}\times]0,\infty[\times\mathbb{G}\times]0,\infty[$ and C^∞ out of the diagonal, is of uniform exponential ε-type $\lambda+2$, if for $q \in \mathbb{N}$ and k-tuple $(i_1,\cdots,i_k) \in \{1,\cdots,n\}^k$ there exists $C_{q,k} > 0$, depending only on k,q and on the Riemannian metric, such that*

$$|(X_{i_1,\varepsilon}\cdots X_{i_k,\varepsilon}\partial_t^q P_\varepsilon)((x,t),(z,\tau))| \leq C_{q,k}(t-\tau)^{-q-k/2+\lambda/2}\frac{e^{-\frac{d_\varepsilon(x,z)^2}{C_{q,k}(t-\tau)}}}{|B_\varepsilon(x,\sqrt{t-\tau})|} \qquad (2.19)$$

for all $x \in \mathbb{G}$ and $t > \tau$.

According with the definition above, the fundamental solution Γ_ε is a kernel of exponential ε-type 2. Precisely, the following result, established in [9] (see also [16] and [10]), holds:

Theorem 2.8. *The fundamental solutions $\Gamma_\varepsilon(x,t)$ of the operators L_ε constitutes a family of kernels of exponential ε-type 2 and there exist constants $C_0, C > 0$ independent of ε such that for each $\varepsilon > 0$, $x \in \mathbb{G}$ and $t > \tau$ one has*

$$C_0^{-1}\frac{e^{-C\frac{d_\varepsilon(x,z)^2}{(t-\tau)}}}{|B_\varepsilon(x,\sqrt{t-\tau})|} \leq \Gamma_\varepsilon((x,t),(z,\tau)) \leq C_0\frac{e^{-\frac{d_\varepsilon(x,z)^2}{C(t-\tau)}}}{|B_\varepsilon(x,\sqrt{t-\tau})|}. \qquad (2.20)$$

Moreover, for any k-tuple $(i_1,\cdots,i_k) \in \{1,\cdots,m\}^k$ one has

$$X_{i_1}\cdots X_{i_k}\partial_t^q\Gamma_\varepsilon \to X_{i_1}\cdots X_{i_k}\partial_t^q\Gamma \quad \text{as } \varepsilon \to 0 \qquad (2.21)$$

uniformly on compact sets and in a dominated way on all $\mathbb{G}$.

Remark 2.9. *In particular from this theorem we can obtain the well known Gaussian estimates of the fundamental solution Γ of the operator L. Indeed Γ is a kernel of exponential type 2 and there exist constants $C_0, C > 0$ such that for each $x \in \mathbb{G}$ and $t > \tau$ one has*

$$C_0^{-1}\frac{e^{-C\frac{d(x,z)^2}{(t-\tau)}}}{|B(x,\sqrt{t-\tau})|} \leq \Gamma((x,t),(z,\tau)) \leq C_0\frac{e^{-\frac{d(x,z)^2}{C(t-\tau)}}}{|B(x,\sqrt{t-\tau})|}. \qquad (2.22)$$

2.3. **The parametrix method.** One of the main instruments that we will use to estimate the fundamental solution is the parametrix method, originally due to Levi and now extremely classical for elliptic and parabolic equations (see [26]). In subriemannian setting the parametrix method have been used to approximate general Hörmander type operators with homogeneous ones: we refer to [44, 45] for the first results, [36] for the subriemannian heat kernel, [15] for estimates in case of low regularity, [3] for a recent self-contained presentation. The method consists in providing an explicit representation of the fundamental solution Γ of an operator L in terms of the fundamental solution of an approximating operator L_{z_1} (with associated fundamental solution Γ_{z_1}). Using the definition of fundamental solution and the fact that $L_{z_1}(\Gamma - \Gamma_{z_1}) = (L_{z_1} - L)\Gamma$, the difference

between the two solutions can be formally represented as

$$(\Gamma - \Gamma_{z_1})((x,t),(z,\tau)) = \int \Gamma_{z_1}((x,t),(y,\theta))(L_{z_1} - L)(\Gamma - \Gamma_{z_1})((y,\theta),(z,\tau)) \, dy \, d\theta$$

$$+ \int \Gamma_{z_1}((x,t),(y,\theta))(L_{z_1} - L)\Gamma_{z_1}((y,\theta),(z,\tau)) \, dy \, d\theta. \quad (2.23)$$

Denoting by

$$H := L_{z_1} - L, \quad (2.24)$$

and $E_{\Gamma_{z_1}}$ the integral operator with kernel Γ_{z_1}, the above expression (2.23) can be written as

$$(I - E_{\Gamma_{z_1}} H)(\Gamma - \Gamma_{z_1}) = E_{\Gamma_{z_1}} H(\Gamma_{z_1}).$$

If the operator $(I - E_{\Gamma_{z_1}} H)$ is invertible, the difference $\Gamma - \Gamma_{z_1}$ can be formally expressed as

$$\Gamma - \Gamma_{z_1} = \sum_{j=0}^{\infty} (E_{\Gamma_{z_1}} H)^{j+1}(\Gamma_{z_1}) = E_{\Gamma_{z_1}} \Phi, \quad \text{with} \quad \Phi := \sum_{j=0}^{\infty} (H E_{\Gamma_{z_1}})^{j} H(\Gamma_{z_1}). \quad (2.25)$$

Roughly speaking the proof is obtained as follows.

1) The first and most delicate part of the proof is to define the approximating operator H and to prove that it is a differential operator of degree $2 - \alpha$ for a suitable positive α. From this fact it follows that the kernel

$$R_1(x,z) := H\Gamma_{z_1}(x,z)$$

is homogeneous of type α with respect to the considered homogeneous space. It is important to note that

$$R_1(x,z) = (L_{z_1} - L)\Gamma_{z_1}(x,z) = L(\Gamma(x,z) - \Gamma_{z_1}(x,z)). \quad (2.26)$$

2) Identifying the operator $H E_{\Gamma_{z_1}}$ with the integral operator E_{R_1} with kernel R_1, the series Φ in (2.25) reduces to

$$\Phi = \sum_{j=0}^{\infty} (E_{R_1})^{j} E_{R_1}. \quad (2.27)$$

Using the fact that the convolution of a kernel of type α with a kernel of type β provides a kernel of type $\alpha + \beta$, it is possible to prove that this series converges uniformly (see for example Lemma 7.3 in [34]).

3) Finally, singular integrals tools lead to the convergence of the derivatives and the function Γ, defined through (2.25), is a fundamental solution.

In the sequel we will consider kernels of type α in the sense of Definition 2.3, when working with subelliptic operators, while kernels of exponential ε-type α in the sense of Definition 2.7, when studying Riemannian heat kernels. The main difficulty to be faced here is that the Riemannian approximation has not a standard notion of homogeneity, since the Riemannian homogeneous dimension n collapses to the subriemannian one Q in the limit. However we have endowed the regularized space with an ε-homogeneous structure (see (2.16) and the remark below) and we will see that this is enough to apply the method in this setting. Therefore, even though our vector fields are homogeneous, our approach is

more similar to the ones [44, 45, 36, 15] where the geometry of the given operator and the approximating one do not coincide.

3. Reproducing formula on a plane

In this section we will prove a first version of Theorem 1.3, under the simplified assumptions that ∂D is a non characteristic plane and that the vector fields $(X_i)_{i=1,\cdots,m}$ are the generators of a Carnot group and have the explicit representation recalled in (2.3). This result will be obtained via a parabolic approximation and a Riemannian regularization. The proof of the same Theorem 1.3 on any smooth hypersurface will be deduced from this result in next section.

3.1. **Geometry of the plane.** Let $\mathbb{G}$ be a Carnot group of step κ. Consider a non characteristic plane M_0. Using the logarithmic coordinates defined in (2.2), it is always possible to represent M_0 as follows:

$$M_0 = \{(x_1, \hat{x}) \in \mathbb{G} : x_1 = 0\}, \tag{3.1}$$

where $x = (x_1, \hat{x})$ is a point of the space, $x_1 \in \mathbb{R}$ and $\hat{x} \in \mathbb{R}^{n-1}$. This choice of coordinates is made in such a way that the vector fields $X_1 = \partial_1$ coincides with the direction normal to the plane, while $\{X_i\}_{i=2,\cdots,n}$ are tangent to M_0 and are represented as in (2.3). Hence the vector fields obtained from X_i by evaluating the coefficients a_{ij} on the points of the plane M_0, are the generators of the first layer $\hat{V}_1$ on the plane, so that

$$\hat{X}_i := X_{i|x_1=0}, \quad i = 2, \cdots, n. \tag{3.2}$$

Thanks to this choice, not only the plane M_0 is non characteristic, but also the planes $M_{z_1} = \{(x_1, \hat{x}) \in \mathbb{G} : x_1 = z_1\}$, for every z_1 sufficiently small, are non characteristic. We note also that assumption (1.5) ensures that the vector fields $\hat{X}_i$ satisfy a Hörmander condition at every point and span a $n - 1$ dimensional space at every point. In analogy with formula (2.7), the homogeneous dimension of the plane is

$$\hat{Q} = \sum_{i=2}^{\kappa} i \, dim(\hat{V}^i).$$

As a consequence

$$\hat{Q} = Q - 1. \tag{3.3}$$

Via the exponential map and definition (2.6) the vector fields $\hat{X}_i$ define a distance

$$\hat{d}(\hat{y}, \hat{x}) := \|\Theta_{\hat{X}_2, \cdots, \hat{X}_n, \hat{y}}(\hat{x})\| \tag{3.4}$$

on M_0. By the Hörmander condition a Laplace operator and its time dependent counterpart are defined on M_0 as

$$\hat{\Delta} := \sum_{i=2}^{m} \hat{X}_i^2, \quad \text{and} \quad \hat{L} := \partial_t - \hat{\Delta}, \tag{3.5}$$

and they have non negative fundamental solutions $\hat{\Gamma}_{\hat{\Delta}}$ and $\hat{\Gamma}$ respectively.

In analogy with Definition 2.7 we give the following definition.

Definition 3.1. *We say that a kernel $\hat{P}$, defined on $\mathbb{R}^{n-1}\times]0,\infty[\times\mathbb{R}^{n-1}\times]0,\infty[$ and C^∞ out of the diagonal, is of exponential type $\lambda + 2$, if for $q \in \mathbb{N}$ and k-tuple $(i_1, \cdots, i_k) \in \{1, \cdots, n\}^k$ there exists $C_{q,k} > 0$, depending only on k, q and on the subriemannian metric, such that*

$$|(\hat{X}_{i_1} \cdots \hat{X}_{i_k} \partial_t^q \hat{P})((\hat{x},t),(\hat{z},\tau))| \leq C_{q,k}(t-\tau)^{-q-k/2+\lambda/2} \frac{e^{-\frac{\hat{d}(\hat{x},\hat{z})^2}{C_{q,k}(t-\tau)}}}{|\hat{B}(\hat{x},\sqrt{t-\tau})|}$$

for all $\hat{x} \in \mathbb{R}^{n-1}$ and $t > \tau$.

Our first result is the following one:

Theorem 3.2. *Assume that $M_0 = \{(x_1,\hat{x}) \in \mathbb{G} : x_1 = 0\}$ is a non characteristic plane and let $T > 0$. Then there exists a constant $C = C(T)$ such that for all $z = (0,\hat{z})$, $x = (0,\hat{x})$ in M_0 and for every t and τ, with $0 < t - \tau < T$, we have*

$$|\hat{\Gamma}((\hat{x},t),(\hat{z},\tau)) - \sqrt{t-\tau}\Gamma(((0,\hat{x}),t),((0,\hat{z}),\tau))| \leq C\hat{\Gamma}((\hat{x},t),(\hat{z},\tau))(t-\tau)^{1/4}. \quad (3.6)$$

In addition $\hat{\Gamma}((\hat{x},t),(\hat{z},\tau)) - \sqrt{t-\tau}\Gamma(((0,\hat{x}),t),((0,\hat{z}),\tau))$ is an operator of exponential type $1/4$ with respect to the vector fields $\{\hat{X}_i\}_{i=2,\cdots,m}$.

Since the kernel $\Gamma((0,\hat{x},t),(0,\hat{z},\tau))$ has the same growth of the kernel $\frac{\hat{\Gamma}((\hat{x},t),(\hat{z},\tau))}{\sqrt{t-\tau}}$, we immediately deduce from the previous theorem the following corollary:

Corollary 3.3. *The kernel $\Gamma(((0,\hat{x}),t),((0,\hat{z}),\tau))$ is an operator of exponential type 1 with respect to the distance $\hat{d}$.*

Theorem 3.2 will be proved with the parametrix method and a Riemannian approximation. Classically, the method is applied for proving the existence of the fundamental solution of a given operator. Extending an approach of [12], in Lemma 3.2 we apply the method to find a relation between the fundamental solutions since we already know that they exist.

Even though the parametrix method has been largely used in the subriemannian setting for internal estimates, the vector fields $\hat{X}_i$ do not provide a subriemannian approximation of the vector fields X_i and the standard parametrix method of Rothschild and Stein cannot be applied starting with the fundamental solution of $\hat{L}$. In order to clarify this fact we start with a concrete example of vector fields:

Example 3.4. *Let us consider the following fields*

$$X_1 = \partial_1, \quad X_2 = \partial_2 + x_1^2\partial_5 + x_3\partial_4, \quad X_3 = \partial_3 + x_4\partial_5. \quad (3.7)$$

Their commutators are $\partial_4 = [X_1, X_2]$, which is a vector field of degree 2, and $\partial_5 = [[X_1, X_2], X_2]$, which is a vector field of degree 3.

If we evaluate the vector fields X_i on the plane $\{x_1 = 0\}$ we obtain

$$\hat{X}_2 = \partial_2 + x_3\partial_4, \quad \hat{X}_3 = \partial_3 + x_4\partial_5,$$

so that

$$X_2 - \hat{X}_2 = x_1^2\partial_5 \text{ is an operator of degree } 1.$$

Consequently, the difference

$$H = \hat{L} - L \;\; \text{is an operator of degree } 2.$$

Hence it is not possible to apply the parametrix method, whose convergence requires H to be an operator of degree strictly less than 2.

Due to these difficulties, we introduce a new version of the parametrix method, using the ε-Riemannian approximation described in Section 2. The whole proof is based on a careful analysis of the Riemannian approximation metric and lies on a delicate interplay between the Riemannian and subriemannian nature of our operators.

3.2. **A Riemannian and frozen approximating operator.** In Section 3.1 we have chosen a point 0 and a constant ε sufficiently small such that for every $z_1 \in \mathbb{R}$, such that $|z_1| \leq \varepsilon^{2\kappa}$, the plane $M_{z_1} = \{(x_1, \hat{x}) \in \mathbb{G} : x_1 = z_1\}$ is non-characteristic. In analogy with (3.2), we define the vector fields $X_{i,z_1} := X_{i|_{x_1 = z_1}}$ as the vector fields whose coefficients are evaluated at the points with first component z_1. Thanks to (2.3), they can be represented as

$$X_{1,z_1} := \partial_1, \quad X_{i,z_1} := X_{i|_{x_1 = z_1}} = \partial_i + \sum_{deg(j) > deg(i)} a_{ij}(z_1, \hat{x}) \partial_j, \quad i = 2, \cdots, n \qquad (3.8)$$

for every $\varepsilon > 0$ and we set, see (2.11) and (2.12),

$$X_{i,z_1,\varepsilon} := X_{i,\varepsilon|_{x_1 = z_1}} \quad i = 1, \cdots, 2n - m, \qquad (X_{1,z_1,\varepsilon} := \partial_1). \qquad (3.9)$$

We introduce now an operator $L_{z_1,\varepsilon}$ which can be split in a tangential and in a normal part on any plane M_{z_1}, and we will use it to approximate with the parametrix method the tangential and the normal part of the operator L_ε. Precisely, we define

$$L_{z_1,\varepsilon} := \partial_t - \sum_{i=1}^{n} X_{i,z_1,\varepsilon}^2 \qquad (3.10)$$

with fundamental solution $\Gamma_{z_1,\varepsilon}$ on the whole space. On every plane M_{z_1} we define the tangential operators

$$\hat{L}_{z_1,\varepsilon} := \partial_t - \hat{\Delta}_{z_1,\varepsilon}, \quad \text{where} \quad \hat{\Delta}_{z_1,\varepsilon} := \sum_{i=2}^{n} X_{i,z_1,\varepsilon}^2, \qquad (3.11)$$

with non negative fundamental solutions $\hat{\Gamma}_{z_1,\varepsilon}$ and $\hat{\Gamma}_{\Delta,z_1,\varepsilon}$ respectively.

Remark 3.5. *Let us explicitly note that $\hat{\Delta}_{z_1,\varepsilon}$ is independent of x_1, hence it commutes with ∂_1. Therefore the operator $L_{z_1,\varepsilon}$ can be represented as*

$$L_{z_1,\varepsilon} = \partial_t - \partial_{11}^2 - \hat{\Delta}_{z_1,\varepsilon}.$$

Since ∂_1 coincides with the direction normal to the plane, the operator splits in the sum of its orthogonal part $\partial_t - \partial_{11}^2$ and its tangential part $\hat{L}_{z_1,\varepsilon}$. Consequently its fundamental solution can be represented as

$$\Gamma_{z_1,\varepsilon}(x_1, \hat{x}, t) = \Gamma_{\perp,z_1,\varepsilon}(x_1, t)\hat{\Gamma}_{z_1,\varepsilon}(\hat{x}, t)$$

where $\hat{\Gamma}_{z_1,\varepsilon}$ is defined above and $\Gamma_{\perp,z_1,\varepsilon}$ is the standard one-dimensional Gaussian function, fundamental solution of $\partial_t - \partial_{11}^2$.

3.3. **Estimates of the approximating operator.**

As recalled in Section 2.3 the first step of the parametrix method is to prove that the difference $X_{i,\varepsilon} - X_{i,z_1,\varepsilon}$ is a differential operator of ε-degree strictly less than 1 around the point z_1 and, as a consequence, that the operator $H_\varepsilon := L_\varepsilon - L_{z_1,\varepsilon}$ (see also (2.17) and (2.24) above) has ε-degree strictly less than 2.

Lemma 3.6. *Let $M_{z_1} = \{(x_1, \hat{x}) \in \mathbb{G} : x_1 = z_1\}$ be a non characteristic plane. For every $z = (z_1, \hat{z}) \in M_{z_1}$, and for every $i \leq n$ and for every h such that $deg(i) + 1 \leq deg(h) \leq \kappa$ (where κ is the step of $\mathbb{G}$) there exists a polynomial $p_{i,h,z_1}(v)$, homogeneous of degree $deg(h) - deg(i) - 1$ as a function of v and z_1, such that, if $v = \Theta_{X_{z_1},z}(x)$,*

$$d\Theta_{X_{z_1},z}(X_i - X_{i,z_1}) = v_1 \sum_{\substack{deg(h)=deg(i)+1}}^{\kappa} p_{i,h,z_1}(v)d\Theta_{X_{z_1},z}(X_{h,z_1}), \qquad (3.12)$$

where $\Theta_{X_{z_1},z}$ has been defined in (2.2). Moreover

$$|p_{i,h,z_1}(v)| \leq C \sum_{j=0}^{deg(h)-deg(i)-1} |z_1|^j \|v\|^{deg(h)-deg(i)-1-j}. \qquad (3.13)$$

Proof. When $i = 1, \cdots, n$, by the definition (2.3) and (3.8) of the vector fields we have, for $deg(i) = \kappa$

$$X_i - X_{i,z_1} = 0, \qquad (3.14)$$

hence the thesis is true, and we have to prove it only for $deg(i) < \kappa$. Using the fact that the translation associated to the vector fields X_{z_1} acts only on the $\hat{x}$ variables, we have

$$d\Theta_{X_{z_1},z}(X_i - X_{i,z_1}) = \sum_{\substack{deg(h)=deg(i)+1}}^{\kappa} \Big(a_{i,h}(x_1, \hat{v}) - a_{i,h}(z_1, \hat{v}) \Big) \partial_h =$$

$$= \sum_{\substack{deg(h)=deg(i)+1}}^{\kappa} (x_1 - z_1) a_{i,h,z_1}^1(v)\partial_h = v_1 \sum_{\substack{deg(h)=deg(i)+1}}^{\kappa} a_{i,h,z_1}^1(v)\partial_h. \qquad (3.15)$$

In the last equality we have denoted $(x_1 - z_1)a_{i,h,z_1}^1(v)$ the polynomial $a_{i,h}(x_1, \hat{v}) - a_{i,h}(z_1, \hat{v})$ and we used the fact that $v_1 = x_1 - z_1$. The polynomial $a_{i,h,z_1}^1(v)$ is homogeneous of degree $deg(h) - deg(i) - 1$ in the variables v and z_1 and we have estimated as

$$|a_{i,h,z_1}^1(v)| \leq C \sum_{j=0}^{deg(h)-deg(i)-1} |z_1|^j \|v\|^{deg(h)-deg(i)-1-j}.$$

If $deg(i) = \kappa - 1$, the proof is completed, by (3.14). For $deg(i) < \kappa - 1$, using again the expression (3.8) for $deg(h) < \kappa$ and (3.14) for $deg(h) = \kappa$, we get

$$d\Theta_{X_{z_1},z}(X_i - X_{i,z_1}) = v_1 \sum_{\substack{deg(h)=deg(i)+1}}^{\kappa} a_{i,h,z_1}^1(v)d\Theta_{X_{z_1},z}(X_{h,z_1})-$$

$$-v_1 \sum_{deg(j)=deg(i)+2}^{\kappa} \sum_{deg(h)=deg(i)+1}^{deg(j)-1} a_{i,h,z_1}^1(v) a_{h,j,z}(v) \partial_{v_j}.$$

Since the Lie group is nilpotent of step κ, after $\kappa - 1$ iteration of this method, we get that there exists a polynomial p_{i,h,z_1} such that (3.16) is satisfied.   □

From this lemma, Corollary 3.7 below immediately follows. The proof is technically very simple, but it is important to note that the $X_{i,\varepsilon} - X_{i,z_1,\varepsilon}$ is a differential operator which has local degree 1 while has local ε-degree $1/2$, in a neighborhood of the point z. This property allows to obtain a better approximation in the Riemannian setting, rather than in the subriemannian setting.

Corollary 3.7. *Let M_0 be a non characteristic plane. Let $\mathcal{S}$ be the strip*

$$\mathcal{S} := \{x = (x_1, \hat{x}) \in \mathbb{G} : |x_1| \leq \varepsilon^{2\kappa}, \ |x_1 - z_1| \leq \varepsilon^{2\kappa}\},$$

where κ is the step of the group. Then $X_{i,\varepsilon} - X_{i,z_1,\varepsilon}$ is a differential operator of ε-degree $1/2$ with respect to the vector fields $X_{i,z_1,\varepsilon}$.

Proof. Applying Lemma 3.6, calling $p_{i,h,z_1,\varepsilon}^0(v) = \varepsilon^{-deg(h)} v_1 p_{i,h,z_1}(v)$ and using the fact that $|v_1| \leq \varepsilon^{2\kappa}$ and $|z_1| \leq 1$ we have

$$|p_{i,h,z_1,\varepsilon}^0(v)| \leq C|v_1|^{1/2} \sum_{deg(h)=deg(i)+1}^{\kappa} \sum_{j=0}^{deg(h)-deg(i)-1} \|v\|^{deg(h)-deg(i)-1-j}.$$

Since $X_{i,\varepsilon} = \varepsilon^{deg(i)} X_i$ and $v_1 p_{i,h,z_1}(v) X_{h,z_1} = p_{i,h,z_1,\varepsilon}^0(v) X_{h,z_1,\varepsilon}$, from Lemma 3.6 we also deduce that

$$d\Theta_{X_{z_1},z}(X_{i,\varepsilon} - X_{i,z_1,\varepsilon}) = \varepsilon^{deg(i)} \sum_{deg(h)=deg(i)+1}^{\kappa} p_{i,h,z_1,\varepsilon}^0(v) d\Theta_{X_{z_1},z}(X_{h,z_1,\varepsilon}). \tag{3.16}$$

If $\|v\| \leq 1$, then

$$|p_{i,h,z_1,\varepsilon}^0(v)| \leq C|v_1|^{1/2}. \tag{3.17}$$

Since $X_{h,z_1,\varepsilon}$ has degree 1, then $p_{i,h,z_1,\varepsilon}^0(v) d\Theta_{X_{z_1},z}(X_{h,z_1,\varepsilon})$ is a differential operator of local ε-degree $1/2$ in the set $\|v\| \leq 1$ with respect to the vector fields $X_{i,z_1,\varepsilon}$.

On the other side, if $\|v\| \geq 1$, we have that X_{h,z_1} is a differential operator of ε-degree h,

$$|p_{i,h,z_1,\varepsilon}^0(v)| \leq C|v_1|^{1/2} \|v\|^{deg(h)}, \tag{3.18}$$

so that $p_{i,h,z_1,\varepsilon}^0(v) d\Theta_{X_{z_1},z}(X_{h,z_1,\varepsilon})$ is a differential operator of ε-degree $1/2$ for $\|v\| \geq 1$ with respect to the vector fields $X_{i,z_1,\varepsilon}$.   □

As a direct consequence of the previous corollary, we can prove that the difference $H_\varepsilon = L_\varepsilon - L_{z_1,\varepsilon}$ is a differential operator of degree strictly less than 2 in a neighborhood of the point z.

Lemma 3.8. *Under the assumptions of Corollary 3.7, $L_\varepsilon - L_{z_1,\varepsilon}$ is an operator of ε-degree $3/2$ with respect to the vector fields $X_{z_1,\varepsilon}$. Precisely there exist polynomials $p^{(1)}_{h,z_1,\varepsilon}$, $p^{(2)}_{i,h,z_1,\varepsilon}$ and a constant C independent of ε satisfying*

$$|p^{(1)}_{h,z_1,\varepsilon}(v)| \le C|v_1|^{1/2} \ \text{for} \ \|v\| \le 1, \quad |p^{(1)}_{h,z_1,\varepsilon}(v)| \le C|v_1|^{1/2}\|v\|^{deg(h)} \ \text{for} \ \|v\| \ge 1$$

and

$$|p^{(2)}_{i,h,z_1,\varepsilon}(v)| \le C|v_1|^{1/2} \ \text{for} \ \|v\| \le 1, \quad |p^{(2)}_{i,h,z_1,\varepsilon}(v)| \le C|v_1|^{1/2}\|v\|^{deg(h)+deg(i)} \ \text{for} \ \|v\| \ge 1,$$

such that

$$d\Theta_{X_{z_1},z}(L_\varepsilon - L_{z_1,\varepsilon}) = \sum_{deg(h)=2}^{\kappa} p^{(1)}_{h,z_1,\varepsilon}(v)d\Theta_{X_{z_1},z}(X_{h,z_1,\varepsilon}) + \qquad (3.19)$$

$$+ \sum_{deg(h)=2}^{\kappa} \sum_{deg(i)=1}^{deg(h)-1} p^{(2)}_{i,h,z_1,\varepsilon}(v)d\Theta_{X_{z_1},z}(X_{i,z_1,\varepsilon}X_{h,z_1,\varepsilon}) - \sum_{i=1}^{m} b_i d\Theta_{X_{z_1},z}(X_{i,\varepsilon})$$

where the coefficients b_i appear in the expression (2.17) of the operator L_ε.

Proof. By the definition of the operators we have:

$$d\Theta_{X_{z_1},z}(L_\varepsilon - L_{z_1,\varepsilon})$$
$$= \sum_{i=1}^{n} d\Theta_{X_{z_1},z}\Big(X_{i,z_1,\varepsilon}(X_{i,\varepsilon} - X_{i,z_1,\varepsilon})\Big) + \sum_{i=1}^{n} \Big(d\Theta_{X_{z_1},z}(X_{i,\varepsilon} - X_{i,z_1,\varepsilon})\Big)^2 + \qquad (3.20)$$
$$+ \sum_{i=1}^{n} d\Theta_{X_{z_1},z}\Big((X_{i,\varepsilon} - X_{i,z_1,\varepsilon})X_{i,z_1,\varepsilon}\Big) - \sum_{i=1}^{m} b_i d\Theta_{X_{z_1},z}(X_{i,\varepsilon})$$

By (3.8), $X_{1,\varepsilon} = X_{1,z_1,\varepsilon} = \partial_1$, so that

$$d\Theta_{X_{z_1},z}(L_\varepsilon - L_{z_1,\varepsilon})$$
$$= \sum_{i=2}^{n} d\Theta_{X_{z_1},z}\Big(X_{i,z_1,\varepsilon}(X_{i,\varepsilon} - X_{i,z_1,\varepsilon})\Big) + \sum_{i=2}^{n} \Big(d\Theta_{X_{z_1},z}(X_{i,\varepsilon} - X_{i,z_1,\varepsilon})\Big)^2 + \qquad (3.21)$$
$$+ \sum_{i=2}^{n} d\Theta_{X_{z_1},z}\Big((X_{i,\varepsilon} - X_{i,z_1,\varepsilon})X_{i,z_1,\varepsilon}\Big) - \sum_{i=1}^{m} b_i d\Theta_{X_{z_1},z}(X_{i,\varepsilon}).$$

Let us consider the first term at the right hand side. By (3.16)

$$d\Theta_{X_{z_1},z}(X_{i,z_1,\varepsilon})d\Theta_{X_{z_1},z}(X_{i,\varepsilon} - X_{i,z_1,\varepsilon}) =$$

$$= d\Theta_{X_{z_1},z}(X_{i,z_1,\varepsilon})\Big(\epsilon^{deg(i)} \sum_{deg(h)=deg(i)+1}^{\kappa} p^0_{i,h,z_1,\varepsilon}(v)d\Theta_{X_{z_1},z}(X_{h,z_1,\varepsilon})\Big) =$$

$$= \sum_{deg(h)=deg(i)+1}^{\kappa} e^{deg(i)-deg(h)}v_1\Big(d\Theta_{X_{z_1},z}(X_{i,z_1,\varepsilon})p_{i,h,z_1}(v)\Big)d\Theta_{X_{z_1},z}(X_{h,z_1,\varepsilon}) +$$

$$+ \sum_{deg(h)=deg(i)+1}^{\kappa} e^{deg(i)-deg(h)} v_1 p_{i,h,z_1}(v) d\Theta_{X_{z_1},z}(X_{i,z_1,\varepsilon} X_{h,z_1,\varepsilon})$$

where we have used the fact that $p^0_{i,h,z_1,\varepsilon}(v) = \varepsilon^{-deg(h)} v_1 p_{i,h,z_1}(v)$. The second term in the right hand side will contribute to the term $p^{(2)}_{i,h,z_1,\varepsilon}(v) d\Theta_{X_{z_1},z}(X_{h,z_1,\varepsilon})$, and the estimates directly follows from the estimates (3.17) and (3.18) of $p^0_{i,h,z_1,\varepsilon}(v)$. The first term in the right hand side will contribute to the terms $p^{(1)}_{h,z_1,\varepsilon}(v) d\Theta_{X_{z_1},z}(X_{h,z_1,\varepsilon})$. Its expression can be estimated arguing as in Corollary 3.7.

The other terms of (3.21) can be handled in a similar way. $\qquad\square$

In analogy with (2.26) we define the kernel

$$R_{1,\varepsilon}((x,t),(z,\tau)) := (L_{z_1,\varepsilon} - L_\varepsilon) \Gamma_{z_1,\varepsilon}((x,t),(z,\tau)) \tag{3.22}$$

for $t > \tau$.

As a consequence of Lemma 3.8 and of the homogeneity of the fundamental solution, we provide an estimate for $R_{1,\varepsilon}$.

Lemma 3.9. *If $M_0 \subset \mathbb{G}$ is a non-characteristic plane, $R_{1,\varepsilon}$ is a family of kernels of ε-uniform exponential type $1/2$ in the set $\{|x_1| \le \varepsilon^{2\kappa}\}$. Precisely for every bounded set there exists a constant C such that for every $x = (x_1, \hat{x})$, $z = (z_1, \hat{z}) \in \mathbb{G}$ such that $|x_1|, |z_1| \le \varepsilon^{2\kappa}$.*

$$|R_{1,\varepsilon}((x,t),(z,\tau))| \le C \frac{\Gamma_{z_1,\varepsilon}((x,2t),(z,2\tau))}{|t-\tau|^{3/4}}, \tag{3.23}$$

with C independent of ε and z_1.

Proof. By the representation formulas obtained in the previous Lemma 3.8, used with $v_1 = x_1 - z_1$, we only have to estimate terms of the type $|x_1 - z_1|^{1/2} X_{i,z_1,\varepsilon} X_{h,z_1,\varepsilon} \Gamma_{z_1,\varepsilon}((x,t),(z,\tau))$. Using (2.19) for any $0 < \varepsilon < 1$ we immediately obtain

$$|R_{1,\varepsilon}((x,t),(z,\tau))| \le C \frac{|x_1 - z_1|^{1/2} \Gamma_{z_1,\varepsilon}((x,2t),(z,2\tau))}{|t-\tau|}.$$

In order to prove (3.23) we note that we can assume that $|x_1 - z_1| > \sqrt{t-\tau}$, since otherwise the assertion is true. In this case we can use the fact that $\rho^{1/4} e^{-\rho^2} \le C e^{-\rho^2/2}$, for a suitable constant C, and the estimate (2.20) of the fundamental solution to ensure that

$$\frac{|x_1 - z_1|^{1/2}}{|t-\tau|^{1/4}} \Gamma_{z_1,\varepsilon}((x,t),(z,\tau)) \le C\Gamma_{z_1,\varepsilon}((x,2t),(z,2\tau)), \tag{3.24}$$

From here the thesis follows at once. $\qquad\square$

3.4. **Convergence of the parametrix method.** The second step of the parametrix method consists in proving that the series Φ, defined in (2.27), is convergent. In order to do this, we first need to obtain an uniform estimate of the distances $d_{z_1,\varepsilon}$. We denote

respectively $d_{z_1,\varepsilon}$ and $d_{z_1,0}$ the distances defined as in (2.15) and (2.6) in terms of the vector fields $X_{i,z_1,\varepsilon}$ and X_{i,z_1}:

$$d_{z_1,0}(x,z) = \|\Theta_{X_{1,z_1},\cdots,X_{n,z_1},z}(x)\|, \ \ d_{z_1,\varepsilon}(x,z) = \|\Theta_{X_{1,z_1,\varepsilon},\cdots,X_{n,z_1,\varepsilon},z}(x)\|_\varepsilon. \qquad (3.25)$$

Under the usual assumption that $M_0 \subset \mathbb{G}$ is a non characteristic plane (so that also M_{z_1} is non characteristic for $|z_1|$ sufficiently small), we have the following lemmata.

Lemma 3.10. *For every* $x = (x_1, \hat{x})$, $z = (z_1, \hat{z})$ *in* $\mathbb{G}$,

$$d(x,z) = d_{z_1,0}(x,z), \quad d_\varepsilon(x,z) = d_{z_1,\varepsilon}(x,z). \qquad (3.26)$$

In addition the distance $\hat{d}$ *defined in Section 3.1 satisfies*

$$\hat{d}(\hat{x}, \hat{z}) = d_{0,0}((0,\hat{x}),(0,\hat{z})). \qquad (3.27)$$

and

$$\hat{d}(\hat{x}, \hat{z}) = d((0,\hat{x}),(0,\hat{z})). \qquad (3.28)$$

Proof. The distance $d(x,z)$ is defined in (2.6) as the norm of the vector v such that

$$x = \exp(v_1 X_1)\exp(\sum_{i=2}^n v_i X_i)(z). \qquad (3.29)$$

Since all the vector fields $(X_i)_{i=2,\cdots,n}$ are tangential to the plane M_{z_1}, the integral curve $t \mapsto \exp(t\sum_{i=2}^n v_i X_i)(z)$ is tangent to the same plane. Therefore, along this curve the vector fields $(X_i)_{i=2,\cdots,n}$ are computed for $x_1 = z_1$ and coincide with the vector fields X_{i,z_1}. This implies that $d(x,z) = d_{z_1,0}(x,z)$. The same argument applies to the second equality in (3.26) to (3.27), and to (3.27). $\qquad\square$

Since we have a good estimate of the kernel $R_{1,\varepsilon}$ only in an ε- neighborhood of the plane M_0, we have to modify the classical parametrix method, restricting the integral in this neighborhood. To this end we consider a cut-off function depending only on the first variable x_1. Precisely, we consider a piecewise function ρ_ε, supported in an ε neighborhood of the origin, defined as follows:

$$\rho_\varepsilon(x_1) = 1 \text{ if } |x_1| \le 2\varepsilon^{2\kappa}, \quad \rho_\varepsilon(x_1) = 0 \text{ elsewhere.} \qquad (3.30)$$

For any suitable kernel K, we define

$$E_{R_{1,\varepsilon}}(K)((x,t),(z,\tau)) := -\int_{\mathbb{R}^n \times [\tau,t]} R_{1,\varepsilon}((x,t),(y,\theta)) \, K((y,\theta),(z,\tau)) \rho_\varepsilon(y_1 - z_1) \, dy d\theta \qquad (3.31)$$

and, in analogy with (2.27), we consider

$$\Phi_\varepsilon((x,t),(z,\tau)) := \sum_{j=0}^\infty (E_{R_{1,\varepsilon}})^j(R_{1,\varepsilon})((x,t),(z,\tau)). \qquad (3.32)$$

We will prove that the series can be is totally convergent on the set

$$\left\{ 0 < t - \tau \le T, \ |x_1|, |z_1| \le \varepsilon^{2\kappa}, \ d_\varepsilon(x,z) + |t - \tau|^{\frac{1}{2}} \ge \delta \right\} \qquad \text{for all } T > 0, \delta > 0,$$

and it is a kernel of uniform exponential ε-type $1/2$, i.e. it satisfies the estimate

$$|\Phi_\varepsilon((x,t),(z,\tau))| \le c(T)(t-\tau)^{-\frac{3}{4}}\Gamma_{z_1,\varepsilon}((x,ct),(z,c\tau)) \qquad 0 < t-\tau \le T, \qquad (3.33)$$

with constants independent of ε and of z_1.

As we mentioned in Section 2.3 the convergence of this series relies on properties of convolutions of kernels. Hence we will need the following property of the operator $E_{R_1,\varepsilon}$, that ensures that the series can be estimated by a power series, so that it is convergent on the mentioned set:

Lemma 3.11. *Let $M_0 \subset \mathbb{G}$ be a non-characteristic plane. For $z \in M_0$, $x \in \mathbb{G}$, with $|x_1| \le \varepsilon^{2\kappa}$ and for $j \in \mathbb{N}$ it holds that*

$$|(E_{R_1,\varepsilon})^j R_1((x,t),(z,\tau))| \le \frac{C^j \beta_j}{2} \frac{\Gamma_{z_1,\varepsilon}((x,c_1t),(z,c_1\tau))}{(t-\tau)^{3/4-j/4}}, \qquad (3.34)$$

$j \in \mathbb{N}$, where $\beta_j := \Gamma^{j+1}(\frac{1}{4})/\Gamma(\frac{j+1}{4})$ and Γ is the Euler Gamma-function.

Proof. We argue as in [36] or [3] and we prove by induction (3.34). Our main concern in the proof of (3.34) is to show that the constants are independent of ε. The estimate for $j = 0$ is already contained in (3.23). Let us assume that estimate (3.34) holds for $j-1 \in \mathbb{N}$. Using the explicit expression of $E_{R_1,\varepsilon}$ contained in (3.31) we have

$$|(E_{R_1,\varepsilon})^j R_1((x,t),(z,\tau))|$$
$$\le \int_{\mathbb{R}^n \times [\tau,t]} \left| R_{1,\varepsilon}((x,t),(y,\theta))\,(E_{R_1,\varepsilon})^{j-1} R_1((y,\theta),(z,\tau))\rho_\varepsilon(y_1-z_1) \right| dy d\theta.$$

Now we apply the estimate (3.23) for R_1 and the inductive assumption (3.34) on $(E_{R_1,\varepsilon})^{j-1}R_1$ to obtain

$$|(E_{R_1,\varepsilon})^j R_1((x,t),(z,\tau))|$$
$$\le \frac{C^j \beta_{j-1}}{2} \int_\tau^t (t-\theta)^{-\frac{3}{4}}(\theta-\tau)^{-\frac{3}{4}+\frac{j-1}{4}} \int_{\mathbb{R}^n} \Gamma_\varepsilon((x,c_1t),(y,c_1\theta))\Gamma_\varepsilon((y,c_1\theta),(z,c_1\tau))\,dy d\theta.$$

By the reproducing property of the fundamental solution, (see [36] or [3]), we have

$$\int_{\mathbb{R}^n} \Gamma_\varepsilon((x,c_1t),(y,c_1\theta))\Gamma_\varepsilon((y,c_1\theta),(z,c_1\tau))\,dy = \Gamma_\varepsilon((x,c_1t),(z,c_1\tau))$$

and, by the change of variable $r = (t-\tau)^{-1}(\theta-\tau)$,

$$\beta_{j-1}\int_\tau^t (t-\theta)^{-\frac{3}{4}}(\theta-\tau)^{-1+\frac{j}{4}}\,d\theta = \beta_{j-1}(t-\tau)^{-\frac{3}{4}+\frac{j}{4}}\int_0^1 (1-r)^{-\frac{3}{4}}r^{-1+\frac{j}{4}}\,dr$$

$$= \beta_{j-1}(t-\tau)^{-3/4+j/4}\frac{\Gamma(\frac{1}{4})\cdot\Gamma(\frac{j}{4})}{\Gamma(\frac{j+1}{4})} = (t-\tau)^{-3/4+j/4}\frac{\Gamma^{j+1}(\frac{1}{4})}{\Gamma(\frac{j+1}{4})} = \beta_j(t-\tau)^{-3/4+j/4}, \tag{3.35}$$

by the definition of β_j. Putting together these terms we obtain

$$|(E_{R_1,\varepsilon})^j R_1((x,t),(z,\tau))| \le \frac{C^j \beta_j}{2}(t-\tau)^{-3/4+j/4}\Gamma_\varepsilon((x,c_1t),(z,c_1\tau)).$$

Thus, (3.34) follows by induction for all $j \in \mathbb{N}$. $\qquad\square$

Remark 3.12. *From the assertion above it follows that the convolution of a family of kernels of uniform exponential ε-type $1/2$ with a family of kernels with uniform exponential ε-type β is a family of kernels with uniform exponential ε-type $\beta + 1/2$.*

For every $\varepsilon > 0$ the operators L_ε and $L_{z_1,\varepsilon}$ are uniformly elliptic, so that the proof of the convergence of the parametrix method is a well known fact (see [26]). In particular $\Gamma_{z_1,\varepsilon}$ provides a good approximation of Γ_ε in a neighborhood of the plane. More precisely

$$\Gamma_\varepsilon((x,t),(z,\tau)) = \Gamma_{z_1,\varepsilon}((x,t),(z,\tau))$$
$$+ \int_{\mathbb{R}^n \times [\tau,t]} \Gamma_{y_1,\varepsilon}((x,t),(y,\theta))\, \Phi_\varepsilon((y,\theta),(z,\tau))\rho_\varepsilon(y_1 - z_1)\, dyd\theta. \tag{3.36}$$

In addition, for $i = 1, \cdots, n$,

$$X_{i,0,\varepsilon}^2 \Gamma_\varepsilon((x,t),(z,\tau)) = X_{i,0,\varepsilon}^2 \Gamma_{z_1,\varepsilon}((x,t),(z,\tau))$$
$$+ \lim_{\delta \to 0^+} \int_{\mathbb{R}^n \times [\tau,t-\delta]} X_{i,0,\varepsilon}^2 \Gamma_{y_1,\varepsilon}((x,t),(y,\theta))\Phi_\varepsilon((y,\theta),(z,\tau))\rho_\varepsilon(y_1 - z_1)\, dyd\theta. \tag{3.37}$$

Using the explicit representation formulas above we can provide the following estimates for $\Gamma_\varepsilon - \Gamma_{z_1,\varepsilon}$ uniform in ε:

Proposition 3.13. *Let M_0 be a non characteristic plane and $x = (x_1, \hat{x}), z = (z_1, \hat{z}) \in \mathbb{G}$ such that $|x_1|, |z_1| \leq \varepsilon^{2\kappa}$. For every $T > 0$ there exists a constant $C = C(T)$ such that for every $\varepsilon > 0$ and for every t, τ with $0 < t - \tau \leq T$ the following inequalities hold*

$$|\Gamma_\varepsilon((x,t),(z,\tau)) - \Gamma_{z_1,\varepsilon}((x,t),(z,\tau))| \leq C(t-\tau)^{1/4}\Gamma_{z_1,\varepsilon}((x,t),(z,\tau)). \tag{3.38}$$

In addition $\Gamma_\varepsilon((x,t),(z,\tau)) - \Gamma_{z_1,\varepsilon}((x,t),(z,\tau))$ is a family of kernels of uniform exponential ε-type $1/4$ with respect to the vector fields $(X_{i,\varepsilon})_{i=2,\cdots,n}$.

For the proof we refer to [36], while the uniformity with respect to ε follows by (2.8).

Proof of Theorem 3.2 . We first prove a Riemannian version of Theorem 3.2. Precisely we show that for all $(0,\hat{z}), (0,\hat{x})$ in M_0 and for every t, τ, with $0 < t - \tau \leq T$ we have

$$|\hat{\Gamma}_{0,\varepsilon}((\hat{x},t),(\hat{z},\tau)) - \sqrt{4\pi(t-\tau)}\Gamma_\varepsilon((0,\hat{x},t),(0,\hat{z},\tau))| \leq C\Gamma_\varepsilon((0,\hat{x},t),(0,\hat{z},\tau))(t-\tau)^{3/4}, \tag{3.39}$$

where C is a constant independent of ε. Indeed,

$$\sqrt{4\pi(t-\tau)}\Gamma_\varepsilon((0,\hat{x},t),(0,\hat{z},\tau)) - \hat{\Gamma}_{0,\varepsilon}((\hat{x},t),(\hat{z},\tau)) =$$
$$= \sqrt{4\pi(t-\tau)}\Gamma_{0,\varepsilon}((0,\hat{x},t),(0,\hat{z},\tau)) - \hat{\Gamma}_{0,\varepsilon}((\hat{x},t),(\hat{z},\tau))$$
$$+ \sqrt{4\pi(t-\tau)}\Big(\Gamma_\varepsilon((0,\hat{x},t),(0,\hat{z},\tau)) - \Gamma_{0,\varepsilon}((0,\hat{x},t),(0,\hat{z},\tau))\Big).$$

By Remark 3.5

$$\Gamma_{0,\varepsilon}((0,\hat{x},t),(0,\hat{z},\tau)) = \Gamma_{\perp,0,\varepsilon}((0,t),(0,\tau))\hat{\Gamma}_{0,\varepsilon}((\hat{x},t),(\hat{z},\tau)),$$

then the first difference at the right hand side is zero since

$$\sqrt{4\pi(t-\tau)}\Gamma_{\perp,0,\varepsilon}((0,t),(0,\tau)) = 1$$

and hence the estimate (3.39) follows by (3.38). Always from Proposition 3.13 it follows that $\sqrt{4\pi(t-\tau)}\Gamma_\varepsilon((0,\hat{x},t),(0,\hat{z},\tau)) - \hat{\Gamma}_{0,\varepsilon}((\hat{x},t),(\hat{z},\tau))$ is a family of kernels of uniform exponential ε-type $3/4$ with respect to the vector fields $(X_{i,\varepsilon})_{i=2,\cdots,n}$. Sending ε to 0 in the assertion (3.39), we obtain that for all $z = (0,\hat{z})$, $x = (0,\hat{x})$ in M_0 and for every t and τ, with $0 < t - \tau < T$, we have

$$|\hat{\Gamma}((\hat{x},t),(\hat{z},\tau)) - \sqrt{t-\tau}\Gamma((x,t),(z,\tau))| \leq C\Gamma((x,t),(z,\tau))(t-\tau)^{3/4},$$

and the left hand side is a kernel of exponential type $3/4$ with respect to the vector fields $(X_i)_{i=2,\cdots,m}$. Using the Gaussian estimate (2.22) of Γ and $\hat{\Gamma}$ together with formula (3.28) we obtain

$$|\hat{\Gamma}((\hat{x},t),(\hat{z},\tau)) - \sqrt{t-\tau}\Gamma((x,t),(z,\tau))| \leq C\hat{\Gamma}((\hat{x},t),(\hat{z},\tau))(t-\tau)^{1/4}$$

and the left hand side is a kernel of exponential type $1/4$ with respect to the vector fields $(\hat{X}_i)_{i=2,\cdots,m}$. Theorem 3.2 follows immediately. $\qquad\square$

3.5. The reproducing formula for homogeneous sub-Laplacians on a plane.

Here we establish the analogous of Theorem 1.3 for homogeneous vector fields expressed as in (2.3), under the assumption that the boundary of D is the plane $\{x_1 = 0\}$. This is done integrating in time the result of Theorem 3.2. Let us first deduce an integral version of Theorem 3.2, based on the reproducing formula of the heat kernel.

Lemma 3.14. *Let $D = \{(x_1,\hat{x}) \in \mathbb{R}^n : x_1 > 0\}$, and assume that its boundary is non characteristic. There exists $C > 0$ such that for any $(0,\hat{x}), (0,\hat{y}) \in \partial D$ and for all t,τ, with $0 \leq \tau \leq t$ we have*

$$\hat{\Gamma}((\hat{x},t),(\hat{y},\tau)) = \tag{3.40}$$

$$= \int_\tau^t \int_{\mathbb{R}^{n-1}} \Gamma((0,\hat{x},t),(0,\hat{z},\theta))\Gamma((0,\hat{z},\theta),(0,\hat{y},\tau))d\hat{z}d\theta + \hat{R}(\hat{x},\hat{y},t-\tau),$$

where

$$|\hat{R}(\hat{x},\hat{y},t)| \leq Ct^{1/4}\hat{\Gamma}((\hat{x},t),(\hat{y},0)), \tag{3.41}$$

and $\hat{R}$ is a kernel of exponential type $5/2$ with respect to the vector fields $\{\hat{X}_i\}_{i=2,\cdots,n}$.

Proof. Let us first prove (3.40). To this end we note that the thesis is true for $t - \tau \geq 1$. Indeed

$$\hat{\Gamma}((\hat{x},t),(\hat{y},\tau)) \leq c(t-\tau)^{1/4}\hat{\Gamma}((\hat{x},t),(\hat{y},\tau))$$

and by the standard Gaussian estimates (2.22) of the fundamental solution and by the relation (3.28) between the distances $\hat{d}$ and d, we obtain

$$\int_\tau^t \int_{\mathbb{R}^{n-1}} \Gamma((0,\hat{x},t),(0,\hat{z},\theta))\Gamma((0,\hat{z},\theta),(0,\hat{y},\tau))d\hat{z}d\theta$$

$$\leq c(t-\tau)^{3/4}\Gamma((0,\hat{x},t),(0,\hat{y},\tau)) \leq c(t-\tau)^{1/4}\hat{\Gamma}((\hat{x},t),(\hat{y},\tau)).$$

If $t - \tau < 1$, by Theorem 3.2, we have

$$\Gamma((0,\hat{x},t),(0,\hat{z},\theta)) = \frac{\hat{\Gamma}((\hat{x},t),(\hat{z},\theta))}{\sqrt{t-\theta}}(1 + O(t-\theta)^{1/4}). \tag{3.42}$$

Thus,

$$\int_\tau^t \int_{\mathbb{R}^{n-1}} \Gamma((0,\hat{x},t),(0,\hat{z},\theta))\Gamma((0,\hat{z},\theta),(0,\hat{y},\tau))d\hat{z}d\theta$$

$$= \int_\tau^t \left(\int_{\mathbb{R}^{n-1}} \hat{\Gamma}((\hat{x},t),(\hat{z},\theta))\hat{\Gamma}((\hat{z},\theta),(\hat{y},\tau))d\hat{z} \right.$$

$$\left. \left(\frac{1}{(t-\theta)^{1/2}} + O\left(\frac{1}{(t-\theta)^{1/4}}\right)\right)\left(\frac{1}{(\theta-\tau)^{1/2}} + O\left(\frac{1}{(\theta-\tau)^{1/4}}\right)\right) \right) d\theta$$

$$= \hat{\Gamma}((\hat{x},t),(\hat{y},\tau)) \int_\tau^t \left(\frac{1}{(t-\theta)^{1/2}} + O\left(\frac{1}{(t-\theta)^{1/4}}\right)\right)\left(\frac{1}{(\theta-\tau)^{1/2}} + O\left(\frac{1}{(\theta-\tau)^{1/4}}\right)\right) d\theta,$$

by the reproducing formula. Now, with the change of variable $r = (t-\tau)^{-1}(\theta-\tau)$, we get

$$\int_\tau^t \left(\frac{1}{(t-\theta)^{1/2}} + O\left(\frac{1}{(t-\theta)^{1/4}}\right)\right)\left(\frac{1}{(\theta-\tau)^{1/2}} + O\left(\frac{1}{(\theta-\tau)^{1/4}}\right)\right) d\theta = 1 + O\left((t-\tau)^{1/4}\right).$$

Therefore, we get

$$\int_\tau^t \int_{\mathbb{R}^{n-1}} \Gamma((0,\hat{x},t),(0,\hat{z},\theta))\Gamma((0,\hat{z},\theta),(0,\hat{y},\tau))d\hat{z}d\theta \qquad (3.43)$$

$$= \hat{\Gamma}((\hat{x},t),(\hat{y},\tau))(1 + O((t-\tau)^{1/4}),$$

so that R satisfies (3.40). A similar argument applied to all derivatives ensures that $\hat{R}$ is a kernel of exponential type $5/2$ with respect to the vector fields $\{\hat{X}_i\}_{i=2,\cdots,n}$ and concludes the proof. $\qquad\square$

3.6. **Proof of Theorem 1.3 for homogeneous vector fields on a plane.** We will provide in Lemma 3.16 below the proof of Theorem 1.3 on a plane and for homogeneous vector fields. This result can be considered the time independent version of Lemma 3.14. It will be established integrating in time the thesis of that Lemma and using the well known fact that the fundamental solutions $\hat{\Gamma}_{\hat{\Delta}}$ of the Laplace type operator (3.5) and Γ_Δ of the Laplace operator (1.2) satisfy respectively

$$\hat{\Gamma}_{\hat{\Delta}}(\hat{x},\hat{z}) = \int_0^{+\infty} \hat{\Gamma}((\hat{x},t),(\hat{z},0))dt, \quad \Gamma_\Delta(x,z) = \int_0^{+\infty} \Gamma((x,t),(z,0))dt. \qquad (3.44)$$

From the previous definition and Corollary 3.3, we deduce the following remark:

Remark 3.15. *The kernel $\Gamma_\Delta((0,\hat{x}),(0,\hat{z}))$ is an operator of type 1 with respect to the distance $\hat{d}$. Indeed the integration (3.44) with respect to the t variable changes kernels of exponential type α in kernels of type α.*

Lemma 3.16. *Let the vector (X_i) be represented as in (2.3), let $M_0 = \{(0,\hat{x}) : \hat{x} \in \mathbb{R}^{n-1}\}$ be a non characteristic plane. For any $(0,\hat{x}),(0,\hat{y}) \in M_0$*

$$\hat{\Gamma}_{\hat{\Delta}}(\hat{x},\hat{y}) = \int_{\mathbb{R}^{n-1}} \Gamma_\Delta((0,\hat{x}),(0,\hat{z}))\Gamma_\Delta((0,\hat{z}),(0,\hat{y}))d\hat{z} + \hat{R}_{\hat{\Delta}}(\hat{x},\hat{y}), \qquad (3.45)$$

where

$$\hat{R}_{\hat{\Delta}}(\hat{x}, \hat{y}) = O(\hat{d}(\hat{x}, \hat{y})^{\frac{1}{2}} \hat{\Gamma}_{\hat{\Delta}}(\hat{x}, \hat{y})). \tag{3.46}$$

In particular $\hat{R}_{\hat{\Delta}}(\hat{x}, \hat{y})$ is a kernel of type 5/2 in the sense of Definition 2.3 with respect to the distance $\hat{d}$ defined on the plane.

Proof. Using (3.44) and integrating both sides of expression (3.40) we obtain:

$$\hat{\Gamma}_{\hat{\Delta}}(\hat{x}, \hat{y}) + \int_0^{+\infty} \hat{R}((\hat{x}, t), (\hat{y}, 0)) dt$$

$$= \int_0^{+\infty} \int_0^t \int_{\mathbb{R}^{n-1}} \Gamma((0, \hat{x}, t - \theta), (0, \hat{z}, 0)) \Gamma((0, \hat{z}, \theta), (0, \hat{y}, 0)) d\hat{z} \, d\theta \, dt.$$

Changing the order of integration, we get that the last term is equal to

$$\int_{\mathbb{R}^{n-1}} \int_0^{+\infty} \left(\int_\theta^{+\infty} \Gamma((0, \hat{x}, t - \theta), (0, \hat{z}, 0)) \, dt \right) \Gamma((0, \hat{z}, \theta), (0, \hat{y}, 0)) \, d\theta d\hat{z}$$

$$= \int_{\mathbb{R}^{n-1}} \int_0^{+\infty} \Gamma_\Delta((0, \hat{x}), (0, \hat{z})) \Gamma((0, \hat{z}, \theta), (0, \hat{y}, 0)) d\theta d\hat{z}$$

and integrating with respect to θ we obtain

$$\int_{\mathbb{R}^{n-1}} \Gamma_\Delta((0, \hat{x}), (0, \hat{z})) \Gamma_\Delta((0, \hat{z}), (0, \hat{y})) d\hat{z}.$$

The estimate of $\hat{R}_{\hat{\Delta}}(\hat{x}, \hat{y})$ directly follows easily integrating in (3.43) and using the estimate of $\hat{\Gamma}((\hat{x}, ct), (\hat{y}, 0))$ provided in Lemma 3.14. Therefore, recalling that $\widehat{Q} = Q - 1$ denotes the homogeneous dimension of the plane, we have

$$\hat{R}_{\hat{\Delta}}(\hat{x}, \hat{y}) = \int_0^{+\infty} \hat{R}((\hat{x}, t), (\hat{y}, 0)) dt \leq$$

$$\leq c \int_0^{+\infty} \hat{\Gamma}((\hat{x}, \tilde{c} t), (\hat{y}, 0)) t^{1/4} dt \leq c \int_0^{+\infty} \frac{e^{-\frac{\hat{d}(\hat{x}, \hat{y})^2}{Ct}}}{t^{\frac{\widehat{Q}}{2} - \frac{1}{4}}} \, dt,$$

where the constants may vary from line to line. Now, with the change of variables $v = -\frac{\hat{d}(\hat{x}, \hat{y})^2}{Ct}$ we get

$$\hat{R}_{\hat{\Delta}}(\hat{x}, \hat{y}) \leq c \int_0^{+\infty} e^{-v} \frac{v^{\frac{\widehat{Q}}{2} - \frac{9}{4}}}{\hat{d}(\hat{x}, \hat{y})^{\widehat{Q} - \frac{5}{2}}} \, dv \leq c \hat{d}(\hat{x}, \hat{y})^{-\widehat{Q} + 2 + \frac{1}{2}} \leq c \hat{d}(\hat{x}, \hat{y})^{\frac{1}{2}} \hat{\Gamma}_{\hat{\Delta}}(\hat{x}, \hat{y}).$$

An analogous inequality holds for any derivative and the result is proved. □

4. Reproducing formula on a smooth hypersurface

4.1. Reduction of a general hypersurface to a plane with a subriemannian structure.

Let us denote by D a smooth, open bounded set in $\mathbb{G}$ and let $0 \in \partial D$ be a non characteristic point. In this section we show that we can always reduce the boundary ∂D to the plane $\{(x_1, \hat{x}) : x_1 = 0\}$, via a change of variables. Indeed, there exists a neighborhood V_0 of 0 such that the subriemannian normal ν satisfies

$$\nu(s) \neq 0 \text{ for every } s \in \partial D \cap V_0.$$

We can also choose an invariant basis $(Z_i)_{i=1,\cdots,n}$ of the tangent space of $\mathbb{G}$ around the point 0 and Z_i coincides with the standard element ∂_i of the tangent basis at the point 0, for every $i = 1, \cdots, n$. In addition, eventually applying a group homomorphism T_0, we can assume that $Z_{1|0} := \partial_{1|0} = \nu(0)$ and that the vector fields $(Z_{i|0})_{i=2,\cdots,m}$ span the horizontal tangent space of ∂D at 0. We also assume that the problem is expressed in canonical coordinates of second type around the point 0 associated to these vector fields. In these coordinates the vector fields admit the representation

$$Z_1 = \partial_1, \quad Z_i = \partial_i + \sum_{deg(j) > deg(i)} a_{i,j}(s)\partial_j, \text{ for } i = 1, \cdots, m \tag{4.1}$$

while the boundary of D can be identified in a neighborhood $V \subset\subset V_0$ with the graph of a regular function w, defined on a neighborhood $\hat{V} = V \cap \mathbb{R}^{n-1}$ of 0:

$$\partial D \cap V = \{(w(\hat{s}), \hat{s}) : \hat{s} \in \hat{V}\}.$$

By the choice of coordinates we have in particular that

$$Z_i w(0) = 0. \tag{4.2}$$

On the set V the function $\Xi(s_1, \hat{s}) = (s_1 - w(\hat{s}), \hat{s})$ is a diffeomorphism. It sends $\partial D \cap V$ to a subset of the plane $\{x_1 = 0\}$:

$$\Xi(\partial D \cap V) = \{(x_1, \hat{x}) : x_1 = 0\}.$$

Through this change of variables the vector fields Z_i can be represented as

$$\begin{aligned} X_1 &= d\Xi(Z_1) = \partial_{x_1}, \\ X_i &= d\Xi(Z_i) = \partial_{x_i} + \sum_{deg(j)>deg(i)} a_{i,j}(x_1 + w(\hat{x}), \hat{x})\partial_{x_j} + Z_i w(\hat{x})\partial_{x_1}, \end{aligned} \tag{4.3}$$

for $i = 1, \cdots, n$, where the polynomials a_{ij} are the same of the ones defined in (2.3).

A neighborhood of 0 in the boundary of D locally becomes in the new coordinates an open subset $M_0 = \Xi(\partial D \cap V)$ of the plane $\{x_1 = 0\}$. We can restrict the vector fields $(X_i)_{i=1,\cdots,m}$ to the tangent to M_0 and we call them $\hat{X}_i$:

$$\hat{X}_i = \partial_i + \sum_{deg(j)=deg(i)+1}^{\kappa} a_{i,j}(w(\hat{x}), \hat{x})\partial_j, \quad i = 2, \cdots, n. \tag{4.4}$$

The vector fields $(\hat{X}_i)_{i=1,\cdots,m}$ still satisfy the assumption (1.5), which ensures that they satisfy the Hörmander finite rank condition [33]. They do not define a general Hörmander structure (see [40]), since they have been obtained from the generators of a Carnot group via a change of variables. It is important to note that the vector fields X_i as well as

the vector fields $\hat{X}_i$ are not homogeneous with respect to the new variables x_i. However we will see in Lemma 4.4 and Lemma 4.5 that at every point they admit approximating vector fields respectively Z_i and $\hat{Z}_i$ homogeneous in the new variables. Hence the local homogeneous dimension of $\mathbb{R}^n$ endowed with the choice of the vector fields X_i is Q. Since the Hörmander condition is satisfied, a Carnot Carathéodory distance d is defined in terms of the vector fields $(X_i)_{i=1}^m$. Thanks to assumption (1.5), the vector fields $\{\hat{X}_i\}_{i=2,\cdots,m}$, defined in (4.4), generate on the plane M_0 a subriemannian structure with local homogeneous dimension $\hat{Q} = Q-1$ and induce a distance $\hat{d}$ on M_0 defined through the exponential map as in (2.6), which satisfies (3.27) and (3.28).

The Laplace type operator, analogous to (1.2) and expressed in terms of the vector fields X_i is denoted by

$$\Delta = \sum_{i=1}^m X_i^2 + \sum_{i=1}^m b_i X_i \tag{4.5}$$

and it has a fundamental solution Γ_Δ, of class C^∞ out of the pole (see for example [44]). The operator analogous to (3.5), expressed in terms of the vector fields $\hat{X}_i$, is

$$\hat{\Delta} = \sum_{i=2}^m \hat{X}_i^2, \tag{4.6}$$

with fundamental solution $\hat{\Gamma}_{\hat{\Delta}}$. In analogy with the definition of type of a kernel with respect to the vector fields (X_i), given in (1.4), we give here the definition of kernel of local type λ with respect to the vector fields $\hat{X}_2, \cdots, \hat{X}_n$:

Definition 4.1. *k is a kernel of local type λ with respect to the vector fields $\hat{X}_2, \cdots, \hat{X}_n$ and the distance $\hat{d}$ if it is a smooth function out of the diagonal and, in any open set V, the following holds: for every p there exists a positive constant C_p such that, for every $\hat{x}, \hat{y} \in \partial D \cap V$, $\hat{x} \neq \hat{y}$,*

$$|\hat{X}_{i_1}, \cdots, \hat{X}_{i_p} k(\hat{x}, \hat{y})| \leq C_p \, \hat{d}(\hat{x}, \hat{y})^{\lambda-p-2} \frac{\hat{d}(\hat{x}, \hat{y})^2}{|B(\hat{x}, \hat{d}(\hat{x}, \hat{y}))|}.$$

Clearly, if the space is homogeneous, the previous definition coincides with Definition 2.3.

4.2. **A freezing procedure.** Here we will show that, when we are studying pointwise properties around a fixed point x_0, we can always reduce our vector fields to homogeneous ones. The proof is made approximating the vector fields with nilpotent ones, adapting to this context the Rothschild and Stein parametrix method. In the classical case the vector fields are lifted to vector fields free up to step κ and then they are reduced to the generators of a free algebra with a freezing method. Here we cannot lift the vector fields to free ones otherwise we would lose assumption (1.5). However, we can use the explicit expression of the vector fields (4.3) to obtain an ad hoc version of the Rothschild and Stein method.

Let D be a smooth, open bounded set in $\mathbb{R}^n$, locally expressed as a graph of a function w. In the previous section we defined a change of variable allowing the description of the

set as

$$\partial D \cap V = \{\Xi_0^{-1}(x_1, \hat{x}) : x_1 = 0\}.$$

In the following remark we perform a similar change of variable for every $z \in \partial D \cap V$.

Remark 4.2. *For every $z \in \partial D \cap V$ we will denote $\nu(z)$ the normal to $\partial D \cap V$ in z and $T_z : (\mathbb{G}, 0) \to (\mathbb{G}, z)$ the group homomorphism such that*

$$T_z(0) = z, \quad dT_z(X_1)_{|0} = \nu(z)$$

and $dT_z(X_2)_{|0}, \cdots, dT_z(X_n)_{|0}$ is a basis of the tangent space to $\partial D \cap V$ at z.

If we fix z the implicit function theorem (see [25], [17]) ensures that there exists a neighborhood $U = I \times \hat{U}$ of 0 and a function $w_z : \hat{U} \to \mathbb{R}$ such that $w_z(0) = 0$ and

$$\{(w_z(\hat{y}), \hat{y}) : \hat{y} \in \hat{U}\} = T_z^{-1}(\partial D \cap V) \cap U,$$

so that $\{T_z(w_z(\hat{y}), \hat{y}) : \hat{y} \in \hat{U}\} \subset \partial D \cap V$. We can always assume that $\nabla_z w_z(0) = 0$. Due to the regularity of the boundary we can find an open set $W \subset V$ such that for every $z \in W \cap \partial D$ the function w_z is defined on the same set $\hat{U}$ with values in the same set I. Clearly $T_0 = id$, and $w_0 = w$, where w is defined above as the defining function of the set ∂D in the original variables.

We prove the following result analogous to [43] in our simplified setting:

Proposition 4.3. *There exist open neighborhoods U of 0 in $\mathbb{R}^n$ and V, W of $0 \in \partial D \subset \mathbb{R}^n$, with $W \subset V$ and, for every z fixed in W, a change of coordinates Ξ_z such that*

- *the function $x \to \Xi_z(x)$ is a diffeomorphism from U on the image*
- *in the new coordinates the vector fields will admit the following representation:*

$$\Xi_z(X_1) = \partial_{y_1}$$

$$d\Xi_z(X_i) = \partial_{y_i} + \sum_{deg(j) > deg(i)} a_{i,j}(y_1 + w_z(\hat{y}), \hat{y})\partial_{y_j} + X_i w_z \partial_{y_1}, \quad i = 2, \cdots, n.$$

Proof. Hence we can define the map

$$E_z : U \to V, \quad E_z(y_1, \hat{y}) := T_z(y_1 + w_z(\hat{y}), \hat{y}).$$

E_z is invertible on its image and sends the plane $\{y_1 = 0\} \cap U$ into a suitable subset of ∂D. The composition $E_0^{-1}E_z$ sends the plane $\{y_1 = 0\}$ into the the plane $\{x_1 = 0\}$, boundary of D. For every $z \in W \cap \partial D$ its inverse function $\Xi_z(x)$ is a diffeomorphism on the image and $\Xi_z(W) \subset U \subset \Xi_z(V)$ The vector fields X_i can be represented as follows in the new coordinates (see also [1], [16]):

$$d\Xi_z(X_1) = \partial_{y_1}$$

$$d\Xi_z(X_i) = \partial_{y_i} + \sum_{deg(j) > deg(i)} a_{i,j}(y_1 + w_z(\hat{y}), \hat{y})\partial_{y_j} + X_i w_z(\hat{y})\partial_{y_1}, \quad i = 2, \cdots, n.$$

$\square$

We can now prove the following result, analogous to Theorem 5 in [44]:

Lemma 4.4. *With the same notations of previous Proposition 4.3, let us call*

$$Z_i = \partial_{y_i} + \sum_{deg(j)>deg(i)} a_{ij}(y)\partial_{y_j},$$

for $i = 1, \cdots, n$. Then we have

$$d\Xi_z(X_i) - Z_i = R_{i,z,\Xi}$$

where $R_{i,z,\Xi}$ are vector fields of local degree $\leq deg(i)-1$ depending smoothly on z. Precisely $R_{i,z,\Xi} = \sum_j r_{ij,z}\partial_j$ where $r_{ij,z} = O(d(0,y)^{deg(j)-deg(i)+1})$.

Proof. It is a direct computation. Indeed the assertion is true for $i = 1$. For every $i > 1$ the difference $d\Xi_z(X_i) - Z_i$ can be expressed as

$$d\Xi_z(X_i) - Z_i = \sum_{deg(j)>deg(i)} \Big(a_{ij}(y_1 + w_z(\hat{y}), \hat{y}) - a_{ij}(y_1, \hat{y})\Big)\partial_{y_j} + X_i w_z(\hat{y})\partial_{y_1}.$$

We first note that, since $w_z(0) = 0$ and we can always think that also $X_i w_z(0) = 0$, then $X_i w_z(\hat{y})\partial_{y_1}$ is an operator of degree 0. Moreover, being a_{ij} homogeneous polynomials, their difference can be represented as a homogeneous polynomial. Precisely there exists a suitable polynomial a_{ij}^1 homogeneous of degree $deg(i) - deg(j) - 1$ such that

$$a_{ij}(y_1 + w_z(\hat{y}), \hat{y}) - a_{ij}(y_1, \hat{y}) = w_z(\hat{y})a_{ij}^1(y_1, y_1 + w_z(\hat{y}), \hat{y}) =$$

$$= O(\|\hat{y}\|^2)a_{ij}^1(y_1, y_1 + w_z(\hat{y}), \hat{y}),$$

since w_z and its gradient vanish at $\hat{y} = 0$. $\square$

A similar relation holds between the vector fields restricted to the boundary:

Lemma 4.5. *Using the same notation of Proposition 4.3 and setting*

$$\hat{Z}_i = \partial_{y_i} + \sum_{deg(j)>deg(i)} a_{ij}(0, \hat{y})\partial_{y_j},$$

we get:

$$d\hat{\Xi}_z(\hat{X}_i) = \hat{Z}_i + \hat{R}_{i,z,\Xi}$$

where $\hat{R}_{i,z,\Xi}$ are vector fields of local degree $\leq deg(i) - 1$ depending smoothly on $z \in W$.

Proof. We omit the proof which is exactly the same as the previous lemma. $\square$

4.3. **Properties of the fundamental solution and its approximating ones.** The vector fields $(X_i)_{i=1,\cdots,n}$ in (4.3), as well as their restriction to the boundary $(\hat{X}_i)_{i=2,\cdots,n}$, are in general non homogeneous in the variables x, but we have proved in the previous section that for every z their images through Ξ_z admit homogeneous approximating vectors fields. Then calling $X_{i,z} = d\Xi_z^{-1}(Z_i)$ for $i = 1, \cdots, n$, and applying the change of variable Ξ to the result of Lemma 4.4, we deduce that for every $i = 1, \cdots, n$ there exists an operator $R_{i,z}$ such that $R_{i,z} \leq deg(i) - 1$ and

$$X_i = X_{i,z} + R_{i,z}. \tag{4.7}$$

Calling $\hat{X}_{i,z} = d\hat{\Xi}_z^{-1}(\hat{Z}_i)$ for $i = 2, \cdots, n$, we obtain from Lemma 4.5 that for every $i = 2, \cdots, n$ there exists a vector field $\hat{R}_{i,z}$ such that $\hat{R}_{i,z} \leq deg(i) - 1$ and

$$\hat{X}_i = \hat{X}_{i,z} + \hat{R}_{i,z}.$$

The associated sub-Laplacian type operators are defined as

$$\Delta_z = \sum_{i=1}^m X_{i,z}^2, \quad \hat{\Delta}_z = \sum_{i=2}^m \hat{X}_{i,z}^2, \quad \Delta_Z = \sum_{i=1}^m Z_i^2, \quad \hat{\Delta}_Z = \sum_{i=2}^m \hat{Z}_i^2, \qquad (4.8)$$

with fundamental solutions $\Gamma_{z,\Delta}$, $\hat{\Gamma}_{z,\hat{\Delta}}$, Γ_{Δ_Z} and $\hat{\Gamma}_{\hat{\Delta}_Z}$ respectively. Note that Γ_{Δ_Z} and $\hat{\Gamma}_{\hat{\Delta}_Z}$ do not depend on the fixed point z.

We can now apply the parametrix method of [36], recalled in (2.24) and (2.25) to estimate the fundamental solutions Γ_Δ and $\hat{\Gamma}_{\hat{\Delta}}$, associated to the operators (4.5) and (4.6) respectively. The argument is similar to the one applied in Section 3 but in this case the proof is standard, since we do not have to take care of the different homogeneous dimensions of the Riemannian and subriemannian structures. Hence we state without proof the following lemma:

Lemma 4.6. *Let us consider the operators defined in (4.8). Then*

$$H = \Delta - \Delta_z \quad \hat{H} = \hat{\Delta} - \hat{\Delta}_z$$

are differential operators of degree 1. As a consequence

$$\Gamma_\Delta(x, z) - \Gamma_{z,\Delta}(x, z) = \Gamma_\Delta(x, z) - \Gamma_{\Delta_Z}(\Xi_z(x), 0)$$

are kernels of type 3, with respect to the vector fields X_i and the distance d. Analogously

$$\hat{\Gamma}_{\hat{\Delta}}(\hat{x}, \hat{z}) - \hat{\Gamma}_{z,\hat{\Delta}}(\hat{x}, \hat{z}) = \hat{\Gamma}_{\hat{\Delta}}(\hat{x}, \hat{z}) - \hat{\Gamma}_{\hat{\Delta}_Z}(\hat{\Xi}_{\hat{z}}(\hat{x}), 0)$$

are kernels of type 3 with respect to the vector fields $\hat{X}_i$ and the distance $\hat{d}$.

We will also denote by $(X_i)^*$ the formal adjoint of X_i.

Remark 4.7. *Let us note that for every $i = 1, \cdots, n$, the vector field X_i is no more self adjoint, but its formal adjoint differs from X_i by an operator of order 0. Indeed there exists a smooth function φ_i such that*

$$(X_i)^* = -X_i + \varphi_i, \quad i = 1, \cdots, n. \qquad (4.9)$$

Indeed

$$(X_i)^* = -X_i - \sum_{deg(j) > deg(i)} \sum_k \partial_{x_1} a_{i,j}(x_1 + w(\hat{x}), \hat{x}) \partial_{x_k} w.$$

In the sequel we will denote X_i^z the derivative with respect to z and X_i^x the one with respect to x of a kernel $K(x, z)$. From Proposition 5.10 in [9] (see also [44], page 295, line 3 from below) we have

Proposition 4.8. *Assume that $f \in C_0^\infty(\mathbb{R}^{n-1})$, and for $x \in \mathbb{R}^n$ define*

$$F(f)(x) := \int_{\mathbb{R}^{n-1}} \Gamma_\Delta(x, (0, \hat{y})) f(\hat{y}) d\hat{y}.$$

For every $i, h = 1, \cdots, m$ there exist kernels $\Gamma_{i,h}(x, y)$ and $S_i(x, y)$, of type 2 with respect to the distance d, such that

$$X_i F(f)(x) =$$

$$= -\int_{\mathbb{R}^{n-1}} \sum_{h=1}^m (X_h^y)^* \Gamma_{i,h}(x, (0, \hat{y})) f(\hat{y}) d\hat{y} - \int_{\mathbb{R}^{n-1}} S_i(x, (0, \hat{y})) f(\hat{y}) d\hat{y}.$$

Lemma 4.9. *Let $f \in C_0^\infty(\mathbb{R}^n)$. Let us call*

$$G(f)(x) := \int_{\mathbb{R}^n} \Gamma_\Delta(x, y) f(y) dy$$

Then there exists a kernel S of type 1 such that the operator $G_1(f) := G(\nabla f)$ can be represented as

$$G(\nabla f) = E_S(f),$$

where E_S is the operator with kernel S.

Proof. For $i = 1, \cdots, m$, we have

$$G(X_i f) = \int \Gamma_\Delta(x, z) X_i^z f(z) dz = \int (X_i^z)^* \Gamma_\Delta(x, z) f(z) dz.$$

Hence we only have to prove that the kernel

$$S := (X_i^z)^* \Gamma_\Delta(x, z)$$

is a kernel of type 1 with respect to the distance d. By (4.9) there exist regular functions φ_i such that

$$(X_i^z)^* = -X_i^z + \varphi_i.$$

On the other side, by (4.7) for every $i = 1, \cdots, n$ there exist and operator $R_{i,z}$ such that $deg(R_{i,z}) \le deg(i) - 1$ and

$$X_i^z = X_{i,z}^z + R_{i,z}^z.$$

Finally in [44], page 295, line 3 from below, it is proved that

$$X_{i,z}^z \Gamma_{z,\Delta}$$

is an kernel of type 1. Now we use the fact that $K = \Gamma_\Delta - \Gamma_{z,\Delta}$ is an operator of type 3, to conclude that

$$S = (X_i^z)^* \Gamma_\Delta = (-X_{i,z}^z - R_{i,z}^z + \varphi_i)(\Gamma_{z,\Delta} + K)$$

is a kernel of local type 1 with respect to the distance d_z associated with the vector fields $X_{i,z}$. On the other side as in Lemma 3.10, the distances d and d_z are equivalent, so that the conclusion follows. $\qquad\square$

4.4. **The reproducing formula for non homogeneous vector fields.** In this section we prove Theorem 1.3. The proof is obtained, via the results of the previous section, by reducing to the analogous result for homogeneous vector fields, already established in Lemma 3.16.

Proof of Theorem 1.3. By Lemma 4.6,

$$\hat{\Gamma}_{\hat{\Delta}}(\hat{x}, \hat{z}) - \hat{\Gamma}_{\hat{\Delta}_Z}(\hat{\Xi}_{\hat{z}}(\hat{x}), 0) \tag{4.10}$$

is a kernel of type 3 with respect to the vector fields $\hat{X}_i$. For the vector fields $(Z_i)_{i=1,\cdots,n}$ and the fundamental solution associated to the corresponding sub-Laplacian type operator, we can apply Lemma 3.16, so that

$$\hat{\Gamma}_{\Delta_Z}(\hat{\Xi}_{\hat{z}}(\hat{x}), 0) - \int_{\mathbb{R}^{n-1}} \Gamma_{\Delta_Z}((0, \hat{x}), (0, \hat{y}))\Gamma_{\Delta_Z}((0, \hat{y}), (0, \hat{z}))d\hat{y}$$

is a kernel of type 5/2 with respect to the vector fields $\hat{X}_{i,z}$. Using Lemma 4.5 we deduce that a kernel has the same type with respect to the vector fields $\hat{X}_i$ and $\hat{X}_{i,z}$. Inserting in (4.10) we get that

$$\hat{\Gamma}_{\hat{\Delta}}(\hat{x}, \hat{z}) - \int_{\mathbb{R}^{n-1}} \Gamma_{\Delta_Z}((0, \hat{x}), (0, \hat{y}))\Gamma_{\Delta_Z}((0, \hat{y}), (0, \hat{z}))d\hat{y} \tag{4.11}$$

is a kernel of type 5/2. Applying again Lemma 4.6 we deduce that the following difference,

$$\int_{\mathbb{R}^{n-1}} \Gamma_{\Delta_Z}((0, \hat{x}), (0, \hat{y}))\Gamma_{\Delta_Z}((0, \hat{y}), (0, \hat{z}))d\hat{y} - \int_{\mathbb{R}^{n-1}} \Gamma_{\Delta}((0, \hat{x}), (0, \hat{y}))\Gamma_{\Delta}((0, \hat{y}), (0, \hat{z}))d\hat{y} =$$

$$\int_{\mathbb{R}^{n-1}} \Gamma_{\Delta_Z}((0, \hat{x}), (0, \hat{y}))\Big(\Gamma_{\Delta_Z}((0, \hat{y}), (0, \hat{z}))d\hat{y} - \Gamma_{\Delta}((0, \hat{y}), (0, \hat{z}))\Big)d\hat{y} +$$

$$+ \int_{\mathbb{R}^{n-1}} \Big(\Gamma_{\Delta_Z}((0, \hat{x}), (0, \hat{y})) - \Gamma_{\Delta}((0, \hat{x}), (0, \hat{y}))\Big)\Gamma_{\Delta}((0, \hat{y}), (0, \hat{z}))d\hat{y},$$

is a kernel of type 3. As a consequence, we deduce from here and (4.11) that

$$\hat{\Gamma}_{\hat{\Delta}}(\hat{x}, \hat{z}) - \int_{\mathbb{R}^{n-1}} \Gamma_{\Delta}((0, \hat{x}), (0, \hat{y}))\Gamma_{\Delta}((0, \hat{y}), (0, \hat{z}))d\hat{y}$$

is a kernel of type 5/2. The proof is complete. $\qquad\square$

5. Poisson kernel and Schauder estimates at the boundary

In this section we will show the existence of a Poisson kernel for the Dirichlet problem, stated in Theorem 1.2. From this we deduce the Schauder estimates at the boundary stated in Theorem 1.1.

Consider a bounded smooth set D and a sub-Laplacian type operator Δ defined in D, as in (1.2), in terms of the homogeneous vector fields defined in (2.3). The corresponding Dirichlet problem is expressed as

$$\Delta u = f \text{ in } D, \quad u = g \text{ on } \partial D, \tag{5.1}$$

for a suitable boundary datum g and a smooth function f defined on D.

As mentioned in Section 4.1, we can locally perform a change of variable, and reduce the domain of the Dirichlet problem to the half space. Hence there is an open set $V \subset \mathbb{R}^n$ such that $D \cap V = \{x = (x_1, \hat{x}) \in V : x_1 > 0\}$ and $\{x_1 = 0\}$ is a non characteristic plane. Under this change of variable, the vector fields $\{X_i\}_{i=1,\cdots,m}$ will take the non homogeneous expression of (4.3). Their restriction to the boundary, denoted by $(\hat{X}_i)$ are defined in (4.4). They induce on the set ∂D a distance $\hat{d}$ defined in (3.27). The corresponding spaces of Hölder continuous functions, will be denoted $\hat{C}^{k,\alpha}$.

We look for a Poisson operator in a neighborhood V of a point $x_0 \in \partial D$. We say that $P : C^\infty(V \cap \partial D) \to C^\infty(V \cap \overline{D})$ is a local Poisson operator for the problem (5.1) if, for every $g \in C^\infty(V \cap \partial D)$, the function $u := P(g)$ satisfies $\Delta u = 0$ in $D \cap V$ and $u(x) = g(x)$ for all $x \in \partial D \cap V$. We will construct an approximate Poisson kernel of the Dirichlet problem, adapting to the present setting a method introduced by Greiner and Stein [30] and Jerison [34]. They used an approximating kernel, defined via pseudodifferential instruments, while we use here the kernel found in Theorem 1.3. We will denote it as follows:

$$\hat{\Gamma}_{\Delta^2}(\hat{x}, \hat{y}) := \int_{\mathbb{R}^{n-1}} \Gamma_\Delta((0,\hat{x}),(0,\hat{z}))\Gamma_\Delta((0,\hat{z}),(0,\hat{y}))d\hat{z}. \tag{5.2}$$

We will now apply the parametrix method presented in Section 2.3 to prove that $\hat{\Gamma}_{\Delta^2}(\hat{x}, \hat{y})$ is an approximation of the Poisson kernel. We first note that it is a kernel of type 2 with respect to the distance $\hat{d}$, since $\Gamma_\Delta((0,\hat{x}),(0,\hat{z}))$ is a kernel of type 1 with respect to same distance, as proved in Remark 3.15.

In analogy with (2.26) we call

$$R_1(\hat{x}, \hat{y}) := \hat{\Delta}\Big(\hat{\Gamma}_{\hat{\Delta}}(\hat{x}, \hat{y}) - \hat{\Gamma}_{\Delta^2}(\hat{x}, \hat{y})\Big).$$

By Theorem 1.3, $\hat{\Gamma}_{\hat{\Delta}} - \hat{\Gamma}_{\Delta^2}$ is a kernel of type 5/2, so that R_1 is a kernel of type 1/2 with respect to the distance $\hat{d}$. As in (2.27) we now call

$$\Phi(\hat{x}, \hat{y}) := \sum_{j=0}^{\infty} (E_{R_1})^j (R_1)(\hat{x}, \hat{y}).$$

We will now prove an uniform estimate for $(E_{R_1})^j(R_1)$, arguing as in Lemma 3.11.

Remark 5.1. *We prove by induction that*

$$|(E_{R_1})^j R_1(\hat{x}, \hat{y})| \leq C^j \frac{\hat{\Gamma}_{\hat{\Delta}}(\hat{x}, \hat{y})}{\hat{d}^{(3-j)/2}(\hat{x}, \hat{y})}. \tag{5.3}$$

The estimate for $j = 0$ is a consequence of the fact that R_1 is a kernel of type 1/2. Let us assume that estimate (5.3) holds for $j - 1 \in \mathbb{N}$. Using the expression of E_{R_1} we have

$$|(E_{R_1})^j R_1(\hat{x}, \hat{y})| \leq C^j \int_{\mathbb{R}^{n-1}} \left| R_1(\hat{x}, \hat{z}) (E_{R_1})^{j-1} R_1(\hat{z}, \hat{y}) \right| d\hat{z}$$

$$\leq C^j \int_{\mathbb{R}^{n-1}} \frac{\hat{\Gamma}_{\hat{\Delta}}(\hat{x}, \hat{z})}{\hat{d}^{3/2}(\hat{x}, \hat{z})} \frac{\Gamma_{\hat{\Delta}}(\hat{z}, \hat{y})}{\hat{d}^{(3-(j-1))/2}(\hat{z}, \hat{y})} d\hat{z} \leq C^j \frac{\hat{\Gamma}_{\hat{\Delta}}(\hat{x}, \hat{y})}{\hat{d}^{(3-j)/2}(\hat{x}, \hat{y})},$$

by the properties of the fundamental solution.

We deduce that the series uniformly converges on any bounded open set V_0 and it is a kernel of the same type as $\frac{\hat{\Gamma}_{\hat{\Delta}}(\hat{x},\hat{y})}{\hat{d}^{3/2}(\hat{x},\hat{y})}$, i.e. it is of type $1/2$. From here and Remark 3.15, which ensures that $\Gamma_\Delta((0,\hat{x}),(0,\hat{y}))$ is a kernel of type 1, we deduce that

$$\int_{\mathbb{R}^{n-1}\cap V_0} \Gamma_\Delta((0,\hat{x}),(0,\hat{z}))\Phi(\hat{z},\hat{y})d\hat{z} \text{ is of type } 3/2 \text{ with respect to the distance } \hat{d}$$

$$E_{\hat{\Gamma}_{\Delta^2}}(\Phi(\hat{x},\hat{y})) \text{ is of type } 5/2 \text{ with respect to the same distance,}$$

(5.4)

where $E_{\hat{\Gamma}_{\Delta^2}}$ denotes the operator with kernel $\hat{\Gamma}_{\Delta^2}$, and we have already noted that it is a kernel of type 2, and applied to Φ provides a kernel of type $5/2$. Again, in analogy with (2.25) we can write $\hat{\Gamma}_{\hat{\Delta}} - \hat{\Gamma}_{\Delta^2} = E_{\hat{\Gamma}_{\Delta^2}}\Phi$. Hence, the fundamental solution of the operator $\hat{\Delta}$ can be represented as

$$\hat{\Gamma}_{\hat{\Delta}}(\hat{x},\hat{y}) = \hat{\Gamma}_{\Delta^2}(\hat{x},\hat{y}) + E_{\hat{\Gamma}_{\Delta^2}}\Phi(\hat{x},\hat{y}),$$

(5.5)

for $\hat{x},\hat{y} \in V_0 \cap \mathbb{R}^{n-1}$.

Let us now prove Theorem 1.2 with

$$R(g)(\hat{y}) := \int_{\mathbb{R}^{n-1}\cap V_0} \int_{\mathbb{R}^{n-1}\cap V_0} \Gamma_\Delta((0,\hat{y}),(0,\hat{s}))\Phi((0,\hat{s}),(0,\hat{z}))\hat{\Delta}g(\hat{z})d\hat{s}d\hat{z}$$

and

$$K : \hat{C}^2(\partial D \cap V_0) \to \hat{C}(\partial D \cap V_0), \quad K = K_1 + R.$$

(5.6)

Proof of Theorem 1.2. Since we are proving a local property, it is not restrictive that the boundary datum g belongs to $C_0^\infty(\partial D \cap V_0)$. Since Γ_Δ is the fundamental solution of Δ, then the function $u = P(g)(x)$ satisfies $\Delta u = 0$ in $V \cap D$. Hence by (5.5), we have

$$P(g)(0,\hat{x}) = \int_{\mathbb{R}^{n-1}} \left(\hat{\Gamma}_{\Delta^2}(\hat{x},\hat{y}) + E_{\hat{\Gamma}_{\Delta^2}}(\Phi(\hat{x},\hat{y}))\right)\hat{\Delta}g(\hat{y})d\hat{y} =$$

$$= \int_{\mathbb{R}^{n-1}} \hat{\Gamma}_{\hat{\Delta}}(\hat{x},\hat{y}))\hat{\Delta}g(\hat{y})d\hat{y} = g(\hat{x}).$$

$\square$

Once proved the existence of a Poisson kernel, the proof of Schauder estimate is based on properties of singular integrals. We follow here the same ideas as in [34] and we prove that the operator P is bounded. Since it can be represented as in (1.10) we will start with the properties of K.

Let us first note that both K_1 and R can be extended to operators with values in $C(D \cap V)$ setting

$$K_1(g)(y) = \int_{\partial D \cap V_0} \Gamma_\Delta(y,(0,\hat{z}))\hat{\Delta}g(\hat{z})d\hat{z}.$$

$$R(g)(y) = \int_{\mathbb{R}^{n-1}\cap V_0} \int_{\mathbb{R}^{n-1}\cap V_0} \Gamma_\Delta(y,(0,\hat{s}))\Phi((0,\hat{s}),(0,\hat{z}))\hat{\Delta}g(\hat{z})d\hat{s}d\hat{z}.$$

As a consequence $K = K_1 + R$ will be considered as an operator acting between the following sets

$$K : \hat{C}^2(\partial D \cap V_0) \to C(D \cap V_0).$$

Remark 5.2. *Let us explicitly note that the spaces $C^{k,\alpha}$ associated with the vector fields X_i defined in (4.3) are equivalent to the spaces $C^{k,\alpha}$ associated with the vector fields*

$$\begin{aligned} Y_1 &= \partial_{x_1}, \\ Y_i &= \partial_{x_i} + \sum_{deg(j)>deg(i)} a_{i,j}(x_1 + w(\hat{x}), \hat{x})\partial_{x_j}, i = 1, \cdots, n \end{aligned} \tag{5.7}$$

since these vectors are linear combinations of the previous ones.

Lemma 5.3. *Let $D = \{(x_1, \hat{x}) \in \mathbb{R} \times \mathbb{R}^{n-1} : x_1 > 0\}$ be a half space with non characteristic boundary. Then for every $V \subset\subset V_0$ there is a constant C_1 such that for every $g \in \hat{C}^{2,\alpha}(\partial D \cap V_0)$*

$$\|K(g)\|_{C^{1,\alpha}(D\cap V)} \le C_1 \|g\|_{\hat{C}^{2,\alpha}(\partial D\cap V_0)}. \tag{5.8}$$

In addition there is a constant C_2 such that if $g \in C_0^\infty(\partial D \cap V_0)$, then

$$K(g) \in \left\{ \varphi : |\varphi(0, \hat{z})| \le C_2 \frac{\hat{d}(\hat{z}, \mathrm{supp}(g))}{|\hat{B}(\hat{z}, \hat{d}(\hat{z}, \mathrm{supp}(g)))|} \ \forall \hat{z} s.t. \ \hat{d}(\hat{z}, \mathrm{supp}(g)) \ge 2\,\mathrm{diam}(\mathrm{supp}(g)) \right\}.$$

Proof. Clearly $\Gamma_\Delta((0, \hat{z}), (0, \hat{y}))$ is a kernel of type 2 with respect to the distance d in the sense of Definition 4.1. Because of inequality (3.28) we deduce that there are constants C_1, C_2 such that

$$C_1 \frac{\hat{d}(\hat{z}, \hat{y})}{|\hat{B}(\hat{z}, d(\hat{z}, \hat{y}))|} \le \Gamma_\Delta((0, \hat{z}), (0, \hat{y})) \le C_2 \frac{\hat{d}(\hat{z}, \hat{y})}{|\hat{B}(\hat{z}, d(\hat{z}, \hat{y}))|} \tag{5.9}$$

so that $\Gamma_\Delta((0, \hat{z}), (0, \hat{y}))$ is a kernel of type 1 with respect to the distance $\hat{d}$ induced on ∂D, while the first derivatives of $\Gamma_\Delta((0, \hat{z}), (0, \hat{y}))$ are singular integrals. As a consequence we obtain (see for example [42])

$$\|E_{\Gamma_\Delta}(\varphi)\|_{C^{1,\alpha}(D\cap V)} \le C\|\varphi\|_{\hat{C}^\alpha(\partial D\cap V_0)}. \tag{5.10}$$

for every $\varphi \in \hat{C}^\alpha(\partial D \cap V_0)$, where E_{Γ_Δ} denotes the operator with kernel $\Gamma_\Delta(z, (0, \hat{y}))$. Therefore $K_1 = E_{\Gamma_\Delta} \circ \hat{\Delta}$ satisfies

$$\|K_1(g)\|_{C^{1,\alpha}(D\cap V)} \le C\|\hat{\Delta}g\|_{\hat{C}^\alpha(\partial D\cap V_0)} \le C\|g\|_{\hat{C}^{2,\alpha}(\partial D\cap V_0)}.$$

Since Φ is a kernel of type $1/2$, its associated operator E_Φ satisfies

$$\|E_\Phi(\hat{\Delta}g)\|_{\hat{C}^{\alpha+1/2}(\partial D\cap V_0)} \le C\|\hat{\Delta}g\|_{\hat{C}^\alpha(\partial D\cap V_0)} \le C\|g\|_{\hat{C}^{2,\alpha}(\partial D\cap V_0)}. \tag{5.11}$$

It follows that

$$\|R(g)\|_{C^{1,\alpha}(D\cap V)} = \|E_{\Gamma_\Delta} E_\Phi(\hat{\Delta}g)\|_{C^{1,\alpha}(D\cap V)} \le \|E_\Phi(\hat{\Delta}g)\|_{C^\alpha(D\cap V)} \le \|g\|_{\hat{C}^{2,\alpha}(\partial D\cap V_0)},$$

In particular (5.8) directly follows. Also the decay property of K immediately follows, since

$$d(\hat{z}, \hat{y}) \ge d(\hat{z}, \mathrm{supp}\, g)$$

for all $\hat{y} \in \mathrm{supp}\, g$ and for all $\hat{z}$ such that $\hat{d}(\hat{z}, \mathrm{supp}\, g) \ge 2\,\mathrm{diam}(\mathrm{supp}\, g)$. $\qquad\square$

Arguing as in Remark 5.2 we have the following

Remark 5.4. *There are C^∞ functions such that the Laplace type operator Δ can be expressed as*

$$\Delta = Y_1^2 + \sum_{i=2}^m (Y_i - Z_i w Y_1)^2 + b_1 Y_1 + \sum_{i=2}^m b_i (Y_i - Z_i w Y_1) =$$

$$= Y_1^2 + \sum_{i=2}^m (Y_i - Z_i w Y_1)^2 + \Big(b_1 - \sum_{i=2}^m Z_i w\Big) Y_1 + \sum_{i=2}^m b_i Y_i =$$

$$= (1 + \sum_{i=2}^m (Z_i w)^2) Y_1^2 + \sum_{i=2}^m Y_i^2 - \sum_{i=2}^m Z_i w (Y_i Y_1 + Y_1 Y_i)$$

$$+ \Big(b_1 - \sum_{i=2}^m Z_i w + \sum_{i=2}^m (Y_i - Z_i w Y_1) Z_i w\Big) Y_1 + \sum_{i=2}^m b_i Y_i. \tag{5.12}$$

In particular the coefficient $1 + \sum_{i=2}^m (Z_i w)^2$ of Y_1^2 is smooth and bounded from above and below by positive constants.

Let us now conclude the proof of the boundedness of P.

Theorem 5.5. *Let V, V_0 be open sets in $\mathbb{R}^n$, with $V \subset\subset V_0$, let $g \in \hat{C}^{2,\alpha}(\partial D \cap V_0)$. Then there is a constant C_1 such that*

$$\|P(g)\|_{C^{2,\alpha}(D \cap V)} \le C_1 \|g\|_{\hat{C}^{2,\alpha}(\partial D \cap V_0)}. \tag{5.13}$$

Proof. Let us fix V_1 such that $V \subset\subset V_1 \subset\subset V_0$. Thanks to the previous lemma we only have to prove that the operator

$$\widetilde{K} : C^{1,\alpha}(D \cap V_0) \cap \left\{ \varphi : |\varphi(0, \hat{z})| \le C \frac{\hat{d}(\hat{z}, \mathrm{supp}(g))}{|\hat{B}(\hat{z}, \hat{d}(\hat{z}, \mathrm{supp}(g)))|}, \right.$$

$$\left. \forall \hat{z} \text{ s.t. } \hat{d}(\hat{z}.\mathrm{supp}(g)) \ge 2 \, \mathrm{diam}(\mathrm{supp}(g)) \right\} \to C^{2,\alpha}(D \cap V_0)$$

defined as

$$\widetilde{K}(\varphi)(x) := \int_{\mathbb{R}^{n-1}} \Gamma_\Delta(x, (0, \hat{z})) \varphi(0, \hat{z}) d\hat{z}$$

satisfies

$$\|\widetilde{K}(\varphi)\|_{C^{2,\alpha}(D \cap V)} \le \|\varphi\|_{C^{1,\alpha}(D \cap V_0)}. \tag{5.14}$$

It is standard to recognize that for every $i, j = 2, \cdots, m$, $Y_i Y_j \widetilde{K}$ it is bounded as operator with values in $C^\alpha(D \cap V_0)$ (see for example [30], [42]).

Hence we have to estimate the normal derivative. Let us begin with the derivatives $Y_i Y_1 \widetilde{K}$ with $i = 2, \cdots, m$. Let $\psi \in C_0^\infty(V_1)$ such that $\psi \equiv 1$ in a neighborhood of V and

let $x \in V$ and $\varphi \in C^{1,\alpha}(D \cap V_0)$. By Proposition 4.8 there exist kernels $(\Gamma_{1,i}(x,y))_{i=1,\cdots,m}$, $S_1(x,y)$ of type 2 such that

$$\partial_1 \widetilde{K}(\varphi)(x) = \int_{\mathbb{R}^{n-1}} \sum_{i=1}^{m} (Y_i^z)^* \Gamma_{1,i}(x,(0,\hat{z}))\varphi(0,\hat{z})d\hat{z} + \int_{\mathbb{R}^{n-1}} S_1(x,(0,\hat{z}))\varphi(0,\hat{z})d\hat{z}$$

$$= -\int_{\mathbb{R}^{n-1}} \partial_1^z \Gamma_{1,1}(x,(0,\hat{z}))\varphi(0,\hat{z})d\hat{z}$$

$$- \int_{\mathbb{R}^{n-1}} \sum_{i=2}^{m} \Gamma_{1,i}(x,(0,\hat{z}))Y_i^z\varphi(0,\hat{z})d\hat{z} + \int_{\mathbb{R}^{n-1}} S_1(x,(0,\hat{z}))\varphi(0,\hat{z})d\hat{z}.$$

Let us estimate the first term, using the fact that $\partial_1^z = \partial_\nu^z$,

$$\int_{\mathbb{R}^{n-1}} \partial_1^z \Gamma_{1,1}(x,(0,\hat{z}))\varphi(0,\hat{z})d\hat{z}$$

$$= -\int_{\mathbb{R}^{n-1}\cap V_1} <\nu, \nabla\Gamma_{1,1}(x,(0,\hat{z}))> \varphi(0,\hat{z})\psi(0,\hat{z})d\hat{z}$$

$$- \int_{\mathbb{R}^{n-1}} <\nu, \nabla\Gamma_{1,1}(x,(0,\hat{z}))> \varphi(0,\hat{z})(1-\psi(0,\hat{z}))d\hat{z}$$

$$= -\int_{V_1\cap D} <\nabla\Gamma_{1,1}(x,z), \nabla(\varphi\psi)(z)> dz$$

$$- \int_{\mathbb{R}^{n-1}\backslash V_1} <\nu, \nabla\Gamma_{1,1}(x,(0,\hat{z}))> \varphi(0,\hat{z})(1-\psi(0,\hat{z}))d\hat{z}.$$

If $x \in V$ the last integral contains a C^∞ kernel since $\psi = 1$, on a closed set which contains V in the interior. Thus, applying standard singular integral theory to all terms in the expression of $\partial_1 \widetilde{K}$ we obtain

$$\|\partial_1 \widetilde{K}(\varphi)\|_{C^{1,\alpha}(D\cap V)} \leq C\|\varphi\|_{C^{1,\alpha}(D\cap V_0)}.$$

Analogously for every $i = 2, \cdots, m$ we have

$$\|\partial_1 Y_i \widetilde{K}(\varphi)\|_{C^\alpha(D\cap V)} \leq C\|\varphi\|_{C^{1,\alpha}(D\cap V_0)}.$$

Finally we note that $\Delta\widetilde{K}(x,(0,\hat{y})) = 0$, consequently the estimate of $Y_1^2\widetilde{K}$ follows by difference from the estimates of all the other second derivatives and the expression (5.12).

Assertion (5.14) is proved, so that the thesis follows. $\qquad\square$

From here it immediately follows:

Corollary 5.6. *Assume that the same assumptions as in Theorem 5.5 are satisfied. If $V \subset\subset V_0$, $k \in \{0,1\}$, $f \in C^{k,\alpha}(V_0)$, and*

$$G(f) := E_{\Gamma_\Delta}(f) - P((E_{\Gamma_\Delta}(f))_{|\partial D\cap V_0}),$$

there exists a constant C such that

$$\|G(f)\|_{C^{2,\alpha}(V)} \leq C\|f\|_{C^\alpha(V_0)} \quad and \quad \|G(\nabla f)\|_{C^{k+1,\alpha}(V)} \leq C\|f\|_{C^{k,\alpha}(V_0)}. \tag{5.15}$$

Proof. The first inequality follows from properties of singular integrals (see [47]) and the boundedness of P established in Theorem 5.5. The last inequality follows applying Lemma 4.9. Indeed there exists a kernel S of type 1 such that

$$E_{\Gamma_\Delta}(\nabla f) = E_S(f).$$

Consequently

$$G(\nabla f) = E_S(f) - P((E_S(f))_{|\partial D \cap V_0}),$$

and the assertion follows at once. $\qquad\square$

Let $D = \{(x_1, \hat{x}) \in \mathbb{R} \times \mathbb{R}^{n-1} : x_1 > 0\}$ be a half space as above and consider the problem

$$\begin{cases} \Delta u = f & \text{in } D, \\ \quad u = g & \text{on } \partial D. \end{cases} \tag{5.16}$$

From Theorem 1.2 next theorem easily follows .

Theorem 5.7. *If $f \in C_0^\infty(V_0)$ and $g \in C_0^\infty(\partial D \cap V_0)$ and*

$$G(f) = E_{\Gamma_\Delta}(f) - P(E_{\Gamma_\Delta}(f))_{|\partial D \cap V_0}),$$

then the function $u = G(f) + P(g)$ solves the problem

$$\Delta u = f \text{ in } D, \quad u = g \text{ on } \partial D \cap V_0.$$

As a consequence of the previous theorem, we immediately get an approximate representation formula for a smooth function u.

Lemma 5.8. *Let $V \subset\subset V_0$ and let $u \in C_0^\infty(V)$. Let us call $\Delta u = f$ and $g = u_{|\partial D \cap V_0}$, and let $\varphi \in C_0^\infty(V_0)$, $\varphi = 1$ on V. Then*

$$u = \varphi v + E_{\Gamma_\Delta}\left(f(1-\varphi) + v\sum_{i=1}^m b_i X_i \varphi\right) - E_S(v\nabla\varphi), \tag{5.17}$$

where $v = G(f) + P(g)$ and b_i are the coefficients of the operator Δ in (1.2).

Proof. Setting $v = G(\Delta u) + P(u|_{\partial D \cap V_0})$ we have by Theorem 5.7

$$\begin{cases} \Delta(u - \varphi v) & = f(1-\varphi) + \nabla v \nabla \varphi + v\Delta\varphi & \text{in } V_0 \cap D, \\ u - \varphi v & = 0 & \text{on } \partial(V_0 \cap D). \end{cases}$$

where we have extended $u - \varphi v$ on the whole space with 0. We deduce by (1.2)

$$u = \varphi v + E_{\Gamma_\Delta}\left(f(1-\varphi) + \nabla v \nabla \varphi + v\Delta\varphi\right) =$$

$$= \varphi v + E_{\Gamma_\Delta}\left(f(1-\varphi) + v\sum_i b_i X_i \varphi\right) - E_{\Gamma_\Delta}(\nabla(v\nabla\varphi)).$$

Now applying Lemma 4.9 we obtain

$$u = \varphi v + E_{\Gamma_\Delta}\left(f(1-\varphi) + v\sum_i b_i X_i\, \varphi\right) - E_S(v\nabla\varphi).$$

$\qquad\square$

5.1. **Schauder estimates.** We can now complete the proof of the Schauder estimates, stated in the introduction:

Proof of Theorem 1.1. Let u be a solution of $\Delta u = f$ and $u_{|\partial D} = g$. We will prove the a priori estimates for u under the assumption that $f \in C^\infty(\bar{D})$, $g \in C^\infty(\partial D)$ and we will obtain the thesis for $f \in C^\alpha(\bar{D})$, $g \in \hat{C}^{2,\alpha}(\partial D)$ by a density argument. For smooth data, by [37] there exists a unique solution $u \in C^\infty(D)$, smooth up to the boundary at non characteristic points.

We first note that
$$\|u\|_\infty \le \|g\|_\infty$$
via the maximum principle. In addition, extending g in the interior of D to a function of class $C^{2,\alpha}$ such that $\|g\|_{C^{2,\alpha}(D)} \le \|g\|_{\hat{C}^{2,\alpha}(\partial D)}$, we see that $u - g$ is a solution of $\Delta(u - g) = f - \Delta g$ in D and $u - g = 0$ on ∂D, hence the Moser iteration technique (see [41]) ensures that there exists a value β such that $u - g \in C^\beta(\bar{D})$, and
$$\|u\|_{C^\beta(\bar{D})} \le C\big(\|f\|_{C^\alpha(\bar{D})} + \|g\|_{\hat{C}^{2,\alpha}(\partial D)}\big). \tag{5.18}$$
We can choose a non characteristic point, say $0 \in \partial D$, and denote V_0 a neighborhood of 0 such that the subriemannian normal
$$\nu(s) \ne 0 \text{ for every } s \in \partial D \cap V_0.$$
Then we can perform the change of variable described in Section 4.1 on a set $V \subset\subset V_0$. Through this change of variables the vector fields X_i can be represented as in (4.3)
$$d\Xi(X_1) = \partial_{x_1}, \quad d\Xi(X_i) = \partial_{x_i} + \sum_{deg(j) > deg(i)} a_{i,j}(x_1 + w(\hat{x}), \hat{x})\partial_{x_j} + X_i w(\hat{x})\partial_{y_1},$$
so that the results of the previous section apply. Let $\varphi \in C_0^\infty(V)$, let V_1 be an open set such that $V \subset\subset V_1$, $\varphi_1 \in C_0^\infty(V_1)$ and identically 1 on V. Define
$$v := G(\Delta(\varphi u)) + P((\varphi u)_{|\partial D \cap V}) = G\Big(f\varphi + \nabla\varphi\nabla u + \Delta\varphi u\Big) + P((\varphi u)_{|\partial D \cap V}). \tag{5.19}$$
By (5.17) we get
$$\varphi u = \varphi_1 v + E_{\Gamma_\Delta}\Big(f(1 - \varphi_1) - v\sum_i b_i X_i \varphi_1\Big) + E_S(v\nabla\varphi_1). \tag{5.20}$$
Then, from previous expressions and using (5.15), for nested open sets $V \subset\subset V_3 \subset\subset V_2 \subset\subset V_1 \subset\subset V_0$ and for every $\gamma \le \alpha$ we get that
$$\|\varphi u\|_{C^{1,\gamma}(V_3 \cap D)} \le C\big(\|v\|_{C^{1,\gamma}(V_2 \cap D)} + \|f\|_{C^\alpha(V_2 \cap D)}\big) \tag{5.21}$$
$$\le C\big(\|u\|_{C^\gamma(V_1 \cap D)} + \|f\|_{C^\alpha(\bar{D})} + \|g\|_{\hat{C}^{2,\alpha}(\partial D)}\big).$$
In particular, using this inequality and the uniform estimate of $\|\varphi u\|_{C^\beta(\bar{D})}$ provided by (5.18) we get for $V \subset\subset V_4 \subset\subset V_3$
$$\|\varphi u\|_{C^\alpha(V_4 \cap D)} \le C\|\varphi u\|_{C^{1,\beta}(V_3 \cap D)} \le C\big(\|f\|_{C^\alpha(\bar{D})} + \|g\|_{\hat{C}^{2,\alpha}(\partial D)}\big).$$

Having an estimate of $\|\varphi u\|_{C^\alpha}$ we apply again (5.18) with $\gamma = \alpha$ and we have for $V \subset\subset V_5 \subset\subset V_4$

$$\|\varphi u\|_{C^{1,\alpha}(V_5 \cap D)} \leq C\big(\|f\|_{C^\alpha(\bar{D})} + \|g\|_{\hat{C}^{2,\alpha}(\partial D)}\big).$$

Finally, we iterate the same argument applying again (5.15) to (5.19) and (5.20). Therefore we get

$$\|\varphi u\|_{C^{2,\alpha}(V \cap D)} \leq C\big(\|\varphi u\|_{C^{1,\alpha}(V_5 \cap D)} + \|f\|_{C^\alpha(\bar{D})} + \|g\|_{\hat{C}^{2,\alpha}(\partial D)}\big) \leq$$
$$\leq C\big(\|f\|_{C^\alpha(\bar{D})} + \|g\|_{\hat{C}^{2,\alpha}(\partial D)}\big).$$

This concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

6. An example

To clarify our approach in finding a Poisson kernel, we apply it to the special case of the Heisenberg group $\mathbb{H}^n$.

6.1. **The Heisenberg group.** The Heisenberg group $\mathbb{H}^n$, $n \geq 2$, can be identified with $\mathbb{R}^{2n+1}$ with the choice of vector fields

$$X_1 = \partial_{x_1}, \ X_2 = \partial_{x_2} + x_1 \partial_{x_{2n+1}}$$
$$X_i = \partial_{x_i} - \frac{1}{2} x_{i+1} \partial_{x_{2n+1}} \text{ if } i \text{ is odd} \quad X_i = \partial_{x_i} + \frac{1}{2} x_{i-1} \partial_{x_{2n+1}} \text{ if } i \text{ is even}$$

with $i \in \{3, \cdots, 2n\}$. Note that, as in Section 2.1, we are using exponential canonical coordinates of second type around a fixed point, so that the vector fields exhibit the structure (2.3). They satisfy the condition

$$[X_i, X_{i+1}] = \partial_{x_{2n+1}} =: X_{2n+1}$$

if i is odd, so that the Hörmander condition is satisfied. The expression of the gauge norm on the space $\mathbb{H}^n$, according to definition (2.5), is

$$\|x\| = \sum_{i=1}^{2n} |x_i| + |x_{2n+1}|^{1/2}.$$

The homogeneous dimension of the space according to definition (2.7) is $Q = 2n + 2$, so that there exist constants C_1, C_2 such that

$$C_1 r^{2n+2} \leq |B(x,r)| \leq C_2 r^{2n+2} \qquad \forall \, r > 0.$$

In order to provide an example, we consider here the heat operator

$$L = \partial_t - \sum_{i=1}^{2n} X_i^2,$$

where we have discarded the first order terms present in (2.18). The fundamental solution of the heat kernel has been found by many authors (see [28], [2]). Due to invariance with respect to the group law, we will consider the expression of the fundamental solution with pole in $(0,0)$. In particular, using (2.22), the estimate of the distance, and the estimate

of the measure of the ball we find the following Gaussian estimate of the fundamental solution. There exist constants $C_0, C_1 > 0$ such that for each $x \in \mathbb{H}^n$ and $t > 0$ one has

$$C_0^{-1} \frac{e^{-\frac{C_1}{t}\left(\sum_{i=1}^{2n} |x_i|^2 + |x_{2n+1}|\right)}}{t^{n+1}} \le \Gamma((x,t),(0,0)) \le C_0 \frac{e^{-\frac{1}{C_1 t}\left(\sum_{i=1}^{2n} |x_i|^2 + |x_{2n+1}|\right)}}{t^{n+1}}. \qquad (6.1)$$

We refer to [28] for an asymptotic pointwise estimate of the heat kernel in a neighborhood of the pole.

6.2. **Restriction of the heat kernel to a non characteristc plane.** As in (3.1), we consider the manifold

$$M = \{x \in \mathbb{H}^n \ : \ x_1 = 0\}.$$

The plane is non characteristic, and the vector field $X_1 = \partial_1$ coincides with the direction normal to the plane. The generators of the first layer of the tangent space of the plane can be represented, according to (3.2), as the restrictions of the operators X_i to the tangent plane to M. In this way we obtain the vector fields

$$\hat{X}_2 = \partial_{x_2}, \quad \hat{X}_i = \partial_{x_i} - \frac{1}{2} x_{i+1} \partial_{x_{2n+1}} \text{ if } i \text{ is odd} \quad \hat{X}_i = \partial_{x_i} + \frac{1}{2} x_{i-1} \partial_{x_{2n+1}} \text{ if } i \text{ is even},$$

$i \ge 3$. Note that $\hat{X}_2$ commutes with all the other vector fields, while the vector fields $(\hat{X}_i)_{i=3,\cdots,2n}$ generate an Heisenberg algebra $\mathfrak{h}^{n-1}$. As a consequence M coincides with $\mathbb{H}^{n-1} \times \mathbb{R}$. In particular the assumption (1.5) is satisfied, and our result can be applied. The homogeneous dimension of the plane, defined in (3.3), becomes $\hat{Q} = Q - 1 = 2n + 1$, and, denoting by $\hat{x} = (x_2, \cdots, x_{2n+1})$ a point of the plane, the induced norm becomes

$$\|\hat{x}\| = \sum_{i=2}^{2n} |x_i| + |x_{2n+1}|^{1/2}.$$

The tangential heat operator is represented as

$$\hat{L} = \partial_t - \sum_{i=2}^{2n} \hat{X}_i^2,$$

and it has a non negative fundamental solutions $\hat{\Gamma}$. From the Gaussian estimates of the fundamental solution we obtain the existence of constants C_0 and C_1 such that

$$C_0^{-1} \frac{e^{-\frac{C_1}{t}\left(\sum_{i=2}^{2n} |x_i|^2 + |x_{2n+1}|\right)}}{t^{n+1/2}} \le \hat{\Gamma}((\hat{x},t),(0,0)) \le C_0 \frac{e^{-\frac{1}{C_1 t}\left(\sum_{i=2}^{2n} |x_i|^2 + |x_{2n+1}|\right)}}{t^{n+1/2}}. \qquad (6.2)$$

Putting together estimates (6.1) and (6.2) we obtain a relation between of the restriction of Γ to the plane M and $\hat{\Gamma}$:

Remark 6.1. *There exist constants $C_0, C_1 > 0$ such that for each $\hat{x} \in M$ and $t > 0$ one has*

$$C_0 \frac{\hat{\Gamma}((\hat{x},t),(0,0))}{t^{1/2}} \le \Gamma((0,\hat{x},t),(0,0)) \le C_1 \frac{\hat{\Gamma}((\hat{x},t),(0,0))}{t^{1/2}}. \qquad (6.3)$$

Theorem 3.2 can be considered a refined version of this inequality, with a precise estimate of the difference

$$\Gamma((0,\hat{x},t),(0,0)) - \frac{\hat{\Gamma}((\hat{x},t),(0,0))}{t^{1/2}} .$$

6.3. **Restriction of the Laplace fundamental solution to a non characteristc plane.** The next step in our proof is to integrate in time, and obtain from Theorem 3.2 an estimate of the restriction to the plane of the fundamental solution of the Laplace equation, contained in Theorem 1.3.

The Laplace operator on the whole space $\mathbb{H}^n$ and the Laplacian on the plane can be represented respectively as

$$\Delta = \sum_{i=1}^{2n} X_i^2 \text{ and } \hat{\Delta} = \sum_{i=2}^{2n} \hat{X}_i^2,$$

As in the whole paper, we again denote their fundamental solutions by Γ_Δ and $\hat{\Gamma}_{\hat{\Delta}}$. In particular Γ_Δ satisfies the estimate

$$\frac{C_0^{-1}}{\left(\sum_{i=1}^{2n}|x_i|^2 + |x_{2n+1}|\right)^{Q-2}} \leq \Gamma_\Delta(x,0) \leq \frac{C_0}{\left(\sum_{i=1}^{2n}|x_i|^2 + |x_{2n+1}|\right)^{Q-2}},$$

(see (1.3)). Restricting on the plane we obtain

$$\frac{C_0^{-1}}{\|\hat{x}\|^{\hat{Q}-1}} = \frac{C_0^{-1}}{\|\hat{x}\|^{Q-2}} \leq \Gamma_\Delta((0,\hat{x}),(0,0)) \leq \frac{C_0}{\|\hat{x}\|^{Q-2}} = \frac{C_0}{\|\hat{x}\|^{\hat{Q}-1}}$$

We recall from Section 2 that the convolution of a kernel of type α with a kernel of type β provides a kernel of type $\alpha + \beta$. More precisely, for every α, β there exists a constant C_1 such that

$$\int_M \frac{1}{\|\hat{x}-\hat{y}\|^{\hat{Q}-\alpha}} \frac{1}{\|\hat{y}\|^{\hat{Q}-\beta}} d\hat{y} \leq C_1 \frac{1}{\|\hat{x}\|^{\hat{Q}-\alpha-\beta}} .$$

Denoting, as in equation (5.2),

$$\hat{\Gamma}_{\Delta^2}(x,\hat{y}) = \int_M \Gamma_\Delta(x,(0,\hat{z}))\Gamma_\Delta((0,\hat{z}),(0,\hat{y}))d\hat{z} ,$$

we immediately obtain that

$$\hat{\Gamma}_{\Delta^2}((0,\hat{x}),\hat{y}) \leq C_1 \frac{1}{\|\hat{x}\|^{\hat{Q}-2}} \leq C_1 \hat{\Gamma}_{\hat{\Delta}}(\hat{x},0) .$$

Theorem 1.3 provides a precise version of this estimate, proving that

$$\hat{\Gamma}_{\hat{\Delta}}(\hat{x},\hat{y}) = \hat{\Gamma}_{\Delta^2}((0,\hat{x}),\hat{y}) + O\Big(\big(\hat{d}(\hat{x},\hat{y})\big)^{1/2} \hat{\Gamma}_{\hat{\Delta}}(\hat{x},\hat{y})\Big). \tag{6.4}$$

6.4. **The Poisson Kernel.** As clarified in the description before Theorem 1.2 and proved in Section 5, equation (6.4) is the main step to prove that a Poisson kernel can be defined as in (1.10):

$$P(g)(x) = \int_M \hat{\Gamma}_{\Delta^2}(x, \hat{y}) \hat{\Delta} g(\hat{y}) d\hat{y}. \tag{6.5}$$

Indeed, by the properties of the fundamental solution, $\Delta P(g)(x) = 0$ for every smooth function g defined on the plane. Moreover, equality (6.4) ensures that the integral operator associated to $\hat{\Gamma}_{\Delta^2}((0, \hat{x}), \hat{y})$ is the inverse operator of the tangential Laplacian $\hat{\Delta}$. Then $Pg(0, \hat{x}) = g(\hat{x})$.

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