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ONE-DIMENSIONAL SYMMETRY FOR THE SOLUTIONS OF A THREE-DIMENSIONAL WATER WAVE PROBLEM

ELEONORA CINTI, PIETRO MIRAGLIO, AND ENRICO VALDINOCI

Abstract. We prove a one-dimensional symmetry result for a weighted Dirichlet-to-Neumann problem arising in a model for water waves in dimension 3. More precisely we prove that minimizers and bounded monotone solutions depend on only one Euclidean variable. The analogue of this result for the 2-dimensional case (and without weights) was established in [16]. In this paper a crucial ingredient in the proof is given by an energy estimate for minimizers obtained via a comparison argument.

1. Introduction

1.1. **A water wave model.** In this paper we establish one-dimensional symmetry results for solutions of a Dirichlet to Neumann problem which arise in a model for water waves.

A classical water wave model is that of considering an ideal fluid with density ϱ and velocity V in the spatial region $\mathbb{R}^2\times[0,H]$ (that is the "sea", which is assumed to be of depth $H>0$). For convenience, one can endow $\mathbb{R}^2\times[0,H]$ with coordinates $x\in\mathbb{R}^2$ and $y\in[0,H]$ (we consider the level $\{y=H\}$ as the "bottom of the sea" and the level $\{y=0\}$ as the "surface of the sea"; in this notation, y represents the "depth of the sea"). The fact that the fluid is incompressible gives that $\operatorname{div}(\varrho V)=0$ and the irrotationality condition that $V=\nabla v$ in $\mathbb{R}^2\times(0,H)$. Assuming that the bottom of the sea is made of solid material, the impenetrability of the matter gives that the vertical velocity vanishes along $\{y=H\}$. Then, given the values of v along the surface of the sea (and denoting such datum by u), one is interesting in finding the vertical velocity on the surface, possibly weighted by the density of the fluid (this vertical velocity is, roughly speaking, responsible for the formation of a wave starting from the rest position of a "flat sea"). The problem turns out to be linear with respect to the datum u, so it is convenient to denote the vertical velocity on the surface by $\mathcal{L}u$ (with a minus sign that we introduce for later convenience). In this setting, writing $V:=(V_1,V_2,V_3)\in\mathbb{R}^3$, and denoting by v_y the derivative of v with respect to the vertical variable y, the problem can be formulated

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as

$$\begin{cases} 0 = \mathrm{div} V = \mathrm{div}(\varrho \nabla v) & \text{in } \mathbb{R}^2 \times (0, H) \\ u = v\big|_{y=0} \\ 0 = V_3\big|_{y=H} = v_y\big|_{y=H} \\ \mathcal{L}u = -\varrho v_y\big|_{y=0}. \end{cases} \tag{1.1}$$

When $\varrho := 1$ and $H \to +\infty$ (which is the case of a fluid with constant density and an "infinitely deep sea"), the problem in (1.1) is related to the square root of the Laplacian, see e.g. [13], but for finite values of H the operator described in (1.1) is nonlocal, but also not of purely fractional type. Hence, in the sequel, we normalize the domain by setting $H := 1$.

1.2. **Dirichlet to Neumann operators.** More specifically, in this paper, motivated by (1.1), we consider the slab $\mathbb{R}^n \times [0,1]$ with coordinates $x \in \mathbb{R}^n, y \in [0,1]$ and a function $u : \mathbb{R}^n \to \mathbb{R}$. We then consider $v = v(x,y)$ as the bounded extension of $u : \mathbb{R}^n \to \mathbb{R}$ in the slab $\mathbb{R}^n \times (0,1)$ satisfying the following problem with density y^a, where $a \in (-1,1)$:

$$\begin{cases} \mathrm{div}(y^a \nabla v) = 0 & \text{in } \mathbb{R}^n \times (0,1) \\ v(x,0) = u(x) & \text{on } \mathbb{R}^n \times \{y = 0\} \\ v_y(x,1) = 0 & \text{on } \mathbb{R}^n \times \{y = 1\}. \end{cases} \tag{1.2}$$

Then, in view of the physical description in (1.1), the problem in (1.2) naturally leads to the study of the Dirichlet to Neumann operator $\mathcal{L}_a$ given by

$$\mathcal{L}_a u(x) = -\lim_{y \to 0} y^a v_y(x,y)^1$$

We observe that the operator $\mathcal{L}_a$ is closely related to the fractional Laplacian $(-\Delta)^s$ with $s = (1-a)/2$, see e.g. [13], though it is not equal to any purely fractional operator. The case $a = 0$, which corresponds to v being the harmonic extension of u in $\mathbb{R}^n \times (0,1)$, was considered in [16], where the authors write explicitly the Fourier symbol of the operator $\mathcal{L}_0$ in this specific case and show that, for large frequencies, the Fourier symbol of $\mathcal{L}_0$ is asymptotic to the Fourier symbol of the half-Laplacian (observe that for $a = 0$ we have $s = 1/2$).

We study here the one-dimensional symmetry of certain bounded solutions to the problem

$$\mathcal{L}_a u = f(u) \qquad \text{in } \mathbb{R}^n, \tag{1.3}$$

where $f \in C^{1,\gamma}(\mathbb{R})$, with $\gamma > \max\{0, -a\}$, $n = 3$ and $a \in (-1,1)$. The results obtained extend a known result in dimension $n = 2$ with $a = 0$, obtained by de la Llave and the third author in [16]. With the extension (1.2),

[1] In this paper we will always work with the extended problem satisfied by v (see problem (1.4) below), hence we do not actually need to define the operator $\mathcal{L}_a$ nor to discuss under which conditions we have uniqueness for solutions to (1.2). We have chosen to introduce the Dirichlet-to-Neumann operator for the sake of completeness and to make a comparison with some well known related results for nonlocal equations involving the fractional Laplacian.

our problem can be formulated in the following local way:

$$\begin{cases} \operatorname{div}(y^a \nabla v) = 0 & \text{in } \mathbb{R}^n \times (0,1) \\ -\lim_{y\to 0} y^a v_y = f(u) & \text{on } \mathbb{R}^n \times \{y=0\} \\ v_y(x,1) = 0 & \text{on } \mathbb{R}^n \times \{y=1\}. \end{cases} \tag{1.4}$$

We will show that, if v is a minimizer (in the sense of Definition 1.1 below) or a bounded monotone solution for problem (1.4), then there exist a function $v_0 : \mathbb{R} \to \mathbb{R}$ and a vector $\omega \in S^2$ such that

$$v(x,y) = v_0(\omega \cdot x, y) \qquad \text{for every } x \in \mathbb{R}^3,\ y > 0.$$

In particular, the trace u of v on $\{y=0\}$ exhibits one-dimensional symmetry, i.e. it is a function of only one Euclidean variable. See also Chapter 3 of [4] for additional discussions.

It is interesting to point out that the results that we give here are new even in the case $a := 0$, corresponding to uniform density of the fluid. Nevertheless, we provided a general setting for the problem in (1.4) and we believe that such generality is worthwhile for a series of reasons:

- From a pure mathematical perspective, weights of the type y^a belong to the Mouckenhoupt[2] class A_2, see [23, 24], which plays a special interest in the analysis of partial differential equations with weights, since, in a sense, these weights constitute the fundamental example of nontrivial, possibly singular or degenerate, weights, for which a "good elliptic theory" is still possible;
- With respect to fractional operators, it is important to study different values of a, corresponding to different values of the fractional parameter (and, in this case, the value $a := 0$ is often a fundamental threshold dividing "local" and "nonlocal" behaviors at large scales, see e.g. Theorem 1.5 in [27]);
- In applications, weights of the type y^a can model laminated materials (see e.g. [14]) and, in the context of fluids, describe situations in which the density of the fluid only depends on the depth;
- In other situations, equations as in (1.4) can be related to models in biological mathematics, in which v represents for instance the density of a given population: in this setting, many real-world experiments have confirmed that different populations exhibit anomalous diffusion, and that the diffusion parameters vary from one species to another (see e.g. [21] and the references therein), therefore it is relevant for concrete models to study nonlocal equations for all the parameter values;
- Most importantly, in our perspective, different values of a allow us more easily to (at least formally) interpolate between classical partial

[2]As customary, one says that a weight w belongs to the Mouckenhoupt class A_2 if there exists $C > 0$ such that, for all balls B, it holds that

$$\fint_B w(x)\,dx \fint_B \frac{1}{w(x)}\,dx \leqslant C,$$

with $\fint$ denoting average. Roughly speaking, Mouckenhoupt weights may be singular or degenerate, but they cannot be "too singular or too degenerate", in an integral sense. Also, w belongs to A_2 if and only if so does $1/w$.

differential equations (in a sense, corresponding to the case $a \to 1$) and strongly nonlocal equations (corresponding to the case $a \to -1$). Hence, since the case $a := 0$ is of course extremely important to address, it is also crucial to comprise in the analysis all the values $a \in (-1, 1)$, so to develop a much better intuition of the problem and to permit the use of continuity and bifurcation methods. In this way, the investigation of nonlocal problems produces results for classical questions which would have not been available with other techniques. For instance, a very neat example in which fractional methods lead to new and important results in classical cases is embodied by the recent work [22], in which the Authors brilliantly exploit nonlocal tools developed e.g. in [28] and [12] to obtain symmetry result in a Peierls-Nabarro model;

- The investigation of fractional problems in the full range of the fractional exponent cases $s \in \left(\frac{1}{2}, 1\right)$, $s = \frac{1}{2}$ and $s \in \left(0, \frac{1}{2}\right)$ is also important to understand the different behaviors of the energy contributions (see e.g. [27]). As a matter of fact, typically, when $s = \in \left(\frac{1}{2}, 1\right)$, in spite of the nonlocal character of the problem, the major contribution is "of local type", in the sense that it comes from a very well delimited region of the space in which "all the action takes place". Conversely, when $s \in \left(0, \frac{1}{2}\right)$ the major contribution comes "from infinity" and long-range interactions become predominant. In this spirit, the case $s = \frac{1}{2}$ keeps a balance between these two energy tendencies and, in fact, when $s = \frac{1}{2}$ the energy contributions typically "repeat themselves at each diadic scale", and, in practice, this special additional invariance often produces logarithmic energy terms that are characteristic for the case $s = \frac{1}{2}$.

1.3. **Connection with the Allen-Cahn equation and a conjecture by De Giorgi.** As we mentioned above, our operator $\mathcal{L}_a$ is related to the s-Laplacian $(-\Delta)^s$, for $s := (1-a)/2$, and the fractional Laplacian can be seen as a Dirichlet-to-Neumann operator for a local problem in the halfspace $\mathbb{R}^{n+1}_+$. More precisely, Caffarelli and Silvestre in [13] proved that one can study a semilinear nonlocal problem of the form

$$(-\Delta)^s u = f(u) \qquad \text{in } \mathbb{R}^n, \tag{1.5}$$

by studying the associated local problem

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla v) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ -\lim\limits_{y\to 0} y^a v_y = f(u) & \text{on } \mathbb{R}^n \times \{y = 0\}. \end{cases} \tag{1.6}$$

The problem of finding one-dimensional symmetry results for monotone solutions to (1.5) is the counterpart, in the fractional setting, of a celebrated conjecture stated in 1978 by E. De Giorgi about bounded and monotone solutions of the classical Allen-Cahn equation $-\Delta u = u - u^3$, see [15]. The nonlinearity $u - u^3$, whose primitive (up to a sign) has a double well potential structure, arises in the study of phase transitions problem.

In dimension $n = 2$ the fractional De Giorgi conjecture has been proved in [11] for $s = \frac{1}{2}$ and in [10, 28, 30] for every $s \in (0, 1)$. The same result in dimension $n = 3$ has been established by Cabré and the first author in [6]

and [7] with respectively $s = \frac{1}{2}$ and $s \in (\frac{1}{2}, 1)$. Then, Savin proved in [26] the conjecture in dimensions $4 \leqslant n \leqslant 8$ for $s \in (\frac{1}{2}, 1)$ and with the additional assumption

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1. \tag{1.7}$$

Very recently, the conjecture has been proved in dimension $n = 3$ and with $s \in (0, \frac{1}{2})$ independently and with different methods by Dipierro, Farina and the third author in [18] (using an improvement of flatness result by [19]) and by Cabré, Serra and the first author in [8] (by a different approach which relies on some sharp energy estimates and a blow-down convergence result for stable solutions).

In another very recent result, Figalli and Serra proved in [22] the conjecture for monotone solutions of the half-Laplacian in dimension four without assumption (1.7) and we plan to further investigate this new method in the setting of water waves.

Focusing on the dimension that we take into account in this paper, i.e. $n = 3$, besides its obvious physical relevance we want to stress an important difference between the water wave problem and the fractional De Giorgi conjecture. As mentioned above, a different approach is needed to prove the one-dimensional symmetry of solutions to (1.5) when the parameter s crosses the value $\frac{1}{2}$. This is due to fact that the optimal energy estimates for solutions of (1.5) change depending whether s is above or below $1/2$, as shown in [7]. In particular, only when $s \in [1/2, 1)$ these energy estimates are enough to apply a Liouville type result and hence to obtain one-dimensional symmetry. As we are going to see, this does not happen in our case, since the framework is $\mathbb{R}^n \times (0, 1)$ and the weight y^a is integrable between 0 and 1. This fact gives us some energy estimates that do not depend on a and allows us to prove one-dimensional symmetry of certain solutions to (1.3) with the same method for all the powers $a \in (-1, 1)$, namely $s \in (0, 1)$.

For similar results in further dimensions, both in the classical and in the nonlocal case, see also [2, 3, 7, 10, 11, 20, 26, 28, 30].

We stress that these types of nonlocal or fractional problems usually present several sources of additional difficulties with respect to the classical cases, such as:

- Lack of explicit barriers and impossibility of performing straightforward calculations;
- Slow decay of the solutions at infinity;
- Long range interactions and contributions coming from infinity;
- Infinite energy amounts;
- Formation of new types of interfaces (such as "nonlocal minimal surfaces").

In general, we also stress that nonlocal operators may present important differences with respect to the classical ones, also at a very basic level (see e.g. the introductory discussion in Section 2.1 of [1]).

1.4. **Variational formulation.** As one can easily observe, problem (1.4) has a variational structure. Let $B_R \subset \mathbb{R}^n$ denote the ball of radius R

centered at 0, and C_R the cylinder

$$C_R := B_R \times (0,1). \tag{1.8}$$

The (localized) energy functional associated to problem (1.4) is given by

$$\mathcal{E}_R(v) = \int_{C_R} y^a |\nabla v|^2 \, dx \, dy + \int_{B_R \times \{y=0\}} G(v) \, dx,$$

where the associated potential G is such that $G' = -f$.

We can now give the definitions of *minimizer* and of *stable solution* for problem (1.4) (problem (1.3) respectively) in a standard way.

Definition 1.1. *We say that a bounded $C^1(\mathbb{R}^n \times (0,1))$ function v is a minimizer for* (1.4) *if*

$$\mathcal{E}_R(v) \leqslant \mathcal{E}_R(w)$$

for every $R > 0$ and for every bounded competitor w such that $v \equiv w$ on $\partial B_R \times (0,1)$.
We say that a bounded $C^1(\mathbb{R}^n)$ function u is a minimizer for (1.3) *if its extension v satisfying* (1.2) *is a minimizer for* (1.4)*.*

Definition 1.2. *We say that a bounded solution v of* (1.4) *is stable if*

$$\int_{\mathbb{R}^n \times [0,1]} y^a |\nabla \xi|^2 \, dx \, dy - \int_{\mathbb{R}^n \times \{y=0\}} f'(u) \xi^2 \, dx \geqslant 0$$

for every function $\xi \in C^1_0(\mathbb{R}^n \times [0,1])$.

We say that a bounded function u is a stable solution for (1.3) *if its extension v satisfying* (1.2) *is a stable solution for* (1.4)*.*

Clearly, if v is a minimizer for (1.4) then, in particular, it is a stable solution. As we will observe later on in Section 3 (see Remark 3.2), also a monotone solution is stable, hence stability is a weaker notion of both minimality and monotonicity.

The one-dimensional symmetry result in two dimensions for the particular case $a = 0$ obtained in [16] follows as a corollary of a more general result (see Theorem 1 in [16]), which states that a bounded monotone solution satisfying a certain energy estimate is necessarily one-dimensional.

More precisely, Theorem 1 in [16] requires the existence of a positive constant C such that:

$$\int_{C_R} |\nabla_x v(x,y)|^2 \, dx \, dy \leqslant CR^2, \tag{1.9}$$

where ∇_x denotes the gradient in the x-variables.

This is trivially true in the case $n = 2$, thanks to the fact that the gradient of v is bounded, by standard elliptic estimates (see [25]).

1.5. **Main results.** The first result of this paper generalizes Theorem 1 of [16] to the class of stable solutions. As it will become clear from the proof, this generalization in itself is not too difficult but it will be technically crucial for the purpose of this paper, and in particular to prove the one-dimensional symmetry of monotone solutions in $\mathbb{R}^3$. Therefore, we state explicitly this result as follows:

Theorem 1.3. *Let $f \in C^{1,\gamma}(\mathbb{R})$, with $\gamma > \max\{0, -a\}$, and let v be a bounded and stable solution of (1.4).*

Suppose that there exists $C > 0$ such that

$$\int_{C_R} y^a |\nabla_x v(x,y)|^2 \, dx \, dy \leqslant CR^2 \tag{1.10}$$

for any $R \geqslant 2$.

Then, there exist $v_0 : \mathbb{R}^2 \to \mathbb{R}$ and $\omega \in \mathrm{S}^{n-1}$ such that

$$v(x) = v_0(\omega \cdot x, y) \qquad \text{for any } (x,y) \in \mathbb{R}^{n+1}_+. \tag{1.11}$$

In particular, the trace u of v on $\{y = 0\}$ can be written as $u(x) = u_0(\omega \cdot x)$.

Moreover, $u_0' > 0$ or $u_0' \equiv 0$.

It is an open problem whether the energy estimate (1.10) holds for stable solutions. In the following two results, we establish it for minimizers and for monotone solutions that, as observed before, are in particular stable solutions.

Next result is an energy estimate for minimizers in any dimension n. This type of results are essential in order to check energy conditions as in (1.10) and so apply Theorem 1.3.

Theorem 1.4 (**Energy estimate for minimizers**). *Let $f \in C^{1,\gamma}(\mathbb{R})$, with $\gamma > \max\{0, -a\}$, and let v be a bounded minimizer for problem (1.4).*

Then, we have

$$\mathcal{E}_R(v) \leqslant CR^{n-1}, \tag{1.12}$$

for any $R \geqslant 2$.

When $n = 3$ we can prove the same estimate for bounded solutions whose traces on $\{y = 0\}$ are monotone in some direction.

Theorem 1.5 (**Energy estimate for monotone solutions for $n = 3$**). *Let $f \in C^{1,\gamma}(\mathbb{R})$, with $\gamma > \max\{0, -a\}$, and let v be a bounded solution of (1.4) with $n = 3$ such that its trace $u(x) = v(x, 0)$ is monotone in some direction.*

Then, we have

$$\mathcal{E}_R(v) \leqslant CR^2, \tag{1.13}$$

for any $R \geqslant 2$.

As a consequence of Theorems 1.3, 1.4, and 1.5 we deduce the following result, which can be seen as the main result of this paper and provides the one-dimensional symmetry for minimizers and monotone solutions of a three-dimensional water wave problem.

Theorem 1.6. *Let $f \in C^{1,\gamma}(\mathbb{R})$, with $\gamma > \max\{0, -a\}$ and let $n = 3$. Assume that one of the two following condition is satisfied:*

- *v is a bounded minimizer for problem (1.4);*
- *v is a bounded solution of (1.4) such that its trace $u(x) = v(x, 0)$ is monotone in some direction.*

Then, there exist $v_0 : \mathbb{R}^2 \to \mathbb{R}$ and $\omega \in S^2$ such that:

$$v(x,y) = v_0(\omega \cdot x, y) \qquad \text{for all } (x,y) \in \mathbb{R}^4_+.$$

In particular, the trace u of v on $\{y = 0\}$ can be written as $u(x) = u_0(\omega \cdot x)$.

1.6. **Technical comments and strategy of the proofs.** It is interesting to point out that the results of this paper are new not only in the three-dimensional case, but also in the two-dimensional case when $a \neq 0$. As mentioned above, in the two-dimensional case studied in [16] the energy estimate (1.10) follows easily by standard elliptic estimates which ensure that the gradient of any bounded solution to (1.4) is bounded. Of course, just using an L^∞ bound on the gradient of the solution, would imply that the energy in cylinders C_R grows like R^n, which, for $n = 3$ would not be enough to apply Theorem 1.3.

Our first energy estimate for minimizers (Theorem 1.4) is obtained via a comparison argument, similar to the one used in [2], based on the construction of a competitor which is constant in the smaller cylinder C_{R-1}.

The proof of the energy estimate for monotone solutions is, instead, more involved and it follows the strategy of [6, 7], in which a similar estimate is proved for the fractional Laplacian.

Observe that in Theorem 1.5 we restrict the statement to the case $n = 3$. This is due to the fact that, after taking the limit at $\pm\infty$ in the direction of monotonicity of the solution, we reduce our problem to the classification of stable solutions in one dimension less. Such a classification (more precisely the one-dimensional symmetry and the monotonicity of the limit functions) is known only in dimension 2.

We point out that several important differences arise comparing the settings in [16] and in [6, 7] with the one considered in this paper. In particular:

- In [16], only the two-dimensional case is taken into account (and only the nonsingular and nondegenerate case $a := 0$). The lower dimensionality assumption is important in [16] since it gives for free the appropriated bounds on the energy growth. Of course, the three-dimensional case considered here has a direct application to the real-world problem;
- In [6, 7], the case of purely fractional operators are taken into account, while the operators treated here are nonlocal, but nonfractional as well, and these special features require here, among the other technical bounds, new energy estimates and a new set of regularity results, that are tailored for the case under consideration. On the other hand, as a byproduct of the sharp energy estimates that we find, we are able to obtain symmetry results for all values of $a \in (-1, 1)$ (while the energy estimates in [6, 7] cannot be applied beyond the range $(-1, 0]$, thus reflecting the important difference between the water wave problem studied here and the fractional Laplace problem in [6, 7]).

1.7. **Organization of the paper.** The paper is organized as follows:

- In Section 2 we collect some preliminary results on regularity and gradient estimates for solutions to (1.4);
- In Section 3 we give the proof of Theorem 1.3, which is based on two preliminary results: a characterization of stability (Lemma 3.1) and a Liouville type theorem (Lemma 3.3);

- In Section 4 we prove the energy estimate for minimizers (Theorem 1.4);
- In Section 5 we prove the energy estimate for monotone solutions (Theorem 1.5) which needs several ingredients (mainly Lemma 5.2 and Lemma 5.6).

2. Regularity results and gradient bounds for solutions to (1.4)

In this section we collect some regularity results and gradient estimates for solutions to problem (1.4).

We start by observing that the weight y^a, with $a \in (-1,1)$ belongs to the so-called Mouckenhoupt class A_2 and hence the theory developed by Fabes, Jerison, Kenig, and Serapioni [23, 24] applies to the operator $\operatorname{div}(y^a\nabla)$.

More precisely in [23, 24] a Poincaré inequality, a Harnack inequality, and the Hölder regularity for weak solutions of $\operatorname{div}(y^a\nabla) = 0$ are established . This theory gives interior regularity for solutions of our problem (1.4). In the sequel we will need regularity up to the boundary $\{y = 0\} \cup \{y = 1\}$ and some global L^∞ estimates for the derivatives of solutions to (1.4). For these results some care is needed, due to the presence of the weight y^a.

We define the weighted Sobolev spaces (recall (1.8))

$$L^2(C_R, y^a) := \{v : C_R \to \mathbb{R} \,|\, y^a v^2 \in L^1(C_R)\}.$$

$$H^1(C_R, y^a) := \{v : C_R \to \mathbb{R} \,|\, y^a(v^2 + |\nabla v|^2) \in L^1(C_R)\}.$$

In the sequel we will consider the following localized (in the x-variable) linear problem

$$\begin{cases} \operatorname{div}(y^a\nabla v) = 0 & \text{in } C_R \\ \partial_y v = 0 & \text{on } B_R \times \{y = 1\} \\ -y^a\partial_y v = g & \text{on } B_R \times \{y = 0\}. \end{cases} \tag{2.1}$$

We start by giving the definition of weak solution for (2.1).

Definition 2.1. *Let $R > 0$, and let $g \in L^1(B_R)$. We say that a function $v \in H^1(C_R, y^a)$ is a weak solution of problem (2.1) if*

$$\int_{C_R} y^a \nabla v \cdot \nabla \xi \, dx\, dy - \int_{B_R \times \{y=0\}} g\, \xi \, dx = 0,$$

for every $\xi \in C_0^\infty(B_R \times [0,1])$.

Later on, we will need the following duality principle which is the analogue, for our problem, of Proposition 3.6 in [9] (see also [13]).

Lemma 2.2. *Let $g \in C(\mathbb{R}^n)$, $v \in C^2(\mathbb{R}^n \times (0,1))$, and $y^a\partial_y v \in C(\mathbb{R}^n \times [0,1])$. If v is a classical solution of*

$$\begin{cases} \operatorname{div}(y^a\nabla v) = 0 & \text{in } \mathbb{R}^n \times (0,1) \\ \partial_y v = 0 & \text{on } \mathbb{R}^n \times \{y = 1\} \\ -y^a\partial_y v = g & \text{on } \mathbb{R}^n \times \{y = 0\}, \end{cases}$$

then the function $w = -y^a \partial_y v$ is a classical solution of the Dirichlet problem

$$\begin{cases} \operatorname{div}(y^{-a}\nabla w) = 0 & \text{in } \mathbb{R}^n \times (0,1) \\ w = 0 & \text{on } \mathbb{R}^n \times \{y = 1\} \\ w = g & \text{on } \mathbb{R}^n \times \{y = 0\}. \end{cases}$$

The result in Lemma 2.2 follows by a simple computation and we refer to [13] for its proof.

We can now give a regularity result for the localized linear problem (2.1).

Proposition 2.3. *Let $g \in L^\infty(B_R)$ and let v be a bounded weak solution of (2.1).*

Then, there exists $\beta \in (0,1)$ (depending only on n and a) such that $v \in C^\beta(C_{R/2})$ with the following estimate

$$\|v\|_{C^\beta(\overline{C_{R/2}})} \leqslant c_R^1,$$

for some c_R^1 depending on n, a, R, $\|g\|_{L^\infty(B_R)}$, $\|v\|_{L^\infty(C_R)}$.

Moreover, we have

$$\|y^a \partial_y v\|_{L^\infty(\overline{C_{R/2}})} \leqslant c_R^2, \tag{2.2}$$

for some c_R^2 depending on the same quantities as above.

Proof. To prove the C^β regularity of the solution v in $B_{R/2} \times [0,1)$, we follow the argument used by Cabré and Sire to prove Lemma 4.5 in [9]. We need to modify such argument since, in our case, the solution of (2.1) is not directly related to the fractional Laplacian and a localization method needs to be exploited. First, we set $\bar{g} = g\,\eta$ where $\eta \in C_0^\infty(\mathbb{R}^n)$ is a cut-off function which is identically 1 in $B_{\frac{3}{4}R}$, so that $\bar{g}$ is now defined on the whole $\mathbb{R}^n$ and agrees with g in $B_{\frac{3}{4}R}$. Let now $\bar{v}$ be the bounded solution of

$$\begin{cases} \operatorname{div}(y^a \nabla \bar{v}) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ -y^a \partial_y \bar{v} = \bar{g} & \text{on } \mathbb{R}^n \times \{y = 0\}, \end{cases}$$

which is precisely the local problem in the halfspace $\mathbb{R}^{n+1}_+$ associated to the nonlocal equation $(-\Delta)^{\frac{1-a}{2}} \bar{u} = \bar{g}$, where $\bar{u} = \bar{v}(x,0)$ ($\bar{v}$ is the so-called Caffarelli-Silvestre extension of $\bar{u}$, see [13]). By Remark 3.10 in [9], we have that $\bar{v}$ is continuous and bounded in $\overline{\mathbb{R}^{n+1}}_+$. Hence, by Proposition 2.9 in [29], we have that $\bar{u} \in C^\beta(\mathbb{R}^n)$ for some $\beta \in (0,1)$ depending only on n and a.

Let now $\widetilde{v} := v - \bar{v}$. Then, in $C_{\frac{3}{4}R} \subset \mathbb{R}^{n+1}_+$, the function $\widetilde{v}$ solves

$$\begin{cases} \operatorname{div}(y^a \nabla \widetilde{v}) = 0 & \text{in } C_{\frac{3}{4}R} \\ -y^a \partial_y \widetilde{v} = 0 & \text{on } B_{\frac{3}{4}R} \times \{y = 0\}. \end{cases}$$

Since now we have reduced our problem to a problem with zero Neumann condition on $\{y = 0\}$ we can do an even reflection of the solution $\widetilde{v}$ with respect to $\{y = 0\}$ in order to get a bounded weak solution of

$$\operatorname{div}(|y|^a \nabla \widetilde{v}) = 0 \quad \text{in } B_{\frac{3}{4}R} \times (-1,1).$$

Now, we can apply the regularity theory in [24] (we recall that the weight $|y|^a$ belongs to the Muckenhoupt class A_2) to get that $\widetilde{v}$, and thus v, is $C^\beta(\overline{B_R}\times[0,1))$ for some $\beta\in(0,1)$ depending only on n and a.

The C^β regularity for v up to the top boundary $\{y=1\}$ follows in a standard way, again by even reflection with respect to $\{y=1\}$, observing that the weight y^a is non degenerate for $y=1$ and we have zero Neumann condition on this part of the boundary. This conclude the proof of the first part of the statement.

We now prove (2.2). By Lemma 2.2, the function $w:=-y^a\partial_y v$ solves

$$\begin{cases}\operatorname{div}(y^{-a}\nabla w)=0 & \text{in } C_R\\ w=0 & \text{on } B_R\times\{y=1\}\\ w=g & \text{on } B_R\times\{y=0\}.\end{cases}$$

We introduce the function

$$\overline{w}:=P_{\bar s}(\cdot,y)*\bar g$$

where $\bar g=g\,\eta$ is defined in the first part of the proof, $\bar s$ is such that $1-2\bar s=-a$ and $P_{\bar s}$ is the Poisson kernel for the fractional Laplacian (see Proposition 3.7 and Remark 3.8 in [9]). We have that $\overline{w}\in L^\infty(\overline{\mathbb{R}^{n+1}_+})$ and satisfies

$$\begin{cases}\operatorname{div}(y^{-a}\nabla\overline{w})=0 & \text{in } \mathbb{R}^{n+1}_+\\ w=\bar g & \text{on } \mathbb{R}^n\times\{y=0\}.\end{cases}$$

Now, we can define $\widetilde{w}:=w-\overline{w}$. Arguing as in the first part of the proof, we have that $\widetilde{w}$ has zero (weighted) Neumann condition on $\{y=0\}$ and hence its odd reflection across $\{y=0\}$ satisfies

$$\operatorname{div}(|y|^{-a}\nabla\widetilde{w})=0\quad\text{in } B_{\frac34R}\times(-1,1).$$

Using again the regularity theory in [24] (we recall that the weight $|y|^{-a}$ belongs to the Muckenhoupt class A_2) we get that $\widetilde{w}$ is $C^\beta(\overline{B_{R/2}}\times[0,1))$ with $\beta\in(0,1)$ depending only on n and a. Hence the function $w=\widetilde{w}+\overline{w}$ is bounded in $\overline{B_{R/2}}\times[0,1]$ with a bound that only depends on the quantities specified in the statement of the proposition. This concludes the proof.

□

As a consequence of Proposition 2.3, we get the following estimate for solutions to the semilinaer (localized) problem.

Corollary 2.4. *Let f be a function in $C^{1,\gamma}(\mathbb{R})$, with $\gamma>\max\{0,-a\}$ and let v be a bounded solution of*

$$\begin{cases}\operatorname{div}(y^a\nabla v)=0 & \text{in } C_R\\ \partial_y v=0 & \text{on } B_R\times\{y=1\}\\ -y^a\partial_y v=f(v) & \text{on } B_R\times\{y=0\}.\end{cases}\tag{2.3}$$

Then, there exists $\beta\in(0,1)$ (depending only on n and a) such that $v\in C^\beta(C_{R/2})$ with the following estimates

$$\|v\|_{C^\beta(\overline{C_{R/2}})}\leqslant c_R^1,$$

for some c depending on n, a, R, $\|f\|_{C^{1,\gamma}}$, $\|v\|_{L^\infty(C_R)}$.

Moreover, we have

$$\|y^a \partial_y v\|_{L^\infty(\overline{C_{R/2}})} \leqslant c_R^2, \tag{2.4}$$

for some c depending on n, a, R, $\|f\|_{C^{1,\gamma}}$, $\|v\|_{L^\infty(C_R)}$.

Proof. It is enough to observe that, since $f \in C^{1,\gamma}$ and v is bounded then $f(v)$ is bounded and hence Proposition 2.3 applies to v. □

In the following proposition, we establish global gradient estimates for solutions to (1.4) (the semilinear problem in the infinite slab), which will be crucial to establish our main result.

Proposition 2.5. *Let f be a function in $C^{1,\gamma}(\mathbb{R})$, with $\gamma > \max\{0, -a\}$ and let v be a bounded solution of* (1.4).

Then,

$$\|\nabla_x v\|_{L^\infty(\mathbb{R}^n \times [0,1])} + \|y^a \partial_y v\|_{L^\infty(\mathbb{R}^n \times [0,1])} \leqslant C_1, \tag{2.5}$$

for some C_1 depending only on n, a, $\|f\|_{C^{1,\gamma}}$, $\|v\|_{L^\infty}$.

Proof. We start with the estimate for $|\nabla_x v|$. Let us define the function

$$v_1(x,y) := \frac{v(x+he,y) - v(x,y)}{|h|^\beta}$$

where $e \in S^{n-1}$, $h \in \mathbb{R}$ and β given by Proposition 2.3 (and, without loss of generality, possibly reducing β, we can assume that β is of the form $1/k$ for some integer k). By Corollary 2.4, we have that $v \in C^\beta(\overline{C_{R/2}})$ and hence that v_1 is bounded in $C_{R/4}$. In addition, v_1 solves

$$\begin{cases} \operatorname{div}(y^a \nabla v_1) = 0 & \text{in } C_{R/4} \\ \partial_y v_1 = 0 & \text{on } B_{R/4} \times \{y = 1\} \\ -y^a \partial_y v_1 = \frac{f(v((x+he,0)) - f(v(x,0))}{|h|^\beta} & \text{on } B_{R/4} \times \{y = 0\}. \end{cases} \tag{2.6}$$

Since $f \in C^{1,\gamma}(\mathbb{R})$ and $v \in C^\beta(\overline{C_{R/2}})$, the right-hand side in the third equation of (2.6) is bounded and we can apply Proposition 2.3 to v_1 in the cylinder $C_{R/4}$. Hence we obtain that v_1 is $C^\beta(\overline{C_{R/8}})$ and, using Lemma 5.6 in [5] and the fact that the direction e is arbitrary, that v is in $C^{2\beta}(\overline{C_{R/8}})$. We have that

$$\|v\|_{C^{2\beta}(\overline{C_{R/8}})} \leqslant c_R$$

with c_R depending on n, a, R, $\|f\|_{C^{1,\gamma}(\mathbb{R})}$, $\|v\|_{L^\infty(C_R)}$.

Now, we can iterate this procedure for a finite number (namely, $k-1$) of times such that $k\beta \geqslant 1$ (this is possible since β is a fix strictly positive number depending only on the quantities specified in Proposition 2.3). In this way, we deduce that $\|\nabla_x v\|_{L^\infty(\overline{C_{R/8^k}})}$ is bounded. Moreover, since problem (1.4) is invariant under translations in the x-direction, we can obtain uniform estimates for $\|\nabla_x v\|_{L^\infty}$ in any (closed) cylinder $\overline{C_{R/8^k}(z,0)} = \overline{B_{R/8^k}(z)} \times [0,1]$ with $z \in \mathbb{R}^n$. Observe that the bound c_R^1 in Proposition 2.3 depends on the radius but not on the center of the balls B_R. Hence, by a covering argument we obtain the global bound (2.5).

To prove the second part of the statement, we use the bound (2.2) of Corollary 2.4. Again, after fixing the radius $R = 1$ and using a covering

argument as before, we deduce that $\|y^a \partial_y v\|_{L^\infty(\mathbb{R}^n \times [0,1])} \leqslant C_1$, with C_1 depending only on n, a, $\|f\|_{C^{1,\gamma}}$, $\|v\|_{L^\infty}$, which concludes the proof. □

3. Proof of Theorem 1.3

In this section we establish Theorem 1.3 with a proof based on two main ingredients. The first one is the following characterization of stability, which is the analogue for our problem of Lemma 6.1 in [10].

Lemma 3.1. *Let d be a bounded, Hölder continuous function on $\mathbb{R}^n$. Then the inequality*

$$\int_{\mathbb{R}^n \times (0,1)} y^a |\nabla \eta|^2 \, dx \, dy + \int_{\mathbb{R}^n \times \{y=0\}} d(x) \eta^2 \, dx \geqslant 0 \tag{3.1}$$

holds true for any $\eta \in C_0^1(\mathbb{R}^n \times [0,1])$ if and only if there exists a Hölder continuous function $\varphi \in H^1_{loc}(\mathbb{R}^n \times [0,1], y^a)$, such that

$$\begin{cases} \operatorname{div}(y^a \nabla \varphi) = 0 & \text{in } \mathbb{R}^n \times (0,1) \\ -y^a \partial_y \varphi + d(x)\varphi = 0 & \text{on } \mathbb{R}^n \times \{y=0\} \\ \partial_y \varphi = 0 & \text{on } \mathbb{R}^n \times \{y=1\} \end{cases} \tag{3.2}$$

with

$$\varphi > 0 \text{ in } \mathbb{R}^n \times [0,1]. \tag{3.3}$$

Proof. We first assume the existence of φ and we prove (3.1). Taken a test function η as in the statement of Lemma 3.1, we can multiply (3.2) by $\frac{\eta^2}{\varphi}$ and then integrate over $\mathbb{R}^n \times (0,1)$. We obtain:

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n \times (0,1)} \operatorname{div}(y^a \nabla \varphi) \frac{\eta^2}{\varphi} \\ &= -\int_{\mathbb{R}^n \times \{y=0\}} y^a \partial_y \varphi \frac{\eta^2}{\varphi} - 2 \int_{\mathbb{R}^n \times (0,1)} y^a \frac{\eta}{\varphi} \nabla \eta \nabla \varphi + \int_{\mathbb{R}^n \times (0,1)} y^a \frac{|\nabla \varphi|^2 \eta^2}{\varphi^2} \\ &\geqslant -\int_{\mathbb{R}^n \times \{y=0\}} d(x) \eta^2 - \int_{\mathbb{R}^n \times (0,1)} y^a |\nabla \eta|^2, \end{aligned}$$

where in the last estimate, we have used the boundary data of (3.2) and Cauchy-Schwarz inequality. This establishes (3.1).
The other implication is more delicate to prove. We first define

$$Q_R(\xi) := \int_{C_R} y^a |\nabla \xi|^2 \, dx \, dy + \int_{B_R \times \{y=0\}} d(x) \xi^2 \, dx$$

and we take λ_R as the infimum of $Q_R(\xi)$ in the set

$$\begin{aligned} S_R &:= \left\{ \xi \in H^1(C_R, y^a) : \xi \equiv 0 \text{ on } \partial B_R \times (0,1), \int_{B_R} \xi^2 = 1 \right\} \\ &\subset H_0(C_R, y^a) = \{ \xi \in H^1(C_R, y^a) : \xi \equiv 0 \text{ on } \partial B_R \times (0,1) \}. \end{aligned}$$

From the stability assumption and Definition 1.2, we know that $\lambda_R \geqslant 0$. We want to prove that λ_R is strictly decreasing in R, in order to deduce that

$$\lambda_R > 0. \tag{3.4}$$

To show that λ_R is decreasing in R, we observe that from the hypothesis λ_R is nonincreasing and Q_R is bounded below in S_R, since d is a bounded function. Now, if we take a minimizing sequence $(\xi_k)_k \subset S_R$, we have that $(\nabla \xi_k)$ is uniformly bounded in $L^2(C_R, y^a)$. Using also the compactness of the inclusion $H_0(C_R, y^a) \subset L^2(B_R)$ (see the proof of Lemma 4.1 in [10]), we can state that the infimum of Q_R in S_R is achieved by a function $\varphi_R \in S_R$. We observe also that, up to take $|\varphi_R|$ instead of φ_R, we can choose $\varphi_R \geqslant 0$. We remark that the function φ_R solves

$$\begin{cases} \operatorname{div}(y^a \nabla \varphi_R) = 0 & \text{in } C_R \\ -y^a \partial_y \varphi_R + d(x)\varphi_R = \lambda_R \varphi_R & \text{on } B_R \times \{y = 0\} \\ \partial_y \varphi_R = 0 & \text{on } B_R \times \{y = 1\} \\ \varphi_R = 0 & \text{on } \partial B_R \times (0,1). \end{cases} \tag{3.5}$$

Hence, from the strong maximum principle, we have that $\varphi_R > 0$ in C_R. Now, we take $R_1 < R_2$ and our goal is to show that $\lambda_{R_1} > \lambda_{R_2}$. Since $\lambda_{R_1} \geqslant \lambda_{R_2}$ due to the inclusion of the domains, we argue by contradiction and suppose that $\lambda_{R_1} = \lambda_{R_2}$. The strategy is then to integrate by parts the quantity

$$\int_{C_{R_1}} \varphi_{R_2} \operatorname{div}(y^a \nabla \varphi_{R_1})$$

in order to obtain a contradiction. Indeed, by using (3.5) and the fact that $\lambda_{R_1} = \lambda_{R_2}$, we find that

$$\int_{\partial B_{R_1} \times (0,1)} y^a \varphi_{R_2} \frac{\partial \varphi_{R_1}}{\partial \nu} = 0. \tag{3.6}$$

Since $\varphi_{R_2} > 0$ and $\frac{\partial \varphi_{R_1}}{\partial \nu} < 0$ on $\partial B_R \times (0,1)$, the identity in (3.6) cannot hold true, thus we have reached the desired contradiction. Hence λ_R is strictly decreasing in R and the proof of (3.4) is complete.

Using the definition of λ_R and the fact that λ_R is strictly positive, we obtain that

$$Q_R(\xi) \geqslant \lambda_R \int_{B_R} \xi^2 \geqslant -\delta_R \int_{B_R} d(x)\xi^2 \qquad \text{for all } \xi \in S_R$$

with $0 < \delta_R := \frac{\lambda_R}{\|d\|_\infty}$, and therefore

$$Q_R(\xi) \geqslant \varepsilon_R \int_{C_R} y^a |\nabla \xi|^2, \tag{3.7}$$

with $\varepsilon_R := 1 - \frac{1}{1+\delta_R} > 0$. Now we are able to prove that, fixed $c_R > 0$, there exists a solution φ_R to the problem

$$\begin{cases} \operatorname{div}(y^a \nabla \varphi_R) = 0 & \text{in } C_R \\ -y^a \partial_y \varphi_R + d(x)\varphi_R = 0 & \text{on } B_R \times \{y = 0\} \\ \partial_y \varphi_R = 0 & \text{on } B_R \times \{y = 1\} \\ \varphi_R = c_R & \text{on } \partial B_R \times (0,1). \end{cases} \tag{3.8}$$

Setting $\varphi_R := \psi_R + c_R$, this problem is equivalent to the following one

$$\begin{cases} \operatorname{div}(y^a \nabla \psi_R) = 0 & \text{in } C_R \\ -y^a \partial_y \psi_R + d(x)\psi_R + c_R d(x) = 0 & \text{on } B_R \times \{y=0\} \\ \partial_y \psi_R = 0 & \text{on } B_R \times \{y=1\} \\ \psi_R = 0 & \text{on } \partial B_R \times (0,1). \end{cases}$$

We notice that we can solve the latter system by minimizing in the space $H_0(C_R, y^a)$ the functional

$$\begin{aligned} D(\xi) &= \int_{C_R} \frac{1}{2} y^a |\nabla \xi|^2 + \int_{B_R \times \{y=0\}} \left[\frac{1}{2} d(x) \xi^2 + c_R d(x) \xi \right] \\ &= \frac{1}{2} Q_R(\xi) + c_R \int_{B_R \times \{y=0\}} d(x) \xi. \end{aligned}$$

Since this functional is bounded from below and coercive in $H_0(C_R, y^a)$, thanks to (3.7), and since the inclusion $H_0(C_R, y^a) \subset L^2(B_R)$ is compact, there exists a minimizer of D in $H_0(C_R, y^a)$.

We want now to show that φ_R is strictly positive. To do this, we consider its negative part φ_R^-. By definition, it vanishes on $\partial B_R \times (0,1)$, and we can compute that $Q_R(\varphi_R^-)$=0.

Since the first eigenvalue λ_R of Q_R is positive, we have that $\varphi_R^- \equiv 0$ and so $\varphi_R \geqslant 0$. Hence, using the Hopf Lemma (see Lemma 4.11 in [9]), we deduce that $\varphi_R > 0$ in C_R.

Now that we have found a positive solution of (3.8), next step is proving that for a fixed $\delta > 0$

$$\sup_{C_R} \varphi_S \leqslant \widetilde{c}_R \qquad \text{for all } S > R + \delta, \tag{3.9}$$

for some $\widetilde{c}_R > 0$ (we stress that $\widetilde{c}_R$ depends on R but not on S). To do that, we choose c_R in (3.8) such that $\varphi_R(0) = 1$. [3] Let now φ_S be a solution of (3.8) in C_S, with $S > R + \delta$. We take a family of half balls $\{\mathcal{B}^+_{r,i}\}_i \subset \mathbb{R}^n \times [0,1]$, centered in $(x,0)$ with $x \in \overline{B}_R$, of radius $r \in (0, \frac{\delta}{4})$, in such a way that they cover $\overline{B}_R \times \{y=0\}$ and they have finite mutual intersection. They are in finite number and we call this number k (such number depends on R, but not on S). Since these balls cover $\overline{B}_R \times \{y=0\}$, there exists $j \in \{1, \dots, k\}$ such that $0 \in \mathcal{B}^+_{r,j}$. Since $\mathcal{B}^+_{4r,j} \subset C_S$, we can use the Harnack inequality of Lemma 4.9 in [9] and obtain:

$$\sup_{\mathcal{B}^+_{r,j}} \varphi_S \leqslant K_R \inf_{\mathcal{B}^+_{r,j}} \varphi_S \leqslant K_R \qquad \text{for every } S > R + \delta,$$

where K_R is a constant depending only on R. Now, using again the Harnack inequality in every ball $\mathcal{B}^+_{r,i}$ of the covering, and using the fact that the balls intersect two-by-two, we obtain the boundedness of φ_S over $B_R \times \{y=0\}$. Thanks to the Neumann condition on the top of the slab, we can extend this bound, using the maximum principle, to the whole cylinder of radius R, obtaining (3.9).

[3] To see that this is possible, consider φ_R^1 to be the solution of (3.8) with $c_R = 1$. Hence, by the Hopf Lemma, $\varphi_R^1(0) \neq 0$. It is then enough to divide φ_R^1 by the value $\varphi_R^1(0)$ to get a solution of (3.8) (corresponding to $c_R = (\varphi_R^1(0))^{-1}$) which takes value 1 at 0.

Using now the regularity result for the linear problem established in Lemma 2.3, we have a uniform bound on $\|\varphi_S\|_{C^\beta(B_{R/2}\times[0,1])}$ for every $S > R+\delta$, therefore we can find a subsequence of (φ_S) that converges locally to a function $\varphi \in C^\beta_{\rm loc}(\mathbb{R}^n \times [0,1])$ that is positive and solves (3.2).

□

Remark 3.2. Let v be a solution of (1.4) such that $\partial_{x_n} v(x,y) > 0$ for any $(x,y) \in \mathbb{R}^n \times [0,1)$. Then, we can apply Lemma 3.1 with the choice $d := -f'(u)$ and $\varphi := \partial_{x_n} v$, to deduce that v is stable. The stability of monotone solutions for this kind of problems has already been observed in Lemma 7 in [16] for the case $a = 0$. We stress that in this paper we also need the existence of a positive solution to the linearized problem as a necessary (and not only sufficient) condition for stability.

The second ingredient in the proof of Theorem 1.3 is the following Liouville-type result, which is the analogue of Theorem 4.10 in [9]. For its proof, we refer to Section 4.4 of [9], where a similar result is proven for some semilinear equations in the half-space. In this case, the adaptation to our framework is straightforward.

Lemma 3.3. *Let φ be a positive function in $L^\infty_{loc}(\mathbb{R}^n \times [0,1])$, $\sigma \in H^1_{loc}(\mathbb{R}^n \times [0,1], y^a)$ such that:*

$$\begin{cases} -\sigma \operatorname{div}(y^a \varphi^2 \nabla \sigma) \leqslant 0 & \text{in } \mathbb{R}^n \times (0,1) \\ y^a \sigma \frac{\partial \sigma}{\partial \nu} \leqslant 0 & \text{on } \mathbb{R}^n \times (\{y=0\} \cup \{y=1\}) \end{cases}$$

in the weak sense. If in addition:

$$\int_{B_R \times (0,1)} y^a \big(\varphi \sigma\big)^2 \leqslant C R^2$$

holds for every $R > 1$, then σ is constant.

We can now give the

Proof of Theorem 1.3. Let v be a stable solution of (1.4). Lemma 3.1 implies that there exists a positive function φ that solves (3.2) with $d(x) = -f'(u(x))$. We can define the functions

$$\sigma_i = \frac{\partial_{x_i} v}{\varphi} \qquad \text{for } i = 1, \dots, n.$$

Our goal is to prove that they are constant. For every fixed i, since $\varphi^2 \nabla \sigma_i = \varphi \nabla v_{x_i} - v_{x_i} \nabla \varphi$ and using that both v_{x_i} and φ satisfy the linearized problem (3.2) with $d(x) = -f'(u(x))$, we deduce

$$\operatorname{div}(y^a \varphi^2 \nabla \sigma_i) = 0.$$

Moreover, using again that v_{x_i} and φ satisfy the same linearized problem (in particular they have the same Neumann condition on $\{y=0\}$), we have

$$y^a \sigma_i \partial_y \sigma_i = y^a \frac{v_{x_i}}{\varphi^2} v_{x_i y} - y^a \frac{v_{x_i}^2}{\varphi^2} \frac{\varphi_y}{\varphi} = 0 \qquad \text{on } (\{y=0\} \cup \{y=1\}) \times \mathbb{R}^n.$$

Finally, assumption (1.10) gives

$$\int_{C_R} y^a(\varphi\sigma_i)^2 = \int_{C_R} y^a|\partial_{x_i} v|^2 \leqslant CR^2,$$

and hence we can apply Lemma 3.3 to deduce that σ_i is constant for every $i \in \{1, \dots, n\}$ and we call these constants c_i. If $c_i = 0$ for every $i = 1, \dots n$ then v only depends on y (it is constant in the x-variables). Otherwise, the solution v only depends on the variable y and on the one parallel to the vector $(c_1, \dots, c_n, 0)$. We call this horizontal variable $\tilde{x}$:

$$\widetilde{x} = \frac{\sum_{i=1}^n c_i x_i}{(\sum_{i=1}^n c_i^2)^{\frac{1}{2}}}.$$

We have thus proven that the trace u of v on $\{y = 0\}$ is a function of only one Euclidean variable and hence can be written in the form

$$u(x_1, \dots, x_n) = u_0(\widetilde{x}),$$

where u_0 is a function defined on $\mathbb{R}$. We can compute the derivative of u_0 to get

$$u_0' = \left(\sum_{i=1}^n c_i^2\right)^{\frac{1}{2}} \varphi.$$

If $c_i = 0$ for every $i \in \{1, \dots, n\}$ then $u_0' \equiv 0$, otherwise $u_0' > 0$. This concludes the proof of Theorem 1.3.

□

4. Energy estimate for minimizers

This section is devoted to the proof of Theorem 1.4. Here we prove the energy estimate (1.12) for solutions of (1.3) which minimize the associated energy, and we argue in an arbitrary dimension n (instead of taking $n = 3$ as we are going to do in the next section). It is worth noting that even if the estimate has no dimensional constraint in the case of minimal solutions, this will not give one-dimensional symmetry of minimizers in further dimensions by applying our method, unless one is willing to take additional assumption on the energy growth of the solutions. Indeed, in order to prove Theorem 1.6 we will use Theorem 1.3, which requires the energy in C_R to grow like R^2. We consider v as a bounded minimizer of the functional:

$$\mathcal{E}_R(v) = \frac{1}{2}\int_{C_R} y^a|\nabla v|^2\, dx\, dy + \int_{B_R\times\{y=0\}} G(u)\, dx,$$

such that $v(x, 0) = u(x)$. The function v solves (1.4), and the potential G is such that $G'(u) = -f(u)$. In particular, the potential G is naturally defined up to an additive constant. To appropriately gauge such constant, we set

$$c_u := \min\{G(s) \text{ s.t. } s \in [\inf u, \sup u]\}$$

in order to replace $G(u)$ with $G(u) - c_u$ and work with a positive potential. Moreover, we define τ as the minimum point of G: in this way, it holds that $G(\tau) = c_s$.

As we are going to see, we can directly prove Theorem 1.4 using a comparison argument with a suitable choice of the competitor.

Proof of Theorem 1.4. Since v is a minimizer of $\mathcal{E}$, for every admissible competitor w (i.e. $w = v$ on $\partial B_R \times (0,1)$) we have

$$\mathcal{E}_R(v) \leqslant \mathcal{E}_R(w).$$

We define

$$w(x,y) := \eta_R(x)\tau + (1-\eta_R(x))v(x,y), \tag{4.1}$$

where $\eta_R : \mathbb{R}^n \to [0,1]$ is a smooth function that is equal to 1 inside B_{R-1} and that vanishes outside B_R. In this way, w is constantly equal to τ in the cylinder $B_{R-1} \times [0,1]$ and it is equal to $v(x,y)$ on the lateral boundary $\partial B_R \times [0,1]$, so w is an admissible competitor.
Now we recall the fact that if u is a bounded solution of (1.3), then from Proposition 2.5

$$\|\nabla_x v\|_{L^\infty(\mathbb{R}^n\times[0,1])} \leqslant C \qquad \text{and} \qquad \|y^a \partial_y v\|_{L^\infty(\mathbb{R}^n\times[0,1])} \leqslant C. \tag{4.2}$$

Using (4.2) and the definition (4.1) of w, we can control the energy of w as

$$\begin{aligned}\mathcal{E}_R(w) &\leqslant C\int_{C_R\setminus C_{R-1}} y^a|\nabla v|^2\,dx\,dy + C\int_{C_R\setminus C_{R-1}} y^a\Big\{|v|^2+\tau^2\Big\}\,dy\\ &\quad + \int_{(B_R\setminus B_{R-1})\times\{y=0\}} \{G(w)-c_u\}\,dx\\ &\leqslant CR^{n-1}\left(\int_0^1 y^a\,dy + \int_0^1 y^{-a}\,dy\right) + CR^{n-1} \leqslant CR^{n-1}.\end{aligned}$$

This concludes the proof of Theorem 1.4. □

5. Energy estimates for monotone solutions

The main goal of this section is to prove the energy estimate of Theorem 1.5 in dimension three for monotone solutions of (1.3). We first give the following result, which is the counterpart of Corollary 6 in [16] in presence of a weight.

Lemma 5.1. *Let v be a solution of (1.4) such that $\partial_{x_n} v(x,0) > 0$ for any $x \in \mathbb{R}^n$.*

Then, $\partial_{x_n} v(x,y) > 0$ for any $(x,y) \in \mathbb{R}^n \times [0,1)$.

Proof. We start by observing that the weak and strong maximum principle hold for weak solutions of problem (1.4). This follows exactly as in Remark 4.2 in [9], with the only difference that now we have a Neumann condition on the bottom boundary $\{y = 1\}$. In this part of the boundary it is then enough to apply Hopf's Lemma to a possible minimum of the solution v to get the result. With maximum principles at hand, the proof of the desired result follows exactly the proof of Lemma 5 in [16]. □

Let now $n = 3$ and let v be a solution of (1.4) whose trace u on $\{y = 0\}$ is monotone in the last direction x_3. By Lemma 5.1, v is monotone in x_3 in the whole slab $\mathbb{R}^3 \times [0,1]$, hence we can define two limit profiles of v as

$$\begin{aligned}\overline{v}(x',y) &:= \lim_{x_3\to+\infty} v(x,y),\\ \underline{v}(x',y) &:= \lim_{x_3\to-\infty} v(x,y),\end{aligned} \tag{5.1}$$

where $x' = (x_1, x_2)$. Notice that $\overline{v}$ and $\underline{v}$ are defined in $\mathbb{R}^2 \times [0,1]$, namely we reduced the problem by one dimension by taking the limit in x_3. This fact allows us to deduce good energy estimates for both $\overline{v}$ and $\underline{v}$ which, in turn, implies the one-dimensional symmetry and the monotonicity of $\overline{u}$ and $\underline{u}$ on $\{y = 0\}$. With these properties for $\underline{v}$ and $\overline{v}$, we are able to characterize the potential G associated to equation (1.3) (see Lemma 5.5 below).

Then, the proof of the energy estimates for monotone solutions follows by these two steps:

- if v is a bounded monotone solution to (1.4), then it is in particular a minimizer for the associated energy in a restricted class of functions S_R (basically functions $\widetilde{w}$ such that $\underline{v} \leqslant \widetilde{w} \leqslant \overline{v}$);
- the characterization of G implies that the competitor w constructed in the previous section belongs to the class S_R.

Some of these results are well known in the classical case or for the fractional Laplacian. Here, we need to prove them for our water waves problem, which offers a series of specific complications also due to the fact that the Poisson Kernel is not explicit. For the sake of completeness, we are going to explain all the details in this section.

Using Theorem 1.3, we are able to prove some important properties of the two limit profiles. This also gives a characterization of the potential G as a corollary.

Lemma 5.2. *Let $f \in C^{1,\gamma}(\mathbb{R})$ with $\gamma > \max\{0, -a\}$ and v a bounded solution of (1.4) whose trace u on $\{y = 0\}$ is monotone in x_3.*

Then $\underline{v}$ and $\overline{v}$ are bounded and stable solutions of (1.4) with $n = 2$, and each of them is either constant or one-dimensional and monotone in the (x_1, x_2)-plane.

From Lemma 5.2, one also obtains:

Corollary 5.3. *Set $m = \inf \underline{u} \leqslant \widetilde{m} = \sup \underline{u}$ and $\widetilde{M} = \inf \overline{u} \leqslant M = \sup \overline{u}$. Then $G > G(m) = G(\widetilde{m})$ in $(m, \widetilde{m})$, $G'(m) = G'(\widetilde{m}) = 0$ and $G > G(M) = G(\widetilde{M})$ in $(M, \widetilde{M})$, $G'(M) = G'(\widetilde{M}) = 0$.*

Proof of Lemma 5.2. We prove the desired result for $\overline{v}$, clearly the same proof can be replied for $\underline{v}$.

The fact that $\overline{v}$ is a solution follows from seeing it as the limit of a sequence of functions in four variables, that is $\overline{v}(x', y) = \lim_{t\to\infty} v^t(x', x_3, y)$ where $v^t(x', x_3, y) = v(x', x_3 + t, y)$. By Corollary 2.4, we have that v^t uniformly converges up to subsequences to $\overline{v}$ in the C^β sense on compact sets.
Now we want to prove that

$$\overline{v} \quad \text{is stable} \tag{5.2}$$

and then apply Theorem 1.3. By Remark 3.2, we have that if v is a monotone solution of (1.4) in dimension $n = 3$, then v is stable in $\mathbb{R}^3 \times (0,1)$, hence

$$\int_{\mathbb{R}^3\times(0,1)} y^a |\nabla \xi|^2 + \int_{\mathbb{R}^3\times\{y=\}} f'(u)\xi^2 \geqslant 0, \tag{5.3}$$

for all $\xi \in C_0^\infty(\mathbb{R}^3 \times (0,1))$. Following an idea in [6], we define a special test function ξ in order to get the stability inequality for $\overline{v}$. We take $\rho > 0$ and

a function $\phi_\rho \in C_0^\infty(\mathbb{R}, [0,1])$ such that $\phi_\rho = 0$ in $(-\infty, \rho) \cup (2\rho+2, +\infty)$ and $\phi_\rho = 1$ in $(\rho+1, 2\rho+1)$. For every $\eta \in C_0^\infty(\mathbb{R}^2 \times (0,1))$ we define $\xi(x,y) = \eta(x',y)\phi_\rho(x_3)$. So (5.3) becomes, after dividing it by $\alpha_\rho = \int_{\mathbb{R}} \phi_\rho^2$:

$$\int_{\mathbb{R}^2\times(0,1)} y^a |\nabla \eta|^2 \, dx' \, dy + \int_{\mathbb{R}^2\times(0,1)} y^a \eta^2 \, dx' \, dy \int_{\mathbb{R}} \frac{(\phi_\rho')^2}{\alpha_\rho} \, dx_3 + \\ - \int_{\mathbb{R}^2\times\{y=0\}} \eta^2 \, dx' \, dy \int_{\mathbb{R}} f'(v) \frac{\phi_\rho^2}{\alpha_\rho} \, dx_3 \geqslant 0.$$

When $\rho \to +\infty$ the second term vanishes, because of the definition of ϕ_ρ. In the third term, thanks to the fact that $f \in C^1(\mathbb{R})$, we have that $f'(v) \to f'(\overline{v})$, hence

$$\int_{\mathbb{R}^2\times(0,1)} y^a |\nabla \eta|^2 - \int_{\mathbb{R}^2\times\{y=0\}} f'(\overline{v}) \eta^2 \geqslant 0.$$

So we proved (5.2). In order to conclude that $\overline{v}$ is one-dimensional and monotone in the (x_1, x_2)-plane it is enough to apply Theorem 1.3, after observing that, from Proposition 2.5, $|\nabla_x \overline{v}| \in L^\infty(\mathbb{R}^2 \times [0,1])$ and hence assumption (1.10) is satisfied. □

Before proving Corollary 5.3, we define the notion of *layer* solution of (1.3) and we give a sufficient condition for the potential G to have a double-well structure.

Definition 5.4. *We say that v is a layer solution for (1.4) if it satisfies (1.4),*

$$v_{x_n}(x,0) > 0 \qquad \text{for all } x \in \mathbb{R}^n$$

and

$$\lim_{x_n \to \pm\infty} v(x,0) = \pm 1 \qquad \text{for every } x' \in \mathbb{R}^{n-1},$$

where $x = (x', x_n) \in \mathbb{R}^n$.

The following result gives a necessary condition to the existence of layer solutions of (1.4) with $n = 1$.

Lemma 5.5. *Let $f \in C^{1,\gamma}(\mathbb{R})$, with $\gamma > \max\{0, -a\}$ and $G' = f$. Let v be a bounded layer solution of*

$$\begin{cases} \operatorname{div}(y^a \nabla v) = 0 & \text{in } \mathbb{R}^n \times (0,1) \\ -y^a v_y(x,0) = f(v) & \text{on } \mathbb{R}^n \times \{y=0\} \\ v_y(x,1) = 0 & \text{on } \mathbb{R}^n \times \{y=1\}. \end{cases} \tag{5.4}$$

Then

$$G'(1) = G'(-1) = 0. \tag{5.5}$$

Moreover, if $n = 1$, we also have

$$G > G(1) = G(-1) \quad \text{in } (-1,1). \tag{5.6}$$

Potentials satisfying (5.5), (5.6) are called "double-well potentials".

Proof of Lemma 5.5. The proof combines some ideas contained in the proofs of Lemmas 4.8 and 5.3 in [9].

First we prove that $G'(1) = G'(-1) = 0$, which holds in any dimension n. We take η smooth and nonnegative with compact support in $B_1 \times [0,1)$ and with strictly positive integral over B_1. For $R > 0$ we define $\eta_R := \eta(\frac{x}{R}, y)$. We slide the function v in the x_n direction by considering

$$v^t(x,y) = v(x', x_n + t, y),$$

which is also a solution of (5.4). So we have:

$$\begin{aligned} 0 &= \int_{C_R} \operatorname{div}(y^a \nabla v^t)\eta_R = \int_{B_R\times\{y=0\}} f(u^t)\eta_R - \int_{C_R} y^a \nabla v^t \cdot \nabla \eta_R \\ &= \int_{B_R\times\{y=0\}} f(u^t)\eta_R - \int_{B_R\times\{y=0\}} y^a u^t \partial_y \eta_R + \int_{C_R} v^t \operatorname{div}(y^a \nabla \eta_R). \end{aligned}$$

We have that the first integral converges as $t \to \infty$ to $f(1)R^n \int_{B_1} \eta$ and the other two integrals are bounded by CR^{n-1}. Hence $|f(1)| \leqslant \frac{C}{R}$ for every R and we get $f(1) = 0$ by letting $R \to \infty$. In the same way we can prove that $G'(-1) = 0$.

We prove now the second part of the statement. Let $n = 1$; we claim that

$$\int_0^1 \frac{t^a}{2}\big(v_x^2(x,t) - v_y^2(x,t)\big)\,dt = G(u(x,0)) - G(1). \tag{5.7}$$

First, we define the function w as:

$$w(x) = \int_0^1 \frac{t^a}{2}\big(v_x^2 - v_y^2\big)(x,t)\,dt.$$

We remark that w is well defined and bounded thanks to (2.5). In addition, Proposition 2.5 allows us to derive under the integral sign in the definition of $w(x)$, so we can compute the derivative of w as

$$\begin{aligned} \partial_x w(x) &= \int_0^1 t^a (v_x v_{xx} - v_y v_{xy})(x,t)\,dt \\ &= \lim_{y\to 0} y^a v_y(x,y) v_x(x,y) = \frac{d}{dx} G(u(x,0)), \end{aligned} \tag{5.8}$$

where we have used an integration by parts and the fact that v is a solution of (5.4). Using (5.8), we obtain that

$$w(x) - [G(v(x,0)) - G(1)] = C \tag{5.9}$$

for some constant C. Our next goal is proving that $C = 0$.

To this end, first we point out the estimate

$$|w(x)| \leqslant C \int_0^1 t^a |\nabla v(x,t)|^2\,dt.$$

We prove now that for every fixed $R > 0$

$$\|\nabla_x v\|_{L^\infty(C_R(x,0))} + \|y^a \partial_y v\|_{L^\infty(C_R(x,0))} \longrightarrow 0 \qquad \text{as } x \to +\infty, \tag{5.10}$$

where $C_R(x,0) = B_R(x,0) \times (0,1) \subset \mathbb{R} \times (0,1)$.

Suppose by contradiction that there exist $R > 0$, $y \in \mathbb{R}$, $\varepsilon > 0$ and a sequence $t_m \to +\infty$ such that:

$$\|\nabla_x v\|_{L^\infty(C_R(x,0))} + \|y^a \partial_y v\|_{L^\infty(C_R(x,0))} \geqslant \varepsilon \qquad \text{for every } m. \tag{5.11}$$

Notice that v^{t_m} is a solution of (5.4) for every m. Also, the sequence v^{t_m} is uniformly bounded. Consequently, we obtain $C^\beta(\overline{C_S})$ estimates for v^{t_m} from Proposition 2.3, and we stress that these estimates are uniform in m for every $S > 0$. From this fact we have that, up to subsequences, v^{t_m} converges to a bounded function $v \in C^\beta_{\mathrm{loc}}(\mathbb{R} \times [0,1])$ such that

$$\operatorname{div}(y^a \nabla v) = 0.$$

Since $v \equiv 1$, we get a contradiction with (5.11) and we obtain (5.10).
Letting now $x \to +\infty$ and using (5.10), we deduce that $C = 0$. Moreover taking the limit for $x \to -\infty$, we also have that

$$G(1) = G(-1). \tag{5.12}$$

Now, we are left with proving that

$$G > G(1) \qquad \text{in } (-1,1).$$

In order to do that, we want to prove that for every $x \in [0,1)$

$$\int_0^x \frac{t^a}{2}\big(v_y^2(t,y) - v_x^2(t,y)\big)\,dt < G(u(0,y)) - G(1). \tag{5.13}$$

We define for every $y \in \mathbb{R}$ and $x \in [0,1]$

$$\eta(x,y) := \int_0^x \frac{t^a}{2}\big(v_y^2(t,y) - v_x^2(t,y)\big)\,dt$$

and

$$\varphi(x,y) := G(u(x,0)) - G(1) - \eta(x,y).$$

First, we observe that φ can not be constant. Indeed, since $\eta(x,0) = 0$, the fact that φ is constant would imply that G is also constant. This would give $f \equiv 0$ and so u would be constant, in contradiction with the monotonicity property $u_x > 0$.
Using the fact that $\operatorname{div}(y^a \nabla v) = 0$, we can compute the derivatives of φ as

$$\partial_y \varphi(x,y) = -\frac{y^a}{2}(v_x^2 - v_y^2)$$
$$\text{and} \qquad \partial_x \varphi(x,y) = y^a v_x(x,y) v_y(x,y).$$

Hence, after some computations (see also the proof of Lemma 5.3 in [9]) we see that $\varphi(x,y)$ is bounded and satisfies

$$\operatorname{div}(y^a \nabla \varphi(x,y)) = -a y^{2a-1} v_x^2(x,y) \tag{5.14}$$

in $\mathbb{R} \times (0,1)$. Our last claim is that

$$\varphi \text{ is strictly positive on } \mathbb{R} \times [0,1). \tag{5.15}$$

Notice that this claim implies (5.13). In order to prove (5.15), we assume by contradiction that there exists (x_0, y_0) in $\mathbb{R} \times [0,1)$ such that $\varphi(x_0,y_0) \leqslant 0$.

Let us divide the proof in two cases, considering at a first attempt $a \geqslant 0$. From (5.14) it follows that

$$\operatorname{div}(y^a \nabla \varphi(x,y)) \leqslant 0 \tag{5.16}$$

and, using also the Hopf Lemma 4.11 in [9], we can say that $y_0 = 0$. Hence there exists $x_0 \in \mathbb{R}$ such that

$$G(u(x_0, 0)) - G(1) \leqslant 0.$$

But $\psi(x) = G(u(x, 0)) - G(1)$ goes to zero when $x \to \pm\infty$, so we can take x_0 as a global minimum for ψ. It follows that

$$0 = \frac{d}{dx} G(u(x_0, 0)) = \lim_{y\to 0} y^a v_y(x_0, y) v_x(x_0, y)$$

and from the monotonicity property of v, see Remark 3.2, we have

$$0 = -\lim_{y\to 0} y^a v_y(x_0, y).$$

By the maximum principle, the point $(x_0, 0)$ is also the minimum of $\varphi(x, y) = G(u(x, 0)) - G(1) - \eta(x, y)$. Since φ is the extension of ψ satisfying (5.16), we have that $(x_0, 0)$ is a strict minimum for φ and we get a contradiction by considering

$$\begin{aligned} 0 > -y^a \partial_y \varphi(x_0, y)_{|_{\{y=0\}}} &= y^a \partial_y \eta(x, y)_{|_{\{y=0\}}} \\ = \lim_{y\to 0} \frac{y^{2a}}{2} \big(v_y^2(x_0, y) - v_x^2(x_0, y)\big) &= \lim_{y\to 0} \frac{y^{2a}}{2} v_x^2(x_0, y) \geqslant 0. \end{aligned}$$

This contradiction proves (5.15) in the case $a \geqslant 0$. Now we deal with the case $a < 0$. First, we compute

$$\operatorname{div}(y^{-a} \nabla \varphi(x, y)) = -a y^{-1} v_y^2(x, y). \tag{5.17}$$

Recalling that we are supposing that a negative minimum of φ is achieved at $(x_0, y_0) \in \mathbb{R}^n \times [0, 1)$, we want to show that $y_0 = 0$. Since now a is negative, we have to add an extra term from (5.17) and consider

$$0 = \operatorname{div}(y^{-a} \nabla \varphi) + a y^{-1} v_y^2 = \operatorname{div}(y^{-a} \nabla \varphi) + \Big(a y^{-a-1} \frac{v_y}{v_x}\Big) \varphi_x.$$

From the fact that this last operator is uniformly elliptic with continuous coefficients in compact sets of $\mathbb{R}^n \times (0, 1)$, it follows that $y_0 = 0$. Now, we can obtain a contradiction by considering

$$\begin{aligned} 0 \geqslant -\liminf_{y\to 0^+} y^{-a} \partial_y \varphi(x_0, y) &= y^a \partial_y \eta(x, y)_{|_{\{y=0\}}} \\ = \liminf_{y\to 0} \frac{1}{2} \big(v_x^2(x_0, y) - v_y^2(x_0, y)\big) &= \frac{1}{2} v_x^2(x_0, 0) > 0. \end{aligned}$$

Notice that we have used also the fact that, from (2.5), $|v_y(x_0, y)| \leqslant C y^{-a} \to 0$ as $y \to 0^+$. This proves (5.15) also when a is negative and finishes the proof. □

Now that we have characterized the potential G in presence of a layer solution, we are able to deduce Corollary 5.3 from Lemma 5.2.

Proof of Corollary 5.3. We want to find a layer solution of (1.4) in order to use the characterization given by Lemma 5.5. Our candidate is the function $\overline{w}$ defined as

$$\overline{w} := 2\left(\frac{\overline{v} - \widetilde{M}}{M - \widetilde{M}}\right) - 1. \tag{5.18}$$

We take the function $h(\overline{w}) := \frac{2f(\overline{v})}{M-\widetilde{M}}$, and we call H the potential associated to h, so $H' = -h$. Then, $\overline{w}$ is a solution of problem (1.4) with $n = 2$ with f replaced by the new nonlinearity $h(\overline{w})$. By Lemma 5.2 $\overline{w}$ is either constant or one-dimensional in the $\{x_1, x_2\}$-plane, and it is monotone if it is not constant. According to Definition 5.4, we have that $\overline{w}$ is a layer solution of problem (1.4) (with the new nonlinearity h). Now we can apply Lemma 5.5, and obtain that H is a double-well potential. Restating this result for G, we have that G is forced to satisfy $G'(\widetilde{M}) = G'(M) = 0$ and $G > G(\widetilde{M}) = G(M)$ in $(\widetilde{M}, M)$.
Using $\underline{v}$ instead of $\overline{v}$, we can prove with the same argument that $G'(\widetilde{m}) = G'(m) = 0$ and $G > G(\widetilde{m}) = G(m)$ in $(m, \widetilde{m})$. □

As a final step before giving the proof of Theorem 1.5, we need to prove the following result, which ensures that if v is a bounded monotone solution for problem (1.4), then it is a minimizer in a particular class of functions. This result can be seen as the counterpart of Proposition 6.2 in [7] for the case into consideration here, in which we have to take into account the singularity and degeneracy of the weights, the different domain of the equation and the different boundary conditions.

Lemma 5.6. *Let $f \in C^{1,\gamma}(\mathbb{R})$, with $\gamma > \max\{0, -a\}$, v a bounded solution of (1.4) with $n = 3$, such that its trace $u(x) = v(x, 0)$ is monotone in its third variable.*

Then

$$\mathcal{E}_R(v) \leqslant \mathcal{E}_R(w)$$

for every $w \in H^1(C_R, y^a)$ such that $w = v$ on $\partial B_R \times (0, 1)$ and $\underline{v} \leqslant w \leqslant \overline{v}$ in C_R.

Proof. The proof of this property follows the proof of Proposition 6.2 in [7] and is based on two results:

(i) Uniqueness of solutions to the problem

$$\begin{cases} \operatorname{div}(y^a \nabla w) = 0 & \text{in } C_R \\ w = v & \text{on } \partial B_R \times (0,1) \\ -y^a \partial_y w = f(w) & \text{on } B_R \times \{y = 0\} \\ \partial_y w = 0 & \text{on } B_R \times \{y = 1\} \\ \underline{v} \leqslant w \leqslant \overline{v} & \text{in } B_R \times (0,1) \end{cases} \tag{5.19}$$

We give here a proof of this result that uses the idea of sliding the function v in the x_n-direction. Keeping in mind that $\underline{u}$ and $\overline{u}$ are respectively the trace of $\underline{v}$ and $\overline{v}$ on $\{y = 0\}$, let w be a solution of (5.19).

By Hopf Lemma (see Lemma 4.11 in [9]) and the maximum principle, we have that

$$\underline{v} < w < \overline{v} \qquad \text{in } \overline{C_R}. \tag{5.20}$$

Now we slide the function v in the direction of monotonicity x_n. We take

$$v^t(x, y) := v(x_1, \dots, x_{n-1}, x_n + t, y).$$

Since $v_t \to \overline{v}$ uniformly in $\overline{C}_R$ and by (5.20), we have that $w < v^t$ in $\overline{C}_R$ for t large enough. We take s as

$$s := \inf\big\{t > 0 \quad \text{s.t.} \quad w < v^t \quad \text{in } \overline{C}_R\big\}.$$

We need to prove that

$$s = 0. \tag{5.21}$$

Suppose by contradiction that $s > 0$. Then we would have $w \leqslant v^s$ in all $\overline{B_R} \times [0,1]$ and there must be a point $(\bar{x}, \bar{y})$ in which the two functions coincide. But $(\bar{x}, \bar{y}) \notin \partial B_R \times (0,1)$ because along $\partial B_R \times (0,1)$ it holds that $w = v$, and we have the monotonicity hypothesis on v. So $(\bar{x}, \bar{y})$ must be either in C_R or in $B_R \times \{y = 0\} \cup B_R \times \{y = 1\}$, but we get a contradiction either with the maximum principle or with the Hopf Lemma applied to the positive function $v^s - w$. Hence we proved (5.21).

(ii) Existence of a minimizer for $\mathcal{E}_R$ in the set:

$$S_R = \{w \in H^1(C_R, y^a) : w \equiv v \text{ on } \partial B_R \times (0,1),\ \underline{v} \leqslant w \leqslant \overline{v} \text{ in } C_R\}.$$

This result is the analogue of the one obtained in Lemma 4.1 of [10] for layer solutions of the fractional Laplacian. The proof can be adapted by substituting -1 and $+1$, which are the limits of the layer solution in [10], with $\underline{v}$ and $\overline{v}$, which are respectively a subsolution and a supersolution for problem (1.4).

We already know that v is a solution to problem (5.19) and, in view of point (i), we have uniqueness of this solution. So this solution must coincide with the minimizer for $\mathcal{E}_R$ in S_R. □

Now we are able to prove the energy estimate of Theorem 1.5 and to deduce the one-dimensional symmetry of monotone solutions from it.

Proof of Theorem 1.5. We follow the idea in the proof of Theorem 5.2 of [2] and Theorem 1.3 of [6], that is we show that the comparison function $\overline{w}$ defined in the previous section satisfies

$$\underline{v} \leqslant \overline{w} \leqslant \overline{v}. \tag{5.22}$$

In this way, we have that $\overline{w}$ belongs to S_R, which is the class of functions where v minimizes the energy. We recall that $\overline{w}$ is defined as:

$$\overline{w}(x,y) = \tau\eta_R(x) + \big(1 - \eta_R(x)\big)v(x,y). \tag{5.23}$$

If we prove that $\tau \in [\sup \underline{u}, \inf \overline{u}]$, we also have (5.22) from the maximum principle.

In order to prove this, we use Corollary 5.3. We set $m = \inf \underline{u}$, $\widetilde{m} = \sup \underline{u}$ and $\widetilde{M} = \inf \overline{u}$, $M = \sup \overline{u}$. We have

$$G > G(m) = G(\widetilde{m}) \quad \text{in } (m, \widetilde{m}) \quad \text{if } \underline{u} \text{ is not constant;}$$

$$G > G(M) = G(\widetilde{M}) \quad \text{in } (\widetilde{M}, M) \quad \text{if } \overline{u} \text{ is not constant.}$$

Suppose that $m \neq \widetilde{M}$: In all possible cases we have $\widetilde{m} \leqslant \widetilde{M}$, so there exists τ in $[\widetilde{m}, \widetilde{M}]$ such that $G(\tau) = c_u$, where c_u is the infimum of G in the range of u. Hence

$$\sup \underline{v} = \sup \underline{u} = \widetilde{m} \leqslant \tau \leqslant \widetilde{M} = \inf \overline{u} = \inf \overline{v}$$

and (5.22) is proved. Hence, $\overline{w} \in S_R$ and from Lemma 5.6 we can conclude that (1.13) holds true.

We have still to consider the special case in which $m = \widetilde{M}$. From Corollary 5.3, it follows that $G \geqslant G(m) = G(\widetilde{M})$ in (m, M) and all the solutions v^t are obtained by translation of v. We can define a new competitor as

$$\widetilde{w}(x,y) = \underline{v}(x,y)\eta_R(x) + \big(1 - \eta_R(x)\big)v(x,y),$$

with η_R the same cut-off function as before. Using again the gradient bounds for v and the fact that $\underline{v}$ depends only on two variables (on y and on one variable in the (x_1, x_2)-plane), we deduce that

$$\mathcal{E}_R(\widetilde{w}) \leqslant CR^2.$$

Clearly $\widetilde{w} \in S_R$, so we can use Lemma 5.6 also in this case and obtain the energy estimate for monotone solutions. □

From the energy estimates in (1.12) and (1.13) we obtain the one-dimensional symmetry of both minimizers and monotone solutions by a direct application of Theorem 1.3.

Proof of Theorem 1.6. Either if v is a bounded minimizer or is a bounded solution whose trace on $\{y = 0\}$ is monotone, we have from Theorem 1.5 and Theorem 1.4 that

$$\mathcal{E}_R(v) \leqslant CR^2.$$

This energy estimate is enough to apply Theorem 1.3 and to obtain that there exist $v_0 : \mathbb{R}^2 \to \mathbb{R}$ and $\omega \in \mathrm{S}^2$ such that

$$v(x,y) = v_0(\omega \cdot x, y) \qquad \text{for any } (x,y) \in \mathbb{R}^4_+.$$ □

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