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# SECANT VARIETIES OF THE VARIETIES OF REDUCIBLE HYPERSURFACES IN $\mathbb{P}^n$

M.V. CATALISANO, A.V. GERAMITA\*, A. GIMIGLIANO, B. HARBOURNE, J. MIGLIORE,  
U. NAGEL, AND Y.S. SHIN

*In fond memory of A.V. Geramita, 1942–2016*

ABSTRACT. Given the space  $V = \mathbb{P}^{\binom{d+n-1}{n-1}-1}$  of forms of degree  $d$  in  $n$  variables, and given an integer  $\ell > 1$  and a partition  $\lambda$  of  $d = d_1 + \cdots + d_r$ , it is in general an open problem to obtain the dimensions of the  $(\ell-1)$ -secant varieties  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  for the subvariety  $\mathbb{X}_{n-1,\lambda} \subset V$  of hypersurfaces whose defining forms have a factorization into forms of degrees  $d_1, \dots, d_r$ . Modifying a method from intersection theory, we relate this problem to the study of the Weak Lefschetz Property for a class of graded algebras, based on which we give a conjectural formula for the dimension of  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  for any choice of parameters  $n, \ell$  and  $\lambda$ . This conjecture gives a unifying framework subsuming all known results. Moreover, we unconditionally prove the formula in many cases, considerably extending previous results, as a consequence of which we verify many special cases of previously posed conjectures for dimensions of secant varieties of Segre varieties. In the special case of a partition with two parts (i.e.,  $r = 2$ ), we also relate this problem to a conjecture by Fröberg on the Hilbert function of an ideal generated by general forms.

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*\*While this paper was being refereed, Tony Geramita passed away. On behalf of Tony’s many friends and colleagues from all walks of life, the six remaining authors dedicate this paper to him.*

## 1. INTRODUCTION

Let  $S = \mathbb{k}[x_1, \dots, x_n] = \bigoplus_{i \geq 0} [S]_i$ , where  $\mathbb{k}$  is an algebraically closed field of characteristic zero. In 1954 Mammarna [31] introduced the variety of reducible plane curves. He was seeking to generalize work of C. Segre [39] (for conics), N. Spampinato [41] (for plane cubics) and G. Bordiga [7] (for plane quartics) as well as other works mentioned in his ample bibliography.

Here we generalize the idea further. Let  $\lambda = [d_1, \dots, d_r]$  be a partition of  $d = \sum_{i=1}^r d_i$ , which we will write as  $\lambda \vdash d$ , where  $d_1 \geq d_2 \geq \dots \geq d_r \geq 1$  and  $r \geq 2$ .

Consider the variety  $\mathbb{X}_{n-1,\lambda} \subset \mathbb{P}([S]_d) = \mathbb{P}^{N-1}$  of  $\lambda$ -reducible forms, where  $N = \binom{d+n-1}{n-1}$ . That is,

$$\mathbb{X}_{n-1,\lambda} = \{[F] \in \mathbb{P}^{N-1} \mid F = F_1 \cdots F_r \text{ for some } 0 \neq F_i \in [S]_{d_i}\}.$$

The object of this paper is to study the dimension of the  $(\ell - 1)$ -secant variety of  $\mathbb{X}_{n-1,\lambda}$ , which we denote by  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$ . So  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  is the closure of the union of the linear spans of all subsets of  $\ell$  distinct points of  $\mathbb{X}_{n-1,\lambda}$ . We will give a new approach to this problem.

We have

$$\dim \mathbb{X}_{n-1,\lambda} = \sum_{i=1}^r \binom{d_i + n - 1}{n - 1} - r.$$

Since all forms in two variables are products of linear forms, we always assume  $n \geq 3$ ,  $d \geq r \geq 2$ , and  $\ell \geq 2$ . We can (and will) view a general point of  $\mathbb{X}_{n-1,\lambda}$  as the product of general forms in  $S$  of degrees  $d_1, \dots, d_r$  respectively.

Since, as it is easy to see, no hyperplane contains  $\mathbb{X}_{n-1,\lambda}$ ,  $\ell \leq N$  general points of  $\mathbb{X}_{n-1,\lambda}$  span a linear space of dimension  $\ell - 1$  (i.e., a secant  $(\ell - 1)$ -plane), so by a simple parameter count we have  $\dim \sigma_\ell(\mathbb{X}_{n-1,\lambda}) \leq \ell \cdot \dim \mathbb{X}_{n-1,\lambda} + \ell - 1$ . But it is possible that  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  fills its ambient space  $\mathbb{P}^{N-1}$ ; this clearly happens, for instance, if  $\ell \geq N$ . We combine the two possibilities to obtain an upper bound for the actual dimension of  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$ , typically referred to as the *expected dimension*:

$$\exp.\dim \sigma_\ell(\mathbb{X}_{n-1,\lambda}) = \min \left\{ \binom{d + n - 1}{n - 1} - 1, \ell \cdot \dim \mathbb{X}_{n-1,\lambda} + \ell - 1 \right\}.$$

The *defect*,  $\delta_\ell$ , is the expected dimension minus the actual dimension. When this is positive, we say that  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  is *defective*. An important part of our work will be to identify when  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  is defective and to compute the defect.

Secant and join varieties of the Veronese, Segre and Grassmann varieties have been extensively studied. The recent intense activity in studying these varieties has certainly benefited from the numerous fascinating applications in Communication Theory, Complexity Theory and Algebraic Statistics as well as from the connections to classical problems in Projective Geometry and Commutative Algebra. (For a partial view of these applications consider the following references and their bibliographies: [2], [3], [8], [12], [13], [17], [29], [37], [44], [46].)

However, very little is known about the secant varieties of the varieties of reducible hypersurfaces. Here also there are useful applications in the study of vector bundles on surfaces and connections to the classical Noether-Severi-Lefschetz Theorem for general hypersurfaces in projective space (see [9] [15], [36]).

The first significant results about the secant varieties of  $\lambda$ -reducible forms were obtained by Arrondo and Bernardi in [5] for the special partition  $\lambda = [1, \dots, 1]$  (they refer to  $\mathbb{X}_{n-1,\lambda}$  for this particular  $\lambda$  as the variety of *split* or *completely decomposable* forms). They find the dimensions of secant varieties in this case for a very restricted, but infinite, family of

examples. This was followed by work of Shin [42] who found the dimension of the secant line variety to the varieties of split plane curves of every degree.

This latter result was further generalized by Abo [1], again for split curves, to a determination of the dimensions of all the higher secant varieties. Abo also dealt with some cases of split surfaces in  $\mathbb{P}^3$  and split cubic hypersurfaces in  $\mathbb{P}^n$ , for any  $n$ .

In all the cases considered, the secant varieties have the expected dimension. Arrondo and Bernardi have speculated that the secant varieties for split hypersurfaces always have the expected dimension. We verify this for  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  ( $\lambda = [1, \dots, 1] \vdash d$ ) as long as  $2\ell \leq n$ , which strengthens [5, Proposition 1.8]. We also note that their speculation is a special case of our Conjecture 1.3 (a).

The parameters for this work are  $n \geq 3$ ,  $\ell \geq 2$  and any partition  $\lambda = [d_1, \dots, d_r]$  with  $r \geq 2$  positive parts. All previous results assume  $d_1 = 1$  (i.e., the split variety case [1, 5, 42]) or  $n = 3$  [14], or  $r = 2$  [6, 9]. We extend all of this previous work significantly. See, for example, Theorem 1.2, as an immediate consequence of which we obtain complete answers in many new cases, including each of the following: for all  $n \gg 0$  (in fact  $n \geq 2\ell$ ), fixing any  $\ell$  and an arbitrary partition  $\lambda$ ; for all  $\ell \gg 0$  (in fact  $\ell \geq \binom{s+n-1}{n-1}$ ), fixing any  $n$  and  $d_2 \geq \dots \geq d_r \geq 1$ , where  $s = d_2 + \dots + d_r$ ; and for all  $d_1 \gg 0$  (in fact  $d_1 \geq (s-1)(n-1)$ ), fixing any  $n$  and  $d_2 \geq \dots \geq d_r \geq 1$ , and any  $\ell$  *not* in the interval  $(\frac{n}{2}, n)$ . We also propose a conjecture (see Conjecture 1.1), which, if true, gives a complete answer in all remaining cases and which has led us to many of our results.

All approaches to finding the dimension of the secant varieties to a given variety  $\mathbb{X} \subset \mathbb{P}^{n-1}$  begin with Terracini's Lemma, including ours. These all require a good understanding of the tangent space to  $\mathbb{X}$  at a general point. Successful applications of Terracini's Lemma begin by identifying this tangent space as a graded piece of some relatively nice ideal (which we will call the *tangent space ideal*). To apply Terracini's Lemma, one then needs a way to deal with the sum of tangent space ideals at a finite set of general points of  $\mathbb{X}$ . For the Veronese, Segre and Grassmann varieties, the quotient by this ideal sum in the appropriate polynomial ring typically is artinian. The standard method for dealing with the sum of such ideals (which, per se, have no geometric content) is to pass, using Macaulay duality, to a consideration of the intersection of the *perps* of the tangent space ideals (see. e.g. the discussion in [22]). In the classical cases considered above, one obtains a union of special 1-dimensional ideals corresponding to zero dimensional projective schemes. One then uses geometric methods to get information about the union of the schemes defined by the perps of the tangent space ideals.

This clever use of Macaulay duality had its first notable success with the work of Alexander and Hirschowitz, who completed (after almost one hundred years) the solution of Waring's Problem for general forms (see [3]). Other work in this direction for these classical varieties can be found in [9], [13], [12], [2], [27].

In the case of the varieties of reducible hypersurfaces, the method described above no longer works. In this case, the tangent space ideals already define very nice schemes of dimension  $\geq 0$ , namely arithmetically Cohen-Macaulay codimension 2 subschemes of  $\mathbb{P}^n$ , and their Macaulay duals are artinian! Thus, one is forced to deal with the sum of the tangent space ideals, i.e., with the *intersection* of the codimension 2 schemes defined by the tangent space ideals at general points.

This is the novelty of our approach: to deal with this intersection we use a version of the so-called diagonal trick from intersection theory, and we show how the so-called Lefschetz properties come in to play in order to study improper intersections.

As we describe in detail, finding this dimension amounts to viewing the intersection of the aforementioned codimension two subschemes in  $\mathbb{P}^n$  as the result of consecutive hyperplane sections of their join in  $\mathbb{P}^{n\ell-1}$ , where the hyperplanes cut out the diagonal. The dimension of the secant variety can then be read off from the Hilbert function in degree  $d$  of the coordinate ring of the intersection of such schemes, although “intersection” must be suitably interpreted in the artinian situation. Algebraically, we are interested in the Hilbert function in degree  $d$  of  $S/(I_{P_1} + \cdots + I_{P_\ell})$  (the  $I_{P_i}$  being the tangent space ideals at general points; see Proposition 2.6), but the geometric notions from intersection theory and hyperplane sections guide our approach.

A key to our method is the observation that we can replace the hyperplanes cutting out the diagonal by truly general hyperplanes. This allows us to compute the dimension of the secant variety  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  in the case where the subschemes meet properly, which occurs precisely when  $2\ell \leq n$ . In the case of an improper intersection of the tangent spaces, i.e.  $2\ell > n$ , we conjecture that the general hyperplanes induce multiplication maps that all have maximal rank. For a single hyperplane such behavior has been dubbed the Weak Lefschetz Property in [24]. Assuming this conjectured property of the hyperplane sections, we obtain a formula for the dimension of the secant variety, which is surprisingly uniform. This single formula proposes the dimension for *any* choice of  $n, \ell$  and  $\lambda$ . We will establish it in some cases. It is a conjecture in the rest, but we know of no cases of known results with which it does not agree.

To be more precise, for  $0 \leq j \leq d$  and every given  $\ell, n$  and partition  $\lambda$  of  $d$ , we define integers  $a_j(\ell, n, \lambda)$  by an explicit formula (see Definition 5.9). Our formula for the dimension of the secant varieties is the following, which we state as a conjecture so that it can be applied for all  $n, \ell$  and  $\lambda$ , in addition to the many cases which we prove below.

**Conjecture 1.1.** *Let  $\lambda = [d_1, \dots, d_r]$  be a partition of  $d$  with  $r \geq 2$  parts. Then:*

- (a) *The secant variety  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  fills its ambient space if and only if there is some integer  $j$  with  $s = d_2 + \cdots + d_r \leq j \leq d$  such that  $a_j(\ell, n, \lambda) \leq 0$ .*
- (b) *If  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  does not fill its ambient space, then it has dimension*

$$\begin{aligned} \dim \sigma_\ell(\mathbb{X}_{n-1,\lambda}) &= \ell \cdot \dim \mathbb{X}_{n-1,\lambda} + \ell - 1 \\ &\quad - \sum_{k=2}^{\ell} (-1)^k \binom{\ell}{k} \binom{d_1 - (k-1)(d_2 + \cdots + d_r) + n - 1}{n-1} \\ &\quad - \binom{\ell}{2} \binom{2d_2 - d + n - 1}{n-1} - \ell(\ell-1) \binom{d_1 + d_2 - d + n - 1}{n-1}. \end{aligned}$$

The fact that this conjecture is a consequence of the indicated Lefschetz property is shown in Theorem 5.11. Throughout this paper we use the convention that a binomial coefficient  $\binom{a}{b}$  is zero if  $a < 0$ . Thus, for example, the last and the penultimate term in the above dimension formula are zero if  $r \geq 3$ . (A heuristic approach to the formula in Conjecture 1.1(b) can be found in Remark 3.15.)

Although stated differently, previous results imply that Conjecture 1.1 is true if  $n = 3$  and  $\lambda = [1, \dots, 1]$  (see [1]), or if  $n = 3$ ,  $\lambda$  is arbitrary, and  $\ell = 2$  (see [14]). Here, we prove

this conjecture in further cases, most of which are summarized in the following theorem. Note that part (b)(i) of the following result was proved in [9, Theorem 5.1] using different language.

**Theorem 1.2.** *Let  $\lambda = [d_1, \dots, d_r]$  be a partition of  $d = d_1 + s$  into  $r \geq 2$  parts, where  $s = d_2 + \dots + d_r$ . Then Conjecture 1.1 is true in the following cases:*

- (a)  $\ell \leq \frac{n}{2}$  or  $\ell \geq \binom{s+n-1}{n-1}$ ;
- (b)  $r = 2$  and either
  - (i)  $\ell \leq \frac{n+1}{2}$ , or
  - (ii)  $\lambda = [d-1, 1]$ , or
  - (iii)  $n = 3$ ; and
- (c)  $r \geq 3$  and  $n \leq \ell \leq 1 + \frac{d_1+n-1}{s}$ .

We prove Theorem 1.2(a) in Remark 5.12(ii) and Proposition 5.13(c). See Theorem 6.8 for parts (b)(i, iii), Theorem 6.11 for part (b)(ii), and Corollary 5.15 for part (c).

We also show that if  $\ell = \frac{n+1}{2}$  (Proposition 5.16) or  $r = 2$  (Proposition 6.6), then the number predicted by Conjecture 1.1 is at least an upper bound for  $\dim \sigma_\ell(\mathbb{X}_{n-1, \lambda})$ .

Notice that the dimension formula in Conjecture 1.1 involves a series of comparisons, checking whether  $a_j(\ell, n, \lambda) > 0$  for all  $j = s, s+1, \dots, d$ . Accordingly, it is worthwhile to point out more explicitly some of the consequences it suggests. Again, the following is stated as a conjecture even though in the different settings of the above theorem these results are proven.

**Conjecture 1.3.** *Let  $\lambda = [d_1, \dots, d_r]$  be a partition of  $d$  with  $r \geq 2$  parts. Then:*

- (a) *If  $d_1 < d_2 + \dots + d_r$  (and thus  $r \geq 3$ ), then  $\sigma_\ell(\mathbb{X}_{n-1, \lambda})$  is not defective.*
- (b) *If  $d_1 \geq d_2 + \dots + d_r$ , then the secant variety  $\sigma_\ell(\mathbb{X}_{n-1, \lambda})$  is defective if and only if it does not fill its ambient space.*

This conjecture highlights the role of the ‘‘partition dividing hyperplane’’  $d_1 = d_2 + \dots + d_r$  in the space of partitions with  $r$  positive parts, as introduced in [14] and discussed here in Remark 3.14. Of course, Conjecture 1.3 might be true even if the more specific formulation given in Conjecture 1.1 is not. Moreover, while Conjecture 1.1 implies most of Conjecture 1.3, it is not yet clear that Conjecture 1.1 implies all of Conjecture 1.3; see Proposition 5.13 and Remark 5.14. In particular, notice that Conjecture 1.3(a) is an immediate consequence of Conjecture 1.1, but we can show that Conjecture 1.3(b) follows from Conjecture 1.1 only in certain cases (see Proposition 5.13).

Our results on defectiveness show unconditionally:

**Theorem 1.4.** *Let  $\lambda = [d_1, \dots, d_r]$  be a partition of  $d$  with  $r \geq 2$  parts and let  $s = d_2 + \dots + d_r$ . Then:*

- (a) *If  $d_1 < s$  (and hence  $r \geq 3$ ) and  $2\ell \leq n$ , then  $\sigma_\ell(\mathbb{X}_{n-1, \lambda})$  is not defective.*
- (b) *If  $d_1 \geq s$  and  $2\ell \leq n$ , then  $\sigma_\ell(\mathbb{X}_{n-1, \lambda})$  is defective if and only if it does not fill its ambient space.*
- (c) *If  $\ell \geq n$  and  $d_1 \geq (n-1)(s-1)$ , then  $\sigma_\ell(\mathbb{X}_{n-1, \lambda})$  always fills its ambient space, while if  $2\ell \leq n$ , then  $\sigma_\ell(\mathbb{X}_{n-1, \lambda})$  fills its ambient space if and only if one of the following conditions is satisfied:*
  - (i)  $n = 4$ ,  $\ell = 2$ , and  $\lambda \in \{[1, 1], [2, 1], [1, 1, 1]\}$  or
  - (ii)  $n = 2\ell \geq 6$  and  $\lambda = [1, 1]$ .

We give the proof near the end of Section 5.

We use our results in the case  $r = 2$  to study the *variety of reducible forms of degree  $d$  in  $n$  variables*

$$\mathbb{X}_{n-1,d} = \bigcup_{k=1}^{\lfloor \frac{d}{2} \rfloor} \mathbb{X}_{n-1,[d-k,k]}.$$

We show that

$$\dim \mathbb{X}_{n-1,d} = \dim \mathbb{X}_{n-1,[d-1,1]}$$

and that all other irreducible components of  $\mathbb{X}_{n-1,d}$  have dimension that is smaller than the dimension of  $\mathbb{X}_{n-1,[d-1,1]}$ . Thus, one can hope that  $\mathbb{X}_{n-1,[d-1,1]}$  determines the dimension of the secant variety of  $\mathbb{X}_{n-1,d}$ . Indeed, we establish:

**Theorem 1.5.** *If  $2\ell \leq n$ , then*

$$\dim \sigma_\ell(\mathbb{X}_{n-1,d}) = \dim \sigma_\ell(\mathbb{X}_{n-1,[d-1,1]}).$$

*Moreover,  $\sigma_\ell(\mathbb{X}_{n-1,d})$  is defective if and only if it does not fill its ambient space.*

We prove this result in Section 7 as a consequence of Theorem 7.4.

Note that the dimension of  $\sigma_\ell(\mathbb{X}_{n-1,[d-1,1]})$  is known for all  $\ell, n$  and  $d$  (see Theorem 6.11 or [6, Proposition 4.4]). Thus, we know exactly when  $\sigma_\ell(\mathbb{X}_{n-1,[d-1,1]})$  is defective (see Theorem 7.4). We suspect that  $\dim \sigma_\ell(\mathbb{X}_{n-1,d}) = \dim \sigma_\ell(\mathbb{X}_{n-1,[d-1,1]})$  is true for all  $n$  and  $\ell$ .

As another application we consider Segre varieties. We show that results on secant varieties to  $\mathbb{X}_{n-1,\lambda}$  imply non-defectivity of secant varieties to a Segre variety (see Theorem 8.4).

In Section 2 we recall the basic facts about the variety of reducible hypersurfaces and how Terracini's lemma is applied. We consider the coordinate ring of the join, which is arithmetically Cohen-Macaulay of codimension  $2\ell$  in  $\mathbb{P}^{n\ell-1}$ , with known minimal free resolution and Hilbert function. We discuss how the algebra that determines the dimension of  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  is obtained by successive hyperplane sections (i.e. reduction by general linear forms). As long as these linear forms are regular elements, the intersection is proper and in Section 3 we give formulas for the dimension and defect. (As an aside, we point out that the varieties  $\mathbb{X}_{n-1,\lambda}$  are not generally arithmetically Cohen-Macaulay. For example, by direct computation they are not for  $n = 3$  and  $\lambda$  either  $[1, 1, 1]$  or  $[1, 2]$ , but we do not know about their secant varieties.)

In Section 4 we summarize our results in the case  $\ell = 2$ , i.e. for the secant line variety. Proper intersection corresponds to  $n \geq 4$ . For the remaining case,  $n = 3$ , we recall the results of [14]. This gives us a bridge from the proper intersections to the improper intersections and gives the idea of how the Lefschetz property is applied in general.

In Section 5 we work out, in general, the connection between the computation of the dimension for arbitrarily large  $\ell$ , corresponding to improper intersections, and the study of Lefschetz Properties. Indeed, based on experiments, we conjecture that the coordinate ring of a certain join variety has enough Lefschetz elements if  $2\ell > n$  (see Conjecture 5.8). If this conjecture is true, then Conjecture 1.1 follows (see Theorem 5.11). However, Conjecture 1.1 is weaker than the conjecture on the existence of enough Lefschetz elements.

In Section 6 we focus on the case  $r = 2$ . We show that Conjecture 1.1 is a consequence of Fröberg's Conjecture on the Hilbert function of ideals generated by generic forms. In Section 7 we study the variety of reducible forms. We conclude in Section 8 by showing how our results imply cases of conjectures raised in [2] about defectivity of Segre Varieties.



## 2. INTERSECTIONS AND THE DIMENSION OF SECANT VARIETIES

After recalling some background and introducing our notation, we lay out our method for computing the desired dimension of a secant variety. It is inspired by a technique from intersection theory. The method will be applied in later sections, where we treat the case of proper and improper intersections separately and carry out the needed computations.

**Notation 2.1.** Let  $S = \mathbb{k}[x_1, \dots, x_n] = \bigoplus_{i \geq 0} [S]_i$  be the standard graded polynomial ring, where  $\mathbb{k}$  is an algebraically closed field of characteristic zero. Let  $\lambda = [d_1, d_2, \dots, d_r]$  be a *partition of  $d$  into  $r \geq 2$  parts*, i.e.,  $\lambda \vdash d$ ,  $d_i \in \mathbb{N}$ ,  $d_1 \geq d_2 \geq \dots \geq d_r > 0$  and  $\sum_{i=1}^r d_i = d$ .

If we set  $N = \binom{d+n-1}{n-1}$  then the *variety of reducible forms in  $[S]_d$  of type  $\lambda$*  (or *the variety of reducible hypersurfaces in  $\mathbb{P}^{n-1}$  of type  $\lambda$* ) is, as noted above:

$$\mathbb{X}_{n-1, \lambda} := \{[F] \in \mathbb{P}([S]_d) = \mathbb{P}^{N-1} \mid F = F_1 \cdots F_r, \deg F_i = d_i\}.$$

The map  $([F_1], \dots, [F_r]) \mapsto [F] = [F_1 \cdots F_r]$  induces a finite morphism

$$(2.1) \quad \mathbb{P}([S]_{d_1}) \times \cdots \times \mathbb{P}([S]_{d_r}) \longrightarrow \mathbb{X}_{n-1, \lambda},$$

and so we have

$$(2.2) \quad \dim \mathbb{X}_{n-1, \lambda} = \left[ \sum_{i=1}^r \binom{d_i + n - 1}{n - 1} \right] - r.$$

As discussed above, given a positive integer  $\ell \leq N$ , the variety  $\sigma_\ell(\mathbb{X}_{n-1, \lambda})$  is the subvariety of  $\mathbb{P}^{N-1}$  consisting of the closure of the union of secant  $\mathbb{P}^{\ell-1}$ 's to  $\mathbb{X}_{n-1, \lambda}$ ; for  $\ell \geq N$ ,  $\sigma_\ell(\mathbb{X}_{n-1, \lambda})$  is simply  $\mathbb{P}^{N-1}$ . Following the classical terminology,  $\sigma_2(\mathbb{X}_{n-1, \lambda})$  is called the *secant line variety* of  $\mathbb{X}_{n-1, \lambda}$  and  $\sigma_3(\mathbb{X}_{n-1, \lambda})$  the *secant plane variety* of  $\mathbb{X}_{n-1, \lambda}$ .

Our main interest in this paper is the calculation of the dimensions of the varieties  $\sigma_\ell(\mathbb{X}_{n-1, \lambda})$ .

**Remark 2.2.** Notice that for  $n = 2$  the question is a triviality since *every* hypersurface of degree  $d$  in  $\mathbb{P}^1$  is reducible of type  $\lambda = [1, 1, \dots, 1] \vdash d$ . Thus, we assume throughout  $n \geq 3$ .

The fundamental tool for the calculation of dimensions of secant varieties is the following celebrated result [45].

**Proposition 2.3** (Terracini's Lemma). *Let  $P_1, \dots, P_\ell$  be general points on  $\mathbb{X}_{n-1, \lambda}$  and let  $T_{P_i}$  be the (projectivized) tangent space to  $\mathbb{X}_{n-1, \lambda}$  at the point  $P_i$ .*

*The dimension of  $\sigma_\ell(\mathbb{X}_{n-1, \lambda})$  is the dimension of the linear span of  $\bigcup_{i=1}^\ell T_{P_i}$ .*

As mentioned above, we have the following definitions.

**Definition 2.4.** The *expected dimension* of  $\sigma_\ell(\mathbb{X}_{n-1, \lambda})$ , written  $\exp. \dim(\sigma_\ell(\mathbb{X}_{n-1, \lambda}))$ , is

$$\min\{N - 1, \ell \cdot \dim \mathbb{X}_{n-1, \lambda} + (\ell - 1)\}.$$

The *defect* of  $\sigma_\ell(\mathbb{X}_{n-1, \lambda})$  is

$$\delta_\ell = \exp. \dim(\sigma_\ell(\mathbb{X}_{n-1, \lambda})) - \dim(\sigma_\ell(\mathbb{X}_{n-1, \lambda})) \geq 0.$$

We say that  $\sigma_\ell(\mathbb{X}_{n-1, \lambda})$  is *defective* if  $\delta_\ell > 0$ .

**Remark 2.5.** Thus  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  is defective if and only if  $\dim \sigma_\ell(\mathbb{X}_{n-1,\lambda}) < N - 1$  and  $\dim \sigma_\ell(\mathbb{X}_{n-1,\lambda}) < \ell(\dim(\mathbb{X}_{n-1,\lambda})) + (\ell - 1)$ . We will see that in some cases it will be easier to write an expression for  $\delta_\ell$  than it will be to show that it is positive.

Clearly, to be able to effectively use Terracini's Lemma it is essential to have a good understanding of the tangent spaces to  $\mathbb{X}_{n-1,\lambda}$  at general (hence smooth) points. We do that now.

Let  $P = [F] \in \mathbb{X}_{n-1,\lambda}$  be a general point. Then  $F = F_1 \cdots F_r$  where the  $F_i$  are irreducible forms of degree  $d_i$  in  $S$ . Let  $G_i = F/F_i$ , and so  $\deg G_i = d - d_i$ . Consider the ideal  $I_P \subset S$ , where  $I_P = (G_1, \dots, G_r)$ .

**Proposition 2.6.** *In the notation of the preceding paragraph, we have*

$$T_P = \mathbb{P}([I_P]_d).$$

*Proof.* This proposition is well known and proofs can be found in several places (see e.g. [9] Prop. 3.2).  $\square$

We refer to the variety in  $\mathbb{P}^{n-1}$  defined by  $I_P$  as the *variety determining the (general) tangent space to  $\mathbb{X}_{n-1,\lambda}$* .

As an immediate corollary of Propositions 2.3 and 2.6 we have the following:

**Corollary 2.7.** *Let  $P_1, \dots, P_\ell$  be  $\ell$  general points of  $\mathbb{X}_{n-1,\lambda}$  and  $I = I_{P_1} + \cdots + I_{P_\ell}$ . Then*

$$\dim(\sigma_\ell(\mathbb{X}_{n-1,\lambda})) = \dim_{\mathbb{k}} [I]_d - 1.$$

This means that  $\dim(\sigma_\ell(\mathbb{X}_{n-1,\lambda}))$  is determined by the Hilbert function in degree  $d$  of the ring  $S/(I_{P_1} + \cdots + I_{P_\ell})$ , which, when  $\sigma_\ell$  does not fill its ambient space, is the coordinate ring of the intersection of  $\ell$  varieties, in  $\mathbb{P}^{n-1}$ , determining tangent spaces to  $\mathbb{X}_{n-1,\lambda}$ .

**Remark 2.8.** Let  $P_1, \dots, P_\ell$  be general points on  $\mathbb{X}_{n-1,\lambda}$ . Suppose that

$$N - 1 \leq \ell \dim \mathbb{X}_{n-1,\lambda} + (\ell - 1);$$

i.e., the expected dimension of  $\mathbb{X}_{n-1,\lambda}$  is  $N - 1$ . Then

$$\sigma_\ell(\mathbb{X}_{n-1,\lambda}) \text{ is defective} \Leftrightarrow \dim_{\mathbb{k}} [S/(I_{P_1} + \cdots + I_{P_\ell})]_d > 0.$$

In this case,  $\delta_\ell = \dim_{\mathbb{k}} [S/(I_{P_1} + \cdots + I_{P_\ell})]_d$ .

Now that we have seen the ideal that enters into the use of Terracini's Lemma, it remains to give a nicer description of the ideal  $I_P$  determining the tangent space at the point  $P$ .

**Proposition 2.9.** *Let  $P$  be a general point of  $\mathbb{X}_{n-1,\lambda}$ ,  $P = [F_1 \cdots F_r]$  where  $\deg F_i = d_i$ . Put  $F = F_1 \cdots F_r$ . Then we have*

$$I_P = (F/F_1, \dots, F/F_r) = \bigcap_{1 \leq i < j \leq r} (F_i, F_j).$$

*Proof.* The first equality is given in Proposition 2.6. The second equality is well known (see for example [35], Thm. 2.3).  $\square$

Thus the ideal  $I_P$ , for a general point  $P \in \mathbb{X}_{n-1,\lambda}$ , is of codimension 2 in  $S$  and is a finite intersection of interrelated codimension 2 complete intersection ideals. Such ideals (and their generalization to the situation where the complete intersection ideals have higher codimension) have been studied in several papers for many different reasons (see e.g. [19], [20], [21] and [10]).

We now derive the graded minimal free resolution of the ideal  $I_P$ .

**Lemma 2.10.** *Let  $R = \mathbb{k}[Y_1, \dots, Y_r]$ ,  $M = Y_1 \cdots Y_r$ ,  $M_i = M/Y_i$ , where  $r \geq 2$ . If  $I = (M_1, \dots, M_r)$  then  $I = \bigcap_{1 \leq i < j \leq r} (Y_i, Y_j)$  and the minimal graded free resolution of  $I$  is*

$$0 \rightarrow R^{r-1}(-r) \xrightarrow{A} R^r(-(r-1)) \rightarrow R \rightarrow R/I \rightarrow 0.$$

*Proof.* Consider the matrix  $A$ , defined by

$$A^t = \begin{bmatrix} Y_1 & -Y_2 & 0 & \cdots & 0 \\ Y_1 & 0 & -Y_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Y_1 & 0 & 0 & \cdots & -Y_r \end{bmatrix}_{(r-1) \times r}.$$

The ideal generated by the maximal minors of  $A$  is  $I$ . It has codimension 2 in  $R$ . Thus, the claim follows from the Hilbert-Burch theorem.  $\square$

**Remark 2.11.** Let  $R = \mathbb{k}[Y_1, \dots, Y_r]$  be as above and let  $S = \mathbb{k}[x_1, \dots, x_n]$ . Let  $F_1, \dots, F_r$  be general homogeneous polynomials in  $S$  of degrees  $d_1, \dots, d_r$  respectively. Let  $F = \prod_{i=1}^r F_i$ ,  $\deg F = d = \sum_{i=1}^r d_i$  and  $G_i = F/F_i$ ,  $\deg G_i = d - d_i$ , for  $1 \leq i \leq r$ . Let

$$\varphi : R \rightarrow S$$

be defined by  $\varphi(Y_i) = F_i$ . Then, with  $I$  as in Lemma 2.10,  $\varphi(I) = (G_1, \dots, G_r) = \bigcap_{1 \leq i < j \leq r} (F_i, F_j) = J$ .

By the generality in the choice of the  $F_i$ ,  $J$  is Cohen-Macaulay of codimension 2 (see Proposition 2.9) and its Hilbert-Burch matrix has transpose

$$\begin{bmatrix} F_1 & -F_2 & 0 & \cdots & 0 \\ F_1 & 0 & -F_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ F_1 & 0 & 0 & \cdots & -F_r \end{bmatrix}_{(r-1) \times r}.$$

The minimal free graded resolution of  $S/J$  is

$$(2.3) \quad 0 \rightarrow S^{r-1}(-d) \rightarrow \bigoplus_{i=1}^r S(-(d-d_i)) \rightarrow S \rightarrow S/J \rightarrow 0.$$

It is a simple consequence of this resolution that the artinian reduction of  $S/J$  is level with socle degree  $d - 2$  and Cohen-Macaulay type  $(r - 1)$ .  $\square$

**Remark 2.12.** Observe that  $J$  is the ideal  $I_P$  for the point  $P = [F]$  on  $\mathbb{X}_{n-1,\lambda}$  if we assume  $d_1 \geq d_2 \geq \cdots \geq d_r$ . In this paper the main goal is to consider sums of such ideals, i.e. ideals of the form  $(G_1, \dots, G_r)$  arising from general forms of prescribed degrees as described above, and to compute the dimension of the component in degree  $d$ . The fact that they

correspond to general points on  $\mathbb{X}_{n-1,\lambda}$  is not needed for most of our computations. Thus, to emphasize the focus on the ideals rather than the points, we will write  $I_{(1)} + \cdots + I_{(\ell)}$  in place of  $I_{P_1} + \cdots + I_{P_\ell}$  when the geometric context is not needed, and retain the latter only when the geometry is important (e.g., Remark 3.15).

**Remark 2.13.** Let us recall a few results about the Hilbert series of a standard graded ring.

Let  $A = \bigoplus_{i=0}^{\infty} [A]_i$ . The *Hilbert series* of  $A$  is the formal power series

$$\mathbf{HS}(A) = \sum_{i=0}^{\infty} (\dim[A]_i) t^i.$$

It is a simple matter to show the following two facts, which we will use often in what follows:

(a) If  $L$  is a linear non-zerodivisor in  $A$  then

$$\mathbf{HS}(A/LA) = (1-t)\mathbf{HS}(A).$$

$$(b) \mathbf{HS}(\mathbb{k}[x_1, \dots, x_n]) = \frac{1}{(1-t)^n} \text{ and } \mathbf{HS}(\mathbb{k}[x_1, \dots, x_n](-a)) = \frac{t^a}{(1-t)^n}.$$

Of course (b) is a simple consequence of (a).

(c) We can apply these observations to the minimal free resolution (2.3) in Remark 2.11 in order to conclude that

$$(2.4) \quad \mathbf{HS}(S/J) = \frac{1}{(1-t)^n} \left[ 1 - \sum_{i=1}^r t^{d_i} + (r-1)t^d \right].$$

(d) If  $A$  and  $B$  are graded  $\mathbb{k}$ -algebras, then

$$\mathbf{HS}(A \otimes_{\mathbb{k}} B) = \mathbf{HS}(A) \cdot \mathbf{HS}(B).$$

Consider a partition  $\lambda = [d_1, \dots, d_r]$ ,  $\lambda \vdash d$ . In the polynomial ring  $\mathbb{k}[x_1, \dots, x_n]$  choose general homogeneous forms  $F_1, \dots, F_r$  of degrees  $d_1, \dots, d_r$  and, as in Remark 2.11, let  $F = \prod_{i=1}^r F_i$ ,  $G_i = F/F_i$  and  $I = (G_1, \dots, G_r) = \bigcap_{1 \leq i < j \leq r} (F_i, F_j)$ .

Inasmuch as we are interested in the secant variety  $\sigma_{\ell}(\mathbb{X}_{n-1,\lambda})$  we form  $\ell$  sets of general polynomials as above in  $\mathbb{k}[x_1, \dots, x_n]$ . Call the elements of the  $j$ -th set

$$\{F_{j,1}, \dots, F_{j,r}\},$$

where  $\deg F_{j,k} = d_k$ . As in Remark 2.11, for  $1 \leq j \leq \ell$  form  $M_j = \prod_{i=1}^r F_{j,i}$ , and  $G_{j,1}, \dots, G_{j,r}$  where  $G_{j,k} = M_j/F_{j,k}$ .

Set

$$I_{(j)} = (G_{j,1}, \dots, G_{j,r}) = \bigcap_{1 \leq i < k \leq r} (F_{j,i}, F_{j,k}), \quad 1 \leq j \leq \ell.$$

Notice that each of the quotients  $S/I_{(j)}$  has the same Hilbert function and minimal free resolution as that of  $S/J$  given in Remark 2.11. Furthermore, each ideal  $I_{(j)}$  defines a variety determining the tangent space to  $\mathbb{X}_{n-1,\lambda}$  at the point  $P_j = [F_{j,1}F_{j,2} \cdots F_{j,r}]$ .

We can perform the same construction as above, but this time choosing each set of  $r$  general polynomials in different polynomial rings, i.e., consider  $\{F_{j,1}, \dots, F_{j,r}\}$  as polynomials in the ring  $\mathbb{k}[x_{j,1}, \dots, x_{j,n}]$ . We can form the sum of these ideals (extended) in

$$T = \mathbb{k}[x_{1,1}, \dots, x_{1,n}, \dots, x_{\ell,1}, \dots, x_{\ell,n}],$$

setting  $\tilde{I} = I_{(1)}^{(e)} + \cdots + I_{(\ell)}^{(e)}$  (i.e., the sum of the extended ideals).

**Theorem 2.14.** *The ring*

$$B = T/\tilde{I} \cong S/I_{(1)} \otimes_{\mathbb{k}} \cdots \otimes_{\mathbb{k}} S/I_{(\ell)}$$

is Cohen-Macaulay of dimension  $\ell(n-2)$ . Its minimal graded free resolution over  $T$  is the tensor product (over  $\mathbb{k}$ ) of the minimal graded free resolutions of  $S/I_{(j)}$  over  $S$  for  $1 \leq j \leq \ell$ .

*Proof.* This is a consequence of the Künneth formulas. See [33, Lemma 3.5] and its proof.  $\square$

Note that  $B$  is the coordinate ring of the join of  $\ell$  varieties, each of which has codimension 2 in  $\mathbb{P}^{n-1}$ , so their join is in  $\mathbb{P}^{n\ell-1}$ . The so-called diagonal trick gives

$$S/(I_{(1)} + \cdots + I_{(\ell)}) \cong B/\Delta B,$$

where the diagonal  $\Delta$  is generated by the  $(\ell-1)n$  linear forms  $x_{1,j} - x_{i,j}$  with  $1 < i \leq \ell$  and  $1 \leq j \leq n$ . Observe that the saturation of  $I_{(1)} + \cdots + I_{(\ell)}$  defines the intersection of the indicated varieties in  $\mathbb{P}^{n-1}$ , provided this intersection is not empty.

A key to our approach is the fact that replacing the linear forms generating the diagonal by truly general linear forms gives a quotient ring with the same Hilbert function as  $S/(I_{(1)} + \cdots + I_{(\ell)})$ . To illustrate the idea, fix a polynomial ring  $R$  in  $m$  variables, and let  $L \in R$  be a general linear form. Since we have a surjection  $R \rightarrow R/(L)$ , if  $\{F_1, \dots, F_t\}$  is a set of general forms in  $R$  of degrees  $d_1, \dots, d_t$ , then the restriction,  $\{\bar{F}_1, \dots, \bar{F}_t\}$ , to  $R/(L)$  can be viewed again as a set of general forms of degrees  $d_1, \dots, d_t$  in  $m-1$  variables. Furthermore, given a prescribed construction of an ideal in  $m$  variables using general forms of prescribed degrees, the restriction to  $R/(L)$  of this ideal can be viewed as an application of the same construction to an ideal of general forms of the same degrees but in  $m-1$  variables.

In our setting, if  $[I_P]_d = [(G_1, \dots, G_r)]_d$  is the vector space determining the tangent space to  $\mathbb{X}_{n-1,\lambda}$  at a general point  $P$  (see Proposition 2.6), then  $[\bar{I}_P]_d = [(\bar{G}_1, \dots, \bar{G}_r)]_d$  is the degree  $d$  component of an ideal that determines the tangent space at a general point of the variety  $\mathbb{X}_{n-2,\lambda}$ . The analogous statement also holds for an ideal of the form  $I_{P_1} + \cdots + I_{P_\ell}$ .

Returning to the above notation, let  $\mathcal{L}$  be a set of  $(\ell-1)n$  general linear forms in  $T$ . Then we have the following useful observation.

**Lemma 2.15.** *The algebras  $S/(I_{(1)} + \cdots + I_{(\ell)})$  and  $B/\mathcal{L}B \cong T/(\mathcal{L}, \tilde{I})$  have the same Hilbert series. I.e.*

$$\mathbf{HS}(S/(I_{(1)} + \cdots + I_{(\ell)})) = \mathbf{HS}(B/\mathcal{L}B) .$$

*Proof.* Each ideal  $I_{(j)} \subset S$  corresponds to a choice of a general point on  $\mathbb{X}_{n-1,\lambda}$ . Thus, it is generated by the  $r$  products of  $r-1$  distinct forms that are created using  $r$  general forms of degrees  $d_1, d_2, \dots, d_r$  in variables  $x_1, x_2, \dots, x_n$  (see Proposition 2.9). The same is true for the summand  $I_{(j)}^{(e)}$  of  $\tilde{I}$ , although these forms are in a new set of variables. Since the linear forms in  $\mathcal{L}$  are general, the residue classes of the forms defining  $I_{(j)}^{(e)}$  modulo  $\mathcal{L}$  are again general forms in  $S$ . It follows that the image  $\overline{I_{(j)}^{(e)}}$  of  $I_{(j)}^{(e)}$  in  $T/\mathcal{L}T \simeq S$  also corresponds to a general point on  $\mathbb{X}_{n-1,\lambda}$ . Thus, the ideals  $I_{(1)} + \cdots + I_{(\ell)}$  and  $\overline{I_{(1)}^{(e)}} + \cdots + \overline{I_{(\ell)}^{(e)}}$  have the same Hilbert function and hence the same Hilbert series.  $\square$

**Remark 2.16.** Lemma 2.15 is, in a sense, the key to the results in this paper. In combination with Corollary 2.7 it shows that computing the dimension of  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$ , for arbitrary  $n, \ell$  and  $\lambda$ , is equivalent to finding the coefficient of  $t^d$  in the Hilbert series of  $B/\mathcal{L}B$ . We emphasize here that the only requirement for the linear forms in  $\mathcal{L}$  is that they be general. We do not need them to be regular elements. The next section will handle the case where they are regular elements, and subsequent sections deal with the case where some of the linear forms are not regular elements.

For our discussion of  $\dim_{\mathbb{k}}[B/\mathcal{L}B]_d$  and more generally the Hilbert series of  $B/\mathcal{L}B$ , it is helpful to consider two cases. We refer to them as *proper* and *improper* intersections.

Consider varieties  $V_1, \dots, V_s \subset \mathbb{P}^{n-1}$ . Then their intersection is defined by the saturation of  $I_{V_1} + \dots + I_{V_s}$  and satisfies

$$\text{codim}(I_{V_1} + \dots + I_{V_s}) \leq \text{codim } I_{V_1} + \dots + \text{codim } I_{V_s}.$$

Abusing notation slightly (in the case where  $\text{codim}(I_{V_1} + \dots + I_{V_s}) = n$ , i.e., the intersection is the empty set), we say that the varieties  $V_1, \dots, V_s \subset \mathbb{P}^{n-1}$  *intersect properly* if

$$\text{codim}(I_{V_1} + \dots + I_{V_s}) = \text{codim } I_{V_1} + \dots + \text{codim } I_{V_s}.$$

Otherwise, they *intersect improperly*.

In particular, this means that, fixing  $n$  and the partition  $\lambda$ , if the intersection of the varieties  $V(I_{(1)}), \dots, V(I_{(\ell)})$  is the empty set for some  $\ell = \ell_0$ , then these varieties intersect improperly for all  $\ell > \ell_0$ .

We close this section with a fact we will have opportunities to apply later. We can partially order partitions of an integer  $d > 0$  as follows. Given partitions  $\lambda_1 = [d_1, \dots, d_p]$  and  $\lambda_2 = [e_1, \dots, e_q]$  of the same integer  $d > 0$ , write  $\lambda_1 \geq \lambda_2$  if for each  $i \geq 0$  we have  $\sum_{j \leq i} d_j \geq \sum_{j \leq i} e_j$  (where we regard  $d_j$  and  $e_j$  as being 0 if  $j$  is out of range). Write  $\lambda_1 > \lambda_2$  if  $\lambda_1 \geq \lambda_2$  but  $\sum_{j \leq i} d_j > \sum_{j \leq i} e_j$  for some  $i$ . So, for example, if  $q > p$ , then either  $\lambda_1$  and  $\lambda_2$  are incomparable (as happens with  $\lambda_1 = [4, 3, 1]$  and  $\lambda_2 = [5, 1, 1, 1]$ ) or  $\lambda_1 > \lambda_2$  (as happens with  $\lambda_1 = [5, 2, 1]$  and  $\lambda_2 = [5, 1, 1, 1]$ ).

**Lemma 2.17.** *Let  $\lambda_1 = [d_1, \dots, d_p]$  and  $\lambda_2 = [e_1, \dots, e_q]$  be partitions of the same integer  $d$  with  $\lambda_1 > \lambda_2$ . If  $n \geq 3$ , then  $\dim \mathbb{X}_{n-1,\lambda_1} > \dim \mathbb{X}_{n-1,\lambda_2}$ .*

*Proof.* Let  $u$  be the least  $i$  such that  $d_i > e_i$ . (There must be such an  $i$  since  $d_i \leq e_i$  for all  $i$  implies  $\sum_{j \leq i} d_j \leq \sum_{j \leq i} e_j$  for all  $i$ .) Note that if  $u > 1$ , then  $e_{u-1} = d_{u-1} \geq d_u > e_u$  and  $\sum_{j \leq u} d_j > \sum_{j \leq u} e_j$ .

Next, let  $v$  be the least  $i > u$  such that  $\sum_{j \leq i} d_j = \sum_{j \leq i} e_j$ . (There must be such an  $i$  since both sums eventually are equal to  $d$ .) Note that if  $v < q$ , then  $e_v > e_{v+1}$ . This is because if  $v < q$ , then (by definition of  $v$  and the fact that  $\sum_{j \leq u} d_j > \sum_{j \leq u} e_j$ ) we have  $\sum_{j \leq v-1} d_j > \sum_{j \leq v-1} e_j$ , but  $\sum_{j \leq v} d_j = \sum_{j \leq v} e_j$ , so  $e_v > d_v$ , and  $\sum_{j \leq v+1} d_j \geq \sum_{j \leq v+1} e_j$ , so  $d_v \geq d_{v+1} \geq e_{v+1}$ .

Now let  $\lambda_3 = [f_1, \dots, f_r]$  where  $f_u = e_u + 1$ ,  $f_v = e_v - 1$ , and otherwise  $f_j = e_j$ . Then  $f_j$  is nondecreasing since  $e_j$  is and either  $u = 1$  or  $f_{u-1} = e_{u-1} \geq e_u + 1 = f_u$ , and either  $v = q$  or  $f_v = e_v - 1 \geq e_{v+1} = f_{v+1}$ . Moreover,  $\sum_{j \leq i} d_j \geq \sum_{j \leq i} f_j \geq \sum_{j \leq i} e_j$  is true for all  $i$ . It holds for  $i < u$  since  $f_j = e_j = d_j$  in this range. It holds for  $u \leq i < v$  since  $\sum_{j \leq i} d_j > \sum_{j \leq i} e_j$  but  $1 + \sum_{j \leq i} e_j = \sum_{j \leq i} f_j$  in this range. And it holds for  $i \geq v$ , since  $\sum_{j \leq i} e_j = \sum_{j \leq i} f_j$  in this range.

Thus  $\lambda_1 \geq \lambda_3 > \lambda_2$ , so it suffices by induction to show  $\dim \mathbb{X}_{n-1, \lambda_3} > \dim \mathbb{X}_{n-1, \lambda_2}$ . Writing each  $f_j$  in terms of  $e_j$ , this is equivalent to showing  $\binom{e_u+1+n-1}{n-1} + \binom{e_v-1+n-1}{n-1} - 2 > \binom{e_u+n-1}{n-1} + \binom{e_v+n-1}{n-1} - 2$ . This in turn is equivalent to

$$\begin{aligned} \binom{e_u+1+n-2}{n-2} &= \binom{e_u+n-1}{n-2} \\ &= \binom{e_u+n}{n-1} - \binom{e_u+n-1}{n-1} > \binom{e_v+n-1}{n-1} - \binom{e_v-1+n-1}{n-1} = \binom{e_v-1+n-1}{n-2} = \binom{e_v+n-2}{n-2}, \end{aligned}$$

which is true because  $\binom{j+n-2}{n-2}$  is a strictly increasing function of  $j \geq 0$  if  $n-2 \geq 1$ .  $\square$

We now have:

**Corollary 2.18.** *Let  $\lambda = [d_1, \dots, d_r]$  be any partition of  $d$  with  $r \geq 2$ , and let  $\lambda_2 = [d_1, 1, \dots, 1]$  also be a partition of  $d$ . Assume  $n \geq 3$ . If  $\lambda$  is neither  $\lambda_2$  nor  $[d-1, 1]$ , then*

$$\dim \mathbb{X}_{n-1, \lambda_2} < \dim \mathbb{X}_{n-1, \lambda} < \dim \mathbb{X}_{n-1, [d-1, 1]}.$$

*Proof.* This follows from Lemma 2.17 and the fact that  $\lambda_2 < \lambda < [d-1, 1]$ .  $\square$

Notice that a result analogous to Lemma 2.17 is not true for the lexicographic order. For example,  $\lambda_1 = [5, 4, 1, 1, 1] > [5, 3, 3, 2] = \lambda_2$  in the lexicographic order, but  $\dim \mathbb{X}_{2, \lambda_1} = 42 < 43 = \dim \mathbb{X}_{2, \lambda_2}$ . Observe that  $\lambda_1$  and  $\lambda_2$  are not comparable in the partial order used in Lemma 2.17.

### 3. PROPER INTERSECTIONS

In this section we focus on the case where the varieties determining tangent spaces to  $\mathbb{X}_{n-1, \lambda}$  at  $\ell$  general points meet properly. Our main result is Theorem 3.5. The case of improper intersections is the subject of a later section.

We first show that the  $\ell$  varieties determining tangent spaces intersect properly if  $\ell$  is small enough.

**Proposition 3.1.** *Assume  $2\ell \leq n$ . Then:*

- (a) *The  $(\ell-1)n$  general linear forms in  $\mathcal{L}$  are a  $B$ -regular sequence.*
- (b) *The varieties defined by  $I_{(1)}, \dots, I_{(\ell)}$  intersect properly, that is,*

$$\text{codim}(I_{(1)} + \dots + I_{(\ell)}) = 2\ell.$$

*Proof.* By Theorem 2.14, the algebra  $B$  is Cohen-Macaulay of dimension  $\ell(n-2)$ . The assumption on  $\ell$  guarantees  $(\ell-1)n \leq \ell(n-2)$ . Hence  $\mathcal{L}$  is a regular sequence and  $B/\mathcal{L}B$  has dimension  $n-2\ell$ . Now Lemma 2.15 gives  $\text{codim}(I_{(1)} + \dots + I_{(\ell)}) = 2\ell$ .  $\square$

**Remark 3.2.** If  $\ell \leq \frac{n}{2}$ , then the minimal free graded resolution of  $S/(I_{(1)} + \dots + I_{(\ell)})$  has the same graded Betti numbers as the minimal free graded resolution of  $T/\tilde{I}$  since forming a quotient by factoring with a regular sequence does not change the graded Betti numbers of the resolution modules.

**Remark 3.3.** Using the isomorphism of graded modules (see Theorem 2.14)

$$T/\tilde{I} \cong S/I_{(1)} \otimes_{\mathbb{k}} \dots \otimes_{\mathbb{k}} S/I_{(\ell)},$$

it follows that  $\mathbf{HS}(T/\tilde{I}) = (\mathbf{HS}(S/J))^\ell$ , where  $J$  is as given in Remark 2.11.

Furthermore, if  $2\ell \leq n$ , Proposition 3.1(a), Remark 2.13(d), and Lemma 2.15 give

$$\begin{aligned}\mathbf{HS}(S/(I_{(1)} + \cdots + I_{(\ell)})) &= (1-t)^{n(\ell-1)} \cdot \mathbf{HS}(T/\tilde{I}) \\ &= (1-t)^{n(\ell-1)} \cdot [\mathbf{HS}(S/J)]^\ell.\end{aligned}$$

Putting this together with Equation (2.4) in Remark 2.13 we obtain

$$\mathbf{HS}(\mathbb{k}[x_1, \dots, x_n]/(I_{(1)} + \cdots + I_{(\ell)})) = \frac{1}{(1-t)^n} \left[ 1 - \sum_{i=1}^r t^{d-d_i} + (r-1)t^d \right]^\ell.$$

Rewriting this last expression we have, if  $2\ell \leq n$ , that

$$\mathbf{HS}(I_{(1)} + \cdots + I_{(\ell)}) = \frac{1}{(1-t)^n} - \frac{1}{(1-t)^n} \left[ 1 - \sum_{i=1}^r t^{d-d_i} + (r-1)t^d \right]^\ell.$$

If we now put together Corollary 2.7 and Remark 3.3 we obtain:

**Theorem 3.4.** *Let  $\lambda \vdash d$ ,  $\lambda = [d_1, \dots, d_r]$  and suppose that  $2\ell \leq n$ . Put*

$$A = \mathbb{k}[x_1, \dots, x_n]/(I_{(1)} + \cdots + I_{(\ell)}).$$

*Then*

$$\mathbf{HS}(A) = \frac{1}{(1-t)^n} \left[ 1 - \sum_{i=1}^r t^{d-d_i} + (r-1)t^d \right]^\ell.$$

*Moreover, if  $a_d$  denotes the coefficient of  $t^d$  in  $\mathbf{HS}(A)$ , then*

$$\dim \sigma_\ell(\mathbb{X}_{n-1, \lambda}) = \binom{d+n-1}{n-1} - a_d - 1.$$

We now compute the coefficient  $a_d$ , which gives the main result of this section. Although it is not obvious that the right-hand side of the formula for  $\dim \sigma_\ell(\mathbb{X}_{n-1, \lambda})$  given in Theorem 3.5 is less than or equal to  $\binom{d+n-1}{n-1} - 1$ , this follows from the fact that the right-hand side is  $\binom{d+n-1}{n-1} - a_d - 1$ , since  $a_d = \dim_{\mathbb{k}}[A]_d$ .

**Theorem 3.5.** *Let  $\lambda \vdash d$ ,  $\lambda = [d_1, \dots, d_r]$  with  $r \geq 2$ . If  $2\ell \leq n$  then:*

$$\begin{aligned}\dim \sigma_\ell(\mathbb{X}_{n-1, \lambda}) &= \ell \cdot \dim \mathbb{X}_{n-1, \lambda} + \ell - 1 \\ &\quad - \sum_{k=2}^{\ell} (-1)^k \binom{\ell}{k} \binom{d_1 - (k-1)(d_2 + \cdots + d_r) + n - 1}{n-1} \\ &\quad - \binom{\ell}{2} \binom{2d_2 - d + n - 1}{n-1} - \ell(\ell-1) \binom{d_1 + d_2 - d + n - 1}{n-1}.\end{aligned}$$

*Moreover,  $\sigma_\ell(\mathbb{X}_{n-1, \lambda})$  fills its ambient space if and only if one of the following conditions is satisfied:*

- (i)  $n = 4$ ,  $\ell = 2$ , and  $\lambda \in \{[1, 1], [2, 1], [1, 1, 1]\}$  or
- (ii)  $n = 2\ell \geq 6$  and  $\lambda = [1, 1]$ .



*Proof.* Let  $P_1, \dots, P_\ell$  be general points on  $\mathbb{X}_{n-1, \lambda}$  and set

$$A = \mathbb{k}[x_1, \dots, x_n]/(I_{P_1} + \dots + I_{P_\ell}).$$

Then we have seen in Theorem 3.4 that

$$(3.1) \quad \mathbf{HS}(A) = \frac{1}{(1-t)^n} \left[ 1 - \sum_{i=1}^r t^{d-d_i} + (r-1)t^d \right]^\ell.$$

Observing that

$$2(d-d_1) + (d-d_2) \geq 2d_2 + (d_1 + d_3 + \dots + d_r) > d,$$

we get

$$\begin{aligned} \left[ 1 - \sum_{i=1}^r t^{d-d_i} + (r-1)t^d \right]^\ell &= \left[ 1 - \sum_{i=1}^r t^{d-d_i} \right]^\ell + (r-1)\ell \cdot t^d + \dots \\ &= 1 - \ell \sum_{i=1}^r t^{d-d_i} + \sum_{k=2}^{\ell} (-1)^k \binom{\ell}{k} \cdot t^{k(d-d_1)} \\ &\quad + \ell(\ell-1) \cdot t^{d-d_1+d-d_2} + \binom{\ell}{2} \cdot t^{2(d-d_2)} + (r-1)\ell \cdot t^d + \dots \end{aligned}$$

where only the terms whose exponent of  $t$  are potentially at most  $d$  have been written out. Using also

$$\frac{1}{(1-t)^n} = \sum_{j \geq 0} \binom{j+n-1}{j} \cdot t^j,$$

we get from Equation (3.1)

$$\mathbf{HS}(A) = \left[ 1 - \sum_{i=1}^r t^{d-d_i} + (r-1)t^d \right]^\ell \cdot \left[ \sum_{j \geq 0} \binom{j+n-1}{j} \cdot t^j \right].$$

The coefficient of  $t^d$  in  $\mathbf{HS}(A)$  is

$$\begin{aligned} \dim_{\mathbb{k}}[A]_d &= \binom{d+n-1}{n-1} - \ell \sum_{i=1}^r \binom{d_i+n-1}{n-1} + (r-1)\ell \\ &\quad + \sum_{k=2}^{\ell} (-1)^k \binom{\ell}{k} \binom{d_1 - (k-1)(d_2 + \dots + d_r) + n - 1}{n-1} \\ &\quad + \binom{\ell}{2} \binom{2d_2 - d + n - 1}{n-1} + \ell(\ell-1) \binom{d_1 + d_2 - d + n - 1}{n-1}. \end{aligned}$$

This gives

$$\begin{aligned}
\dim \sigma_\ell(\mathbb{X}_{n-1,\lambda}) &= -1 + \dim_{\mathbb{k}}[I_{(1)} + \cdots + I_{(\ell)}]_d \\
&= -1 + \ell \sum_{i=1}^r \binom{d_i + n - 1}{n - 1} - (r - 1)\ell \\
&\quad - \sum_{k=2}^{\ell} (-1)^k \binom{\ell}{k} \binom{d_1 - (k - 1)(d_2 + \cdots + d_r) + n - 1}{n - 1} \\
&\quad - \binom{\ell}{2} \binom{2d_2 - d + n - 1}{n - 1} - \ell(\ell - 1) \binom{d_1 + d_2 - d + n - 1}{n - 1}.
\end{aligned}$$

Then using Formula (2.2), we get

$$\begin{aligned}
\dim \sigma_\ell(\mathbb{X}_{n-1,\lambda}) &= \ell \cdot \dim \mathbb{X}_{n-1,\lambda} + (\ell - 1) \\
&\quad - \sum_{k=2}^{\ell} (-1)^k \binom{\ell}{k} \binom{d_1 - (k - 1)(d_2 + \cdots + d_r) + n - 1}{n - 1} \\
&\quad - \binom{\ell}{2} \binom{2d_2 - d + n - 1}{n - 1} - \ell(\ell - 1) \binom{d_1 + d_2 - d + n - 1}{n - 1},
\end{aligned}$$

as claimed.

It remains to show the characterization, for  $2\ell \leq n$ , of when  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  fills its ambient space.

This clearly occurs if and only if  $[A]_d = 0$ . If  $2\ell < n$ , then  $[A]_d$  cannot be zero because  $A$  is not artinian as  $\dim A = n - 2\ell$ .

Let  $n = 2\ell$ . In this case we can write

$$\mathbf{HS}(A) = \left[ \frac{1}{(1-t)^{2\ell}} \left( 1 - \sum_{i=1}^r t^{d-d_i} + (r-1)t^d \right) \right]^\ell.$$

Remark 2.11 implies that the artinian reduction of  $S/J$  is level of socle degree  $d - 2$ . Hence, each factor

$$\frac{1}{(1-t)^2} \left( 1 - \sum_{i=1}^r t^{d-d_i} + (r-1)t^d \right)$$

is a polynomial of degree  $d - 2$ . It follows that  $\mathbf{HS}(A)$  is a polynomial of degree  $\ell(d - 2)$ . Since  $A$  is artinian, this shows that  $[A]_d = 0$  if and only if  $\ell(d - 2) < d$ . This is equivalent to

$$d < 2 + \frac{2}{\ell - 1}.$$

If  $\ell = 2$  (hence  $n = 4$ ), then we can have  $d = 2$  or  $3$  and so  $\lambda$  can be  $[1, 1]$ ,  $[2, 1]$  or  $[1, 1, 1]$ . If  $\ell > 2$  then we must have  $d = 2$  and  $\lambda = [1, 1]$ .  $\square$

**Remark 3.6.** Note that  $\dim \sigma_\ell(\mathbb{X}_{n-1,\lambda}) < N - 1 < \ell(\dim_{\mathbb{k}}(\mathbb{X}_{n-1,\lambda})) + \ell - 1$  can occur (where  $N = \binom{d+n-1}{n-1}$ ), as happens, for example, when  $n = 4$ ,  $\lambda = [2, 2]$  and  $\ell = 2$ . Thus, when  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  does not fill its ambient space, the defect in Theorem 3.5 need not be given by

$$\sum_{k=2}^{\ell} (-1)^k \binom{\ell}{k} \binom{d_1 - (k-1)(d_2 + \cdots + d_r) + n - 1}{n-1} + \binom{\ell}{2} \binom{2d_2 - d + n - 1}{n-1} + \ell(\ell - 1) \binom{d_1 + d_2 - d + n - 1}{n-1}.$$

In general one must use

$$\sum_{k=2}^{\ell} (-1)^k \binom{\ell}{k} \binom{d_1 - (k-1)(d_2 + \dots + d_r) + n - 1}{n-1} + \binom{\ell}{2} \binom{2d_2 - d + n - 1}{n-1} + \ell(\ell-1) \binom{d_1 + d_2 - d + n - 1}{n-1} - \epsilon,$$

where  $\epsilon = \max(0, \ell(\dim_{\mathbb{k}}(\mathbb{X}_{n-1, \lambda})) + \ell - 1 - (N - 1))$ .

**Remark 3.7.** Observe that the last term in the formula of Theorem 3.5 is zero if and only if  $r \geq 3$  and that the penultimate term is zero unless  $r = 2$  and  $d_1 = d_2$ .

**Remark 3.8.** There is an interesting way to interpret the formula in Theorem 3.5.

Let  $\mathcal{I} \subset S$  be an ideal generated by  $\ell$  general forms of degree  $s = d_2 + \dots + d_r$ , where  $\ell \leq n$ . Let  $\text{Syz}$  be the module of first syzygies of  $\mathcal{I}$ , i.e., the sequence

$$0 \longrightarrow \text{Syz} \longrightarrow S(-s)^{\ell} \longrightarrow \mathcal{I} \longrightarrow 0$$

is exact.

From the Koszul complex we obtain the following resolution of  $\text{Syz}$ ,

$$(3.2) \quad 0 \rightarrow S(-\ell s)^{\binom{\ell}{\ell}} \rightarrow \dots \rightarrow S(-3s)^{\binom{\ell}{3}} \rightarrow S(-2s)^{\binom{\ell}{2}} \rightarrow \text{Syz} \rightarrow 0$$

and so

$$(3.3) \quad \dim_{\mathbb{k}}[\text{Syz}]_d = \sum_{k=2}^{\ell} (-1)^{k-1} \binom{\ell}{k} \binom{d - ks + n - 1}{n-1}.$$

Using Remark 3.8 and Theorem 3.5 we get

**Corollary 3.9.** *Let  $2\ell \leq n$  and  $\lambda = [d_1, \dots, d_r] \vdash d$ . Let  $\mathcal{I} \subset S$  be an ideal generated by  $\ell$  general forms of degree  $s = d_2 + \dots + d_r$  and let  $\text{Syz}$  be the first syzygy module of  $\mathcal{I}$ .*

*Then*

$$\begin{aligned} \dim \sigma_{\ell}(\mathbb{X}_{n-1, \lambda}) &= \ell \dim \mathbb{X}_{n-1, \lambda} + (\ell - 1) - \dim_{\mathbb{k}}[\text{Syz}]_d \\ &\quad - \binom{\ell}{2} \binom{2d_2 - d + n - 1}{n-1} - \ell(\ell - 1) \binom{d_1 + d_2 - d + n - 1}{n-1}. \end{aligned}$$

**Remark 3.10.** It is immediate from Corollary 3.9 that if  $r \geq 3$  then

$$\dim \sigma_{\ell}(\mathbb{X}_{n-1, \lambda}) = \ell \dim \mathbb{X}_{n-1, \lambda} + (\ell - 1) - \dim_{\mathbb{k}}[\text{Syz}]_d$$

and if  $r = 2$  and  $d_1 > d_2$  then

$$\dim \sigma_{\ell}(\mathbb{X}_{n-1, \lambda}) = \ell \dim \mathbb{X}_{n-1, \lambda} + (\ell - 1) - \dim_{\mathbb{k}}[\text{Syz}]_d - \ell(\ell - 1).$$

We now discuss the defectivity of  $\sigma_{\ell}(\mathbb{X}_{n-1, \lambda})$  if  $2\ell \leq n$ . We note that we state additional results in the case  $\ell = 2$  without this restriction on  $n$  in Section 4.

For convenience, we consider the case  $r = 2$  separately.

**Theorem 3.11.** *Let  $r = 2$  with  $\ell \leq \frac{n}{2}$  and  $\lambda = [d_1, d_2]$ . Then  $\sigma_{\ell}(\mathbb{X}_{n-1, \lambda})$  fills its ambient space if and only if  $n = 2\ell$  and  $\lambda = [1, 1]$ , or  $n = 4$ ,  $\ell = 2$  and  $\lambda = [2, 1]$ .*

*In all other cases,  $\sigma_{\ell}(\mathbb{X}_{n-1, \lambda})$  is defective and the defect is*

$$\delta_{\ell} = \begin{cases} 2\ell(\ell - 1) - \epsilon & \text{if } d_1 = d_2, \text{ and} \\ \ell(\ell - 1) - \epsilon + \dim_{\mathbb{k}}[\text{Syz}]_d & \text{if } d_1 > d_2, \end{cases}$$

where an explicit formula for  $\dim_{\mathbb{k}}[\text{Syz}]_d$  is given in (3.3), and

$$\epsilon = \max\{0, \ell \cdot \dim_{\mathbb{k}}(\mathbb{X}_{n-1, \lambda}) + \ell - 1 - (N - 1)\}.$$

*Proof.* Theorem 3.5 implies the first claim, and that  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  does not fill its ambient space in all other cases. Keeping Remark 3.6 in mind, we get  $\delta_\ell$  for the case that  $d_1 = d_2$  from Theorem 3.5, and from Remark 3.10 for the other case.  $\square$

**Remark 3.12.** The paper [9] solves the problem of determining when  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  fills its ambient space in the case when  $r = 2$  and  $2\ell \leq n + 1$ , in quite different language. We should point out, first, that their  $r$  is our  $\ell$  and their  $n$  is our  $n - 1$ . Most of the cases that they need to consider satisfy  $2\ell = n + 1$ , and so do not overlap with Theorem 3.11 (but see Theorem 6.8 (a)). Nevertheless, the second sentence of Theorem 3.11 follows from Theorem 5.1 of [9]. We included it above for reference and because it is such an easy consequence of our current approach.

**Theorem 3.13.** *Let  $\lambda \vdash d$ ,  $\lambda = [d_1, \dots, d_r]$ , where  $r \geq 3$ . Assume  $2\ell \leq n$ .*

(a) *If  $d_1 < d_2 + \dots + d_r$ , then*

$$\dim \sigma_\ell(\mathbb{X}_{n-1,\lambda}) = \ell \cdot \dim \mathbb{X}_{n-1,\lambda} + (\ell - 1) \leq N - 1$$

*and  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  is not defective.*

(b) *If  $d_1 \geq d_2 + \dots + d_r$ , then  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  is defective, with defect*

$$\delta_\ell = \dim_{\mathbb{k}}[\text{Syz}]_d - \epsilon,$$

*where  $\dim_{\mathbb{k}}[\text{Syz}]_d$  is given explicitly in (3.8) and  $\epsilon$  is as given in Remark 3.6.*

*Proof.* We begin with (a). The proof is immediate from Theorem 3.5 since, as partly noted in Remark 3.7, all the summands in that formula vanish except for  $\ell \cdot \dim \mathbb{X}_{n-1,\lambda} + (\ell - 1)$ .

As for (b), Theorem 3.5 shows that  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  does not fill its ambient space. Thus, if  $N - 1 \leq \ell(\dim \mathbb{X}_{n-1,\lambda}) + \ell - 1$ , then  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  is defective, and using Remarks 3.6 and 3.10 we see the defect is  $\delta_\ell = \dim_{\mathbb{k}}[\text{Syz}]_d - \epsilon$ . Suppose now that  $N - 1 > \ell(\dim \mathbb{X}_{n-1,\lambda}) + \ell - 1$ , so  $\epsilon = 0$ . Hence, Remark 3.10 gives that the defect is

$$\delta_\ell = \dim_{\mathbb{k}}[\text{Syz}]_d = \dim_{\mathbb{k}}[\text{Syz}]_d - \epsilon.$$

It remains to show that it is positive. However,  $\text{Syz}$  is a submodule of a free module, and the generators of  $\text{Syz}$  have degree  $2(d_2 + \dots + d_r) \leq d$  (see (3.2)). We conclude that  $\dim_{\mathbb{k}}[\text{Syz}]_d > 0$ , and so  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  again is defective.  $\square$

**Remark 3.14.** Theorems 3.11 and 3.13 give us our first view of the fact that the hyperplane  $d_1 = d_2 + \dots + d_r$  in  $\mathbb{N}^r$  separates two very different kinds of behaviors with respect to defectivity, when  $[d_1, \dots, d_r] = \lambda \vdash d$  is a partition of  $d$  into  $r \geq 2$  parts. This was observed for  $n = 2$  in [14], and it will recur frequently in this paper. As a result, we will follow [14] in referring to this as the *partition dividing hyperplane in  $\mathbb{N}^r$* .

**Remark 3.15.** The formula for  $\dim \sigma_\ell(\mathbb{X}_{n-1,\lambda})$  given in Conjecture 1.1(b) and Theorem 3.5 comes from Corollary 2.7 after interpreting  $\dim_{\mathbb{k}}[I_{P_1} + \dots + I_{P_\ell}]_d = \dim_{\mathbb{k}}([I_{P_1}]_d + \dots + [I_{P_\ell}]_d)$ . The simplest case occurs when the spaces  $[I_{P_i}]_d$  meet pair-wise in just 0. In that case  $\dim_{\mathbb{k}}([I_{P_1}]_d + \dots + [I_{P_\ell}]_d) = \dim_{\mathbb{k}}[I_{P_1}]_d + \dots + \dim_{\mathbb{k}}[I_{P_\ell}]_d = \ell(\dim_{\mathbb{k}}[I_{P_1}]) = \ell(1 + \dim \mathbb{X}_{n-1,\lambda})$  so  $\dim \sigma_\ell(\mathbb{X}_{n-1,\lambda}) = \ell(\dim \mathbb{X}_{n-1,\lambda}) + \ell - 1$ .

Often the pairs will not meet only in 0. To consider that case, we set  $\underline{i} = \{i_1, \dots, i_j\}$  for  $1 \leq i_1 < \dots < i_j \leq \ell$  and say  $|\underline{i}| = j$ . We then define  $V_{\underline{i}} = \cap_{t \in \underline{i}} [I_{P_t}]$ . For  $1 \leq u \leq \ell$ , let  $v_u = \sum_{|\underline{i}|=u} \dim V_{\underline{i}}$ , so  $v_1 = \dim_{\mathbb{k}}[I_{P_1}]_d + \dots + \dim_{\mathbb{k}}[I_{P_\ell}]_d$ ,  $v_2$  is the sum of the dimensions of

the pair-wise intersections of the  $[I_{P_i}]_d$ ,  $v_3$  is the sum of the triple intersections, and so on. Inclusion-exclusion now gives  $\dim_{\mathbb{k}}([I_{P_1}]_d + \cdots + [I_{P_\ell}]_d) = \sum_{1 \leq u \leq \ell} (-1)^{u+1} v_u$ .

Let's look at this in the case that  $\lambda$  is such that  $d_1 \geq s = d_2 + d_3 + \cdots + d_r$ ; i.e.  $\lambda$  is above the "partition dividing hyperplane" in  $\mathbb{N}^r$ . This is an example for which it is not possible that  $[I_{P_{i_1}}]_d \cap [I_{P_{i_2}}]_d = 0$  for  $i_1 \neq i_2$ . To see this, say each  $P_j$  corresponds to the form  $F_{j,1}F_{j,2} \cdots F_{j,r}$ . Then  $I_{P_i} \cap I_{P_j}$  will, for every  $i, j \in \{1, \dots, \ell\}$ ,  $i \neq j$ , contain all the products of the type:  $F_{i,2} \cdots F_{i,r} F_{j,2} \cdots F_{j,r} G$ , where  $G$  is any form of degree  $d - 2s = d_1 - s$ . Thus  $\dim_{\mathbb{k}}[I_{P_{i_1}}]_d \cap [I_{P_{i_2}}]_d \geq \binom{d-2s+n-1}{n-1}$  for each pair  $i_1 \neq i_2$ , so  $v_2 \geq \binom{\ell}{2} \binom{d-2s+n-1}{n-1}$ . Moreover, when  $r = 2$ ,  $I_{P_i} \cap I_{P_j}$  will also contain the forms  $F_{1,i}F_{2,j}$ , for all  $i \neq j \in \{1, \dots, \ell\}$  so in this case  $v_2 \geq \binom{\ell}{2} \binom{d-2s+n-1}{n-1} + \ell(\ell - 1)$ . If  $r = 2$  and  $d_1 = d_2$ , we also have  $F_{1,i}F_{1,j}$  in the intersection, so  $v_2 \geq \binom{\ell}{2} \binom{d-2s+n-1}{n-1} + \ell(\ell - 1) + \binom{\ell}{2}$ .

Similarly, if  $d_1 \geq 2s$ , then  $\dim_{\mathbb{k}}[I_{P_{i_1}} \cap I_{P_{i_2}} \cap I_{P_{i_3}}]_d \geq \binom{d-3s+n-1}{n-1}$ , since

$$F_{i_1,2} \cdots F_{i_1,r} F_{i_2,2} \cdots F_{i_2,r} F_{i_3,2} \cdots F_{i_3,r} H \in [I_{P_{i_1}} \cap I_{P_{i_2}} \cap I_{P_{i_3}}]_d$$

for any  $H \in S_{d-3s}$ . Hence  $v_3 \geq \binom{\ell}{3} \binom{d-3s+n-1}{n-1}$ . In the same way, if  $d_1 \geq ks$ , then  $v_k \geq \binom{\ell}{k} \binom{d-ks+n-1}{n-1}$ .

If the lower bounds on each  $v_u$  above were to equal the corresponding  $v_u$ , then  $\dim_{\mathbb{k}}([I_{P_1}]_d + \cdots + [I_{P_\ell}]_d) = \sum_{1 \leq u \leq \ell} (-1)^{u+1} v_u$  becomes precisely the formula given in Conjecture 1.1(b) and proved in a special case in Theorem 3.5. Of course, because cancellations may occur in an alternating sum, it is possible for the formula in Conjecture 1.1(b) to hold even if some of the lower bounds were to be strictly less than their corresponding  $v_u$ .

#### 4. THE SECANT LINE VARIETY AND PASSAGE TO IMPROPER INTERSECTIONS

A consequence of the work done up to this point is that if  $n \geq 4$  we have the following result for the secant line variety of the variety of reducible forms of prescribed type:

**Theorem 4.1.** *Assume  $\ell = 2$  and  $n \geq 4$ . Then, for all partitions  $\lambda = [d_1, \dots, d_r]$  of  $d$  with  $r \geq 2$ :*

(a)

$$\begin{aligned} \dim \sigma_2(\mathbb{X}_{n-1,\lambda}) &= 2 \dim \mathbb{X}_{n-1,\lambda} + 1 - \binom{d_1 - (d_2 + \cdots + d_r) + n - 1}{n-1} \\ &\quad - \binom{2d_2 - d + n - 1}{n-1} - 2 \binom{d_1 + d_2 - d + n - 1}{n-1}; \end{aligned}$$

(b)  $\sigma_2(\mathbb{X}_{n-1,\lambda})$  fills its ambient space if and only if  $\lambda \in \{[1, 1], [2, 1], [1, 1, 1]\}$  and  $n = 4$ ;

(c) if  $\sigma_2(\mathbb{X}_{n-1,\lambda})$  does not fill its ambient space, then  $\sigma_2(\mathbb{X}_{n-1,\lambda})$  is not defective if and only if  $d_1 < d_2 + \cdots + d_r$ ; and

(d) if  $\sigma_2(\mathbb{X}_{n-1,\lambda})$  is defective, then the defect is

$$\binom{d_1 - (d_2 + \cdots + d_r) + n - 1}{n-1} + \binom{2d_2 - d + n - 1}{n-1} + 2 \binom{d_1 + d_2 - d + n - 1}{n-1} - \epsilon,$$

where  $\epsilon$  is as given in Remark 3.6.

*Proof.* Parts (a) and (b) are immediate from Theorem 3.5. Theorem 1.4 gives part (c). Claim (d) is a consequence of (a).  $\square$

Notice that Theorem 4.1 assumes  $n \geq 4$ , since  $\ell = 2$  but it relies on results that assume  $n \geq 2\ell$ . The main purpose of this section is to understand what is needed to pass beyond the condition  $2\ell \leq n$  with our approach. We begin with a review of the main results of [14], since that paper gives a careful analysis of the case  $\ell = 2$ ,  $n = 3$  using entirely different methods. Then we will see what would be needed in order to pass from the results of the previous section to this case. Having the essential idea in hand, subsequent sections of this paper will carry out the calculations. The main result is a single, explicit (albeit complicated) conjectured formula for the dimension of the secant variety for *any* choice of  $n$ ,  $\lambda$  and  $\ell$  (see Conjecture 1.1 and Theorem 5.11). In particular, the results of the previous sections agree with this conjecture. We are then able to give many consequences, some proven unconditionally, some conjectural.

So first we recall the results of [14], which in particular confirm the importance of the “partition dividing hyperplane” mentioned in Remark 3.14. We will see that, in general, the partitions “below” the hyperplane (i.e., those partitions for which  $d_1 < d_2 + \cdots + d_r$ ) behave quite differently from those “above” the hyperplane (i.e., those partitions for which  $d_1 \geq d_2 + \cdots + d_r$ ). In fact, we expect this to be true in all cases (see Conjecture 1.3 and Proposition 5.13).

Propositions 4.2 and 4.3 can be deduced easily from the results of [14]. We see that the condition for defectivity when  $r \geq 6$  is straightforward, but there are exceptions when  $2 \leq r < 6$ .

**Proposition 4.2.** *Let  $n = 3$  with  $\lambda = [d_1, \dots, d_r] \vdash d$  a partition of  $d$  into  $r \geq 6$  parts and  $s = d_2 + \cdots + d_r$ . Set*

$$p = \sum_{2 \leq i < j \leq r} d_i d_j.$$

(a) *If  $d_1 \geq s$ , then  $\sigma_2(\mathbb{X}_{2,\lambda})$  is always defective, and the defect is*

$$(4.1) \quad \min \left\{ \binom{d_1 - s + 2}{2}, 2p - 3s \right\}.$$

(b) *If  $d_1 < s$ , then*

$$\dim(\sigma_2(\mathbb{X}_{2,\lambda})) = 2 \dim(\mathbb{X}_{2,\lambda}) + 1,$$

*and hence  $\sigma_2(\mathbb{X}_{2,\lambda})$  is not defective.*

**Proposition 4.3.** *Let  $n = 3$  with  $\lambda = [d_1, \dots, d_r] \vdash d$  a partition of  $d$  into  $2 \leq r < 6$  parts and let  $s = d_2 + \cdots + d_r$ .*

(a) *If  $r = 2$ , then the secant line variety fills its ambient space, and so it is never defective.*

(b) *For the following partitions the secant variety  $\sigma_2(\mathbb{X}_{2,\lambda})$  fills its ambient space and so the defect is zero:*

- $r = 3$  and  $\lambda \in \{[d_1, d_2, 1], [d_1, 2, 2], [d_1, 3, 2], [d_1, 4, 2], [d_1, 5, 2], [d_1, 6, 2], [d_1, 3, 3]\}$ ;
- $r = 4$  and  $\lambda \in \{[d_1, 1, 1, 1], [d_1, 2, 1, 1], [d_1, 3, 1, 1], [d_1, 4, 1, 1]\}$ ;
- $r = 5$  and  $\lambda = [d_1, 1, 1, 1, 1]$ .

(c) *Apart from the partitions described above, if  $d_1 \geq s$  and  $r \geq 3$ , then  $\sigma_2(\mathbb{X}_{2,\lambda})$  is always defective. In this case the defect is equal to (4.1) above.*

(d) *If  $d_1 < s$  then  $\sigma_2(\mathbb{X}_{2,\lambda})$  is never defective. Apart from the partitions described in (b), the secant line variety has dimension  $2 \dim(\mathbb{X}_{2,\lambda}) + 1$ .*

**Example 4.4.** Consider  $\lambda = [2, 2, 2, 1]$ . This partition has  $d_1 < s$  and so we are below the partition dividing hyperplane. By Proposition 4.3(d),  $\dim \sigma_2(\mathbb{X}_{2,\lambda}) = 2 \dim \mathbb{X}_{2,\lambda} + 1$  and a computation shows that  $2 \dim \mathbb{X}_{2,\lambda} + 1 = N - 1$ , hence  $\sigma_2(\mathbb{X}_{2,\lambda})$  fills its ambient space for this example.

**Remark 4.5.** In case  $\ell = 2$ ,  $\lambda \vdash d$ ,  $\lambda = [d_1, \dots, d_r]$ , with  $d_1 = s$  we can show that the only time that  $\sigma_2(\mathbb{X}_{2,\lambda})$  is a hypersurface in its ambient space is when  $\lambda = [9, 7, 2], [5, 2, 2, 1]$  and  $[7, 5, 1, 1]$ . It would be interesting to find the equations of these hypersurfaces. These are all defective secant line varieties, and up to this point whenever we have been able to find equations for such defective secant varieties they have been determinants. Is that the situation in this case as well?

Now fix  $n = 3$  and  $\ell = 2$ . Then the codimension of  $\sigma_2(\mathbb{X}_{2,\lambda})$  is given by  $\dim_{\mathbb{k}}[S/(I_{P_1} + I_{P_2})]_d$  (see Corollary 2.7). By Lemma 2.15 and Theorem 2.14, the latter is equal to  $\dim_{\mathbb{k}}[B/\mathcal{L}B]_d$ , where  $B = T/(I_{(1)}^{(e)} + I_{(2)}^{(e)})$  and  $\mathcal{L} \subset T$  is a regular sequence comprised of  $(\ell - 1)n = 3$  general linear forms. By Remark 3.3, we know the Hilbert function of  $B$ . Thus, the important question now is to determine the relation between the Hilbert function of  $B$  and that of  $B/\mathcal{L}B$ .

Since  $\dim B = \ell(n - 2) = 2$  (Theorem 2.14), the first two linear forms in  $\mathcal{L}$  form a  $B$ -regular sequence by Proposition 3.1. Say  $\mathcal{L} = \{L_1, L_2, L_3\}$  and  $L_1, L_2$  form a regular sequence. Thus we have to find the Hilbert function after reducing by one more general linear form,  $L_3$ . Let  $\mathcal{L}' = \{L_1, L_2\}$  and let  $B' = B/\mathcal{L}'B$ .

Note that there is an exact sequence

$$(4.2) \quad B'(-1) \xrightarrow{\times L_3} B' \rightarrow B'/L_3B' \rightarrow 0.$$

Thus, in order to determine the Hilbert function of  $B/\mathcal{L}B \cong B'/L_3B'$  it is enough to know that  $\times L_3$  has maximal rank at each degree. This is exactly what is provided by the Weak Lefschetz Property, as described in the next section. It is the basis for the remaining calculations and the general formulae that they give.

As an example, let us consider a case mentioned in Remark 4.5 and verify that if  $\lambda = [9, 7, 2]$ , and if the maximal rank property holds for  $B'$ , then our results of the previous sections imply that  $\sigma_2(\mathbb{X}_{2,\lambda})$  is a hypersurface. Using Remark 3.3, we get for the Hilbert series of  $B'$

$$\begin{aligned} \mathbf{HS}(B') &= (1 - t)^2 \cdot \mathbf{HS}(B) \\ &= \frac{(1 - t)^6}{(1 - t)^4} \cdot \mathbf{HS}(B) \\ &= \frac{1}{(1 - t)^4} [1 - t^{16} - t^{11} - t^9 + t^{18}]^2 \\ &= 1 + 4t + \dots + 634t^{17} + 635t^{18} + \dots + 4t^{32}. \end{aligned}$$

Thus,

$$h_{17} = \dim_{\mathbb{k}}[B']_{17} = 634 \quad \text{and} \quad h_{18} = \dim_{\mathbb{k}}[B']_{18} = 635.$$

Note that  $d = 18$  in our example. If multiplication by  $L_3$  on  $B'$  has maximal rank, we obtain (see Sequence (4.2)) that for  $S = \mathbb{k}[x_1, x_2, x_3]$ ,

$$\dim_{\mathbb{k}}[S/(I_{P_1} + I_{P_2})]_d = \dim_{\mathbb{k}}[B'/L_3B']_d = \max\{h_d - h_{d-1}, 0\} = 1.$$

Since this is precisely the codimension of  $\sigma_2(\mathbb{X}_{2,\lambda})$ , we see that under the hypothesis that multiplication by  $L_3$  has maximal rank we obtain that indeed  $\sigma_2(\mathbb{X}_{2,\lambda})$  is a hypersurface in its ambient space.

For arbitrary  $n$  and  $\ell > \frac{n}{2}$ , we will need that the maximal rank property holds sequentially, using the right number of linear forms, until we arrive at  $n$  variables. In the following sections we make use of this idea to give a general formula for the dimensions of the secant varieties (see Theorem 5.11 and Conjecture 1.1), and as a consequence we describe the defective cases, assuming that suitable maximal rank properties hold. In many cases we know for different reasons that these maximal rank properties do hold, and in those cases we obtain unconditional (not conjectural) formulas. Given the many special cases and the seemingly disparate results covering them that have been found up to now, we were astonished to find a simple unifying principle that produces a single conjectural formula for the exact dimension of the secant variety for any given  $n$ ,  $\lambda$  and  $\ell$ .

## 5. IMPROPER INTERSECTIONS AND LEFSCHETZ PROPERTIES

We now consider the case in which the  $\ell$  varieties  $V(I_{(j)})$  ( $1 \leq j \leq \ell$ ) that determine tangent spaces at  $\ell$  general points to  $\mathbb{X}_{n-1,\lambda}$  intersect improperly. By Proposition 3.1(b), their intersection is proper if  $2\ell \leq n$ . However, if  $2\ell > n$  then we will see that the intersection is improper, so in particular the diagonal trick from intersection theory becomes more difficult to apply. To make up for this, we will use Lefschetz properties as formally introduced in [24] in order to determine  $\dim_{\mathbb{k}}[S/(I_{(1)} + \cdots + I_{(\ell)})]_d$ . Because we are dealing with general forms, these properties are known in some cases, and we conjecture them in the remaining cases.

More precisely, we conjecture that a general artinian reduction of the coordinate ring of the join of  $V(I_{(1)}), \dots, V(I_{(\ell)})$  has enough Lefschetz elements (see Conjecture 5.8 below).

We continue to use the notation introduced above. In particular,

$$B = S/I_{(1)} \otimes_{\mathbb{k}} \cdots \otimes_{\mathbb{k}} S/I_{(\ell)} \cong T/\tilde{I}$$

is the coordinate ring of the join of  $V(I_{(1)}^{(e)}), \dots, V(I_{(\ell)}^{(e)})$  in  $\mathbb{P}^{\ell n - 1}$ .

As above, let  $\mathcal{L}$  be a set of  $(\ell - 1)n$  general linear forms in  $T$ . Let  $\mathcal{L}' \subset \mathcal{L}$  be a subset consisting of  $\min\{\ell(n - 2), (\ell - 1)n\}$  such forms. Thus,

$$\mathcal{L}' = \begin{cases} \mathcal{L} & \text{if } \ell \leq \frac{n}{2} \\ \subsetneq \mathcal{L} & \text{if } \ell > \frac{n}{2} \end{cases}.$$

Then there is the following useful observation.

**Proposition 5.1.** *If  $2\ell \geq n$ , then:*

- (a) *The linear forms in  $\mathcal{L}'$  form a  $B$ -regular sequence and  $\dim B/\mathcal{L}'B = 0$ .*
- (b)  *$\text{codim}(I_{(1)} + \cdots + I_{(\ell)}) = n$ , and hence the varieties  $V(I_{(1)}), \dots, V(I_{(\ell)})$  intersect improperly if  $2\ell > n$ .*

*Proof.* By Theorem 2.14,  $B$  is Cohen-Macaulay of dimension  $\ell(n - 2)$ . Hence, Claim (a) follows by the generality of the linear forms in  $\mathcal{L}'$ .

Part (a) shows in particular that  $\dim B/\mathcal{L}'B = 0$ . Hence Lemma 2.15 gives

$$\text{codim}(I_{(1)} + \cdots + I_{(\ell)}) = n,$$

and we are done. □



By Lemma 2.15, we are interested in the Hilbert function of  $B/\mathcal{L}B$ . We know the Hilbert function of  $B$ , and thus the Hilbert function of its general artinian reduction  $B/\mathcal{L}'B$  by Proposition 5.1(a). However, if  $\ell > \frac{n}{2}$ , then  $\mathcal{L}' \neq \mathcal{L}$ .

Recall that a linear form  $L$  is a non-zerodivisor of a graded algebra  $A$  if the multiplication by  $L$  on  $A$  is injective. If  $A \neq 0$  is artinian, this cannot be true. However, one may hope that this multiplication still has maximal rank. This has been codified in [24] as follows:

**Definition 5.2.** Let  $A = S/J$  be an artinian graded  $\mathbb{k}$ -algebra. Then  $A$  is said to have the *Weak Lefschetz Property* if, for each integer  $i$ , multiplication by a general linear form  $L \in A$  from  $[A]_i$  to  $[A]_{i+1}$  has maximal rank. In this case, the form  $L$  is called a *Lefschetz element* of  $A$ .

We say  $A$  has the *Strong Lefschetz Property* if, for all  $i$  and  $e$ , multiplication by  $L^e$  from  $[A]_i$  to  $[A]_{i+e}$  has maximal rank.

The first systematic study of these properties in this generality was carried out in [24]. In particular, the Hilbert functions of algebras with either the Weak Lefschetz Property or Strong Lefschetz Property were classified there, and a sharp bound was given on the possible graded Betti numbers. Thus, the presence of these properties leads to strong restrictions on the possible invariants. Many natural families of algebras are expected to have a Lefschetz property. However, it is typically very difficult to establish this. We refer to [23] and [34] for further information and results.

There is a useful numerical characterization of Lefschetz elements. To state it we need some notation.

**Definition 5.3.** Let  $\sum_{i \geq 0} a_i t^i$  be a formal power series, where  $a_i \in \mathbb{Z}$ . Then we define an associated power series with non-negative coefficients by

$$\left| \sum_{i \geq 0} a_i t^i \right|^+ = \sum_{i \geq 0} b_i t^i,$$

where

$$b_i = \begin{cases} a_i & \text{if } a_j > 0 \text{ for all } j \leq i \\ 0 & \text{otherwise.} \end{cases}$$

**Lemma 5.4.** Let  $A$  be a standard artinian graded algebra, and let  $L \in A$  be a linear form. Then the following conditions are equivalent:

- (a)  $L$  is a Lefschetz element of  $A$ .
- (b) The Hilbert function of  $A/LA$  is given by

$$\dim_{\mathbb{k}}[A/LA]_i = \max\{0, \dim_{\mathbb{k}}[A]_i - \dim_{\mathbb{k}}[A]_{i-1}\} \quad \text{for all integers } i.$$

- (c) The Hilbert series of  $A/LA$  is

$$\mathbf{HS}(A/LA) = |(1-t) \cdot \mathbf{HS}(A)|^+.$$

This is immediate from the definitions.

It is natural to ask if an algebra has the Weak Lefschetz Property repeatedly, using more than one linear form. This notion was first introduced by Iarrobino (according to the introduction of [25]) and formalized in [25] and [16].

**Definition 5.5.** An artinian standard graded  $\mathbb{k}$ -algebra  $A$  is said to have the *k-Weak Lefschetz Property* (denoted *k-WLP*) if either  $k = 0$ , or  $k > 0$  and there are linear forms  $L_1, \dots, L_k \in A$  such that  $L_i$  is a Lefschetz element of  $A/(L_1, \dots, L_{i-1})A$  for all  $i = 1, \dots, k$ . In this case,  $\{L_1, \dots, L_k\}$  is called a *k-Lefschetz set* of  $A$ .

By definition, every artinian algebra has the 0-WLP. Moreover, if an algebra  $A$  has the *k-WLP*, then a set of  $k$  general linear forms is a *k-Lefschetz set* of  $A$ . Since all quotients of polynomial rings of at most two variables have the Weak Lefschetz Property by [24], the conditions  $(n - 2)$ -WLP and  $n$ -WLP are equivalent for quotients of  $S$ .

Using Lemma 5.4, the above property can be restated as follows.

**Lemma 5.6.** *An artinian standard graded algebra  $A$  has the  $k$ -WLP if and only if there are linear forms  $L_1, \dots, L_k \in A$  such that*

$$\mathbf{HS}(A/(L_1, \dots, L_i)A) = |(1 - t)^i \cdot \mathbf{HS}(A)|^+ \quad \text{for all } i \leq k.$$

*Proof.* This follows by combining Lemma 5.4 and [18, Lemma 4]. □

We now want to use these concepts to determine the dimension of secant varieties  $\sigma_\ell(\mathbb{X}_{n-1, \lambda})$ , using  $B/\mathcal{L}B$ . Since the linear forms in  $\mathcal{L}$  are general, it is reasonable to ask if, in the case where  $\ell > \frac{n}{2}$ , the linear forms in  $\mathcal{L} \setminus \mathcal{L}'$  form a Lefschetz set of the general artinian reduction  $B/\mathcal{L}'B$ . We illustrate the usefulness of this property.

**Example 5.7.** Consider the case  $n = 4$ ,  $\ell = 3$ , and  $\lambda = [3, 2, 2]$ , so  $d = 7$ . Then Remark 3.3 gives

$$\mathbf{HS}(B) = \frac{(1 - t^4 - 2t^5 + 2t^7)^3}{(1 - t)^{12}}.$$

Since  $|\mathcal{L}'| = 6$ , Proposition 3.1 implies

$$\begin{aligned} \mathbf{HS}(B/\mathcal{L}'B) &= \frac{(1 - t^4 - 2t^5 + 2t^7)^3}{(1 - t)^6} \\ &= 8t^{15} + 48t^{14} + 144t^{13} + 292t^{12} + 456t^{11} + 588t^{10} + 646t^9 + 612t^8 \\ &\quad + 504t^7 + 363t^6 + 228t^5 + 123t^4 + 56t^3 + 21t^2 + 6t + 1 \\ &= \sum_{i=0}^{15} c_i t^i \end{aligned}$$

Note that the passage from  $B$  to  $C = B/\mathcal{L}'B$  corresponds to intersecting the join of  $V(I_{P_1}), V(I_{P_2}),$  and  $V(I_{P_3})$  properly with a linear subspace of codimension three. However,  $C$  is artinian, and thus any further hyperplane sections correspond to improper intersections.

We have checked by computer that  $C$  has the 2-WLP, so let  $L_1, L_2$  be a 2-Lefschetz set. Then, we may assume that  $\mathcal{L} = \mathcal{L}' \cup \{L_1, L_2\}$ , and Lemma 2.15 gives

$$\mathbf{HS}(A) = \mathbf{HS}(B/\mathcal{L}B) = \mathbf{HS}(C/(L_1, L_2)C) = |(1 - t)^2 \cdot \mathbf{HS}(C)|^+.$$

We compute this in two steps. First, we get

$$\begin{aligned} \mathbf{HS}(C/L_1C) &= \sum_{i \geq 0} \max\{0, c_i - c_{i-1}\} t^i \\ &= 34t^9 + 108t^8 + 141t^7 + 135t^6 + 105t^5 + 67t^4 + 35t^3 + 15t^2 + 5t + 1 \\ &= \sum_{i=0}^9 b_i t^i. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} \mathbf{HS}(A) = \mathbf{HS}(C/(L_1, L_2)C) &= \sum_{i \geq 0} \max\{0, b_i - b_{i-1}\} t^i \\ &= 6t^7 + 30t^6 + 38t^5 + 32t^4 + 20t^3 + 10t^2 + 4t + 1. \end{aligned}$$

In particular, the secant variety  $\sigma_3(\mathbb{X}_{3,\lambda})$  has codimension  $\dim_{\mathbb{k}}[A]_d = 6$  in its ambient space. Hence,  $\sigma_3(\mathbb{X}_{3,\lambda})$  is non-defective of dimension 113 and does not fill its ambient space.

Computer experiments suggest that a similar analysis can always be carried out. Thus, we conjecture:

**Conjecture 5.8** (WLP-Conjecture). *The algebra  $B/\mathcal{L}^l B$  has the  $k$ -WLP for  $k = \max\{0, 2\ell - n\}$ .*

Notice that, by definition, the WLP-Conjecture is true if  $\ell \leq \frac{n}{2}$ .

In Section 6 we will see that in the case  $r = 2$  this conjecture is closely related to a well-known conjecture by Fröberg, lending additional evidence to the WLP-Conjecture.

Here we show that if true, the WLP-Conjecture allows us to extend Theorem 3.5 to  $\ell$  with  $2\ell > n$ . In order to express this we need more notation.

**Definition 5.9.** Let  $\lambda = [d_1, \dots, d_r] \vdash d$  be a partition with  $r \geq 2$ , and let  $\ell$  and  $n$  be positive integers. For  $j = 0, \dots, d$ , define integers  $a_j = a_j(\ell, n, \lambda)$  by

$$\begin{aligned} a_j &= \binom{j+n-1}{n-1} - \ell \sum_{i=1}^r \binom{j+d_i-d+n-1}{n-1} + (r-1)\ell \binom{j}{d} \\ &\quad + \sum_{k=2}^{\ell} (-1)^k \binom{\ell}{k} \binom{j-k(d-d_1)+n-1}{n-1} \\ &\quad + \binom{\ell}{2} \binom{j+2d_2-2d+n-1}{n-1} + \ell(\ell-1) \binom{j+d_1+d_2-2d+n-1}{n-1}. \end{aligned}$$

Observe that  $a_j(\ell, n, \lambda) > 0$  if  $0 \leq j < s = d_2 + \dots + d_r$  as, for example,  $\binom{j+d_i-d+n-1}{n-1} = 0$  in this case.

Now we explain the meaning of the numbers  $a_j(\ell, n, \lambda)$ .

**Theorem 5.10.** *Assume that the WLP-Conjecture is true for some  $\ell, n$ , and  $\lambda$ . Let  $P_1, \dots, P_\ell$  be general points on  $\mathbb{X}_{n-1,\lambda}$ , and set  $A = S/(I_{P_1} + \dots + I_{P_\ell})$ . If  $i \leq d$  is a non-negative integer then*

$$\dim_{\mathbb{k}}[A]_i = \begin{cases} 0 & \text{if } a_j \leq 0 \text{ for some } j \text{ with } 0 \leq j \leq i \\ a_i > 0 & \text{otherwise.} \end{cases}$$

In particular, if  $a_j(\ell, n, \lambda) > 0$  for all  $j = 0, \dots, i-1$  and  $a_i(\ell, n, \lambda) \geq 0$ , then

$$a_i(\ell, n, \lambda) = \dim_{\mathbb{k}}[A]_i.$$

*Proof.* To simplify notation set  $a_j = a_j(\ell, n, \lambda)$ .

First consider the case where  $2\ell \leq n$ . Then the proof of Theorem 3.5 gives, for all  $i \leq d$ ,

$$\dim_{\mathbb{k}}[A]_i = a_i,$$

and so the conclusion holds without the hypothesis that  $a_j > 0$  for  $j < i$ . Note that, in this case,  $a_j \geq 0$  for all  $j \leq i$ .

Now assume that  $2\ell > n$ . We have seen in Remark 3.3 that

$$\begin{aligned} \mathbf{HS}(B) &= [\mathbf{HS}(S/I_{P_1})]^\ell \\ &= \frac{1}{(1-t)^{\ell n}} \left[ 1 - \sum_{i=1}^r t^{d-d_i} + (r-1)t^d \right]^\ell. \end{aligned}$$

Since the elements of  $\mathcal{L}'$  provide a regular sequence in  $B$  of length  $\ell(n-2)$  by Proposition 5.1, we get

$$\mathbf{HS}(B/\mathcal{L}'B) = \frac{1}{(1-t)^{2\ell}} \left[ 1 - \sum_{i=1}^r t^{d-d_i} + (r-1)t^d \right]^\ell.$$

Hence Lemmas 2.15 and 5.6 together with the WLP-Conjecture give

$$(5.1) \quad \mathbf{HS}(A) = \left| \frac{1}{(1-t)^n} \left[ 1 - \sum_{i=1}^r t^{d-d_i} + (r-1)t^d \right]^\ell \right|^+.$$

Define integers  $b_j$  by

$$\sum_{j \geq 0} b_j t^j = \frac{1}{(1-t)^n} \left[ 1 - \sum_{i=1}^r t^{d-d_i} + (r-1)t^d \right]^\ell.$$

Then computations as in the proof of Theorem 3.5 provide for  $j \leq d$ ,

$$\begin{aligned} b_j &= \binom{j+n-1}{n-1} - \ell \sum_{i=1}^r \binom{j+d_i-d+n-1}{n-1} + (r-1)\ell \binom{j}{d} \\ &\quad + \sum_{k=2}^{\ell} (-1)^k \binom{\ell}{k} \binom{j-k(d-d_1)+n-1}{n-1} \\ &\quad + \binom{\ell}{2} \binom{j+2d_2-2d+n-1}{n-1} + \ell(\ell-1) \binom{j+d_1+d_2-2d+n-1}{n-1} \\ &= a_j. \end{aligned}$$

Thus, we conclude for all non-negative integers  $i \leq d$

$$\dim_{\mathbb{k}}[A]_i = \begin{cases} 0 & \text{if } a_j \leq 0 \text{ for some } j \text{ with } 0 \leq j \leq i \\ a_i > 0 & \text{otherwise.} \end{cases}$$

□

Notice that when  $2\ell < n$  the intersection of the varieties determining tangent spaces to  $\mathbb{X}_{n-1,\lambda}$  is non-empty, and the  $a_j(\ell, n, \lambda)$  give its Hilbert function. When  $2\ell = n$ , the intersection of the varieties determining tangent spaces to  $\mathbb{X}_{n-1,\lambda}$  becomes empty, but this is still a proper intersection and so the methods of Section 3 continue to apply and result in the values given by the  $a_j(\ell, n, \lambda)$ . As soon as  $2\ell > n$ , however, this intersection remains empty but becomes improper. Nevertheless, the  $a_j(\ell, n, \lambda) \geq 0$  essentially provide the Hilbert function of the “algebraic intersection,” i.e. give the Hilbert function of  $S/(I_{P_1} + \cdots + I_{P_\ell})$ , as formalized in the previous result.

We now have the following extension of Theorem 3.5:

**Theorem 5.11.** *Let  $\lambda = [d_1, \dots, d_r] \vdash d$  be a partition with  $r \geq 2$ . Assume that the WLP-Conjecture is true for some  $\ell$  and  $n$ . Then:*

(a) *The secant variety  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  does not fill its ambient space if and only if*

$$a_j(\ell, n, \lambda) > 0 \quad \text{for all } j = 0, \dots, d.$$

(b) *If  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  does not fill its ambient space, then it has dimension*

$$\begin{aligned} \dim \sigma_\ell(\mathbb{X}_{n-1,\lambda}) &= \ell \cdot \dim \mathbb{X}_{n-1,\lambda} + \ell - 1 \\ &\quad - \sum_{k=2}^{\ell} (-1)^k \binom{\ell}{k} \binom{d_1 - (k-1)(d_2 + \cdots + d_r) + n - 1}{n-1} \\ &\quad - \binom{\ell}{2} \binom{2d_2 - d + n - 1}{n-1} - \ell(\ell-1) \binom{d_1 + d_2 - d + n - 1}{n-1} \end{aligned}$$

*Proof.* Using Theorem 5.10 and

$$\dim \sigma_\ell(\mathbb{X}_{n-1,\lambda}) = \binom{d+n-1}{n-1} - 1 - \dim_{\mathbb{k}}[A]_d,$$

this follows from a computation as in the end of the proof of Theorem 3.5.  $\square$

**Remark 5.12.** (i) The argument in the proof of Theorem 5.10 shows more generally that the WLP-Conjecture allows us to determine the Hilbert function of the ring  $A$  in every degree. However, for finding the dimension of  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$ , it is enough to know this Hilbert function in degree  $d$  only. Thus, even if the WLP-conjecture is not true, it is possible that Conjecture 1.1 is correct.

(ii) As noted above, the WLP-Conjecture is true if  $2\ell \leq n$ . Hence, Theorem 5.11 shows Conjecture 1.1 is true if  $2\ell \leq n$ , thus proving Theorem 1.2(a). We will establish further instances of Conjecture 1.1 in Section 6.

(iii) Complete results for the dimension of  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  have been previously obtained only if  $n = 3$ ,  $\lambda = [1, \dots, 1]$  in [1], or  $n = 3$ ,  $\ell = 2$  in [14]. Both results confirm Conjecture 1.1. Moreover, if  $\lambda = [1, \dots, 1]$  and  $d \geq 3$ , then  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  is not defective, as predicted in Conjecture 1.3. The case  $\lambda = [1, 1]$  is covered by Theorem 1.2. The case  $\ell = 2$  was discussed in Section 4.

Thus, Conjecture 1.1 presents a unified formula for  $\dim \sigma_\ell(\mathbb{X}_{n-1,\lambda})$  in all cases. It is consistent with all the known results that we have checked.

We explore some consequences of our main conjecture, Conjecture 1.1.

As we show in our next result, Conjecture 1.3(a) is an immediate consequence of Conjecture 1.1 and thus holds in the many cases for which we establish Conjecture 1.1. The situation for Conjecture 1.3(b) is more complicated. For certain choices of the parameters  $n$ ,  $\ell$  and  $\lambda$ , our next result shows that Conjecture 1.3(b) is true while for some others it shows that Conjecture 1.1 implies Conjecture 1.3(b). In the remaining cases, for each  $d_2 \geq \dots \geq d_r > 0$  and  $\ell \geq n$ , it shows that there are at most finitely many cases, namely  $s = d_2 + \dots + d_r \leq d_1 < (n-1)(s-1)$ , for which we do not know either that Conjecture 1.3(b) is true or that Conjecture 1.1 implies Conjecture 1.3(b). For these cases we have run numerical tests based on Proposition 5.13, as discussed in more detail below, which support our expectation that Conjecture 1.1 implies Conjecture 1.3(b) in these cases also.

**Proposition 5.13.** *As usual, let  $n \geq 3$ ,  $r \geq 2$ ,  $N = \binom{n+d-1}{n-1}$  and  $\lambda = [d_1, \dots, d_r]$ , where  $d = d_1 + s$  and  $s = d_2 + \dots + d_r$ .*

- (a) *Assume  $d_1 < s$  (and thus  $r \geq 3$ ). Then Conjecture 1.1 implies Conjecture 1.3(a) (i.e., that  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  is not defective).*
- (b) *Now assume  $d_1 \geq s$ .*
  - (i) *If  $2\ell \leq n$ , then Conjecture 1.3(b) is true (i.e., either  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  fills its ambient space,  $\mathbb{P}^{N-1}$ , or it is defective).*
  - (ii) *If  $\frac{n}{2} < \ell \leq n$ , then Conjecture 1.1 implies Conjecture 1.3(b).*
  - (iii) *If  $d_1 < 2s$ , then Conjecture 1.1 implies Conjecture 1.3(b).*
  - (iv) *If  $n \leq \ell$  and  $(n-1)(s-1) \leq d_1$ , then  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  fills its ambient space (and hence Conjecture 1.3(b) is true).*
- (c) *If  $\ell \geq \binom{s+n-1}{n-1}$ , then Conjecture 1.1 is true and  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  fills its ambient space (hence Conjecture 1.3 is true).*
- (d) *If  $\ell \geq \binom{s+n-1}{n-1}/t$ , where  $t$  is the number of occurrences of  $d_1$  in  $\lambda$ , then Conjecture 1.1 implies that  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  fills its ambient space (and hence, for such  $\ell$ , if Conjecture 1.1 is true, then so is Conjecture 1.3.)*

*Proof.* (a) The assumptions imply that the second and third lines of the formula in Conjecture 1.1(b) are zero, so

$$\dim \sigma_\ell(\mathbb{X}_{n-1,\lambda}) = \ell \cdot \dim \mathbb{X}_{n-1,\lambda} + \ell - 1 = \min \left\{ \binom{d+n-1}{n-1} - 1, \ell \cdot \dim \mathbb{X}_{n-1,\lambda} + \ell - 1 \right\},$$

which is the expected dimension.

(b)(i) This follows from Theorem 3.5.

(b)(ii) By definition,  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  is not defective if it fills its ambient space,  $\mathbb{P}^{N-1}$ , so assume  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  does not fill its ambient space. Then  $\dim \sigma_\ell(\mathbb{X}_{n-1,\lambda}) < N-1$  and Conjecture 1.1(b) gives

$$(5.2) \quad \begin{aligned} \dim \sigma_\ell(\mathbb{X}_{n-1,\lambda}) &= \ell \cdot \dim \mathbb{X}_{n-1,\lambda} + \ell - 1 \\ &- \sum_{k=2}^{\ell} (-1)^k \binom{\ell}{k} \binom{d_1 - (k-1)s + n - 1}{n-1} \\ &- \binom{\ell}{2} \binom{2d_2 - d + n - 1}{n-1} - \ell(\ell-1) \binom{d_1 + d_2 - d + n - 1}{n-1}. \end{aligned}$$

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Let  $\text{Syz}$  be the first syzygy module of a complete intersection in  $S$  that is generated by  $\ell \leq n$  forms of degree  $d - d_1 = d_2 + \dots + d_r$ . We observed in Remark 3.8 that

$$\sum_{k=2}^{\ell} (-1)^k \binom{\ell}{k} \binom{d_1 - (k-1)s + n - 1}{n-1} = \dim_{\mathbb{k}}[\text{Syz}]_d.$$

Hence we get

$$\dim \sigma_{\ell}(\mathbb{X}_{n-1,\lambda}) \leq \ell \cdot \dim \mathbb{X}_{n-1,\lambda} + \ell - 1 - \dim_{\mathbb{k}}[\text{Syz}]_d.$$

The Koszul resolution shows that the initial degree of  $\text{Syz}$  is  $2(d - d_1)$ . Since  $\text{Syz}$  is torsion free, it follows that  $[\text{Syz}]_d \neq 0$  if and only if  $d \geq 2(d - d_1)$ , which is equivalent to

$$d_1 \geq d - d_1 = s.$$

Therefore our assumption gives

$$\dim \sigma_{\ell}(\mathbb{X}_{n-1,\lambda}) < \ell \cdot \dim \mathbb{X}_{n-1,\lambda} + \ell - 1,$$

so  $\dim \sigma_{\ell}(\mathbb{X}_{n-1,\lambda}) < \text{exp. dim } \sigma_{\ell}(\mathbb{X}_{n-1,\lambda})$ , and we are done.

(b)(iii) As in the proof of (b)(ii), if  $\sigma_{\ell}(\mathbb{X}_{n-1,\lambda})$  does not fill its ambient space (and so  $\dim \sigma_{\ell}(\mathbb{X}_{n-1,\lambda}) < N - 1$ ) we must show that it is defective. Put

$$g = \sum_{j=2}^{\ell} (-1)^j \binom{\ell}{j} \binom{d_1 - (j-1)s + n - 1}{n-1}.$$

If  $s \leq d_1 < 2s$ , then  $g > 0$ . Thus, the summation in display (5.2) is positive, but it is subtracted so we have  $\dim \sigma_{\ell}(\mathbb{X}_{n-1,\lambda}) < \ell \cdot \dim \mathbb{X}_{n-1,\lambda} + \ell - 1$ , and hence  $\sigma_{\ell}(\mathbb{X}_{n-1,\lambda})$  is defective. Thus in the presence of the restriction on  $d_1$ , Conjecture 1.1 implies Conjecture 1.3(b).

(b)(iv) To see that  $\sigma_{\ell}(\mathbb{X}_{n-1,\lambda})$  fills its ambient space for  $d_1 \geq (n-1)(s-1)$  and  $\ell \geq n$ , it is enough to do so for  $\ell = n$ . Take  $\ell$  general points  $P_j$  on  $\mathbb{X}_{n-1,\lambda}$ . Each ideal  $I_{P_j}$  contains a minimal generator of degree  $d - d_1 = s$ . Thus, by genericity, the ideal  $I = I_{P_1} + \dots + I_{P_{\ell}}$  contains a complete intersection generated by  $n$  forms of degree  $s$ . The socle degree of this complete intersection is  $n(s-1)$ . Thus  $[R/I]_d = 0$  if  $d_1 + s = d > n(s-1)$ ; i.e., if  $d_1 \geq (n-1)(s-1)$ . It follows that for these  $\ell$  and  $d_1$  we get that Conjecture 1.3(b) is true.

(c) Note that  $a_s = a_s(\ell, n, \lambda) = \binom{s+n-1}{n-1} - t\ell$ , hence  $a_s \leq 0$  for  $\ell \geq \binom{s+n-1}{n-1}/t$ , and so also for  $\ell \geq \binom{s+n-1}{n-1}$ . So with the latter hypothesis, to prove Conjecture 1.1 we must show that  $\sigma_{\ell}(\mathbb{X}_{n-1,\lambda})$  fills its ambient space. Notice that  $[S]_s$  has a basis consisting of monomials, hence a basis of forms each of which factors as a product of forms of degrees  $d_2, \dots, d_r$ . Thus we can find points  $P_1, \dots, P_{\ell}$  for which  $I = I_{P_1} + \dots + I_{P_{\ell}}$  spans  $[S]_s$ , so also in degree  $d$  we have  $[I]_d = [S]_d$ . The same is then true for a general choice of  $\ell$  points, and so  $\sigma_{\ell}(\mathbb{X}_{n-1,\lambda})$  indeed fills its ambient space. But then for these  $\ell$ , all parts of Conjecture 1.3 are automatically true as well.

(d) Finally, assume that  $\ell \geq \binom{s+n-1}{n-1}/t$ . Since  $a_s \leq 0$ , Conjecture 1.1 would imply that  $\sigma_{\ell}(\mathbb{X}_{n-1,\lambda})$  fills its ambient space and so Conjecture 1.3 would hold. (We also note that the bound  $\ell \geq \binom{s+n-1}{n-1}/t$  is sharp, since  $\sigma_{\ell}(\mathbb{X}_{n-1,\lambda})$  does not always fill its ambient space for  $\ell < \binom{s+n-1}{n-1}/t$ , as we see for  $\sigma_{\ell}(\mathbb{X}_{n-1,[1,1]})$  by Theorem 1.4(c).)  $\square$

**Remark 5.14.** Proposition 5.13 is the basis for numerical tests that support our belief that all of the cases left open do in fact follow from Conjecture 1.1. In these cases, we have  $n < \ell$  and  $2s \leq d_1 < (n-1)(s-1)$ . As noted above, Conjecture 1.1 predicts: If  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  does not fill its ambient space, then

$$\dim \sigma_\ell(\mathbb{X}_{n-1,\lambda}) = \ell \cdot \dim \mathbb{X}_{n-1,\lambda} + \ell - 1 - g - \binom{\ell}{2} \binom{2d_2 - d + n - 1}{n-1} - \ell(\ell-1) \binom{d_1 + d_2 - d + n - 1}{n-1}.$$

Thus, Conjecture 1.3(b) follows if  $g > 0$ .

We have used Macaulay2 [30] to check for all cases satisfying the above restrictions with  $n, \ell, s \leq 60$  that Conjecture 1.1 implies Conjecture 1.3(b). There were 57,345,933 such cases. Note that typically for each  $s$  and  $d_1$ , there are many possible partitions of  $d = d_1 + s$ ; thus for each of the 57,345,933 cases for which  $g \leq 0$ , we merely checked that  $\ell \dim(\mathbb{X}_{n-1,[d_1,1,\dots,1]}) + \ell \geq N$ , and hence by Proposition 5.13(b)(iii) we see that Conjecture 1.1 implies Conjecture 1.3(b) for each case when  $\lambda = [d_1, 1, \dots, 1]$ . But by Corollary 2.18 this means Conjecture 1.1 implies Conjecture 1.3(b) also for all other partitions  $\lambda$  of  $d = d_1 + s$  for each of these 57,345,933 cases.

**Corollary 5.15.** *Let  $3 \leq n \leq \ell \leq 1 + \frac{d_1+n-1}{s}$  with  $\lambda = [d_1, \dots, d_r] \vdash d$ ,  $r \geq 3$ , and  $s = d_2 + \dots + d_r$ . Then Conjecture 1.1 is true for such  $n, \ell$  and  $\lambda$ .*

*Proof.* Conjecture 1.1(a) asserts that  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  fills its ambient space if and only if  $a_j(\ell, n, \lambda)$  is not positive for some integer  $j$  with  $s \leq j \leq d$ , while Conjecture 1.1(b) applies only when  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  does not fill its ambient space. Since  $d_1 \geq (n-1)(s-1)$  is equivalent to  $1 + \frac{d_1+n-1}{s} \geq n$ , Proposition 5.13 implies  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  fills its ambient space. Thus Conjecture 1.1 is true if we show  $a_d(\ell, n, \lambda) \leq 0$ .

Recall the identity

$$\sum_{k=2}^{\ell} (-1)^k \binom{\ell}{k} \binom{d - ks + n - 1}{n-1} = 0.$$

(See formula 10.13 of <http://www.math.wvu.edu/~gould/Vol.4.PDF>, where  $n, k, r, y, x$  in 10.13 become, respectively, our  $\ell, j, n-1, d+n-1$  and  $-s$ , so the assumption  $n > r$  in 10.13 becomes  $\ell > n-1$  and is thus satisfied. We also note that 10.13 does not assume that  $\binom{a}{b} = 0$  when  $a < 0$ , but our assumption  $\ell \leq 1 + \frac{d_1+n-1}{s}$  is equivalent to  $d - \ell s + n - 1 \geq 0$ . This ensures that  $d - js + n - 1 \geq 0$  hence the convention used in 10.13 agrees with our convention that  $\binom{d-js+n-1}{n-1} = 0$  when  $d - js + n - 1 < n - 1$ .)

Using the identity above, we have

$$\sum_{j=2}^{\ell} (-1)^j \binom{\ell}{j} \binom{d - js + n - 1}{n-1} = - \binom{d + n - 1}{n-1} + \ell \binom{d_1 + n - 1}{n-1},$$

and substituting this into the expression for  $a_d(\ell, n, \lambda)$  given in Definition 5.9 we obtain

$$a_d(\ell, n, \lambda) = -\ell \sum_{i>1} \left( \binom{d_i + n - 1}{n-1} - 1 \right) < 0.$$

□



We are now ready to prove one of the main results of the paper, as mentioned in the introduction.

*Proof of Theorem 1.4.* Part (a) follows from Theorem 3.13(a). Part (b) follows from Theorem 3.11 and Theorem 3.13(b). Part (c) follows from Proposition 5.13(iv) and Theorem 3.5.  $\square$

We conclude this section with a case where we can show that the prediction of Conjecture 1.1 is at least an upper bound.

**Proposition 5.16.** *If  $2\ell = n + 1$ , then the dimension of  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  is at most the number predicted in Conjecture 1.1.*

*Proof.* Using the above notation, put  $A' = B/\mathcal{L}'$ . Let  $L \in A'$  be a general linear element. Notice that the assumption on  $\ell$  gives  $|\mathcal{L}| = |\mathcal{L}'| + 1$ . Thus, we get

$$(5.3) \quad \mathbf{HS}(A) = \mathbf{HS}(B/\mathcal{L}B) = \mathbf{HS}(A'/LA') \geq |(1-t)\mathbf{HS}(A')|^+,$$

which means that the comparison is true coefficientwise.

If  $a_j(\ell, n, \lambda) \leq 0$  for some non-negative  $j \leq d$ , then Conjecture 1.1 says that  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  fills its ambient space, and so the estimate follows.

Otherwise, Equation (5.3) implies

$$\dim_{\mathbb{k}}[A]_d \geq a_d(\ell, n, \lambda),$$

which gives

$$\dim \sigma_\ell(\mathbb{X}_{n-1,\lambda}) = \binom{d+n-1}{n-1} - 1 - \dim_{\mathbb{k}}[A]_d \leq \binom{d+n-1}{n-1} - 1 - a_d(\ell, n, \lambda).$$

Since, the right-hand side is the formula for  $\dim \sigma_\ell(\mathbb{X}_{n-1,\lambda})$  that is predicted by Conjecture 1.1, this completes the argument.  $\square$

**Remark 5.17.** Consider an arbitrary graded  $\mathbb{k}$ -algebra, and let  $L_1, L_2 \in [A]_1$  be two general elements. Then it is *not* necessarily true that

$$\mathbf{HS}(A/(L_1, L_2)) \geq |(1-t)^2 \cdot \mathbf{HS}(A')|^+.$$

Thus, the above argument cannot be easily extended to show that Conjecture 1.1 gives an upper bound for  $\dim \sigma_\ell(\mathbb{X}_{n-1,\lambda})$  for all  $\ell \geq \frac{n+1}{2}$ . Note however that in the following section we will prove that Conjecture 1.1 does give an upper bound if  $r = 2$  by using a different approach.

## 6. FORMS WITH TWO FACTORS AND FRÖBERG'S CONJECTURE

In this section we focus on the case  $r = 2$ , that is, we consider secant varieties to the varieties whose general point corresponds to a product of two irreducible polynomials. We begin by recalling that the dimension of secant varieties, in case  $r = 2$ , is related to a famous conjecture of Fröberg (see [9]). We systematically relate this conjecture to our approach in the previous section. In particular, we will see that Fröberg's Conjecture and the WLP-Conjecture lead to the same prediction for the dimension of the secant variety in the case  $r = 2$ . This allows us to establish further instances of Conjecture 1.1.

Fröberg's Conjecture concerns the Hilbert function of an ideal generated by generic forms. More precisely, it says:

**Conjecture 6.1** (Fröberg’s Conjecture [18]). *Let  $J \subset S = \mathbb{k}[x_1, \dots, x_n]$  be an ideal generated by  $s$  generic forms of degrees  $e_1, \dots, e_s$  in  $S$ . Then the Hilbert series of  $S/J$  is*

$$\mathbf{HS}(S/J) = \left| \frac{\prod_{i=1}^s (1 - t^{e_i})}{(1 - t)^n} \right|^+.$$

There is an equivalent version of Fröberg’s Conjecture that gives a recursion to predict the Hilbert function of such an algebra.

**Conjecture 6.2** (Fröberg’s Conjecture, recursive version). *Let  $J \subset S = \mathbb{k}[x_1, \dots, x_n]$  be an ideal that is generated by generic forms, and let  $f \in S$  be a generic form of degree  $e$ . Then, for all integers  $j$ ,*

$$\dim_{\mathbb{k}}[S/(J, f)]_j = \max\{0, \dim_{\mathbb{k}}[S/J]_j - \dim_{\mathbb{k}}[S/J]_{j-e}\}.$$

Comparing the latter version with Definition 5.2 shows that  $S/J$  has the Weak Lefschetz Property if the above is true and  $e = 1$ . We refer to [32, Proposition 2.1] for further results on the relation between Fröberg’s conjecture and the Lefschetz properties. Here we need only the following observation.

**Proposition 6.3.** *Assume Fröberg’s Conjecture is true for polynomial rings in up to  $n$  variables. If  $J \subset S = \mathbb{k}[x_1, \dots, x_n]$  is an ideal that is generated by at least  $n$  generic forms, then  $S/J$  has the  $n$ -WLP.*

*Proof.* Let  $L \in S$  be a generic linear form. Then, as noted above,  $L$  is a Lefschetz element for  $S/J$ . Since  $S/(J, L)$  is isomorphic to a quotient of a polynomial ring in  $n - 1$  variables modulo an ideal generated by generic forms in these  $(n - 1)$  variables, we can use this argument  $n$  times.  $\square$

Since each quotient of a polynomial ring in at most two variables has the Weak Lefschetz Property by [24], the properties  $n$ -WLP and  $(n - 2)$ -WLP are equivalent if  $n \geq 2$ .

We now relate this to the secant varieties of  $\mathbb{X}_{n-1, \lambda}$ , where  $\lambda$  is a partition with two parts. In this case, we simplify notation and write  $\lambda = [d - k, k]$ , where  $1 \leq k \leq \frac{d}{2}$ .

Our starting point for the case  $r = 2$  is the following observation.

**Lemma 6.4.** *If  $\lambda = [d - k, k]$ , then*

$$\dim \sigma_{\ell}(\mathbb{X}_{n-1, \lambda}) = -1 + \dim_{\mathbb{k}}[I]_d,$$

where  $I \subset S$  is an ideal generated by  $\ell$  generic forms of degree  $d - k$  and  $\ell$  generic forms of degree  $k$ .

*Proof.* This follows from Corollary 2.7 and Proposition 2.9.  $\square$

In the case  $r = 2$ , the definition of the integers  $a_j(\ell, n, \lambda)$  becomes somewhat simpler.

**Remark 6.5.** Assume  $\lambda = [d - k, k]$ , where  $1 \leq k \leq \frac{d}{2}$ . Then

$$\begin{aligned} a_j(\ell, n, \lambda) &= \binom{j + n - 1}{n - 1} - \ell \left[ \binom{j + k - d + n - 1}{n - 1} + \binom{j - k + n - 1}{n - 1} \right] \\ &\quad + \sum_{i=2}^{\ell} (-1)^i \binom{\ell}{i} \binom{j - ik + n - 1}{n - 1} \end{aligned}$$

if  $0 \leq j < d$ , and

$$a_d(\ell, n, \lambda) = \binom{d+n-1}{n-1} - \ell \left[ \binom{k+n-1}{n-1} + \binom{d-k+n-1}{n-1} \right] \\ + \sum_{i=2}^{\ell} (-1)^i \binom{\ell}{i} \binom{d-ik+n-1}{n-1} + \binom{\ell}{2} \binom{2k-d+n-1}{n-1} + \ell^2.$$

Observe that the penultimate term is zero, unless  $k = \frac{d}{2}$ .

Lemma 6.4 allows us to relate Fröberg's Conjecture to our work in the previous sections.

**Proposition 6.6.** *Let  $\lambda = [d-k, k]$  be a partition with two parts. Then, for each  $\ell \geq 2$  and each  $n \geq 3$ , the value for the dimension of the secant variety in Conjecture 1.1 gives an upper bound for the dimension of  $\sigma_\ell(\mathbb{X}_{n-1, \lambda})$ . Moreover, if Fröberg's Conjecture is true for  $S$ , then, for all  $\ell \geq 2$ , the variety  $\sigma_\ell(\mathbb{X}_{n-1, \lambda})$  has the dimension predicted in Conjecture 1.1.*

*Proof.* If  $\ell \leq \frac{n}{2}$ , Conjecture 1.1 is true by Theorem 3.5. Thus, we may assume  $2\ell > n$ .

With  $I$  as in Lemma 6.4, we have

$$\dim \sigma_\ell(\mathbb{X}_{n-1, \lambda}) = \binom{d+n-1}{n-1} - 1 - \dim_{\mathbb{k}}[S/I]_d.$$

The value predicted for  $\dim \sigma_\ell(\mathbb{X}_{n-1, \lambda})$  by Conjecture 1.1 comes, by Theorem 5.10, from the value of  $\dim_{\mathbb{k}}[S/I]_d$  predicted by the WLP-Conjecture 5.8. Thus it is enough to derive from Fröberg's Conjecture 6.1 the same value for  $\dim_{\mathbb{k}}[S/I]_d$  as given by Conjecture 5.8, and to show that this value is a lower bound for the actual value.

Using our earlier notation, observe that  $B/\mathcal{L}'B \cong U/J$ , where  $U$  is a polynomial ring in  $2\ell$  variables and  $J$  is a complete intersection generated by  $\ell$  general forms of degree  $k$  and  $\ell$  general forms of degree  $d-k$ . Hence the Hilbert series of  $U/J$  is

$$\mathbf{HS}(U/J) = \frac{(1-t^{d-k})^\ell (1-t^k)^\ell}{(1-t)^{2\ell}}.$$

Lemma 2.15 shows that  $S/I$  is isomorphic to  $B/\mathcal{L}B$ , which is obtained from  $U/J$  by quotienting by  $(2\ell - n)$  general linear forms. Hence [18, Theorem] gives

$$\mathbf{HS}(S/I) \geq \left| \frac{(1-t^{d-k})^\ell (1-t^k)^\ell (1-t)^{2\ell-n}}{(1-t)^{2\ell}} \right|^+ = \left| \frac{(1-t^{d-k})^\ell (1-t^k)^\ell}{(1-t)^n} \right|^+,$$

the right hand side of which is exactly the Hilbert series of  $S/I$  as predicted by Fröberg's Conjecture 6.1. Moreover, the WLP-Conjecture 5.8 predicts this same Hilbert series for  $S/I$  (see Equation (5.1)), and hence equality holds if Fröberg's Conjecture does.  $\square$

**Remark 6.7.** Notice that in Proposition 6.6 we assumed the correctness of Fröberg's Conjecture only for ideals in  $S$ . If we assume more, namely that this conjecture is true for all ideals in  $\max\{n, 2\ell\}$  variables, then Proposition 6.3 shows that  $B/\mathcal{L}'B$  has the  $(2\ell - n)$ -Weak Lefschetz Property. Hence, in this case Theorem 5.11 immediately gives the conclusion of the above proposition.

We are ready to establish new instances where Conjecture 1.1 holds.

**Theorem 6.8.** *Conjecture 1.1 is true if  $r = 2$  and*

- (a)  $2\ell \leq n + 1$  or
- (b)  $n = 3$  or
- (c)  $\lambda = [1, 1]$ , that is,  $d = 2$ .

*Proof.* We use Proposition 6.6. Fröberg's Conjecture is true for forms in at most three variables by a result of Anick [4]. This gives (b). The conjecture also holds for ideals generated by general linear forms, and thus (c) follows.

Turning to (a), by Theorem 3.5 it suffices to consider the case where  $2\ell = n + 1$ . Then  $I$  is generated by  $n + 1$  general forms in  $n$  variables. For such ideals Fröberg's Conjecture is true because complete intersections of general forms have the Strong Lefschetz Property (see [43, 47, 38, 40] or [26]).  $\square$

**Remark 6.9.** With some work, part (a) of Theorem 6.8 can be shown to be equivalent to Theorem 5.1 of [9]. There, however, all the cases where  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  fills its ambient space are enumerated.

We now begin working out more explicit formulas for some particular partitions as consequences of Proposition 6.6. First we consider balanced partitions.

**Theorem 6.10.** Consider a balanced partition  $\lambda = [\frac{d}{2}, \frac{d}{2}]$  and fix  $n \geq 3$ . Assume that Fröberg's Conjecture holds for the polynomial ring  $S = \mathbb{K}[x_1, \dots, x_n]$ . Put

$$\ell_0 = \frac{1}{2} \left[ \binom{\frac{d}{2} + n - 1}{n - 1} + \frac{1}{2} - \sqrt{\left[ \binom{\frac{d}{2} + n - 1}{n - 1} + \frac{1}{2} \right]^2 - 2 \binom{d + n - 1}{n - 1}} \right].$$

Then

$$\dim \sigma_\ell(\mathbb{X}_{n-1, [\frac{d}{2}, \frac{d}{2}]}) = \begin{cases} \ell \cdot \dim \mathbb{X}_{n-1,\lambda} + \ell - 1 - 2\ell(\ell - 1) < \binom{d+n-1}{n-1} - 1 & \text{if } 2 \leq \ell < \ell_0 \\ \binom{d+n-1}{n-1} - 1 & \text{if } \ell_0 \leq \ell. \end{cases}$$

In particular, the secant variety  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  is defective if and only if it does not fill its ambient space. Furthermore, the defect is

$$\delta_\ell = \begin{cases} 2\ell(\ell - 1) & \text{if } 2 \leq \ell \leq \frac{\binom{d+n-1}{n-1}}{2 \binom{\frac{d}{2}+n-1}{n-1} - 1} \\ \binom{d+n-1}{n-1} - 1 - \dim \sigma_\ell(\mathbb{X}_{n-1, [\frac{d}{2}, \frac{d}{2}]}) & \text{if } \frac{\binom{d+n-1}{n-1}}{2 \binom{\frac{d}{2}+n-1}{n-1} - 1} < \ell < \ell_0. \end{cases}$$

*Proof.* Put  $N = \binom{d+n-1}{n-1}$  and recall that

$$\dim \mathbb{X}_{n-1,\lambda} = 2 \cdot \binom{\frac{d}{2} + n - 1}{n - 1} - 2.$$

Thus, the expected dimension of  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  is

$$\begin{aligned} \exp.\dim \sigma_\ell(\mathbb{X}_{n-1,\lambda}) &= \min\{N - 1, \ell \cdot \dim \mathbb{X}_{n-1,\lambda} + (\ell - 1)\} \\ &= \min\left\{N - 1, 2\ell \cdot \binom{\frac{d}{2} + n - 1}{n - 1} - \ell - 1\right\}. \end{aligned}$$

In particular,

$$(6.1) \quad \exp.\dim \sigma_\ell(\mathbb{X}_{n-1,\lambda}) = \ell \cdot \dim \mathbb{X}_{n-1,\lambda} + (\ell - 1) \text{ if and only if } 2 \leq \ell \leq \frac{N}{2\binom{\frac{d}{2}+n-1}{n-1} - 1}.$$

We now will consider various ranges for the value of  $\ell$  and use Lemma 6.4. For the partition  $\lambda$ , the ideal  $I$  is generated by  $2\ell$  general forms of degree  $\frac{d}{2}$ . Instead of applying Proposition 6.6 directly, it is more convenient to use the recursive approach (see Conjecture 6.2).

Assume first  $2 \leq \ell < \frac{n}{2}$ . Then Theorem 3.5 gives that  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  does not fill its ambient space and has dimension

$$\begin{aligned} \dim \sigma_\ell(\mathbb{X}_{n-1,\lambda}) &= \ell \cdot \dim \mathbb{X}_{n-1,\lambda} + (\ell - 1) - 2\binom{\ell}{2} - \ell(\ell - 1) \\ &= \ell \cdot \dim \mathbb{X}_{n-1,\lambda} + (\ell - 1) - 2\ell(\ell - 1). \end{aligned}$$

This proves the statement if  $\ell < \frac{n}{2}$ .

Assume now  $\frac{n}{2} \leq \ell$ . In order to simplify notation, set  $k = \frac{d}{2}$  and  $t = 2\ell - n \geq 0$ . In this range of  $\ell$ ,  $S/I$  is artinian and

$$\dim_{\mathbb{k}}[S/I]_k = \max \left\{ 0, \binom{k+n-1}{n-1} - 2\ell \right\}.$$

Hence  $[S/I]_k = 0$  if  $2\ell \geq \binom{k+n-1}{n-1}$ , which implies  $[S/I]_d = 0$ . It follows that  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  fills  $\mathbb{P}^{N-1}$  for such  $\ell$ .

We are left to consider  $\ell$  such that  $\frac{n}{2} \leq \ell < \frac{1}{2}\binom{k+n-1}{n-1}$ . Notice that this forces  $k \geq 2$ , that is,  $d \geq 4$ .

For  $i = 0, 1, \dots, t = 2\ell - n$ , let

$$\mathfrak{a}_i = \text{ideal generated by } n+i \text{ general forms of degree } k.$$

Observe that  $\mathfrak{a}_0$  is a complete intersection and  $I = \mathfrak{a}_t$ . Notice that, for all  $i$ ,

$$\dim_{\mathbb{k}}[S/\mathfrak{a}_i]_k = \binom{k+n-1}{n-1} - n - i = \dim_{\mathbb{k}}[S/\mathfrak{a}_0]_k - i.$$

The minimal free resolution of  $S/\mathfrak{a}_0$  has the form

$$\dots \rightarrow S(-d)\binom{n}{2} \rightarrow S(-k)^n \rightarrow S \rightarrow S/\mathfrak{a}_0 \rightarrow 0,$$

where we only display the terms that are non-trivial in degree  $d$ . This shows

$$\dim_{\mathbb{k}}[S/\mathfrak{a}_0]_d = N - n \cdot \binom{k+n-1}{n-1} + \binom{n}{2}.$$

Fröberg's Conjecture 6.2 predicts, for all  $i$ ,

$$\dim_{\mathbb{k}}[S/\mathfrak{a}_{i+1}]_d = \max\{\dim_{\mathbb{k}}[S/\mathfrak{a}_i]_d - \dim_{\mathbb{k}}[S/\mathfrak{a}_i]_k, 0\}.$$

Hence, we get

$$\begin{aligned}
\dim_{\mathbb{k}}[S/I]_d &= \max \left\{ 0, \dim_{\mathbb{k}}[S/\mathfrak{a}_0]_d - t \cdot \dim_{\mathbb{k}}[S/\mathfrak{a}_0]_k + \binom{t}{2} \right\} \\
&= \max \left\{ 0, \left[ N - n \cdot \binom{k+n-1}{n-1} + \binom{n}{2} \right] - t \left[ \binom{k+n-1}{n-1} - n \right] + \binom{t}{2} \right\} \\
&= \max \left\{ 0, N - 2\ell \cdot \binom{k+n-1}{n-1} + \binom{n}{2} + tn + \binom{t}{2} \right\} \\
&= \max \left\{ 0, N - 2\ell \cdot \binom{k+n-1}{n-1} + 2\ell^2 - \ell \right\}.
\end{aligned}$$

It follows that

$$\dim \sigma_{\ell}(\mathbb{X}_{n-1,\lambda}) = \min \left\{ N - 1, 2\ell \binom{k+n-1}{n-1} + \ell - 1 - 2\ell^2 \right\}$$

(note that we subtracted 1 from the dimension of the component of the ideal). Therefore, for  $\ell < \frac{1}{2} \binom{k+n-1}{n-1}$ , the variety  $\sigma_{\ell}(\mathbb{X}_{n-1,\lambda})$  fills  $\mathbb{P}^{N-1}$  if and only if

$$N \leq 2\ell \binom{k+n-1}{n-1} + \ell - 2\ell^2,$$

which means

$$\ell \geq \frac{1}{2} \left[ \binom{k+n-1}{n-1} + \frac{1}{2} - \frac{1}{2} \sqrt{\left[ \binom{k+n-1}{n-1} + \frac{1}{2} \right]^2 - 2N} \right] = \ell_0.$$

An induction on  $n \geq 2$  shows that the radicand is at least  $\frac{1}{4}$ , which also implies

$$\left\lceil \frac{1}{2} \binom{k+n-1}{n-1} \right\rceil \geq \ell_0.$$

We conclude that  $\sigma_{\ell}(\mathbb{X}_{n-1,\lambda})$  fills its ambient space if and only if  $\ell \geq \ell_0$  and that

$$\begin{aligned}
\dim \sigma_{\ell}(\mathbb{X}_{n-1,\lambda}) &= 2\ell \binom{k+n-1}{n-1} + \ell - 1 - 2\ell^2 \\
&= \ell \cdot \dim \mathbb{X}_{n-1,\lambda} + \ell - 1 - 2\ell(\ell - 1)
\end{aligned}$$

if  $\frac{n}{2} \leq \ell \leq \ell_0$ . This concludes finding the dimension of  $\sigma_{\ell}(\mathbb{X}_{n-1,\lambda})$ . Combining the result with Observation (6.1) proves the assertion on the defect.  $\square$

Second, we consider the most unbalanced partition of  $d$  into two parts. Notice that the following result is true unconditionally. Since the partition  $[1, 1]$  has been dealt with in the previous result (see Theorem 6.8), there is no harm in assuming  $d \geq 3$  in the next statement. The fact that  $\sigma_{\ell}(\mathbb{X}_{n-1,[d-1,1]})$  fills its ambient space if and only if  $\ell \geq \ell_0$  was shown in [9, Proposition 5.6]. The dimension of  $\sigma_{\ell}(\mathbb{X}_{n-1,[d-1,1]})$  can also be found in [6, Proposition 4.4]. We give a new proof of these facts using our methods.

Note that, in the following theorem, the formula for  $\dim(\mathbb{X}_{n-1,[d-1,1]})$  is simply the specialization of the formula of Conjecture 1.1 to the case at hand.

**Theorem 6.11.** Assume  $\lambda = [d - 1, 1]$ , where  $d \geq 3$ . Put

$$\ell_0 = \min \left\{ \ell \geq \frac{n}{2} \mid \ell \in \mathbb{Z} \text{ and } \binom{d - \ell + n - 1}{d} \leq \ell(n - \ell) \right\}.$$

Then  $\ell_0 \leq n - 1$  and

$$\dim \sigma_\ell(\mathbb{X}_{n-1, [d-1, 1]}) = \begin{cases} \binom{d+n-1}{n-1} - \binom{d+n-\ell-1}{d} + \ell(n - \ell) - 1 < \binom{d+n-1}{n-1} - 1 & \text{if } 2 \leq \ell < \ell_0 \\ \binom{d+n-1}{n-1} - 1 & \text{if } \ell_0 \leq \ell. \end{cases}$$

Moreover, the secant variety  $\sigma_\ell(\mathbb{X}_{n-1, \lambda})$  is defective if and only if it does not fill its ambient space. In this case, the defect is

$$\delta_\ell = \begin{cases} \binom{d+n-\ell-1}{d} + \ell \cdot \binom{d+n-2}{n-1} - \binom{d+n-1}{n-1} + \ell^2 \geq \ell^2 & \text{if } 2 \leq \ell \leq \frac{\binom{d+n-1}{n-1}}{\binom{d+n-2}{n-1} + n} \\ \binom{d+n-1}{n-1} - 1 - \dim \sigma_\ell(\mathbb{X}_{n-1, [\frac{d}{2}, \frac{d}{2}]}) & \text{if } \frac{\binom{d+n-1}{n-1}}{\binom{d+n-2}{n-1} + n} < \ell < \ell_0. \end{cases}$$

*Proof.* Again we use Lemma 6.4. This time the ideal  $I = I_{(1)} + \dots + I_{(\ell)}$  contains  $\ell$  generic linear forms. Thus,  $I = (x_1, \dots, x_n)$  if  $\ell \geq n$ , and we are done in this case. If  $\ell < n$ , then we get

$$A = S/I \cong \mathbb{k}[x_1, \dots, x_{n-\ell}]/(G_1, \dots, G_\ell),$$

where each  $G_j$  is a generic form of degree  $d - 1$  in  $T = \mathbb{k}[x_1, \dots, x_{n-\ell}]$ . It follows that

$$\begin{aligned} \dim \sigma_\ell(\mathbb{X}_{n-1, \lambda}) &= -1 + \dim_{\mathbb{k}}[I]_d \\ &= -1 + \dim_{\mathbb{k}}[S]_d - \dim_{\mathbb{k}}[A]_d \\ &= -1 + \dim_{\mathbb{k}}[S]_d - \dim_{\mathbb{k}}[T]_d + \dim_{\mathbb{k}}[(G_1, \dots, G_\ell)]_d \\ &= -1 + \binom{d+n-1}{n-1} - \binom{d+n-\ell-1}{d} + \min \left\{ \binom{d+n-\ell-1}{d}, \ell(n-\ell) \right\} \\ &= -1 + \binom{d+n-1}{n-1} - \max \left\{ 0, \binom{d+n-\ell-1}{d} - \ell(n-\ell) \right\}. \end{aligned}$$

In order to see the penultimate equality consider the graded minimal free resolution of  $(G_1, \dots, G_\ell)$ . Its beginning is of the form

$$\dots \rightarrow F \rightarrow T^\ell(-d+1) \rightarrow (G_1, \dots, G_\ell) \rightarrow 0,$$

where  $F$  is a graded free  $T$ -module. It follows that to compute  $\dim_{\mathbb{k}}[(G_1, \dots, G_\ell)]_d$  it is enough to know the number of linearly independent linear syzygies of the ideal  $(G_1, \dots, G_\ell)$ . Since the forms  $G_j$  are generic this number is the least possible by the main result in [28], and the dimension formula follows. It shows that  $\sigma_\ell(\mathbb{X}_{n-1, \lambda})$  fills its ambient space if and only if

$$\binom{d+n-\ell-1}{d} \leq \ell(n-\ell).$$

If  $\ell = n - 1$ , this is true. Hence, the number  $\ell_0$  is well defined and satisfies  $\ell_0 \leq n - 1$ . Furthermore, if  $\sigma_\ell(\mathbb{X}_{n-1, \lambda})$  fills its ambient space, then so does  $\sigma_{\ell+1}(\mathbb{X}_{n-1, \lambda})$ . This completes the argument for finding the dimension of  $\sigma_\ell(\mathbb{X}_{n-1, \lambda})$ .

It remains to discuss the defect of  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$ . Note that

$$\dim \mathbb{X}_{n-1,\lambda} = \binom{d+n-2}{n-1} + n - 1,$$

and so the expected dimension of  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  is

$$\text{exp.dim } \sigma_\ell(\mathbb{X}_{n-1,\lambda}) = \min \left\{ \binom{d+n-1}{n-1} - 1, \ell \cdot \binom{d+n-2}{d-1} + \ell n - 1 \right\}.$$

We need to show that  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  is defective if and only if  $2 \leq \ell < \ell_0 < n$ . Since for such  $\ell$  the variety  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  does not fill its ambient space, this is equivalent to proving

$$\ell \cdot \binom{d+n-2}{d-1} + \ell n - 1 > \binom{d+n-1}{n-1} - \binom{d+n-\ell-1}{d} + \ell(n-\ell) - 1,$$

that is,

$$(6.2) \quad \binom{d+n-\ell-1}{d} - \binom{d+n-1}{n-1} + \ell \cdot \binom{d+n-2}{d-1} > -\ell^2.$$

Notice that

$$\begin{aligned} \binom{d+n-\ell-1}{d} - \binom{d+n-1}{n-1} + \ell \cdot \binom{d+n-2}{d-1} &= \sum_{j=2}^{\ell} (-1)^j \binom{\ell}{j} \binom{d-j+n-1}{n-1} \\ &= \dim_{\mathbb{k}}[\text{Syz}]_d, \end{aligned}$$

where  $\text{Syz}$  is the first syzygy module of a complete intersection in  $S$  that is generated by  $\ell < n$  linear forms (see Remark 3.8). This shows that the left-hand side in Inequality (6.2) is non-negative, and hence establishes that this inequality is true.

In order to determine the positive defect, it is enough to observe that

$$\text{exp.dim } \sigma_\ell(\mathbb{X}_{n-1,\lambda}) \leq \binom{d+n-1}{n-1} - 1$$

if and only if

$$\ell \leq \frac{\binom{d+n-1}{n-1}}{\binom{d+n-2}{n-1} + n}.$$

This concludes the calculation of the defect.  $\square$

Since we discussed the case  $\ell = 2$  in Section 4, we illustrate the last result in the case  $\ell = 3$ .

**Corollary 6.12.** *Consider the secant plane variety  $\sigma_3(\mathbb{X}_{n-1,[d-1,1]})$ .*

- (a)  $\sigma_3(\mathbb{X}_{n-1,[d-1,1]})$  fills its ambient space if and only if
  - (i)  $n \in \{3, 4\}$  and  $d \geq 2$ , or
  - (ii)  $n = 5$  and  $d \in \{2, 3, 4, 5\}$ , or
  - (iii)  $n = 6$  and  $d = 2$ .
- (b) In all other cases  $\sigma_3(\mathbb{X}_{n-1,[d-1,1]})$  is defective with dimension

$$\dim \sigma_3(\mathbb{X}_{n-1,[d-1,1]}) = \begin{cases} 6n - 16 & \text{if } d = 2 \\ \binom{d+n-1}{n-1} - \binom{d+n-4}{d} + 3n - 10 & \text{if } d \geq 3. \end{cases}$$



*Proof.* Consider first  $d = 2$ . Then we can apply Theorem 6.10. However, it is easier to argue directly. In this case, the ideal  $I$  in Lemma 6.4 is generated by  $2\ell = 6$  linear forms. Hence  $[I]_2 = [S]_2$  if and only if  $n \leq 6$ . If  $n \geq 7$ , then we get

$$\begin{aligned} \dim \sigma_3(\mathbb{X}_{n-1,([1,1])}) &= \dim_{\mathbb{k}}[I]_2 - 1 = \binom{n+1}{2} - \binom{n-5}{2} - 1 \\ &= 6n - 16. \end{aligned}$$

Moreover,  $\sigma_2(\mathbb{X}_{n-1,[1,1]})$  is defective if it does not fill its ambient space by Theorem 6.10.

Assume now  $d \geq 3$ . Then Theorem 6.11 shows that  $\sigma_3(\mathbb{X}_{n-1,[d-1,1]})$  fills its ambient space if and only if

$$\frac{n}{2} \leq \ell_0 \leq 3.$$

Furthermore,  $\sigma_3(\mathbb{X}_{n-1,[d-1,1]})$  does not fill its ambient space if  $n \geq 6$  by Theorem 3.5. Hence, it remains to consider the cases  $n \in \{3, 4, 5\}$ .

If  $n = 5$ , this forces  $\ell_0 = 3$ , which means

$$d + 1 \leq 6 \quad \text{and} \quad \binom{d+2}{2} > 6,$$

that is,  $d \in \{3, 4, 5\}$ .

Since  $\ell_0 \leq n - 1$ , we get  $\ell_0 \leq 3$  if  $n \leq 4$ , and thus  $\sigma_3(\mathbb{X}_{n-1,([1,1])})$  fills its ambient space. This shows Part (a). Claim (b) follows by Theorem 6.11.  $\square$

## 7. THE VARIETY OF REDUCIBLE FORMS

Every reducible form of degree  $d$  in  $n$  variables corresponds to a point of the variety

$$\mathbb{X}_{n-1,d} = \bigcup_{k=1}^{\lfloor \frac{d}{2} \rfloor} \mathbb{X}_{n-1,[d-k,k]}.$$

(Notice that this holds even for reducible forms with more than two factors.)

Thus, we call  $\mathbb{X}_{n-1,d}$  the *variety of reducible forms of degree  $d$  in  $n$  variables*. In this section we study its secant varieties. This is based on the results on the secant varieties of the various  $\mathbb{X}_{n-1,[d-k,k]}$ , where  $k$  varies between 1 and  $\lfloor \frac{d}{2} \rfloor$ .

**Remark 7.1.** The variety  $\mathbb{X}_{n-1,d}$  is irreducible if and only if  $d = 2$ . In this case  $\mathbb{X}_{n-1,2} = \mathbb{X}_{n-1,[1,1]}$  and

$$\dim \sigma_\ell(\mathbb{X}_{n-1,2}) = \begin{cases} 2\ell(n - \ell) + \ell - 1 & \text{if } 2\ell < n \\ \binom{n+1}{2} - 1 & \text{if } 2\ell \geq n. \end{cases}$$

Moreover,  $\sigma_\ell(\mathbb{X}_{n-1,2})$  is defective if and only if  $2 \leq \ell < \frac{n}{2}$ . In this case, the defect is given by Theorem 6.10.

We begin by determining the dimension of  $\mathbb{X}_{n-1,d}$ . The next result is an immediate consequence of Corollary 2.18.

**Proposition 7.2.** *For all  $d \geq 2$  and  $n \geq 3$ ,*

$$\dim \mathbb{X}_{n-1,d} = \dim \mathbb{X}_{n-1,[d-1,1]} = \binom{d+n-2}{n-1} + n - 1.$$

Corollary 2.18 also gives that  $\mathbb{X}_{n-1,[d-1,1]}$  is the unique irreducible component of  $\mathbb{X}_{n-1,d}$  that has the same dimension as  $\mathbb{X}_{n-1,d}$ .

As we noted above,  $\mathbb{X}_{n-1,d}$  is not irreducible as soon as  $d > 2$ . Thus, in order to calculate the dimension of  $\sigma_\ell(\mathbb{X}_{n-1,d})$  one must also consider the (embedded)  $\ell$ -joins of the irreducible components of  $\mathbb{X}_{n-1,d}$ .

Recall that if  $\mathbb{X}_1, \dots, \mathbb{X}_\ell$  are irreducible varieties in  $\mathbb{P}^m$  (not necessarily distinct), then the *embedded join* of  $\mathbb{X}_1, \dots, \mathbb{X}_\ell$ , denoted

$$J(\mathbb{X}_1, \dots, \mathbb{X}_\ell)$$

is the Zariski closure of the union of all the linear spaces  $\langle P_1, \dots, P_\ell \rangle \subset \mathbb{P}^m$  where  $P_i \in \mathbb{X}_i$ . If all the  $\mathbb{X}_i = \mathbb{X}$  then this is nothing other than  $\sigma_\ell(\mathbb{X})$ . Furthermore, for any (possibly reducible) variety  $\mathbb{X} \subset \mathbb{P}^m$ , the parameter count mentioned in the introduction gives

$$\dim \sigma_\ell(\mathbb{X}) \leq \min\{m, \ell \cdot \dim \mathbb{X} + \ell - 1\},$$

and the right-hand side is called the *expected dimension* of  $\sigma_\ell(\mathbb{X})$ .

The following Lemma shows that if  $\mathbb{X}_1, \dots, \mathbb{X}_\ell$  are any  $\ell$  irreducible components of  $\mathbb{X}_{n-1,d}$  and  $2\ell \leq n$ , then

$$\dim \sigma_\ell(\mathbb{X}_{n-1,[d-1,1]}) \geq \dim J(\mathbb{X}_1, \dots, \mathbb{X}_\ell).$$

**Lemma 7.3.** *Consider integers  $k_1, k_2, \dots, k_\ell \in \{1, \dots, \lfloor \frac{d}{2} \rfloor\}$ , where  $d \geq 2$ . Let  $I \subset S = \mathbb{k}[x_1, \dots, x_n]$  be an ideal generated by  $2\ell$  general forms of degrees  $k_1, k_2, \dots, k_\ell, d - k_1, d - k_2, \dots, d - k_\ell$ . Let  $J \subset S$  be an ideal generated by  $\ell$  general linear forms and  $\ell$  general forms of degree  $d - 1$ . If  $2\ell \leq n$ , then, for all integers  $j$ ,*

$$\dim_{\mathbb{k}}[S/I]_j \geq \dim_{\mathbb{k}}[S/J]_j.$$

Moreover, if  $k_i > 1$  for some  $i \in \{1, \dots, \ell\}$ , then there is some  $j$  such that this is a strict inequality.

*Proof.* Consider first the case, where  $n = 2\ell$ . If  $n = 2$ , then

$$\dim_{\mathbb{k}}[S/J]_j = \begin{cases} 1 & \text{if } 0 \leq j \leq d - 2 \\ 0 & \text{otherwise,} \end{cases}$$

and the claim follows in this case. Using Hilbert series, this observation can be expressed as

$$(7.1) \quad \frac{1}{(1-t)^2}(1-t^k)(1-t^{d-k}) \geq \frac{1}{(1-t)^2}(1-t)(1-t^{d-1}) \quad \text{whenever } 1 \leq k \leq \frac{d}{2}.$$

Let now  $n \geq 4$ . Then the Hilbert series of  $S/I$  and  $S/J$  are

$$\mathbf{HS}(S/I) = \prod_{i=1}^{\ell} \frac{(1-t^{k_i})(1-t^{d-k_i})}{(1-t)^2}$$

and

$$\mathbf{HS}(S/J) = \prod_{i=1}^{\ell} \frac{(1-t)(1-t^{d-1})}{(1-t)^2}.$$

Thus, Inequality (7.1) gives

$$\mathbf{HS}(S/I) \geq \mathbf{HS}(S/J),$$

as desired.

Finally, assume  $n > 2\ell$ . Then

$$\begin{aligned} \mathbf{HS}(S/I) &= \frac{1}{(1-t)^{n-2\ell}} \cdot \prod_{i=1}^{\ell} \frac{(1-t^{k_i})(1-t^{d-k_i})}{(1-t)^2} \\ &\geq \frac{1}{(1-t)^{n-2\ell}} \cdot \prod_{i=1}^{\ell} \frac{(1-t)(1-t^{d-1})}{(1-t)^2} \\ &= \mathbf{HS}(S/J), \end{aligned}$$

where the estimate follows from the case  $n = 2\ell$  and the fact that the coefficients in the power series expansion of  $\frac{1}{(1-t)^{n-2\ell}}$  are all non-negative.  $\square$

We are ready for the main result of this section.

**Theorem 7.4.** *Assume  $2\ell \leq n$ . Then*

$$\dim \sigma_{\ell}(\mathbb{X}_{n-1,d}) = \dim \sigma_{\ell}(\mathbb{X}_{n-1,[d-1,1]}).$$

Moreover:

- (a) *The variety  $\sigma_{\ell}(\mathbb{X}_{n-1,d})$  fills its ambient space if and only if*
  - (i)  $2\ell = n$  and  $d = 2$ ; or
  - (ii)  $\ell = 2$ ,  $n = 4$ , and  $d = 3$ .
- (b) *If  $\sigma_{\ell}(\mathbb{X}_{n-1,d})$  does not fill its ambient space, then its dimension is*

$$\dim \sigma_{\ell}(\mathbb{X}_{n-1,d}) = \binom{d+n-1}{n-1} - \binom{d+n-\ell-1}{d} + \ell(n-\ell) - 1,$$

and  $\sigma_{\ell}(\mathbb{X}_{n-1,d})$  is defective.

*Proof.* Let  $P_1, \dots, P_{\ell} \in \mathbb{X}_{n-1,d}$  be points such that each  $P_i$  is a general point on some component, say  $\mathbb{X}_{n-1,[d-k_i,k_i]}$ , of  $\mathbb{X}_{n-1,d}$ . Then Terracini's Lemma gives (as in Corollary 2.7)

$$\dim(\sigma_{\ell}(\mathbb{X}_{n-1,d})) = \max \left\{ \dim_{\mathbb{k}} [I_{P_1} + \dots + I_{P_{\ell}}]_d - 1 \mid k_1, k_2, \dots, k_{\ell} \in \left\{ 1, \dots, \left\lfloor \frac{d}{2} \right\rfloor \right\} \right\}.$$

Using the notation of Lemma 7.3, this implies

$$\dim(\sigma_{\ell}(\mathbb{X}_{n-1,d})) = \dim_{\mathbb{k}} [J]_d - 1 = \dim(\sigma_{\ell}(\mathbb{X}_{n-1,[d-1,1]})),$$

as desired.

Part (a) is now a consequence of the second part of Theorem 3.5. We claim that part (b) follows from Theorem 6.11. Indeed, if  $\ell < \ell_0$  then Theorem 6.11 gives the desired dimension. If  $\ell_0 \leq \ell$  then Theorem 6.11 yields that  $\sigma_{\ell}(\mathbb{X}_{n-1,[d-1,1]})$  fills its ambient space.  $\square$

**Remark 7.5.** Lemma 7.3 also implies that

$$\dim \sigma_{\ell}(\mathbb{X}_{n-1,[d-k,k]}) \leq \dim \sigma_{\ell}(\mathbb{X}_{n-1,[d-1,1]}) \quad \text{if } 2 \leq k \leq \frac{d}{2},$$

provided  $\ell \leq \frac{n}{2}$ . We conjecture that this bound is true without the latter restriction. Notice that we have an upper bound for  $\dim \sigma_{\ell}(\mathbb{X}_{n-1,[d-k,k]})$  by Proposition 6.6 and that we know  $\dim \sigma_{\ell}(\mathbb{X}_{n-1,[d-1,1]})$  by Theorem 6.11. This reduces this conjecture to a comparison of two numbers. However, we have been unable to establish the needed estimate.

By Theorem 6.11, we know exactly when the secant variety  $\sigma_{\ell}(\mathbb{X}_{n-1,[d-1,1]})$  fills its ambient space. This gives:

**Theorem 7.6.** *The secant variety  $\sigma_\ell(\mathbb{X}_{n-1,d})$  fills its ambient space if  $\ell \geq \ell_0$ , where*

$$\ell_0 = \min \left\{ \ell \geq \frac{n}{2} \mid \ell \in \mathbb{Z} \text{ and } \binom{d - \ell + n - 1}{d} \leq \ell(n - \ell) \right\}.$$

*In particular,  $\sigma_\ell(\mathbb{X}_{n-1,d})$  fills its ambient space if  $\ell \geq n - 1$ .*

*Proof.* Notice that

$$\dim \sigma_\ell(\mathbb{X}_{n-1,d}) \geq \dim \sigma_\ell(\mathbb{X}_{n-1,[d-1,1]}).$$

Thus, the claim follows from Theorem 6.11 if  $d \geq 3$ . If  $d = 2$ , then  $\ell_0 = \lceil \frac{n}{2} \rceil$ , and we conclude by Remark 7.1.  $\square$

*Proof of Theorem 1.5.* Combine Theorems 7.4 and 7.6.  $\square$

**Remark 7.7.** If  $2\ell > n$ , then Theorems 7.4 and 7.6 do not rule out the possibility that  $\sigma_\ell(\mathbb{X}_{n-1,d})$  fills its ambient space even when  $\sigma_\ell(\mathbb{X}_{n-1,[d-1,1]})$  does not. However, we do not expect this ever happening. In fact, we suspect that the following extension of Theorem 7.4 is true:

$$\dim \sigma_\ell(\mathbb{X}_{n-1,d}) = \dim \sigma_\ell(\mathbb{X}_{n-1,[d-1,1]}),$$

for all  $\ell$ ,  $n$  and  $d$ . If so, then, by Theorem 6.11, the converse of Theorem 7.6 is true and  $\sigma_\ell(\mathbb{X}_{n-1,d})$  is defective whenever it does not fill its ambient space.

## 8. APPLICATION TO SECANT VARIETIES OF SEGRE VARIETIES

The paper [2] by Abo, Ottaviani and Peterson classifies all Segre varieties  $X$  such that the  $\ell$ -secant variety is defective for some  $\ell < 7$  and it raises some questions as to conjecturally what happens in general. Our results verify certain cases of these conjectures.

We first recall some terminology from [2] and a related result from [11]. Assume  $2 \leq n_r \leq \dots \leq n_1$  (this is consistent with our ordering convention, but it is the reverse of what [2] does). Say that  $(n_1, \dots, n_r)$  is *balanced* if  $n_1 - 1 \leq \Pi_{i=2}^r n_i - \sum_{i=2}^r (n_i - 1)$  and *unbalanced* otherwise. Using this terminology, [11] proves that with respect to the Segre embedding of  $X = \mathbb{P}^{n_1-1} \times \dots \times \mathbb{P}^{n_r-1}$  in projective space,  $X$  has a defective  $\ell$ -secant variety for some  $\ell$  if  $(n_1, \dots, n_r)$  is unbalanced. This result, [11, Proposition 3.3], was paraphrased in [2] essentially as follows (see [2, Lemma 4.1]):

**Proposition 8.1.** *With respect to the Segre embedding of  $X = \mathbb{P}^{n_1-1} \times \dots \times \mathbb{P}^{n_r-1}$  in projective space,  $\sigma_\ell(X)$  is defective for  $\ell$  satisfying*

$$\Pi_{i=2}^r n_i - \sum_{i=2}^r (n_i - 1) < \ell < \min\{\Pi_{i=2}^r n_i, n_1\}.$$

By a Segre variety  $X$  being *unbalanced* [2] means  $X = \mathbb{P}^{n_1-1} \times \dots \times \mathbb{P}^{n_r-1}$  where  $(n_1, \dots, n_r)$  is unbalanced. With these definitions, [2, Question 6.6] then asks:

**Question 8.2.** *Is it true for a Segre variety  $X$  such that  $\sigma_\ell(X)$  is defective for some  $\ell$ , that either:*

1.  $X$  is unbalanced; or
2.  $X = \mathbb{P}^2 \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$  with  $n$  odd; or
3.  $X = \mathbb{P}^2 \times \mathbb{P}^3 \times \mathbb{P}^3$ ; or
4.  $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ ?

The conjecture in [2] is that the answer is yes; to rephrase:

**Conjecture 8.3.** *Let  $X$  be a balanced Segre variety but not any of those listed in items 2, 3 or 4 of Question 8.2. Then  $\sigma_\ell(X)$  is not defective for all  $\ell \geq 2$ .*

Our results verify this in various cases, as we now explain.

It is useful to reconsider the morphism (2.1). Let  $X = \mathbb{P}^{n_1-1} \times \dots \times \mathbb{P}^{n_r-1}$ . The points of the Segre embedding of  $X$  in  $\mathbb{P}^{N-1}$ , where  $N = \prod_i n_i$ , are exactly those of the form  $[\dots, a_{i_1 1} a_{i_2 2} \dots a_{i_r r}, \dots]$ , where  $(a_{1j}, \dots, a_{n_j j}) \in \mathbb{P}^{n_j-1}$ . We can regard  $[\dots, a_{i_1 1} a_{i_2 2} \dots a_{i_r r}, \dots]$  as the multi-homogeneous polynomial

$$\begin{aligned} F(\dots, x_{ij}, \dots) &= \sum a_{i_1 1} a_{i_2 2} \dots a_{i_r r} x_{i_1 1} x_{i_2 2} \dots x_{i_r r} \\ &= \prod_{j=1}^r (a_{1j} x_{1j} + \dots + a_{n_j j} x_{n_j j}) \in \mathbb{k}[\dots, x_{ij}, \dots] \end{aligned}$$

of multi-degree  $(1^r) = (1, \dots, 1)$ , where the variables  $x_{ij}$  are indexed by  $1 \leq i \leq n_j$  with  $1 \leq j \leq r$  and  $\mathbb{k}[\mathbb{P}^{n_j-1}] = \mathbb{k}[x_{1j}, \dots, x_{n_j j}]$ . Now suppose that  $n_j = \binom{d_j+n-1}{n-1}$  for each  $j$ , where  $d_1 \geq d_2 \geq \dots \geq d_r$  and  $d = d_1 + \dots + d_r$ . Regarding the coordinate variables  $x_{ij}$  of  $\mathbb{P}^{n_j-1}$  as an enumeration of the monomials  $M_{ij} \in \mathbb{k}[x_1, \dots, x_n]$  of degree  $d_j$  for each  $j$ , we get a map

$$F \mapsto \bar{F} \in \mathbb{k}[x_1, \dots, x_n],$$

where  $\bar{F}$  is the degree  $d$  singly homogeneous polynomial  $\bar{F}(\dots, M_{ij}, \dots)$  obtained by substituting  $M_{ij}$  into  $x_{ij}$ . Note that the map  $\bar{\phantom{F}}$  is actually a linear homomorphism

$$\bar{\phantom{F}} : (\mathbb{k}[\dots, x_{ij}, \dots])_{(1, \dots, 1)} \rightarrow (\mathbb{k}[x_1, \dots, x_n])_d.$$

The projectivizations of  $(\mathbb{k}[\dots, x_{ij}, \dots])_{(1, \dots, 1)}$  and  $(\mathbb{k}[x_1, \dots, x_n])_d$  are  $\mathbb{P}^{N-1}$  and  $\mathbb{P}^{\binom{d+n-1}{n-1}-1}$ , respectively. The restriction of  $\bar{\phantom{F}}$  to the affine cone corresponding to the Segre embedding of  $X$  in  $\mathbb{P}^{N-1}$  is just the surjective morphism induced on affine cones by the morphism (2.1) of  $X$  to  $\mathbb{X}_{n-1, \lambda}$  for  $\lambda = [d_1, \dots, d_r]$ .

Since  $\bar{\phantom{F}} : (\mathbb{k}[\dots, x_{ij}, \dots])_{(1, \dots, 1)} \rightarrow (\mathbb{k}[x_1, \dots, x_n])_d$  is linear, it maps the affine cone of the  $\ell$ -secant variety of  $X$  to the affine cone of the  $\ell$ -secant variety of  $\mathbb{X}_{n-1, \lambda}$ . If  $\sigma_\ell(\mathbb{X}_{n-1, \lambda})$  is not defective, let us say that the  $\ell$ -secant variety  $\sigma_\ell(\mathbb{X}_{n-1, \lambda})$  of  $\mathbb{X}_{n-1, \lambda}$  does not *overly fill* its ambient space if  $\dim \sigma_\ell(\mathbb{X}_{n-1, \lambda}) = \ell \cdot \dim \mathbb{X}_{n-1, \lambda} + \ell - 1$ . Also, given the partition  $\lambda = [d_1, \dots, d_r]$ , let  $X_{n-1, \lambda}$  denote the Segre embedding of  $\mathbb{P}^{n_1-1} \times \dots \times \mathbb{P}^{n_r-1}$ , where  $n_i = \binom{d_i+n-1}{n-1}$ . We thus have:

**Theorem 8.4.** *If  $\sigma_\ell(\mathbb{X}_{n-1, \lambda})$  is not defective and does not overly fill its ambient space, then  $\sigma_\ell(X_{n-1, \lambda})$  is not defective.*

*Proof.* Assume  $\sigma_\ell(\mathbb{X}_{n-1, \lambda})$  does not overly fill its ambient space. If  $\sigma_\ell(X_{n-1, \lambda})$  were defective, then  $\dim \sigma_\ell(X_{n-1, \lambda}) < \ell \cdot \dim X_{n-1, \lambda} + \ell - 1 = \ell \cdot \dim \mathbb{X}_{n-1, \lambda} + \ell - 1$ . Since  $\sigma_\ell(\mathbb{X}_{n-1, \lambda})$  does not overly fill its ambient space, we have  $\ell \cdot \dim \mathbb{X}_{n-1, \lambda} + \ell - 1 = \dim \sigma_\ell(\mathbb{X}_{n-1, \lambda})$  and hence  $\dim \sigma_\ell(X_{n-1, \lambda}) < \dim \sigma_\ell(\mathbb{X}_{n-1, \lambda})$ . But  $\bar{\phantom{F}} : (\mathbb{k}[\dots, x_{ij}, \dots])_{(1, \dots, 1)} \rightarrow (\mathbb{k}[x_1, \dots, x_n])_d$  maps the affine cone of  $\sigma_\ell(X_{n-1, \lambda})$  onto the affine cone of  $\sigma_\ell(\mathbb{X}_{n-1, \lambda})$ , so we must have  $\dim \sigma_\ell(X_{n-1, \lambda}) \geq \dim \sigma_\ell(\mathbb{X}_{n-1, \lambda})$ , hence  $\sigma_\ell(X_{n-1, \lambda})$  cannot be defective.  $\square$

As an example, we have the following corollary which verifies Conjecture 8.3 in a range of cases. Although this particular consequence is known (see [11, Proposition 2.3]), our approach is new.

**Corollary 8.5.** *Let  $1 \leq d_r \leq \dots \leq d_1 < d_2 + \dots + d_r$ ,  $3 \leq r$  and  $4 \leq 2\ell \leq n$  be integers with  $n_i = \binom{d_i+n-1}{n-1}$  and  $a = n_1 \cdots n_r$ . If  $X$  is the Segre embedding of  $\mathbb{P}^{n_1-1} \times \dots \times \mathbb{P}^{n_r-1}$  in  $\mathbb{P}^{a-1}$ , then  $\sigma_\ell(X)$  is not defective. Moreover,  $X$  is balanced for  $d_2 = \dots = d_r \gg 0$ .*

*Proof.* By Theorem 3.13,  $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$  is not defective for  $\lambda = [d_1, \dots, d_r]$  and does not overly fill its ambient space, hence by Theorem 8.4,  $\sigma_\ell(X)$  is not defective. Now we just need to check that  $X$  is balanced when  $d_2 = \dots = d_r \gg 0$ . Since  $\binom{d_1+n-1}{n-1} < \binom{d_2(r-1)+n-1}{n-1}$ , it suffices to show that  $\binom{d_2(r-1)+n-1}{n-1} \leq \binom{d_2+n-1}{n-1}^{r-1} - (r-1)\binom{d_2+n-1}{n-1}$ . But  $\binom{d_2(r-1)+n-1}{n-1}$  is a polynomial in  $d_2$  of degree  $n-1$  while  $\binom{d_2+n-1}{n-1}^{r-1} - (r-1)\binom{d_2+n-1}{n-1}$  is a polynomial in  $d_2$  of degree  $(r-1)(n-1)$ , hence the inequality must hold for  $d_2 \gg 0$ .  $\square$

## 9. FURTHER QUESTIONS AND COMMENTS

In this section we pose some natural questions arising from our work.

It is clear from the previous sections that the major algebraic question left open in the paper is the extent to which sums of at least two generic tangent space ideals to the varieties of reducible hypersurfaces have enough Lefschetz elements (see Conjecture 5.8). One should note that less information than the full Weak Lefschetz Property is needed to establish the conjectured dimension of secant varieties to varieties of reducible forms. This allowed us to use results by Hochster and Laksov [28], Anick [4], and a theorem on complete intersections (see [43, 47, 38, 40] or [26]) in the proofs of Theorems 6.11 and 6.8, respectively. It would be very interesting to have new instances where (partial) Weak Lefschetz Properties are established.

As we have observed earlier in the paper, each variety of reducible hypersurfaces is a finite projection of a Segre embedding of a product of projective spaces. Are there more geometric conclusions (than those we have found in Section 8) that we can draw about the secant varieties of the Segre embedding from the more abundant information we have for the secant varieties of the varieties of reducible hypersurfaces?

Another question is if any of the secant varieties of the varieties of reducible forms are arithmetically Cohen-Macaulay, apart from the trivial cases in which those secant varieties are themselves hypersurfaces in their ambient space.

Again, apart from some very small examples, we do not have equations for the varieties of reducible hypersurfaces, much less for their secant varieties (even in cases where we know the latter are hypersurfaces in their ambient spaces (see Remark 4.5)). It would be interesting to have some intrinsic equations for these varieties or a bound on the degrees of their equations.

Mammana [31] gives a formula for the degree of the variety of reducible plane curves but we have no generalization of that formula for varieties of reducible hypersurfaces beyond the case of plane curves. More generally, one would like to have a formula for the degree of the secant varieties of these varieties. This is not known even for varieties of plane curves, except in the most trivial of cases. It would even be interesting to know the degree when the variety is a hypersurface in its ambient space.

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