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L1-Poincaré and Sobolev inequalities for differential forms in Euclidean spaces

This is the final peer-reviewed author's accepted manuscript (postprint) of the following publication:

*Published Version:*

L1-Poincaré and Sobolev inequalities for differential forms in Euclidean spaces / Baldi, Annalisa; Franchi, Bruno; Pansu, Pierre. - In: SCIENCE CHINA. MATHEMATICS. - ISSN 1674-7283. - STAMPA. - 62:6(2019), pp. 1029-1040. [10.1007/s11425-018-9498-8]

*Availability:*

This version is available at: <https://hdl.handle.net/11585/687337> since: 2021-06-29

*Published:*

DOI: <http://doi.org/10.1007/s11425-018-9498-8>

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This is the final peer-reviewed accepted manuscript of:

**Baldi, Annalisa, Bruno Franchi, e Pierre Pansu. «L1-Poincaré and Sobolev inequalities for differential forms in Euclidean spaces». Science China Mathematics 62, n. 6 (1 giugno 2019): 1029–40.**

The final published version is available online at: <https://doi.org/10.1007/s11425-018-9498-8>

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# $L^1$ -POINCARÉ AND SOBOLEV INEQUALITIES FOR DIFFERENTIAL FORMS IN EUCLIDEAN SPACES

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## 1. INTRODUCTION

The simplest form of Poincaré inequality in an open set  $B \subset \mathbb{R}^n$  can be stated as follows: if  $1 \leq p < n$  there exists  $C(B, p) > 0$  such that for any (say) smooth function  $u$  on  $\mathbb{R}^n$  there exists a constant  $c_u$  such that

$$\|u - c_u\|_{L^q(B)} \leq C(n, p) \|\nabla u\|_{L^p(B)}$$

provided  $\frac{1}{p} - \frac{1}{q} = \frac{1}{n}$ . Sobolev inequality is very similar, but in that case we are dealing with compactly supported functions, so that the constant  $c_u$  can be dropped. It is well known (Federer & Fleming theorem, [5]) that for  $p = 1$  Sobolev inequality is equivalent to the classical isoperimetric inequality (whereas Poincaré inequality correspond to classical *relative* isoperimetric inequality).

Let us restrict for a while to the case  $B = \mathbb{R}^n$ , to investigate generalizations of these inequalities to differential forms. It is easy to see that Sobolev and Poincaré inequalities are equivalent to the following problem: we ask whether, given a closed differential 1-form  $\omega$  in  $L^p(\mathbb{R}^n)$ , there exists a 0-form  $\phi$  in  $L^q(\mathbb{R}^n)$  with  $\frac{1}{p} - \frac{1}{q} = \frac{1}{n}$  such that

$$(1) \quad d\phi = \omega \quad \text{and} \quad \|\phi\|_q \leq C(n, p, h) \|\omega\|_p.$$

Clearly, this problem can be formulated in general for  $h$ -forms  $\omega$  in  $L^p(\mathbb{R}^n)$  and we are lead to look for  $(h-1)$ -forms  $\phi$  in  $L^q(\mathbb{R}^n)$  such that (1) holds. This is the problem we have in mind when we speak about Poincaré inequality for differential forms. When we speak about Sobolev inequality, we have in mind compactly supported differential forms.

The case  $p > 1$  has been fully understood on bounded convex sets by Iwaniec & Lutoborsky ([11]). On the other hand, in the full space  $\mathbb{R}^n$  an easy proof consists in putting  $\phi = d^* \Delta^{-1} \omega$ . Here,  $\Delta^{-1}$  denotes the inverse of the Hodge Laplacian  $\Delta = d^* d + d d^*$  and  $d^*$  is the formal  $L^2$ -adjoint of  $d$ . The operator  $d^* \Delta^{-1}$  is given by convolution with a homogeneous kernel of type 1 in the terminology of [6] and [8], hence it is bounded from  $L^p$  to  $L^q$  if  $p > 1$ . Unfortunately, this argument does not suffice for  $p = 1$  since, by [8], Theorem 6.10,  $d^* \Delta^{-1}$  maps  $L^1$  only into the weak Marcinkiewicz space  $L^{n/(n-1), \infty}$ . Upgrading from  $L^{n/(n-1), \infty}$  to  $L^{n/(n-1)}$  is possible for functions (see [14], [9], [10]), but the trick does not seem to generalize to differential forms.

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1991 *Mathematics Subject Classification.* 58A10, 26D15, 46E35.

*Key words and phrases.* Differential forms, Sobolev-Poincaré inequalities, homotopy formula.  
 TO APPEAR IN SCIENCE CHINA MATHEMATICS

Since the case  $p = 1$  is the most relevant from a geometric point of view, we focus on that case. First of all, we notice that Poincaré inequality with  $p = 1$  fails in top degree unless a global integral inequality is satisfied. Indeed for  $h = n$  forms belonging to  $L^1$  and with nonvanishing integral cannot be differentials of  $L^{n/(n-1)}$  forms, see [15]. In arbitrary degree, a similar integral obstruction takes the form  $\int \omega \wedge \beta = 0$  for every constant coefficient form  $\beta$  of complementary degree. Therefore we introduce the subspace  $L_0^1$  of  $L^1$ -differential forms satisfying these conditions. However, in degree  $n$  assuming that the integral constraint is satisfied does not suffice, as we shall see in Section 4. On the other hand, for instance it follows from [4] that Poincaré inequality holds in degree  $n - 1$ . We refer the reader to [1] for a discussion, in particular in connection with van Schaftingen's [17]) and Lanzani & Stein's [13] results.

We can state our main results. We have:

**Theorem 1.1** (Global Poincaré and Sobolev inequalities). *Let  $h = 1, \dots, n-1$  and set  $q = n/(n-1)$ . For every closed  $h$ -form  $\alpha \in L_0^1(\mathbb{R}^n)$ , there exists an  $(h-1)$ -form  $\phi \in L^q(\mathbb{R}^n)$ , such that*

$$d\phi = \alpha \quad \text{and} \quad \|\phi\|_q \leq C \|\alpha\|_1.$$

Furthermore, if  $\alpha$  is compactly supported, so is  $\phi$ .

We also prove a local version of this inequality.

**Corollary 1.2.** *For  $h = 1, \dots, n-1$ , let  $q = n/(n-1)$ . Let  $B \subset \mathbb{R}^n$  be a bounded open convex set, and let  $B'$  be an open set,  $B \Subset B'$ . Then there exists  $C = C(n, B, B')$  with the following property:*

- (1) *Interior Poincaré inequality. For every closed  $h$ -form  $\alpha$  in  $L^1(B')$ , there exists an  $(h-1)$ -form  $\phi \in L^q(B)$ , such that*

$$d\phi = \alpha|_B, \quad \text{and} \quad \|\phi\|_{L^q(B)} \leq C \|\alpha\|_{L^1(B')}.$$

- (2) *Sobolev inequality. For every closed  $h$ -form  $\alpha \in L^1$  with support in  $B$ , there exists an  $(h-1)$ -form  $\phi \in L^q$ , with support in  $B'$ , such that*

$$d\phi = \alpha \quad \text{and} \quad \|\phi\|_{L^q(B')} \leq C \|\alpha\|_{L^1(B)}.$$

We shall refer to the above inequality as *interior Poincaré* and *interior Sobolev inequality*, respectively. The word “interior” is meant to stress the loss of domain from  $B'$  to  $B$ .

Remarkably, most of the techniques developed here can be adapted, in combination with other *ad hoc* arguments to deal with Poincaré and Sobolev inequalities in the Rumin complex of Heisenberg groups (see [1]).

## 2. KERNELS

Throughout the present note our setting will be the Euclidean space  $\mathbb{R}^n$  with  $n > 2$ .

If  $f$  is a real function defined in  $\mathbb{R}^n$ , we denote by  ${}^v f$  the function defined by  ${}^v f(p) := f(-p)$ , and, if  $T \in \mathcal{D}'(\mathbb{R}^n)$ , then  ${}^v T$  is the distribution defined by  $\langle {}^v T | \phi \rangle := \langle T | {}^v \phi \rangle$  for any test function  $\phi$ .

We remind also that the convolution  $f * g$  is well defined when  $f, g \in \mathcal{D}'(\mathbb{R}^n)$ , provided at least one of them has compact support. In this case the following identities hold

$$(2) \quad \langle f * g | \phi \rangle = \langle g | {}^\vee f * \phi \rangle \quad \text{and} \quad \langle f * g | \phi \rangle = \langle f | \phi * {}^\vee g \rangle$$

for any test function  $\phi$ .

Following [6], Definition 5.3, we recall now the notion of *kernel of type  $\mu$*  and some properties stated below in Proposition 2.2.

**Definition 2.1.** *A kernel of type  $\mu$  is a distribution  $K \in \mathcal{S}'(\mathbb{R}^n)$ , homogeneous of degree  $\mu - n$  that is smooth outside of the origin.*

*The convolution operator with a kernel of type  $\mu$*

$$f \rightarrow f * K$$

*is still called an operator of type  $\mu$ .*

**Proposition 2.2.** *Let  $K \in \mathcal{S}'(\mathbb{R}^n)$  be a kernel of type  $\mu$  and let  $D_j$  denote the  $j$ -th partial derivative in  $\mathbb{R}^n$ .*

- i)  ${}^\vee K$  is again a kernel of type  $\mu$ ;
- ii)  $D_j K$  and  $K D_j$  are associated with kernels of type  $\mu - 1$  for  $j = 1, \dots, n$ ;
- iii) If  $\mu > 0$ , then  $K \in L^1_{\text{loc}}(\mathbb{R}^n)$ .

**Lemma 2.3.** *Let  $g$  be a kernel of type  $\mu > 0$ , and let  $\psi \in \mathcal{D}(\mathbb{R}^n)$  be a test function. Then  $\psi * g$  is smooth on  $\mathbb{R}^n$ .*

*If, in addition,  $R = R(D)$  is an homogeneous polynomial of degree  $\ell \geq 0$  in  $D := (D_1, \dots, D_n)$ , we have*

$$R(\psi * g)(p) = O(|p|^{\mu-n-\ell}) \quad \text{as } p \rightarrow \infty.$$

In particular, if  $\psi \in \mathcal{D}(\mathbb{R}^n)$ , and  $K$  is a kernel of type  $\mu < n$ , then both  $\psi * K$  and all its derivatives belong to  $L^\infty(\mathbb{R}^n)$

**Corollary 2.4.** *If  $K$  is a kernel of type  $\mu \in (0, n)$ ,  $u \in L^1(\mathbb{R}^n)$  and  $\psi \in \mathcal{D}(\mathbb{R}^n)$ , then*

$$(3) \quad \langle u * K | \psi \rangle = \langle u | \psi * {}^\vee K \rangle.$$

In this equation, the left hand side is the action of a distribution on a test function, see formula (8), the right hand side is the inner product of an  $L^1$  vector-valued function with an  $L^\infty$  vector-valued function.

**Remark 2.5.** *The conclusion of Corollary 2.4 still holds if we assume  $K \in L^1_{\text{loc}}(\mathbb{R}^n)$ , provided  $u \in L^1(\mathbb{R}^n)$  is compactly supported.*

**Lemma 2.6.** *Let  $K$  be a kernel of type  $\alpha \in (0, n)$ , then for any  $f \in L^1(\mathbb{R}^n)$  such that*

$$\int_{\mathbb{R}^n} f(y) dy = 0,$$

*we have:*

$$R^{-\alpha} \int_{B(0, 2R) \setminus B(0, R)} |K * f| dx \longrightarrow 0 \quad \text{as } R \rightarrow \infty.$$

*Proof.* If  $R > 1$  we have:

$$\begin{aligned}
R^{-\alpha} \int_{R < |x| < 2R} |K * f| dx &= R^{-\alpha} \int_{R < |x| < 2R} dx \left| \int K(x-y) f(y) dy \right| \\
&= R^{-\alpha} \int_{R < |x| < 2R} dx \left| \int [K(x-y) - K(x)] f(y) dy \right| \\
&\leq R^{-\alpha} \int |f(y)| \left( \int_{R < |x| < 2R} |K(x-y) - K(x)| dx \right) dy \\
&= R^{-\alpha} \int_{|y| < \frac{1}{2}R} |f(y)| (\dots) dy + R^{-\alpha} \int_{4R > |y| > \frac{1}{2}R} |f(y)| (\dots) dy \\
&\quad + R^{-\alpha} \int_{|y| > 4R} |f(y)| (\dots) dy \\
&=: R^{-\alpha} I_1(R) + R^{-\alpha} I_2(R) + R^{-\alpha} I_3(R).
\end{aligned}$$

Consider first the third term above. By homogeneity we have

$$I_3(R) \leq C_K \int_{|y| > 4R} |f(y)| \left( \int_{R < |x| < 2R} (|x-y|^{-n+\alpha} + |x|^{-n+\alpha}) dx \right) dy$$

Notice now that, if  $|y| > 4R$  and  $R < |x| < 2R$ , then  $|x-y| \geq |y| - |x| \geq 4R - R \geq \frac{3}{2}|x|$ . Therefore

$$|x-y|^{-n+\alpha} + |x|^{-n+\alpha} \leq \left\{ \left( \frac{2}{3} \right)^{n-\alpha} + 1 \right\} |x|^{-n+\alpha},$$

and then

$$\int_{R < |x| < 2R} (|x-y|^{-n+\alpha} + |x|^{-n+\alpha}) dx \leq C_\alpha R^\alpha.$$

Thus

$$R^{-\alpha} I_3(R) \leq C_{K,\alpha} \int_{|y| > 4R} |f(y)| dy \longrightarrow 0$$

as  $R \rightarrow \infty$ .

Consider now the second term. Again we have

$$I_2(R) \leq C_K \int_{\frac{1}{2}R < |y| < 4R} |f(y)| \left( \int_{R < |x| < 2R} (|x-y|^{-n+\alpha} + |x|^{-n+\alpha}) dx \right) dy.$$

Obviously, as above,

$$\int_{R < |x| < 2R} |x|^{-n+\alpha} dx \leq C R^\alpha.$$

Notice now that, if  $\frac{1}{2}R < |y| < 4R$  and  $R < |x| < 2R$ , then  $|x-y| \leq |x| + |y| \leq 6R$ .

Hence

$$\int_{\frac{1}{2}R < |y| < 4R} |f(y)| \left( \int_{|x-y| < 6R} |x-y|^{-n+\alpha} dx \right) dy \leq C R^\alpha.$$

Therefore

$$R^{-\alpha} I_2(R) \leq C_K \int_{\frac{1}{2}R < |y| < 4R} |f(y)| dy \longrightarrow 0$$

as  $R \rightarrow \infty$ . Finally, if  $|y| < \frac{R}{2}$  and  $R < |x| < 2R$  we have  $|y| < \frac{1}{2}|x|$  so that, by [8] Proposition 1.7 and Corollary 1.16,

$$\begin{aligned} R^{-\alpha} I_1(R) &\leq C_K \int_{|y| < \frac{1}{2}R} |f(y)| \left( \int_{R < |x| < 2R} \frac{|y|}{|x|^{n-\alpha+1}} dx \right) dy \\ &= C_K \int_{\mathbb{R}^n} |f(y)| |y| \chi_{[0, \frac{1}{2}R]}(|y|) \left( R^{-\alpha} \int_{R < |x| < 2R} \frac{1}{|x|^{n-\alpha+1}} dx \right) dy \\ &\leq C_K \int_{\mathbb{R}^n} |f(y)| |y| \chi_{[0, \frac{1}{2}R]}(|y|) R^{-1} dy =: C_K \int_{\mathbb{R}^n} |f(y)| H_R(|y|) dy. \end{aligned}$$

Obviously, for any fixed  $y \in \mathbb{H}^n$  we have  $(|y|)H_R(|y|) \rightarrow 0$  as  $R \rightarrow \infty$ . On the other hand,  $|f(y)|H_R(|y|) \leq \frac{1}{2}|f(y)|$ , so that, by dominated convergence theorem,

$$R^{-\alpha} I_1(R) \rightarrow 0$$

as  $R \rightarrow \infty$ .

This completes the proof of the lemma.  $\square$

**Definition 2.7.** Let  $f$  be a measurable function on  $\mathbb{R}^n$ . If  $t > 0$  we set

$$\lambda_f(t) = |\{|f| > t\}|.$$

If  $1 \leq p \leq \infty$  and

$$\sup_{t>0} \lambda_f^p(t) < \infty,$$

we say that  $f \in L^{p,\infty}(\mathbb{R}^n)$ .

**Definition 2.8.** Following [3], Definition A.1, if  $1 < p < \infty$ , we set

$$\|u\|_{M^p} := \inf \{C \geq 0; \int_K |u| dx \leq C|K|^{1/p'} \text{ for all } L\text{-measurable set } K \subset \mathbb{R}^n\}.$$

By [3], Lemma A.2, we obtain

**Lemma 2.9.** If  $1 < p < \infty$ , then

$$\frac{(p-1)^p}{p^{p+1}} \|u\|_{M^p}^p \leq \sup_{\lambda>0} \{\lambda^p |\{|u| > \lambda\}|\} \leq \|u\|_{M^p}^p.$$

In particular, if  $1 < p < \infty$ , then  $M^p = L^{p,\infty}(\mathbb{R}^n)$ .

**Corollary 2.10.** If  $1 \leq s < p$ , then  $M^p \subset L_{\text{loc}}^s(\mathbb{R}^n) \subset L_{\text{loc}}^1(\mathbb{R}^n)$ .

*Proof.* If  $u \in M^p$  then  $|u|^s \in M^{p/s}$ , and we can conclude thanks to Definition 2.8.  $\square$

**Lemma 2.11.** Let  $E$  be a kernel of type  $\alpha \in (0, n)$ . Then for all  $f \in L^1(\mathbb{R}^n)$  we have  $f * E \in M^{n/(n-\alpha)}$  and there exists  $C > 0$  such that

$$\|f * E\|_{M^{n/(n-\alpha)}} \leq C \|f\|_{L^1(\mathbb{R}^n)}$$

for all  $f \in L^1(\mathbb{R}^n)$ . In particular, by Corollary 2.10,  $f * E \in L_{\text{loc}}^1$ .

As in [2], Lemma 4.4 and Remark 4.5, we have:

**Remark 2.12.** Suppose  $0 < \alpha < n$ . If  $K$  is a kernel of type  $\alpha$  and  $\psi \in \mathcal{D}(\mathbb{R}^n)$ ,  $\psi \equiv 1$  in a neighborhood of the origin, then the statements of Lemma 2.11 still hold if we replace  $K$  by  $(1 - \psi)K$  or by  $\psi K$ .

## 3. DIFFERENTIAL FORMS AND CURRENTS

Let  $(dx_1, \dots, dx_n)$  be the canonical basis of  $(\mathbb{R}^n)^*$  and indicate as  $\langle \cdot, \cdot \rangle$  the inner product in  $(\mathbb{R}^n)^*$  that makes  $(dx_1, \dots, dx_n)$  an orthonormal basis. We put  $\bigwedge^0(\mathbb{R}^n) := \mathbb{R}$  and, for  $1 \leq h \leq n$ ,

$$\bigwedge^h(\mathbb{R}^n) := \text{span}\{dx_{i_1} \wedge \dots \wedge dx_{i_h} : 1 \leq i_1 < \dots < i_h \leq n\}$$

the linear space of the alternating  $h$ -forms on  $\mathbb{R}^n$ . If  $I := (i_1, \dots, i_h)$  with  $1 \leq i_1 < \dots < i_h \leq n$ , we set  $|I| := h$  and

$$dx^I := dx_{i_1} \wedge \dots \wedge dx_{i_h}.$$

We indicate as  $\langle \cdot, \cdot \rangle$  also the inner product in  $\bigwedge^h(\mathbb{R}^n)$  that makes  $(dx_1, \dots, dx_n)$  an orthonormal basis.

By translation,  $\bigwedge^h(\mathbb{R}^n)$  defines a fibre bundle over  $\mathbb{R}^n$ , still denoted by  $\bigwedge^h(\mathbb{R}^n)$ . A differential form on  $\mathbb{R}^n$  is a section of this fibre bundle.

Through this Note, if  $0 \leq h \leq n$  and  $\mathcal{U} \subset \mathbb{R}^n$  is an open set, we denote by  $\Omega^h(\mathcal{U})$  the space of differential  $h$ -forms on  $\mathcal{U}$ , and by  $d : \Omega^h(\mathcal{U}) \rightarrow \Omega^{h+1}(\mathcal{U})$  the exterior differential. Thus  $(\Omega^\bullet(\mathcal{U}), d)$  is the de Rham complex in  $\mathcal{U}$  and any  $u \in \Omega^h$  can be written as

$$u = \sum_{|I|=h} u_I dx^I.$$

**Definition 3.1.** If  $\mathcal{U} \subset \mathbb{R}^n$  is an open set and  $0 \leq h \leq n$ , we say that  $T$  is a  $h$ -current on  $\mathcal{U}$  if  $T$  is a continuous linear functional on  $\mathcal{D}(\mathcal{U}, \bigwedge^h(\mathbb{R}^n))$  endowed with the usual topology. We write  $T \in \mathcal{D}'(\mathcal{U}, \bigwedge^h(\mathbb{R}^n))$ . If  $u \in L^1_{\text{loc}}(\mathcal{U}, \bigwedge^h(\mathbb{R}^n))$ , then  $u$  can be identified canonically with a  $h$ -current  $T_u$  through the formula

$$\langle T_u | \varphi \rangle := \int_{\mathcal{U}} u \wedge * \varphi = \int_{\mathcal{U}} \langle u, \varphi \rangle dx$$

for any  $\varphi \in \mathcal{D}(\mathcal{U}, \bigwedge^h(\mathbb{R}^n))$ .

From now on, if there is no way to misunderstandings, and  $u \in L^1_{\text{loc}}(\mathcal{U}, \bigwedge^h(\mathbb{R}^n))$ , we shall write  $u$  instead of  $T_u$ .

Suppose now  $u$  is sufficiently smooth (take for instance  $u \in C^\infty(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n))$ ). If  $\phi \in \mathcal{D}(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n))$ , then, by the Green formula

$$\int_{\mathbb{R}^n} \langle du, \phi \rangle dx = \int_{\mathbb{R}^n} \langle u, d^* \phi \rangle dx.$$

Thus, if  $T \in \mathcal{D}'(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n))$ , it is natural to set

$$\langle dT | \phi \rangle = \langle T | d^* \phi \rangle$$

for any  $\phi \in \mathcal{D}(\mathbb{R}^n, \bigwedge^{h+1}(\mathbb{R}^n))$ .

Analogously, if  $T \in \mathcal{D}'(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n))$ , we set

$$\langle d^* T | \phi \rangle = \langle T | d \phi \rangle$$

for any  $\phi \in \mathcal{D}(\mathbb{R}^n, \bigwedge^{h-1}(\mathbb{R}^n))$ .

Notice that, if  $u \in L^1_{\text{loc}}(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n))$

$$\langle u | d^* \phi \rangle = \int_{\mathbb{R}^n} u \wedge * d^* \phi = (-1)^{h+1} \int_{\mathbb{R}^n} u \wedge d^* (* \phi).$$



A straightforward approximation argument yields the following identity:

**Lemma 3.2.** *Let  $u \in L^1(\mathbb{R}^n, \bigwedge^{h+1}(\mathbb{R}^n))$  be a closed form, and let  $K$  be a kernel of type  $\mu \in (0, n)$ . If  $\psi \in \mathcal{D}(\mathbb{R}^n, \Omega^h)$ , then*

$$(4) \quad \int \langle u, d^*(\psi * K) \rangle dx = 0.$$

**Definition 3.3.** *In  $\mathbb{R}^n$ , we define the Laplace-Beltrami operator  $\Delta_h$  on  $\Omega^h$  by*

$$\Delta_h = dd^* + d^*d$$

Notice that  $-\Delta_0 = \sum_{j=1}^{2n} \partial_j^2$  is the usual Laplacian of  $\mathbb{R}^n$ .

**Proposition 3.4** (see e.g. [12] (2.1.28)). *If  $u = \sum_{|I|=h} u_I dx^I$ , then*

$$\Delta u = - \sum_{|I|=h} (\Delta u_I) dx^I.$$

For sake of simplicity, since a basis of  $\bigwedge^h(\mathbb{R}^n)$  is fixed, the operator  $\Delta_h$  can be identified with a diagonal matrix-valued map, still denoted by  $\Delta_h$ ,

$$(5) \quad \Delta_h = -(\delta_{ij} \Delta)_{i,j=1, \dots, \dim \bigwedge^h(\mathbb{R}^n)} : \mathcal{D}'(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n)) \rightarrow D'(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n)),$$

where  $D'(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n))$  is the space of vector-valued distributions on  $\mathbb{R}^n$ .

If we denote by  $\Delta^{-1}$  the matrix valued kernel

$$(6) \quad \Delta_h^{-1} = -(\delta_{ij} \Delta^{-1})_{i,j=1, \dots, \dim \bigwedge^h(\mathbb{R}^n)} : \mathcal{D}'(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n)) \rightarrow D'(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n)),$$

then  $\Delta_h^{-1}$  is a matrix-valued kernel of type 2 and

$$\Delta_h^{-1} \Delta_h \alpha = \Delta_h \Delta_h^{-1} \alpha = \alpha \quad \text{for all } \alpha \in \mathcal{D}(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n)).$$

We notice that, if  $n > 1$ , since  $\Delta_h^{-1}$  is associated with a kernel of type 2  $\Delta_h^{-1} f$  is well defined when  $f \in L^1(\mathbb{H}^n, E_0^h)$ . More precisely, by Lemma 2.11 we have:

**Lemma 3.5.** *If  $1 \leq h < n$ , and  $R = R(D)$  is a homogeneous polynomial of degree  $\ell = 1$  in  $D_1, \dots, D_n$ , we have:*

$$\|f * R(D) \Delta_h^{-1}\|_{M^{n/(n-1)}} \leq C \|f\|_{L^1(\mathbb{R}^n)}$$

for all  $f \in L^1(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n))$ .

By Corollary 2.10, in both cases  $f * R(D) \Delta_h^{-1} \in L_{\text{loc}}^1(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n))$ . In particular, the map

$$(7) \quad \Delta_h^{-1} : L^1(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n)) \longrightarrow L_{\text{loc}}^1(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n))$$

is continuous.

**Remark 3.6.** *By Lemma 2.4, if  $u \in L^1(\mathbb{R}^n, \bigwedge^{h+1}(\mathbb{R}^n))$  and  $\psi \in \mathcal{D}(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n))$ , then*

$$(8) \quad \langle \Delta_h^{-1} u | \psi \rangle = \langle u | \Delta_h^{-1} \psi \rangle.$$

In this equation, the left hand side is the action of a matrix-valued distribution on a vector-valued test function, see formula (6), whereas the right hand side is the inner product of an  $L^1$  vector-valued function with an  $L^\infty$  vector-valued function.

A standard argument yields the following identities:

**Lemma 3.7** (see [2], Lemma 4.11). *If  $\alpha \in \mathcal{D}(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n))$ , then*

- i)  $d\Delta_h^{-1}\alpha = \Delta_{h+1}^{-1}d\alpha, \quad h = 0, 1, \dots, n-1,$
- iv)  $d^*\Delta_{\mathbb{H},h}^{-1}\alpha = \Delta_{\mathbb{H},h-1}^{-1}d^*\alpha \quad h = 1, \dots, n.$

**Lemma 3.8.** *If  $\alpha \in L^1(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n))$ , then  $\Delta_h^{-1}\alpha$  is well defined and belongs to  $L_{\text{loc}}^1(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n))$ . If in addition  $d\alpha = 0$  in the distributional sense, then the following result holds:*

$$d\Delta_h^{-1}\alpha = 0.$$

*Proof.* Let  $\phi \in \mathcal{D}(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n))$  be arbitrarily given. By Lemma 3.7,  $d\Delta_h^{-1}\phi = \Delta_h^{-1}d\phi$ . Thus Remark 3.6 and Lemma 3.2 yield

$$\langle d\Delta_h^{-1}\alpha | \phi \rangle = \langle \Delta_h^{-1}\alpha | d\phi \rangle = \langle \alpha | \Delta_h^{-1}d\phi \rangle = \langle \alpha | d\Delta_h^{-1}\phi \rangle = 0.$$

□

#### 4. $n$ -PARABOLICITY

Recall that a noncompact Riemannian manifold  $M$  is  $p$ -parabolic if for every compact subset  $K$  and every  $\epsilon > 0$ , there exists a smooth compactly supported function  $\chi$  on  $M$  such that  $\chi \geq 1$  on  $K$  and

$$\int_M |d\chi|^p < \epsilon.$$

It is well known that Euclidean  $n$ -space is  $n$ -parabolic (the relevant functions  $\chi$  can be taken to be piecewise affine functions of  $\log r$ , where  $r$  is the distance to the origin). It follows that Sobolev inequality in  $L^n$  cannot hold, and, as we saw in the introduction, that the Poincaré inequality on  $n$ -forms fails as well.

Here, we explain an other consequence of  $n$ -parabolicity.

**Proposition 4.1.** *Let  $\omega$  be a  $k$ -form in  $L^1(\mathbb{R}^n)$ . Assume that  $\omega = d\phi$  where  $\phi \in L^{n/(n-1)}(\mathbb{R}^n)$ . Then, for every constant coefficient  $n-k$ -form  $\beta$ ,*

$$\int_{\mathbb{R}^n} \omega \wedge \beta = 0.$$

*Proof.* Let  $\chi_R$  be a smooth compactly supported function on  $\mathbb{R}^n$  such that  $\chi_R = 1$  on  $B(R)$  and  $\int |d\chi_R|^n \leq \frac{1}{R}$ . Let  $\omega_R = d(\chi_R\phi)$ . Then, since  $\chi_R\phi \wedge \beta$  is compactly supported,

$$\int_{\mathbb{R}^n} \omega_R \wedge \beta = \int_{\mathbb{R}^n} d(\chi_R\phi \wedge \beta) = 0.$$

Write  $\omega_R = d\chi_R \wedge \phi + \chi_R\omega$ . Since

$$\left| \int_{\mathbb{R}^n} d\chi_R \wedge \phi \wedge \beta \right| \leq \|d\chi_R\|_n \|\phi\|_{n/(n-1)} \|\beta\|_\infty \leq \frac{C}{R^{1/n}}$$

tends to 0,

$$\begin{aligned} \int_{\mathbb{R}^n} \omega \wedge \beta &= \lim_{R \rightarrow \infty} \int_{\mathbb{R}^n} \chi_R \omega \wedge \beta \\ &= - \lim_{R \rightarrow \infty} \int_{\mathbb{R}^n} \omega_R \wedge \beta = 0. \end{aligned}$$

□

In other words, the vanishing of all integrals  $\int \omega \wedge \beta$  is a necessary condition for an  $L^1$   $k$ -form to be the differential of an  $L^{n/(n-1)}$   $k-1$ -form.

## 5. MAIN RESULTS

The following estimate provides primitives for globally defined closed  $L^1$ -forms, and can be derived from Lanzani & Stein inequality [13], approximating closed forms in  $L_0^1(\mathbb{R}^n), \bigwedge^h(\mathbb{R}^n)$  by means of closed compactly supported smooth form. The convergence of the approximation is guaranteed by Lemma 2.6.

**Proposition 5.1.** *Denote by  $L_0^1(\mathbb{R}^n), \bigwedge^h(\mathbb{R}^n)$  the subspace of  $L^1(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n))$  of forms with vanishing average, and by  $\mathcal{H}^1(\mathbb{R}^n)$  the classic real Hardy space (see [16], Chapter 3). We have:*

i) *if  $h < n$ , then*

$$\|d^* \Delta_h^{-1} u\|_{L^{n/(n-1)}(\mathbb{R}^n)} \leq C \|u\|_{L^1(\mathbb{R}^n)} \quad \text{for all } u \in L_0^1(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n)) \cap \ker d;$$

ii) *if  $h = n$ , then*

$$\|d^* \Delta_n^{-1} u\|_{L^{n/(n-1)}(\mathbb{R}^n)} \leq C \|u\|_{\mathcal{H}^1(\mathbb{R}^n)} \quad \text{for all } u \in \mathcal{H}^1(\mathbb{R}^n) \cap \ker d$$

We stress that the vanishing average assumption is necessary (see Proposition 4.1).

A standard approximation argument (akin to that of the classical Meyer & Serrin Theorem) yields the following density result.

**Lemma 5.2.** *Let  $B \subset \mathbb{R}^n$  an open set. If  $0 \leq h \leq n$ , we set*

$$(L^1 \cap d^{-1} L^1)(B, \bigwedge^h(\mathbb{R}^n)) := \{\alpha \in L^1(B, \bigwedge^h(\mathbb{R}^n)); d\alpha \in L^1(B, \bigwedge^{h+1}(\mathbb{R}^n))\},$$

*endowed with the graph norm. Then  $C^\infty(B, \bigwedge^h(\mathbb{R}^n))$  is dense in  $(L^1 \cap d^{-1} L^1)(B, \bigwedge^h(\mathbb{R}^n))$ .*

Again through an approximation argument we can prove the following two lemmata:

**Lemma 5.3.** *If  $K = d^* \Delta_c^{-1}$ , then:*

- *$K$  is a kernel of type 1;*
- *if  $\chi$  is a smooth function with compact support in  $B$ , then the identity*

$$\chi = dK\chi + Kd\chi$$

*holds on the space  $(L^1 \cap d^{-1}(L^1))(B, \bigwedge^\bullet(\mathbb{R}^n))$ .*

**Lemma 5.4.** *If  $1 \leq h < n$ , let  $\psi \in L^1(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n))$  be a compactly supported form with  $d\psi \in L^1(\mathbb{R}^n, \bigwedge^{h+1}(\mathbb{R}^n))$ , and let  $\xi \in \bigwedge^{2n-h}$  be a constant coefficient form. Then*

$$\int_{\mathbb{R}^n} d\psi \wedge \xi = 0.$$

We are able now to prove the following (approximate) homotopy formula for closed forms.

**Proposition 5.5.** *Let  $B \Subset B'$  be open sets in  $\mathbb{R}^n$ . For  $h = 1, \dots, n-1$ , take  $q = n/(n-1)$ . Then there exists a smoothing operator  $S : L^1(B', \bigwedge^h(\mathbb{R}^n)) \rightarrow W^{s,q}(B, \bigwedge^h(\mathbb{R}^n))$  for every  $s \in \mathbb{N}$ , and a bounded operator  $T : L^1(B', \bigwedge^h(\mathbb{R}^n)) \rightarrow L^q(B, \bigwedge^{h-1}(\mathbb{R}^n))$  such that, for closed  $L^1$ -forms  $\alpha$  on  $B'$ ,*

$$(9) \quad \alpha = dT\alpha + S\alpha \quad \text{on } B.$$

*In particular,  $S\alpha$  is closed.*

*Furthermore,  $T$  and  $S$  merely enlarge by a small amount the support of compactly supported differential forms.*

*Proof.* Let us fix two open sets  $B_0$  and  $B_1$  with

$$B \Subset B_0 \Subset B_1 \Subset B',$$

and a cut-off function  $\chi \in \mathcal{D}(B_1)$ ,  $\chi \equiv 1$  on  $B_0$ . If  $\alpha \in (L^1 \cap d^{-1})(B', \bigwedge^\bullet(\mathbb{R}^n))$ , we set  $\alpha_0 = \chi\alpha$ , continued by zero outside  $B_1$ . Denote by  $k$  the kernel associated with  $K$  in Lemma 5.3. We consider a cut-off function  $\psi_R$  supported in a  $R$ -neighborhood of the origin, such that  $\psi_R \equiv 1$  near the origin. Then we can write  $k = k\psi_R + (1 - \psi_R)k$ . Thus, let us denote by  $K_R$  the convolution operator associated with  $\psi_R k$ . By Lemma 5.3,

$$(10) \quad \begin{aligned} \alpha_0 &= dK\alpha_0 + K_d\alpha_0 \\ &= dK_R\alpha_0 + K_R d\alpha_0 + S\alpha_0, \end{aligned}$$

where  $S_0$  is defined by

$$S\alpha_0 := d((1 - \psi_R)k * \alpha_0) + (1 - \psi_R)k * d\alpha_0.$$

We set

$$T_1\alpha := K_R\alpha_0, \quad S_1\alpha := S\alpha_0.$$

If  $\beta \in L^1(B_1, \bigwedge^h(\mathbb{R}^n))$ , we set

$$T_1\beta := K_R(\chi\beta)|_B, \quad S_1\alpha := S\alpha_0|_B.$$

We notice that, provided  $R > 0$  is small enough, the values of  $T_1\beta$  do not depend on the continuation of  $\beta$  outside  $B_1$ . Moreover

$$K_R d\alpha_0|_B = K_R d(\chi\alpha)|_B = K_R(\chi d\alpha)|_B = T_1(d\alpha),$$

since  $d(\chi\alpha) \equiv \chi d\alpha$  on  $B_0$ . Thus, by (10),

$$\alpha = dT_1\alpha + T_1 d\alpha + S_1\alpha \quad \text{in } B.$$

Assume now that  $d\alpha = 0$ . Then

$$\alpha = dT_1\alpha + S_1\alpha \quad \text{in } B.$$

Write  $\phi = T_1\alpha \in L^1(B_0, \bigwedge^{h-1}(\mathbb{R}^n))$ . By difference,  $d\phi = \alpha - S_1\alpha \in L^1(B_0, \bigwedge^{h-1}(\mathbb{R}^n))$ .

The next step will consist of proving that  $\phi \in L^q(B_0, \bigwedge^{h-1}(\mathbb{R}^n))$ , “iterating” the previous argument. Let us sketch how this iteration will work: let  $\zeta$  be a cut-off function supported in  $B_0$ , identically equal to 1 in a neighborhood  $\mathcal{U}$  of  $B$ , and set  $\omega = d(\zeta\phi)$ . Obviously, the form  $\zeta\phi$  (and therefore also  $\omega$ ) are defined on all  $\mathbb{R}^n$  and are compactly supported in  $B_0$ . In addition,  $\omega$  is closed. Suppose for a while we are able to prove that

- a)  $\omega \in L^1(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n))$ ;
- b)  $\|K_0\omega\|_{L^q(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n))} \leq C\|\alpha\|_{L^1(B', \bigwedge^h(\mathbb{R}^n))}$ ,

and let us show how the argument can be carried out.

First we stress that, if  $R$  is small enough, then when  $x \in B$ ,  $K_R\omega(x)$  depends only on the restriction of  $d\phi$  to  $\mathcal{U}$ , so that the map

$$\alpha \rightarrow K_R\omega|_B$$

is linear.

In addition, notice that  $\omega = \chi\omega$ , so that, by (10),

$$d(\zeta\phi) = \omega = dK_R\omega + S\omega.$$

Therefore in  $B$

$$\alpha - S_1\alpha = d\phi = d(\zeta\phi) = dK_R\omega + S_0\omega,$$

and then in  $B$

$$\begin{aligned}\alpha &= d(K_R\omega|_B) + S_1\alpha|_B + S\omega|_B \\ &=: d(K_R(\chi\omega)|_B) + S\alpha = dT\alpha + S\alpha.\end{aligned}$$

First notice that the map  $\alpha \rightarrow \omega = \omega(\alpha)$  is linear, and hence  $T$  and  $S$  are linear maps. In addition, by b),

$$\|T\alpha\|_{L^q(B, \bigwedge^{h-1}(\mathbb{R}^n))} \leq \|K_R(\chi\omega)\|_{L^q(\mathbb{R}^n), \bigwedge^h(\mathbb{R}^n)} = \|K_R(\omega)\|_{L^q(\mathbb{R}^n, \bigwedge^h(\mathbb{R}^n))} \leq C \|\alpha\|_{L^1(B', \bigwedge^h(\mathbb{R}^n))}.$$

As for the map  $\alpha \rightarrow S\alpha$  we have just to point out that, when  $x \in B$ ,  $S\alpha(x)$  can be written as the convolution of  $\alpha_0$  with a smooth kernel with bounded derivatives of any order, and the proof is completed.  $\square$

Interior Poincaré and Sobolev inequalities follow now from the approximate homotopy formula for closed forms (9).

**Corollary 5.6** (Interior Poincaré and Sobolev inequalities). *Let  $B \Subset B'$  open sets in  $\mathbb{R}^n$ , and assume  $B$  is convex. For  $h = 1, \dots, n-1$ , let  $q = n/(n-1)$ . Then for every closed form  $\alpha \in L^1(B', \bigwedge^h(\mathbb{R}^n))$ , there exists an  $(h-1)$ -form  $\phi \in L^q(B, \bigwedge^{h-1}(\mathbb{R}^n))$ , such that*

$$d\phi = \alpha|_B \quad \text{and} \quad \|\phi\|_{L^q(B, \bigwedge^{h-1}(\mathbb{R}^n))} \leq C \|\alpha\|_{L^1(B', \bigwedge^h(\mathbb{R}^n))}.$$

Furthermore, if  $\alpha$  is compactly supported, so is  $\phi$ .

*Proof.* By Proposition 5.5, the  $h$ -form  $S\alpha$  defined in (9) is closed and belongs to  $L^q(B, \bigwedge^h(\mathbb{R}^n))$ , with norm controlled by the  $L^1$ -norm of  $\alpha$ . Thus we can apply Iwaniec & Lutoborski's homotopy ([11], Proposition 4.1) to obtain a differential  $(h-1)$ -form  $\gamma$  on  $B$  with norm in  $W^{1,q}(B, \bigwedge^{h-1}(\mathbb{R}^n))$  controlled by the  $L^q$ -norm of  $S\alpha$  and therefore from the  $L^1$ -norm of  $\alpha$ . Set  $\phi := T\alpha + \gamma$ . Clearly

$$d\phi = dT\alpha + d\gamma = dT\alpha + S\alpha = \alpha.$$

Then, by Proposition 5.5,

$$\|\phi\|_{L^q(B, \bigwedge^{h-1}(\mathbb{R}^n))} \leq C(\|\alpha\|_{L^1(B, \bigwedge^h(\mathbb{R}^n))} + \|S\alpha\|_{L^q(B, \bigwedge^h(\mathbb{R}^n))}) \leq C\|\alpha\|_{L^1(B, \bigwedge^h(\mathbb{R}^n))}.$$

$\square$

#### ACKNOWLEDGMENTS

A. B. and B. F. are supported by the University of Bologna, funds for selected research topics, and by MAnET Marie Curie Initial Training Network, by GNAMPA of INdAM (Istituto Nazionale di Alta Matematica “F. Severi”), Italy, and by PRIN of the MIUR, Italy.

P.P. is supported by MAnET Marie Curie Initial Training Network, by Agence Nationale de la Recherche, ANR-10-BLAN 116-01 GGAA and ANR-15-CE40-0018 SRGI. P.P. gratefully acknowledges the hospitality of Isaac Newton Institute, of EPSRC under grant EP/K032208/1, and of Simons Foundation.

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