Research Article

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On the Relationship between Primal/Dual Cell Complexes of the Cell Method and Primal/Dual Vector Spaces: an Application to the Cantilever Elastic Beam with Elastic Inclusion

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Abstract: The Cell Method (CM) is an algebraic numerical method based on the use of global variables: the configuration, source and energetic global variables. The configuration variables with their topological equations, on the one hand, and the source variables with their topological equations, on the other hand, define two vector spaces that are a bialgebra and its dual algebra. The operators of these topological equations are generated by the outer product of the geometric algebra, for the primal vector space, and by the dual product of the dual algebra, for the dual vector space. The topological equations in the primal cell complex are coboundary processes on even exterior discrete p–forms, whereas the topological equations in the dual cell complex are coboundary processes on odd exterior discrete p–forms. Being expressed by coboundary processes in two different vector spaces, compatibility and equilibrium can be enforced at the same time, with compatibility enforced on the primal cell complex and equilibrium enforced on the dual cell complex. By way of example, in the present paper compatibility and equilibrium are enforced on a cantilever elastic beam with elastic inclusion. In effect, the CM shows its maximum potentialities right in domains made of several materials, as, being an algebraic approach, can treat any kind of discontinuities of the domain easily.

Keywords: Cell Method; Topological equations; Geometric Algebra; Bialgebra; Coboundary Process; Elastic Beams; Elastic Inclusions

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1 Introduction

The physical variables can be classified as field and global variables [1–6]. The use of global variables instead of field variables allows us to obtain a purely algebraic approach to physical laws, called the direct algebraic formulation of the Cell Method (CM) [2–5, 7–38]. The term “direct” emphasizes that this formulation is not induced by the differential formulation, as is the case for the so-called discrete formulations.

The range of applicability of differential formulation is restricted to regions without material discontinuities or concentrated sources, whereas that of the algebraic formulation is not restricted to such regions [3]. Consequently, the global variables involved in obtaining the direct algebraic formulation non necessarily must be differentiable functions. This makes the CM particularly useful for modeling domains made of several materials, such as masonry walls [32, 33, 39] heated floors [21] and composite materials [23, 40]. Another interesting field of application of the CM is fracture mechanics in brittle materials [3, 21–23, 39–47].

By performing densities and rates of the global variables, it is then always possible to obtain the differential formulation from the direct algebraic formulation.

A further criterion for classifying the physical variables is based on the role they play in a theory. According to this second criterion, all physical variables belong to one of the following three classes [2, 4, 17, 22, 42]:

- Configuration variables, describing the field configuration (displacements for solid mechanics, spatial velocity for fluidodynamics, electric potential for electrostatics, temperature for thermal conduction, and so forth).
- Source variables, describing the field sources (forces for solid mechanics and fluidodynamics, masses for geodesy, electric charges for electrostatics, electric currents for magnetostatics, heat sources for thermal conduction, and so forth).

Table 1: Examples of discrete p-forms

Variable	Potential of a vector field	Line integral of a vector	Flux	Mass content
Evaluated on	0-cells (points)	1-cells (lines)	2-cells (surfaces)	3-cells (volumes)
Discrete *p*-form	discrete 0-form $\Phi[\mathbf{P}]$	discrete 1-form $\Phi[\mathbf{L}]$	discrete 2-form $\Phi[\mathbf{S}]$	discrete 3-form $\Phi[\mathbf{V}]$

- Energetic variables, resulting from the multiplication of a configuration variable by a source variable (elastic energy density for solid mechanics, kinetic energy for dynamics, electrostatic energy for electrostatics, magnetostatic energy for magnetostatics, heat for thermal conduction, and so forth).

The terms "configuration variables" and "source variables" were introduced by Tonti in 1972 and are of special importance for the philosophy of the CM [4, 20]. The configuration variables and source variables correspond to the geometric variables and force variables, respectively, used by Penfield and Haus [48].

The equations used to relate the configuration variables of the same physical theory to each other and the source variables of the same physical theory to each other are known as topological equations. They can also be defined as those equations that express a relationship between a variable associated with a space element and a variable associated with the boundary of the same space element. Let $\mathbf{M}$ be a space element and let $\partial\mathbf{M}$ be its boundary, broadly speaking a topological equation:

$$A[\mathbf{M}] = \pm B[\partial\mathbf{M}] \quad (1)$$

is therefore expressed by one of the two following maps [17]:

$$t_1 : \partial\mathbf{M} \to \mathbf{M} \quad (2)$$

$$t_2 : \mathbf{M} \to \partial\mathbf{M} \quad (3)$$

The equations that relate configuration to source variables, of the same physical theory, are known as constitutive equations, or material equations. They are phenomenological equations and specify the behavior of a material, a substance, or a media. The constitutive relations can be reversible or irreversible.

The equations providing the configuration of a system, once the sources are assigned, are called the fundamental equations, and the related problem is called the fundamental problem. The set of fundamental equations defines the fundamental system of equations.

The topological equations of a fundamental problem are always maps of the type t_1 in Eq. (2). They can be described in algebraic topology [49–52] by using discrete p–forms [2], which are the algebraic versions of the exterior differential forms [49, 53–55].

In algebraic topology, it is usual to introduce cell complexes, mainly in the restricted form of simplicial complexes, and to consider the vertices, edges, surfaces, and volumes of a cell complex as p–cells, that is, cells of different dimensions. A physical variable ϕ associated with one set of p–cells of a cell-complex defines a discrete p–form (or a discrete form of degree p). The potential of a vector field, line integral of a vector, flux and mass content are discrete forms of degree 0, 1, 2, and 3, respectively (Table 1).

The discrete p–forms generalize the notion of field functions, because, in a discrete p–form $\Phi[\mathbf{P}]$, $\Phi[\mathbf{L}]$, $\Phi[\mathbf{S}]$, or $\Phi[\mathbf{V}]$, we associate the value of a physical variable with the space elements of degree p, where $p = 0, 1, 2, 3$, whereas the field functions, $f(\mathbf{P})$, always associate the value of a physical variable with the points of the domain. As a consequence, $\Phi[\mathbf{P}]$, $\Phi[\mathbf{L}]$, $\Phi[\mathbf{S}]$, and $\Phi[\mathbf{V}]$ are domain functions, or set functions, that is, functions whose input is the set of all subsets of a set, whereas $f(\mathbf{P})$ is a point function.

Moreover, in algebraic topology the topological equations of a fundamental problem are described by using the coboundary operators: a coboundary operator is any map from a subset of n p–cells to a subset of m $(p+1)$-cells [56–60]:

$$d^p : \sum_{i=1}^{n} \mathbf{e}_p^i \to \sum_{j=1}^{m} \mathbf{e}_{p+1}^j \quad (4)$$

where $\mathbf{e}_p^i$ is the i-th p-cell and $\mathbf{e}_{p+1}^j$ is the j-th $(p+1)$-cell.

When $m = 1$ and n equals the number of cofaces of the $(p+1)$–cell, the coboundary operator is indicated with the symbol δ:

$$\delta^p : \sum_{i=1}^{n} \mathbf{e}_p^i \to \mathbf{e}_{p+1} \quad (5)$$

and δ^p defines the coboundary of $\mathbf{e}_{p+1}$.

Let η^i be the weight (an integer number) of the i–th p–cell, $\mathbf{e}_p^i$. This weight induces a weight on the $(p+1)$–cells of the cochains of $\mathbf{e}_p^i$, by a process that is called the coboundary process, where a cochain complex, $(A^\bullet, d^\bullet)$, is formally defined as a sequence of Abelian

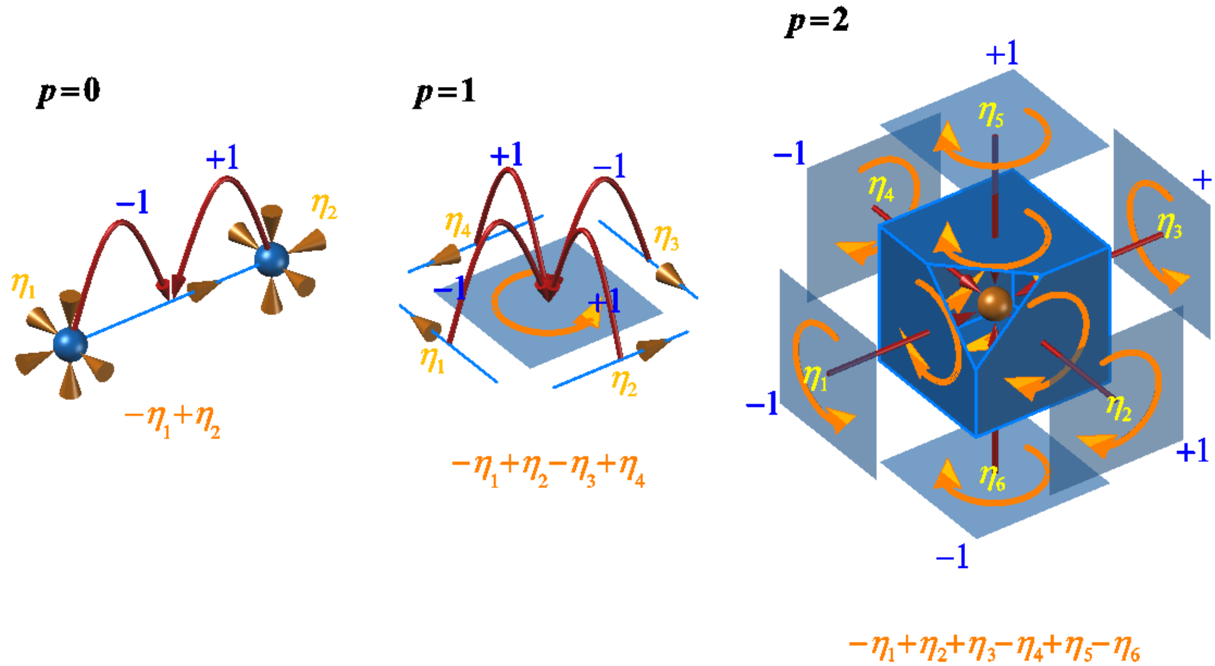

Figure 1: How to find the weight of a $(p+1)$ –cell starting from the weights of its faces.

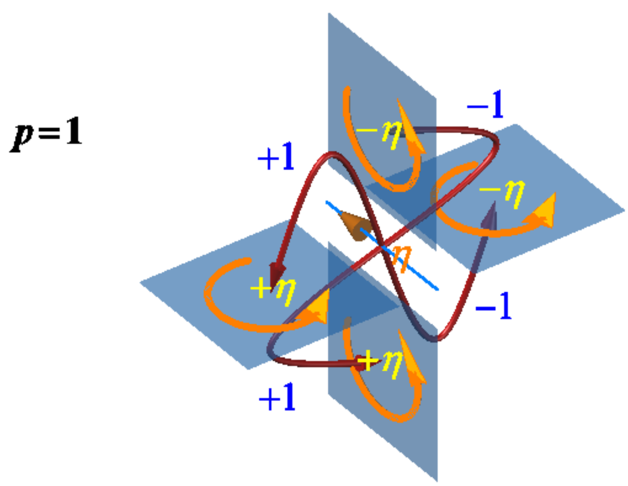

Figure 2: The coboundary process.

groups, or modules, $\ldots, A^{n-2}, A^{n-1}, A^{n}, A^{n+1}, A^{n+2}, \ldots$ connected by homomorphisms (the coboundary operators):

$$d^n : A^n \to A^{n+1} \tag{6}$$

such that the composition of any two consecutive maps is zero for all n:

$$d^{n+1} \circ d^n = 0 \quad \forall n; \tag{7}$$

$$\ldots \xrightarrow{d^{n-2}} A^{n-1} \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} A^{n+1} \xrightarrow{d^{n+1}} A^{n+2} \xrightarrow{d^{n+2}} \ldots \tag{8}$$

Each $(p+1)$ –cell collects the weights that are spread on the $(p+1)$ –cell itself by its faces, after having multiplied the weights by the mutual incidence numbers (Figure 1).

The incidence number is equal to (the pictorial view of the coboundary process is provided in Figure 2 for $n = 1$ and $p = 1$):

- 0, if the $(p-1)$-cell is not on the boundary of the p–cell;
- +1, if the $(p-1)$-cell is on the boundary of the p–cell and the orientations of the p–cell and $(p-1)$-cell are compatible;
- −1, if the $(p-1)$-cell is on the boundary of the p–cell and the orientations of the p–cell and $(p-1)$-cell are not compatible.

The coboundary process can be defined as the action of the n p–cells, which spread their own weights on their cofaces, in accordance with the mutual incidence numbers.

When the constitutive relations are reversible, it is also possible to find the sources once the configuration of the system is assigned. In this latter case, the solving system is called the system of the dual fundamental problem, or dual fundamental system, and its equations are called the dual fundamental equations.

The topological equations of a dual fundamental problem are still maps of the type t_1. Therefore, even the topological equations of the dual fundamental problem can be described in algebraic topology by using discrete p–forms and coboundary operators. Consequently, the coboundary process plays a key role in both the fundamental problems in physics. In particular, in [1, 2, 15, 17, 18, 20] we discussed the role played by the coboundary process performed on discrete p–forms of degree 2, 1, and 0, both in space and space/time domains.

2 Why and how to use two cell complexes in the CM

In the algebraic setting, the global physical variables have a natural association with one of the four space elements, **P**, **L**, **S**, and **V** (Figure 3), and one of the two time elements, **I** and **T** (Figure 4) [1, 2, 22, 39, 41]. Moreover, the space and time elements are provided with two kinds of orientations, inner and outer, in relation of duality between them. Thus, global physical variables are associated with oriented space and time elements.

A more accurate analysis of this association shows that some global variables are associated with space or time elements provided with inner orientations, whereas some other global variables are associated with space or time elements provided with outer orientations. If we consider all the combinations between oriented space and oriented time elements related to physical variables, we see that there are 32 possible couples of space/time elements in physics. These 32 couples can, in turn, be divided into two groups (Figure 5), each consisting of 16 elements. In the first group, there are the couples of time and space elements that are endowed with the same kind of orientation, either inner or outer, whereas, in the second group, there are the couples of time and space elements that are endowed with opposite kinds of orientation, one inner and the other outer.

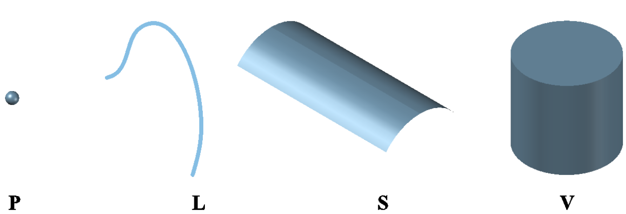

Figure 3: The four space elements and their notations.

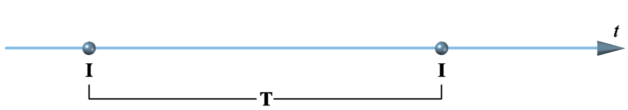

Figure 4: The two time elements and their notations.

In Figure 5, a point, a line, a surface, and a volume endowed with inner orientations are denoted by putting bars over their symbols ($\bar{\mathbf{P}}$, $\bar{\mathbf{L}}$, $\bar{\mathbf{S}}$, and $\bar{\mathbf{V}}$), whereas a point, a line, a surface, and a volume endowed with outer orientations are denoted by putting tildes over their symbols ($\tilde{\mathbf{P}}$, $\tilde{\mathbf{L}}$, $\tilde{\mathbf{S}}$, and $\tilde{\mathbf{V}}$).

first group

	$\bar{\mathbf{I}}$	$\bar{\mathbf{T}}$
$\bar{\mathbf{P}}$	$[\bar{\mathbf{I}},\bar{\mathbf{P}}]$	$[\bar{\mathbf{T}},\bar{\mathbf{P}}]$
$\bar{\mathbf{L}}$	$[\bar{\mathbf{I}},\bar{\mathbf{L}}]$	$[\bar{\mathbf{T}},\bar{\mathbf{L}}]$
$\bar{\mathbf{S}}$	$[\bar{\mathbf{I}},\bar{\mathbf{S}}]$	$[\bar{\mathbf{T}},\bar{\mathbf{S}}]$
$\bar{\mathbf{V}}$	$[\bar{\mathbf{I}},\bar{\mathbf{V}}]$	$[\bar{\mathbf{T}},\bar{\mathbf{V}}]$

	$\tilde{\mathbf{I}}$	$\tilde{\mathbf{T}}$
$\tilde{\mathbf{P}}$	$[\tilde{\mathbf{I}},\tilde{\mathbf{P}}]$	$[\tilde{\mathbf{T}},\tilde{\mathbf{P}}]$
$\tilde{\mathbf{L}}$	$[\tilde{\mathbf{I}},\tilde{\mathbf{L}}]$	$[\tilde{\mathbf{T}},\tilde{\mathbf{L}}]$
$\tilde{\mathbf{S}}$	$[\tilde{\mathbf{I}},\tilde{\mathbf{S}}]$	$[\tilde{\mathbf{T}},\tilde{\mathbf{S}}]$
$\tilde{\mathbf{V}}$	$[\tilde{\mathbf{I}},\tilde{\mathbf{V}}]$	$[\tilde{\mathbf{T}},\tilde{\mathbf{V}}]$

second group

	$\tilde{\mathbf{I}}$	$\tilde{\mathbf{T}}$
$\bar{\mathbf{P}}$	$[\tilde{\mathbf{I}},\bar{\mathbf{P}}]$	$[\tilde{\mathbf{T}},\bar{\mathbf{P}}]$
$\bar{\mathbf{L}}$	$[\tilde{\mathbf{I}},\bar{\mathbf{L}}]$	$[\tilde{\mathbf{T}},\bar{\mathbf{L}}]$
$\bar{\mathbf{S}}$	$[\tilde{\mathbf{I}},\bar{\mathbf{S}}]$	$[\tilde{\mathbf{T}},\bar{\mathbf{S}}]$
$\bar{\mathbf{V}}$	$[\tilde{\mathbf{I}},\bar{\mathbf{V}}]$	$[\tilde{\mathbf{T}},\bar{\mathbf{V}}]$

	$\bar{\mathbf{I}}$	$\bar{\mathbf{T}}$
$\tilde{\mathbf{P}}$	$[\bar{\mathbf{I}},\tilde{\mathbf{P}}]$	$[\bar{\mathbf{T}},\tilde{\mathbf{P}}]$
$\tilde{\mathbf{L}}$	$[\bar{\mathbf{I}},\tilde{\mathbf{L}}]$	$[\bar{\mathbf{T}},\tilde{\mathbf{L}}]$
$\tilde{\mathbf{S}}$	$[\bar{\mathbf{I}},\tilde{\mathbf{S}}]$	$[\bar{\mathbf{T}},\tilde{\mathbf{S}}]$
$\tilde{\mathbf{V}}$	$[\bar{\mathbf{I}},\tilde{\mathbf{V}}]$	$[\bar{\mathbf{T}},\tilde{\mathbf{V}}]$

Figure 5: Classification of the space and time elements related to the physical variables.

The physical variables of the first group are those of the mechanical theories, whereas the physical variables of the second group are those of the field theories.

As far as the field theories are concerned, it was found that the configuration variables of the fundamental problem of any field theory are associated with space elements endowed with an inner orientation, $\bar{\mathbf{P}}$, $\bar{\mathbf{L}}$, $\bar{\mathbf{S}}$, and $\bar{\mathbf{V}}$, whereas the source variables are associated with space elements endowed with an outer orientation, $\tilde{\mathbf{P}}$, $\tilde{\mathbf{L}}$, $\tilde{\mathbf{S}}$, and $\tilde{\mathbf{V}}$.

This becomes a key point in computational physics, when we relate it with the discussion on the inner and outer orientations of a vector space and its dual vector space. In fact, by providing the elements of a vector space with an inner orientation, the elements of the dual vector space turn out to be automatically provided with an outer orientation, as a consequence of the Riesz representation theorem. Now, due to the geometrical interpretation of the elements of the vector spaces, given by the geometric algebra [61–72], we can associate the elements of the two vector spaces with the geometrical elements of two cell complexes, where the elements of the second cell complex (the dual cell complex, or dual complex) are the orthogonal complements of the corresponding elements in the first cell complex (the primal cell complex, or primal complex). Due to this association, by providing the elements of the first cell complex with an inner (or an outer) orientation, we induce an outer (or an inner) orientation on the second cell complex. This suggests us two considerations:

- Due to the inner rather than outer orientations of the configuration and source variables, the space of the configuration variables may be viewed as a real (or complex) inner product Hilbert space [65], H, and the space of the source variables may be viewed as its dual space, consisting of all continuous linear functionals from H into the field $\mathbb{R}$ (or the field $\mathbb{C}$). For example, a force (which is a source variable) is a covector on the space of the configuration variables because the force acts on the displacement vector (which is a configuration variable) by originating the real scalar that represents the work of the force.

- Since the source variables require an outer orientation, a proper description of a given physical phenomenon requires to use two cell complexes in relation of duality, not just one, as usually was done in computational physics before the introduction of the CM. In fact, it is true that the inner orientation of the elements of a vector space also induces an outer orientation on the elements of the same vector space and this may allow us to think that a single cell complex would be sufficient. Nevertheless, the association between the two orientations of the same cell complex is not automatic. There are always two possible criteria for establishing the correspondence between the two orientations, which depend on the orientation of the embedding space. Conversely, the relationship between inner (or outer) orientation of a cell complex and outer (or inner) orientation of its dual cell complex is derived from the Riesz representation theorem and does not depend on the orientation of the embedding space. Therefore, choosing to use two cell complexes, the one the dual of the other, instead of one single cell complex, is motivated by the need to provide a description of vector spaces that is independent of the orientation of the embedding space.

The second consideration and all that follows from it (see Section 3) were pointed out in [20] for the first time. Previously, the use of two cell complexes was presented as just a geometrical feature of the CM, whereas it is an unavoidable choice, due to the structure of bialgebra of the algebraic formulation.

Moreover, we can also observe that the cell complexes are generalizations of the oriented graphs. Consequently, all the properties of the dual graphs naturally extend to the dual cell complexes. In particular, it is worth noting that the dual graphs depend on a particular embedding. Since even the orthogonal complements (that is, the isomorphic dual vectors) and the outer orientation depend on the embedding, we will associate the outer orientation with the dual cell complex and will retain the inner orientation for the primal cell complex (Figure 6). This is why we will not take into consideration the possibility of providing the primal cell complex with an outer orientation and the dual cell complex with an inner orientation.

In doing so, the elements of the first cell complex in space and first cell complex in time are associated with those variables that require an inner orientation of the cell complex, whereas the elements of the second cell complex in space and second cell complex in time are associated with those variables that require an outer orientation of the cell complex. This together with the relationship between global variables and orientations provide the explanation of why source variables are always associated with the elements of the dual complex only [2, 39, 43–47] and configuration variables are always associated with the elements of the primal complex only. This last point was raised by Tonti in several papers and books – see, for example, [4] – but never solved before [20].

The most natural way for building the two cell complexes is starting from a primal cell complex made of simplices and providing this first cell complex with an arbitrary inner orientation. The set of the dual elements can then be chosen as any arbitrary set of staggered elements whose outer orientations provide the (known) inner orientations of the primal p–cells [2, 39, 43, 47]. In this sense, we can say that the outer orientations of the dual p–cells are induced by the inner orientations of the primal p–cells (Figure 6). We have spoken of "any" arbitrary set of staggered elements because, since the dual elements are equipped with the strong topology, there is not a unique way for defining the dual elements. In particular, they may also overlap. When they do not overlap, each p–space element of the dual cell complex can be put in dual correspon-

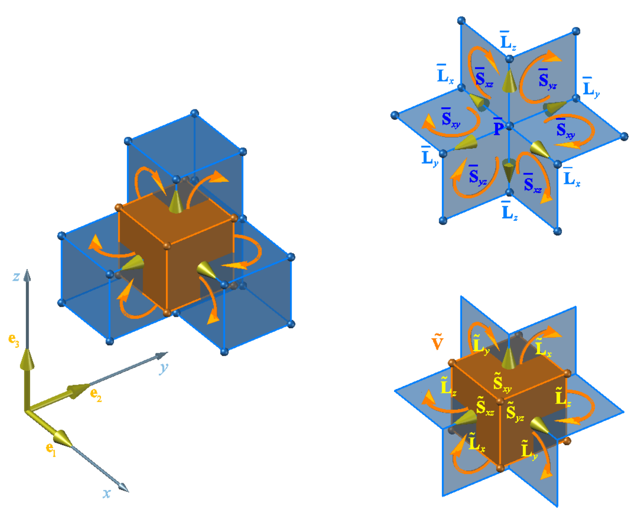

Figure 6: Relation of duality in three-dimensional space, between inner orientations of the primal cells and outer orientations of the dual cells.

dence with one $(n-p)$ –space element of the primal cell complex, staggered with respect to the former one, where n is the dimension of the space. In particular, in three-dimensional space:

- each node of the dual complex is contained in one volume of the primal complex,
- each edge of the dual complex intersects a face of the primal complex,
- each face of the dual complex is intersected by one edge of the primal complex,
- each volume of the dual complex contains one node of the primal complex.

3 The Classification diagram of the Global Variables as a Plot of a Bialgebra and Its Dual Algebra

By associating the configuration variables with the primal p–cells, the set of topological equations between global configuration variables defines a geometric algebra on the space of global configuration variables, provided with a geometric product. The operators of these topological equations are generated by the outer product of the geometric algebra, which is equal to the exterior product of the enclosed exterior algebra. The dual algebra of the enclosed exterior algebra is the space of global source variables, associated with the dual p–cells, and is provided with a dual product that is compatible with the exterior product of the exterior algebra. The topological equations between global source variables arise from the adjoint operators of the primal operators. Finally, the pairing between the exterior algebra and its dual gives rise to the energetic variables, by the interior product. Since the reversible constitutive relations may be written in terms of energetic variables, because energy is the potential of the reversible constitutive relations, the reversible constitutive relations realize the pairing between the exterior algebra and its dual.

In algebraic topology, the topological equations on the primal cell complex are coboundary processes on even exterior discrete p–forms, whereas the topological equations on the dual cell complex are coboundary processes on odd exterior discrete p–forms.

When we deduce the field variables from the corresponding global variables, the exterior discrete forms become exterior differential forms. In particular, while the configuration variables can be described by exterior differential forms of even kind, the source variables can be described by differential forms of odd kind (that is, twisted differential forms). Moreover, the elements of the group G are vectors instead of scalars, and the corresponding differential form is a vector valued differential form.

The association between the physical variables, with their topological equations, and two vector spaces, which are a bialgebra and its dual algebra, suggests us to store the global variables in a classification diagram made of two columns, that is, the column of the primal vector space, composed of the configuration variables with their topological equations, and the column of the dual vector space, composed of the source variables with their topological equations (Figure 7). This classification diagram is formally identical to the classification diagram introduced by [5], but holds an additional meaning beyond the mere classification of variables. As a matter of fact, the two columns of the classification diagram given in [5] are composed by the configuration variables, the one, and the source variables, the other, but are not put in relationship with the primal vector space and the dual vector space, as was done in Figure 7.

The configuration variables are arranged from top to bottom in their column (Figure 7), in order of increasing multiplicity of the associated space element, thus realizing a downward cochain. Conversely, the source variables are arranged from bottom to top in their column, in order of increasing multiplicity of the associated space element, thus realizing an upward cochain. With this choice, each primal p–cell is at the same level of its dual $(n-p)$ –cell.

The solid lines indicate the constitutive relations between (primal) configuration variables and (dual) source

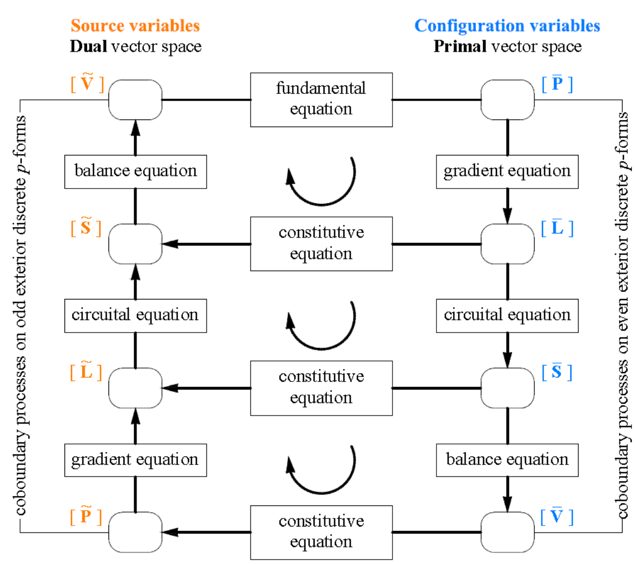

Figure 7: Classification diagram of the physical variables in the fundamental problem.

variables. They also represent the pairing between configuration and source variables.

Note that the structure of the classification diagram is the same both for the global and the field variables of every physical theory of the macrocosm. The importance of this diagram stands just in its ability of providing a concise description of physical variables, without distinguishing between the physical theories.

As observed in [4], even the variables and the equations of relativistic quantum mechanics for particles with integer spins can be arranged in a diagram, which is formally similar to the classification diagram. This leads us to assume that even the operators used in quantum mechanics for describing the microcosm can be associated with space and time elements.

When the physical phenomenon evolves in time, we have so many classification diagrams of the type shown in Figure 7 as the time instants are. Since it is not possible to draw a classification diagram for each time instant, we simply double the diagram in Figure 7 and shift it to the rear (Figure 8).

The choice of two mutually dual cell complexes also allows us to improve the description of global variables in computational physics. In fact, in the spirit of geometric algebra, where the oriented space elements are p–vectors generated by the exterior product, the attitude vectors of the p–cells are given by the inner orientation of their dual elements, the $(n-p)$ –cells. This means that two mutually dual cell complexes allows us to describe all the attributes of the p–vectors, that is, attitude vector, orientation, and magnitude. Conversely, by using just one cell complex, we cannot describe the attitude vector, but only the (unoriented) attitude.

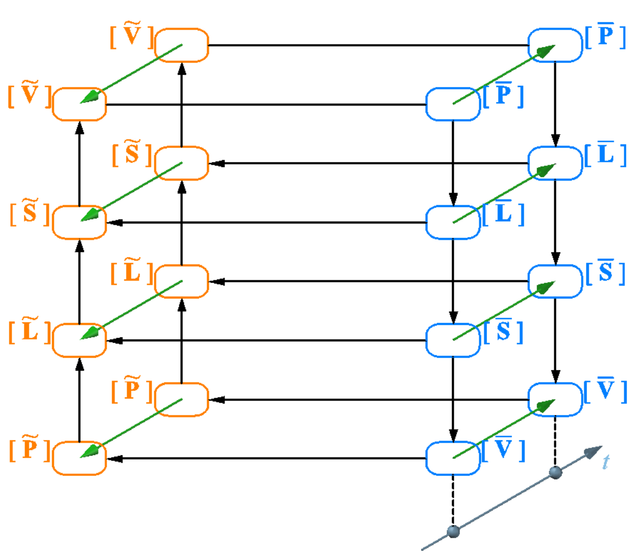

Figure 8: Space-times classification diagram of the physical variables.

In conclusion, by associating the global variables with the elements of two mutually dual cell complexes, the consequence is twofold:

- The set of the configuration variables, together with their topological equations, is a particular case of bialgebra. This leads us to enforce compatibility and equilibrium at the same time, with compatibility enforced on the primal cell complex and equilibrium enforced on the dual cell complex.
- The description of both the configuration and the source variables is improved, by allowing us to automatically take into account the attitude vectors of the p–vectors, which is impossible when the outer orientation of cell complexes is ignored.

By overturning the point of view, that is, by assuming these two conclusions as our starting point, and not as the consequence, we can find in these properties, in particular the first one, the reason why the configuration variables are associated with space elements endowed with a kind of orientation and the source variables are associated with space elements endowed with the other kind of orientation. In effect, the fact that the equilibrium operators in the fundamental problem of a given physical theory are adjoint operators of the compatibility operators does not depend on the used computational tool. It does not even depend on computation. It is a general property of the fundamental problem and, consequently, we can take it as our starting point.

In particular, by assuming for the orientation of volumes their positive orientation, the inward orientation, the relationships between equilibrium operators on source variables, grad*, div*, and curl*, and compatibility operators on configuration variables, grad, div, and curl, are

$$\mathrm{div}^* = \mathrm{grad}^T; \tag{9}$$

$$\mathrm{curl}^* = \mathrm{curl}^T; \tag{10}$$

$$\mathrm{grad}^* = \mathrm{div}^T; \tag{11}$$

whereas, by assuming for the orientation of volumes their negative orientation, the outward orientation (as usual), Eq. (9) is changed in

$$\mathrm{div}^* = -\mathrm{grad}^T; \tag{12}$$

Due to the relationship between a basis of a given vector space and its dual basis, the adjoints in Eqs. (9 – 11) indicate that it is always possible to choose the orientation of volumes in the way that the set of configuration variables, with their topological equations, is a bialgebra. Being elements of a space vector, the configuration variables are provided with inner orientations and their covectors—which, in this case, are the source variables—are provided with outer orientations.

Finally, the possibility of formulating a dual fundamental problem when the constitutive laws are reversible suggests us that, in this second case, the role of bialgebra is played by the source variables, together with the dual exterior product (leading to the topological equations between source variables). Thus, the source variables are now provided with inner orientations, whereas the configuration variables of the dual exterior algebra are provided with outer orientations. In this second case, we will denote the source variables as the dual configuration variables and the configuration variables as the dual source variables. The classification diagram for the dual fundamental problem is shown in Figure 9.

Consequently, for the computational solution of the dual fundamental problems, we have to associate the source variables (dual configuration variables) with the elements of the primal cell complex and the configuration variables (dual source variables) with the elements of the dual cell complex. This is always possible as, in a relation of mutual duality, defining which one of the two vector spaces is the exterior algebra and which one is the dual exterior algebra is just a convention.

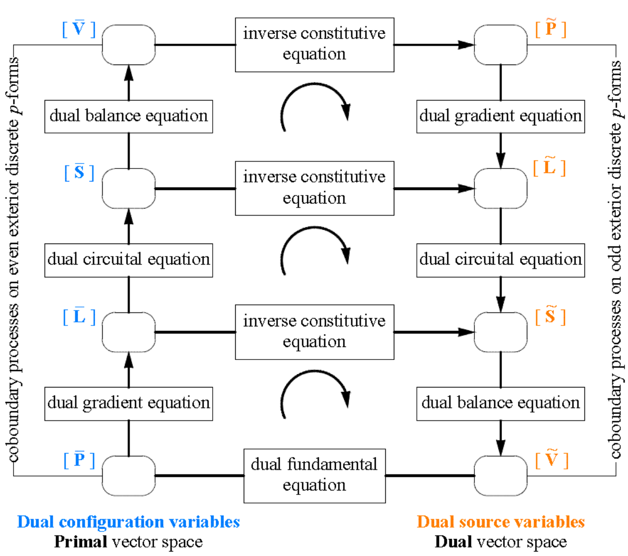

Figure 9: Classification diagram of the physical variables in the dual fundamental problem.

4 A numerical example

As we have already stated in Section 1, the algebraic formulation of the CM is more convenient than the differential formulation for treating domains made of several materials. In fact, there is a remarkable difference between global and field variables when the domain of the physical problem is composed of more than one medium: while global variables are continuous through the interface of two different media, their variations can be discontinuous. Consequently, even field variables, which are densities and rates, are generally discontinuous. The same can be said for any kind of discontinuities of the domain or the sources of the physical problem (some examples of continuity of the global variables are collected in Figure 10).

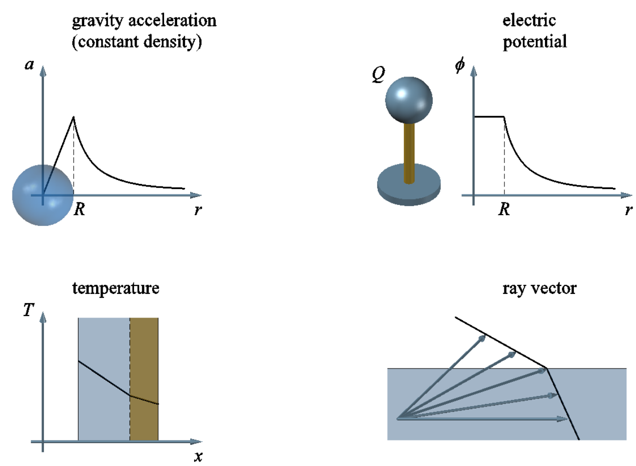

Figure 10: Continuity of the global variables associated with points in domains made of more than one medium.

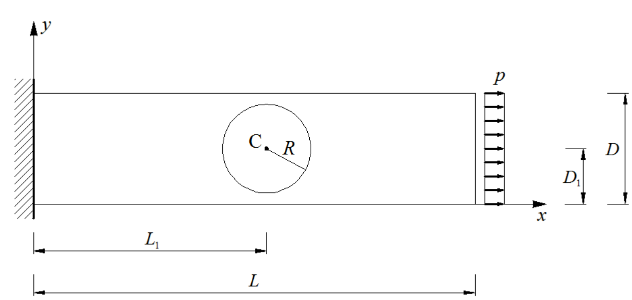

Figure 11: Geometry and loading condition of the cantilever elastic beam.

By way of example, we will employ the CM for investigating the displacement and stress fields in the cantilever elastic beam with elastic inclusion shown in Figure 11, a typical example of domain discontinuity since the mechanical properties of the inclusion differ from those of the matrix (some applications of the CM to problems with concentrated sources are collected in [39, 44–46].

The sample geometry depicted in Figure 11 is the same as that of the cantilever beam with a vertical force $P = 1kN/m$ distributed along the free right edge that we studied and compared with FEM and GDQFEM analyses in [36]. This second time, we applied an uniformly distributed horizontal load $p = 10kN/m^2$ along the free right edge (Figure 11).

The beam is $L = 4m$ long and $D = 1m$ high. The inclusion radius is $R = 0.4m$ and the location of the inclusion centre, C, is defined by $L_1 = 2.11m$ and $D_1 = 0.5m$ (Figure 11). The Young modulus and Poisson ratio of the matrix are $E = 2 \cdot 10^7 Pa$ and $\nu = 0.3$, respectively, while the mechanical properties of the circular inclusion are $E = 2 \cdot 10^{10} Pa$ and $\nu = 0.3$.

The primal cell complex used for CM analysis is a triangular mesh of Delaunay, refined along the boundary of the circular inclusion as shown in Figure 12. Once a primal cell complex has been provided, there are many ways for building a dual cell complex. For the two-dimensional domain in Figure 12, we used a barycentric dual cell complex: the dual polygons were obtained by connecting the barycenter of every triangle with the mid-points of the edges of the triangle. In so doing, the dual of each primal 1-cell (a primal side) is not a straight line.

The main effects on the stress and strain fields produced by the difference of stiffness between the matrix and the inclusion are depicted in Figures 13-20:

- The strain field is not uniform. Consequently, the horizontal displacements, u_x, are not described by linear functions in the variable x (Figure 13). In particular, since the inclusion is stiffer than the matrix, the partial derivatives $\partial u_x(x, y)/\partial x$ provide us with values that decrease when computed for points near or within the inclusion (Figure 13). Moreover, the vertical displacements u_y are almost equal to zero near and within the inclusion, whereas they are de-

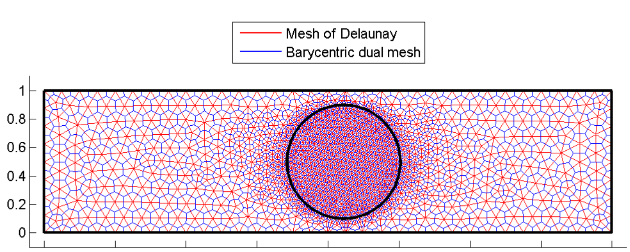

Figure 12: Primal mesh of Delaunay and barycentric dual mesh.

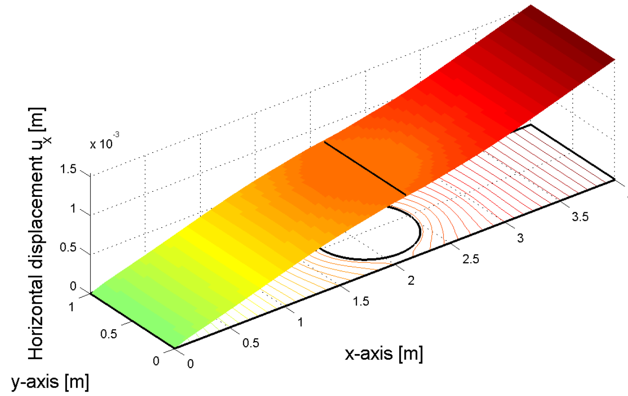

Figure 13: 3D plot of the horizontal displacements u_x.

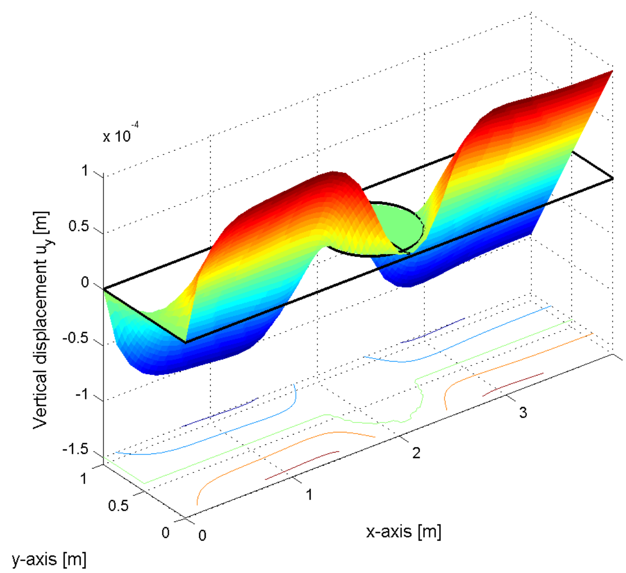

Figure 14: 3D plot of the vertical displacements u_y.

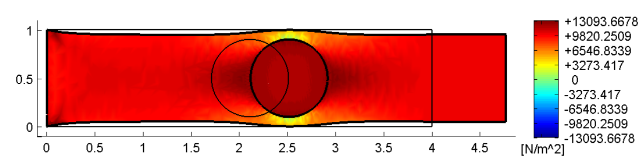

Figure 15: Normal stresses σ_x plotted on the deformed configuration (thin line: undeformed configuration; thick line: deformed configuration, amplification factor of the displacements: $k = 500$).

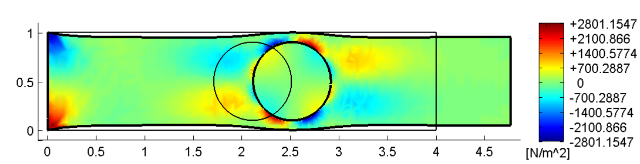

Figure 16: Shear stresses τ_{xy} plotted on the deformed configuration (thin line: undeformed configuration; thick line: deformed configuration, amplification factor of the displacements: $k = 500$).

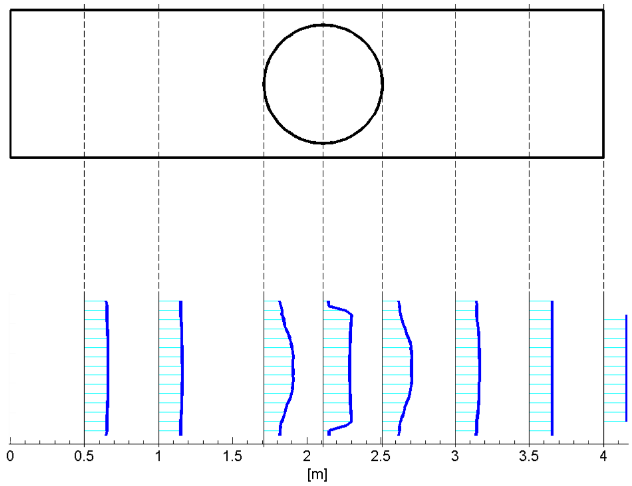

Figure 17: Plot of the normal stresses σ_x along some significant cross-sections (unit of measurement of σ_x: N/m^2, amplification factor of σ_x: $k = 1.5567 \cdot 10^{-5}$).

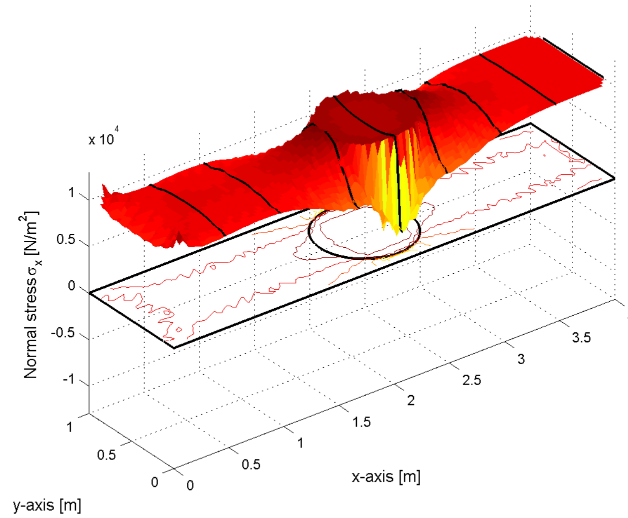

Figure 19: 3D plot of the normal stresses σ_x.

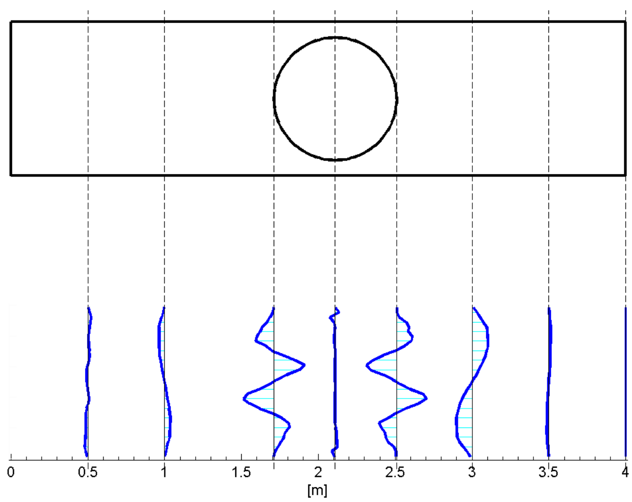

Figure 18: Plot of the shear stresses τ_{xy} along some significant cross-sections (unit of measurement of τ_{xy}: N/m^2, amplification factor of τ_{xy}: $k = 2.0562 \cdot 10^{-4}$.

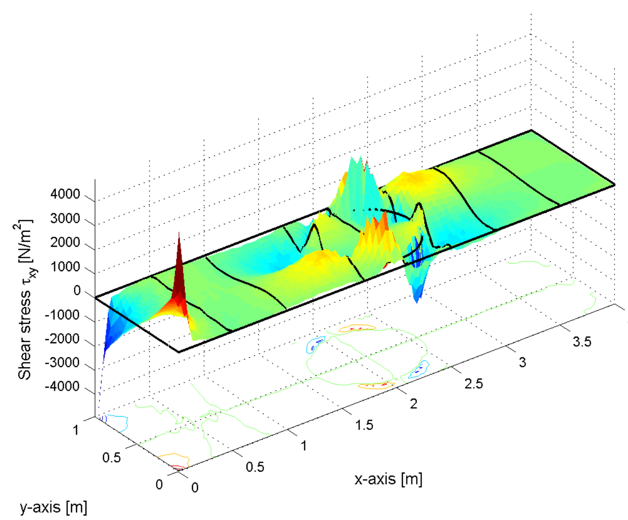

Figure 20: 3D plot of the shear stresses τ_{xy}.

scribed by skew-symmetric linear functions far from the inclusion and the constraint (Figure 14). This means that the Poisson effect on the cross-section is greater far from the inclusion (and the constraint) than near or within the inclusion (Figures 15-16).

- The isolines of the horizontal displacements u_x, shown by the contour plot in Figure 13, indicate that the cross-sections remain almost plane far from the inclusion, whereas they warp in the x-direction the more they are near to the inclusion. Moreover, near the inclusion the isolines follow the contour of the inclusion. Thus, the warping of the cross-sections is shaped by the inclusion.
- The axial normal stresses σ_x are uniformly distributed along the free right edge (cross-section $x = 4$ in Figure 17), in order to comply with the boundary conditions. Elsewhere, they cease to be uniformly distributed but remain described by symmetric functions along the cross-sections (Figure 17, 19). The main variations of σ_x occur along the bimaterial cross-sections: in particular, σ_x increases where the local stiffness is higher and decreases where the local stiffness is lower (Figures 15, 17, 19). The concentration of the higher values of σ_x in the area of the inclusion is particularly evident in the 3D plot of Figure 19. Moreover, the areas subtended to the graphs in Figure 17 are equal to each other: they equal the external axial load for reasons of equilibrium along the x-axis.

- The shear stresses τ_{xy} are equal to zero only along the free right edge (cross-section $x = 4$ in Figure 18), in order to comply with the boundary conditions. Elsewhere, the function describing the variation of the shear stresses τ_{xy} along the cross-section is a skew-symmetric function (Figures 18, 20), for reasons of equilibrium along the x-axis (the areas subtended to the graphs in Figure 18 must be equal to zero). Particularly, the peaks of τ_{xy} are higher near the external constraint and along the boundary of the circular inclusion, outside the inclusion (Figures 16, 18, 20). By excluding the ends from our analysis, we can thus conclude that the stiffness discontinuity along the boundary of the circular inclusion is one of the principal factors that originates a shear stress different from zero in a problem of axial load. If the constitutive behavior of the beam was elastic-plastic, the peaks of τ_{xy} would plasticize the material near the external constraint and the circular boundary. Moreover, the alternation of positive and negative peaks along the circular boundary (Figures 16, 20) makes the probability of having some sliding along this boundary very high. In real applications, we can discriminate whether we are faced with a problem of creep or crack propagation near the circular boundary by performing a micro-seismic analysis with increasing external load [73].

5 Conclusions

The classification diagram of the CM, originally obtained on the basis of physical considerations on the associations between physical variables and geometry, has a structure of bialgebra. In particular, the operators of the classification diagram are generated by the outer product of the geometric algebra and the exterior product of the dual algebra of the enclosed exterior algebra. The classification itself of the physical variables takes on a deeper meaning, by allowing us to associate the configuration variables with the geometric interpretation for the elements of a vector space and the source variables with the geometric interpretation for the elements of the dual vector space in the bialgebra.

The most relevant consequence of having compatibility and equilibrium operators belonging to a geometric algebra and the dual algebra of the enclosed exterior algebra, respectively, is that compatibility and equilibrium are enforced at the same time, each one in its own vector space. This makes exact both compatibility and equilibrium, the truly strength of the CM.

We have provided the results given by the CM for a cantilever elastic beam with a stiffer elastic inclusion, loaded by an axial load. Being an algebraic formulation, the CM does not require differentiable functions and is able to treat the material discontinuity of the cantilever very easily. The results clearly show the perturbation effect on the displacement and stress fields caused by the inclusion, which modifies the well known solution of De Saint Venant. In particular, the CM captures the warping of the cross-sections and the effect of stress concentration due to the stiffer inclusion, both for the normal and the shear stresses, giving very accurate representations along the boundary of the inclusion. The analysis of the shear stresses is particularly interesting: due to the stiffness difference, the inclusion generates some shear stresses along its boundary, while the stress field on the cross-section of an elastic beam without inclusion, subjected to axial loads, is composed by normal stresses only. The amplification of the shear stresses along the boundary of the inclusion may be the cause of locale damages along the boundary itself.

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