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Published Version:
Tamargo, L.H., Martinez, D.C., Rotolo, A., Governatori, G. (2019). An axiomatic characterization of temporalised belief revision in the law. ARTIFICIAL INTELLIGENCE AND LAW, 27(4), 347-367 [10.1007/s10506-019-09241-4].

Availability:
This version is available at: https://hdl.handle.net/11585/683225 since: 2019-12-18

Published:
DOI: http://doi.org/10.1007/s10506-019-09241-4

An Axiomatic Characterization of Temporalised Belief Revision in the Law

Abstract. This paper presents a belief revision operator that considers time intervals for modelling norm change in the law. This approach relates techniques from belief revision formalisms and time intervals with temporalised rules for legal systems. Our goal is to formalise a temporalised belief base and corresponding timed derivation, together with a proper revision operator. This operator may remove rules when needed or adapt intervals of time when contradictory norms are added in the system. For the operator, both constructive definition and an axiomatic characterisation by representation theorems are given.

Keywords. Norm Change, Belief Revision, Temporal Reasoning

1. Introduction and Motivation

One peculiar feature of the law is that it necessarily takes the form of a dynamic normative system [29,28]. In simple terms, the dynamics of law are mainly due to several actions of lawmakers that produce or change legal norms, such as norm enactment, derogation, annulment and abrogation, among others (for a rather complete list, see [18,20]). In particular, while with norm enactment rules are introduced in the system as provisions of new, different norms for society, with operations such as derogation, annulment and abrogation rules are somehow—and in different ways—eliminated from the system, or made inapplicable, or are no longer in force. The evolution, modification and adaptation of the law is intrinsically complex and it is not free of conflicts. In particular, the introduction of new rules may cause some re-interpretation of existing rules. Suppose that a municipality establishes that all taxis licensed since 2015 must be all-yellow, and a couple of years later the city adds a new rule establishing that all taxis with license starting in 2018 must be all-black. Hence, the yellow-taxi rule only applies for passenger cars with a valid license from 2015 to 2017. However, this is true only years later, after the introduction of the black-taxi rule.

Since conflicts may arise with existing rules, a consistent revision of the rules of the system is frequent and mandatory. Hence, formal models of belief revision should be suitable to capture this intrinsic feature of law.

Despite the importance of norm-change mechanisms, the logical investigation of legal dynamics was for long time underdeveloped. In the eighties a pioneering research effort was devoted by Alchourrón, Gärdenfors and Makinson [5] to develop a logical model (AGM) for also modeling norm change. As is well-known, the AGM framework distinguishes three types of change operation over theories. Contraction is an operation that removes a specified sentence ϕ from a given theory Γ (a logically closed set of sentences) in such a way that Γ is set aside in favor of another theory Γ^-_ϕ which is a subset of Γ not containing ϕ. Expansion operation adds a given sentence ϕ to Γ so that the resulting theory Γ_{ϕ}^{+} is the smallest logically closed set that contains both Γ and ϕ. Revision operation adds ϕ to Γ but it is ensured that the resulting theory Γ_{ϕ}^{*} be consistent. Alchourrón, Gärdenfors and Makinson argued that, when Γ is a code of legal norms, contraction corresponds to norm derogation (norm removal) and revision to norm amendment. AGM framework has the advantage of being very abstract, as it works with theories consisting of simple logical assertions. For this reason, it can capture basic aspects of the dynamics of legal systems, such as the change obligations and permissions [9,19].

Some research has been carried out to reframe AGM ideas within richer rule-based logical systems [32,30]. However, also these attempts suffer from some drawbacks of standard AGM, among them the fact that the proposed frameworks fail to handle the temporal aspects of norm change: indeed, legal norms are qualified by temporal properties, such as the time when the norm comes into existence and belongs to the legal system or the time when the norm is in force. Since all these properties can be relevant when legal systems change, [19] argues that failing to consider the temporal aspects of legal dynamics poses a serious limit to correctly model norm change in the law.

Unlike rich but complex frameworks such as the one of [19], this paper claims that belief revision techniques—which are based on an abstract and elegant machinery—can be reconciled with the need to consider several temporal patterns of legal reasoning. In this work we are thus interested in the formalisation of a belief revision operator applied to an epistemic model that considers rules and time. We enrich a simple logic language with an interval-based model of time, to represent temporal dimensions such as the effectiveness of norms, i.e., when norms are applicable. The revision operator may remove rules when needed or adapt intervals of time when newer, contradictory norms are introduced in the system. For the operator, both constructive definition and an axiomatic characterization by representation theorems are given.

The layout of the paper is as follows. Section 2 shows an example to motivate the main ideas of our framework. Section 3 proposes the notions of temporalised belief base and temporalised derivation. Section 4 presents a set of properties that the temporalised belief revision operators should satisfy, by means of postulates. Section 5 introduces both a complete construction for legal revision operator based on temporalised belief base and their characterization regarding the presented postulates through a representation theorem. Section 6 reports on related work. Finally, in Section 7 conclusions are offered and ideas for future work are given. The proof for the representation theorem can be found in Appendix.

2. The Problem and a Motivating Example

As we have briefly mentioned in the introduction (see also Section 6), belief revision, and specifically the AGM paradigm, has been advocated to be an elegant and abstract model for legal change. Its has been however argued that standard belief techniques do not capture the following aspects of the law [19]:

1. the law usually regulate its own changes by setting specific norms whose peculiar objective is to change the system by stating what and how other existing norms should be modified;
2. since legal modifications are derived from these peculiar norms, they can be in conflict and so are defeasible;

3. legal norms are qualified by temporal properties, such as the time when the norm comes into existence and belongs to the legal system or the time when the norm is in force.

The general temporal model, as proposed in [19] assumes that all legal norms are qualified by different temporal parameters:

- the time when the norm comes into existence and belongs to the legal system,
- the time when the norm is in force,
- the time when the norm produces legal effects (it is applicable), and
- the time when the normative effects (conclusions) hold.

Indeed, it is common legislative practice that, once a legal provision is enacted (for example, the Italian 2018 budget law was enacted on 23 December 2017), its force can for instance be postponed to a subsequent time (for example, the Italian 2018 budget law was in force since 1 January 2018). Similarly, a part of a certain provision, which is in force since a certain time t, can be effective (i.e., can be applied) since a different time t' (for example, the Italian 2018 budget law, which was in force since 1 January 2018, at art. 1, par. 253 states that par. 252 will be applicable since 1 January 2019), or any provision can produce effects that hold retroactively (for example, art. 1 of Italian 2018 budget law, par. 629, states that certain tax effects cover cases since December 2017).

In this paper we concentrate on issue 3 in the list above, i.e., how to integrate belief revision with time in the law. As regards issue 2, we do not work directly on rule-based defeasible reasoning, but we define a revision operator that may remove rules when needed or adapt intervals of time when contradictory norms are introduced in the system: for instance, if n is effective from 2001 and 2008 and a contradictory norm n' is added at 2006, we know that n is still effective from 2001 and 2005.

Let us now present a concrete example that will serve to motivate the main ideas of our proposal. It involves information and rules referring to intervals of time in which some taxes applies.

EXAMPLE 1. *Consider the following pieces of information regarding a legislative attempt to ease tax pressure for people that have been unemployed.*

(a) A citizen was unemployed from 1980 to 1985.
(b) If unemployed from 1980 to 1983, then a tax exemption applies from 1984 to 1986, in order to increase individual savings.
(c) New authorities in government revoke tax exemption for years 1985 and 1986.
(d) Tax exemption reinstated for the year 1985 due to agreements with labor unions.

However, later on the legislator approved a new provision establishing that finally there is no tax-exemption for all citizens for the years 1985 and 1986.

Here some rules are produced and, as it happens in legislative bodies, norms change later according to the political and economical context. Rule (a) provides time-bounded information: only between 1980 and 1985 the status of being unemployed holds for a given citizen. Rule (b) states that if some property (unemployed) holds between 1980 and 1983, then other property (tax exemption) holds between 1984 and 1986. Rule (c) establishes that this is no longer valid for a certain interval of time. This means that, from now on, rule (b) of tax exemption should not be applied in its original text. In other words, the intervals of rule (b) are *revised* according to new political positions. Finally, rules are revised again as a consequence of labor unions, only to be revoked later. In this example the general rule of tax-exemption is revised several times. This revision is actually about the moments in which this benefit can be applied. In fact, rule (c) solely demands a revision of the interval for tax exemption. Hence, it cannot be the case that there is a rule in the normative system that entails a tax exemption for 1985 and 1986. From (c) and (b), it can be concluded that the benefit is only applied to 1984. Therefore, (b) should be not used anymore and a new rule for 1984 should be introduced.

This is naturally a process of belief revision. Our interest is the formalization of a belief revision operator that can address the evaluation of *timed rules* representing legal norms. Technical aspects of temporalised knowledge are considered in the following section.

3. Legal System as Temporalised Belief Base

The problem of representing temporal knowledge and temporal reasoning arises in many disciplines, including Artificial Intelligence. A usual way to do this is to determine a *primitive* to represent time, and its corresponding *metric relations*. There are in the literature two traditional approaches to reasoning with and about time: a point based approach, as in [19], and an interval based approach as in [6,12]. In the first case, the emphasis is put on *instants* of time (e.g., timestamps) and a relation of precedence among them. In the second case, time is represented as continuous sets of instants in which something relevant occurs. These intervals are identified by the starting and ending instants of time.

In this work, time intervals (like in [8,12]) are considered. This design decision has been taken because it simplifies the construction of a revision operator which we will introduce below. That is, following the semantics of the temporalised rules proposed in [19] and explained in Section 3 (in an adapted version), the revision operator in many cases only consists in modifying the intervals to maintain the consistency.

The above-mentioned temporal machinery is able to explicitly model two temporal dimensions among those mentioned above in Section 2, that is the time of norm effectiveness—i.e. when a norm can produce legal effects—and the time when the norm effects hold [19].

3.1. Preliminaries and Notation

We will adopt a propositional language $\mathbb{L}$ with a complete set of boolean connectives: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$. Each formula in $\mathbb{L}$ will be denoted by lowercase Greek characters: $\alpha, \beta, \delta, \ldots, \omega$. We will say that α is the complement of $\neg\alpha$ and vice versa. The characters σ will be reserved to represent cut function for a change operator. We also use a consequence operator, denoted $Cn(\cdot)$, that takes sets of sentences in $\mathbb{L}$ and produces new sets of sentences. This operator $Cn(\cdot)$ satisfies *inclusion* ($A \subseteq Cn(A)$), *idempotence* ($Cn(A) = Cn(Cn(A))$), and *monotony* (if $A \subseteq B$ then $Cn(A) \subseteq Cn(B)$). We will assume that the consequence operator includes classical consequences and verifies the standard properties of *supraclassicality* (if α can be derived from A by deduction in classical logic, then $\alpha \in Cn(A)$), *deduction* ($\beta \in Cn(A \cup \{\alpha\})$ if and only if $(\alpha \rightarrow \beta) \in Cn(A)$) and *compactness* (if $\alpha \in Cn(A)$ then $\alpha \in Cn(A')$ for some finite subset A' of A). In general, we will write $\alpha \in Cn(A)$ as $A \vdash \alpha$.

Note that the AGM model [5] represents epistemic states by means of belief sets, that is, sets of sentences closed under logical consequence. Other models use belief bases; i.e., arbitrary sets of sentences [14,25,33]. Our epistemic model is based on an adapted version of belief bases which have additional information (time intervals). The use of belief bases makes the representation of the legal system state more natural and computationally tractable. That is, following [27, page 24] and [33], we consider that legal systems sentences could be represented by a finite number of sentences that correspond to the explicit beliefs on the legal system. That norm change can be better captured by base revision was also discussed by [19].

3.2. Time Interval

We will consider a universal finite *set of time labels* $\mathbb{T} = \{t_1, \ldots, t_n\}$ strictly ordered; each time label will represent a unique time instant. Simplifying the notation, we assume that $t_i - 1$ is the immediately previous instant to the instant t_i and $t_i + 1$ is the immediately posterior instant to the instant t_i.

Like in [22] we propose temporalised literals, however, we use intervals. We will consider an interval like a finite ordered sequence of time labels $t_i, \ldots, t_j$ where i, j are natural numbers ($i \leq j$) and $t_i, \ldots t_j \in \mathbb{T}$ denoting instances of time or *timepoints*. The discreteness of the flow of time is appropriate for modelling norms dynamics since norms usually refer to time in the spectrum of hours, days, months and years. Generally speaking, the law itself view time as determined by discrete steps. Thus, let $\alpha \in \mathbb{L}$, we have expressions of the type $\alpha^{interval}$, where *interval* can be as follow:

- $[t_i, t_i]$: meaning that α holds at time t_i. Following [19] α is transient (holding at precisely one instant of time). For simplicity $[t_i, t_i] = [t_i]$.
- $[t_i, \infty]$: meaning that α holds from t_i. Following [19] α is (indefinitely) persistent from t_i.
- $[t_i, t_j]$: meaning that α holds from time t_i to t_j with $t_i < t_j$.

Then we will consider a set of time intervals $\mathbb{I}$ which contains intervals as those described previously. Thus, for simplicity, we can have expressions like α^J where $J \in \mathbb{I}$. Intervals in $\mathbb{I}$ will be denoted by uppercase Latin characters: $A, B, C, \ldots, Z$. Then, throughout this work we will say that α^J is a *temporalised sentence* meaning the sentence α has an effectiveness time indicated by J. We preserve then the semantics of classical propositional logic to a timed context. A temporalised sentence $\alpha^{[t_a, t_b]}$ is true when its non-temporalised expression α is true in every time point t between t_a and t_b. In other words, α holds at $[t_a, t_b]$.

Naturally, two intervals may not be disjoint, as defined next.

Definition 1 (Contained interval)**.** *Let $R, S \in \mathbb{I}$ be two intervals. We say that R is contained in S, denoted $R \subseteq S$ if and only if for all $t_i \in R$ it holds that $t_i \in S$.*

Definition 2 (Overlapped interval)**.** *Let $R, S \in \mathbb{I}$ be two intervals. We say that R and S are overlapped, denoted $R \top S$ if and only if there exists $t_i \in R$ such that $t_i \in S$.*

EXAMPLE 2. *Let $R, S, V \in \mathbb{I}$ where $R = [t_3, t_7]$, $S = [t_4, t_6]$ and $V = [t_5, t_9]$ with $t_3, t_4, t_5, t_6, t_7, t_9 \in \mathbb{T}$. Then $S \subseteq R$, $R \top V$ and $S \top V$.*

3.3. Temporalised Belief Base

As rules are part of the knowledge, they are subject of temporal effectiveness too. In this perspective we can have expressions like

$$\alpha^{[t_a,t_b]} \rightarrow \beta^{[t_c,t_d]}$$

meaning that the rule can derive that β holds from time t_c to t_d if we can prove that α holds from time t_a to t_b. In the same way a conclusion can persist, this applies as well to rules and then to derivations. Note that the implication itself is not decorated with intervals, but α and β are. This means that the implication always holds at $[-\infty,\infty]$ and hence again the classical semantics of first order logic is preserved. Thus, if the implication holds (since it is not conditioned in time) and α holds at $[t_a,t_b]$ then β holds at $[t_c,t_d]$.

EXAMPLE 3. *The provision from Example 1 "If unemployed from 1980 to 1983, then a tax exemption applies from 1984 to 1986" can be formalised as follows:*

$$\mathit{Unemployed}^{[1980,1983]} \rightarrow \mathit{Tax\_Exemption}^{[1984,1986]}.$$

Thus, it is possible to define *temporalised belief base* which will contain temporalised literal and temporalised rules (see Example 4). This base represents a legal system in which each temporalised sentence defines a norm whose time interval determines the effectiveness time.

EXAMPLE 4. *The set*

$$\begin{aligned}\mathbb{K} = \{&\alpha^{[t_1,t_3]}, \alpha^{[t_4]}, \alpha^{[t_1,t_4]} \rightarrow \beta^{[t_4,t_6]},\\ &\beta^{[t_5,t_6]}, \beta^{[t_6,t_8]}, \beta^{[t_{10}]}, \delta^{[t_{11}]},\\ &\delta^{[t_{11}]} \rightarrow \beta^{[t_{15},t_{20}]}, \omega^{[t_2,t_8]},\\ &\omega^{[t_4]} \rightarrow \beta^{[t_6,\infty]}, \varepsilon^{[t_1,\infty]}\}\end{aligned}$$

is a valid temporalised belief base for a legal system.

Note that, sentence ε is valid (or true) from t_1. This type of belief base representation implies that a sentence can appear more than once in a temporalised belief base; but from the point of view of the temporalised sentences stored in the temporalised belief base there is no redundancy because each temporalised sentence has different time intervals. For instance, consider Example 4, where α appears twice, but with different intervals. In this case, we will say that α is **intermittent** and it means that α is held from t_1 to t_3 and it holds in the instant t_4. Besides, if the intervals of a sentence are overlapped ($\beta^{[t_5,t_6]}$, $\beta^{[t_6,t_8]}$ in Example 4), despite that the time interval of the sentence intuitively be only one ($[t_5,t_8]$), we decided to maintain all versions because this makes more natural modelling the dynamics of the legal system.

3.4. Temporalised Derivation

Note that a norm can explicitly be in a temporalised belief base, as $\alpha^{[t_5]} \in \mathbb{K}$ in Example 4. However, a norm can implicitly be represented in a temporal belief base if some conditions hold. For instance, in Example 4, norm β is implicitly represented with $\omega^{[t_2,t_8]}$, $\omega^{[t_4]} \rightarrow \beta^{[t_6,\infty]}$ due to the antecedent of the rule is held in t_4 by the temporalised sentence $\omega^{[t_2,t_8]}$. Next, the notion of temporalised derivation for a sentence is defined to capture this intuition. To do this, we first define a temporalised derivation in a time instant and then we give a definition of temporalised derivation in time interval.

Definition 3 (Temporalised derivation in a time instant)**.** *Let $\mathbb{K}$ be a set of temporalised sentences and $\alpha^{[t_i]}$ be a temporalised sentence. We say that $\alpha^{[t_i]}$ is derived from $\mathbb{K}$, denoted $\mathbb{K} \vdash^t \alpha^{[t_i]}$, if and only if:*

- $\alpha^J \in \mathbb{K}$ *and* $t_i \in J$*, or*
- $\beta^H \rightarrow \alpha^P \in \mathbb{K}$ *and* $t_i \in P$ *and* $\mathbb{K} \vdash^t \beta^{[t_j]}$ *for all* $t_j \in H$.

Definition 4 (Temporalised derivation in a time interval)**.** *Let $\mathbb{K}$ be a set of temporalised sentences and $\alpha^{[t_i,t_j]}$ be a temporalised sentence. We say that $\alpha^{[t_i,t_j]}$ is derived from $\mathbb{K}$ (denoted $\mathbb{K} \vdash^t \alpha^{[t_i,t_j]}$) if and only if $\mathbb{K} \vdash^t \alpha^{[t_p]}$ for all $t_p \in [t_i,t_j]$.*

Computing the temporalised derivation of a sentence through checking each instant of the intervals is useful in special cases where implicit sentences need temporalised sentences with overlapped intervals as antecedents. To determine the time interval of the implicitly derived temporal sentence, the temporal consequence will be defined below.

Definition 5 (Temporalised consequence)**.** *Let $\mathbb{K}$ be a set of temporalised sentences and $\alpha^{[t_i,t_j]}$ be a temporalised sentence. We say that $\alpha^{[t_i,t_j]}$ is a temporalised consequence of $\mathbb{K}$ ($\alpha^{[t_i,t_j]} \in Cn^t(\mathbb{K})$) if and only if $\mathbb{K} \vdash^t \alpha^{[t_i,t_j]}$.*

EXAMPLE 5. *Consider again the temporalised belief base of Example 4. Then, $\mathbb{K} \vdash^t \beta^{[t_4,\infty]}$, that is, $\beta^{[t_4,\infty]} \in Cn^t(\mathbb{K})$; and $\mathbb{K} \vdash^t \alpha^{[t_1,t_4]}$, that is, $\alpha^{[t_1,t_4]} \in Cn^t(\mathbb{K})$.*

Following Definition 4, notice that the **interval of an implicitly derived sentence** will be the interval of the consequent of the rule that derives the conclusion of the proof. For instance, suppose that $\mathbb{K} = \{\gamma^{[t_2,t_5]}, \gamma^{[t_3,t_4]} \rightarrow \varepsilon^{[t_6,t_9]}\}$ then the time interval of ε is $[t_6,t_9]$. Thus, a temporalised sentence $\alpha^{[t_i,t_j]}$ is **valid** (or true) in $\mathbb{K}$ if $\mathbb{K} \vdash^t \alpha^{[t_i,t_j]}$.

In this proposal, a ***contradiction*** arises when two complementary sentences can be derived with time intervals overlapped. For instance, suppose $\mathbb{K} = \{\alpha^{[t_2,t_9]}, \neg\alpha^{[t_1,t_3]}\}$, in this case there exist a contradiction. However, consider $\mathbb{K} = \{\alpha^{[t_5]}, \neg\alpha^{[t_1,t_3]}\}$, in this case, we will say that $\mathbb{K}$ does not have contradictions. Moreover, we will say that a temporalised belief base is ***temporally consistent*** if the base does not have contradictions. The temporalised belief base of Example 4 is temporally consistent.

REMARK 1. *If $\mathbb{K}$ represents a legal system then $\mathbb{K}$ should be temporally consistent.*

A set of postulates for the revision operator is needed. These are formalised in the following section. Later on, we will provide a theorem linking postulates and construction of this operator.

4. Temporal Belief Revision Postulates

Following [26,33], in this section, we will characterize a prioritized legal revision through rationality postulates. Later, in following section, we will give a construction for this operator.

We modify the notion of *safe element* proposed in [1] for contraction; in that work, the authors consider an order among sentences and define a contraction operator over belief sets. Here, we follow a different route and consider a prioritised revision operator defined as acting over belief bases containing temporalised sentences. This choice is also motivated by the fact that legal changes typically prevail over existing norms and modify them. So, we will consider a temporalised belief base $\mathbb{K}$ and a temporalised sentence α^J that we would like to add to $\mathbb{K}$. As done in [1], in what follows we consider an element $\beta^P \in \mathbb{K}$ to be *safe with respect to the revision* of $\mathbb{K}$ by α^J iff β^P is not a sentence that can produce effects in favour of a possible temporally contradiction with α^J where J and P are overlapped time interval. In the rest of the paper, when no confusion arise we will write "is safe" instead of "is safe with respect to revision" by α^J in $\mathbb{K}$; and, if $\beta^J \in \mathbb{K}$ and $t_i \in J$ then we will say that $\beta^{[t_i]} \in \mathbb{K}$.

The temporalised postulates are:

(TBR-1) **Success**: $\alpha^J \in \mathbb{K} * \alpha^J$.

(TBR-2) **Inclusion**: If $\beta^{[t_i]} \in \mathbb{K} * \alpha^J$ then $\beta^{[t_i]} \in \mathbb{K} \cup \{\alpha^J\}$.

(TBR-3) **Consistence**: If α is consistent then $\mathbb{K} * \alpha^J$ is temporally consistent.

(TBR-4) **Uniformity**: If for all $\mathbb{K}' \subseteq \mathbb{K}$, $\{\alpha^J\} \cup \mathbb{K}'$ is temporally inconsistent if and only if $\{\beta^J\} \cup \mathbb{K}'$ is temporally inconsistent then $\mathbb{K} \cap (\mathbb{K} * \alpha^J) = \mathbb{K} \cap (\mathbb{K} * \beta^J)$.

(TBR-5) **Safe retainment**: $\beta^P \in \mathbb{K} * \alpha^J$ if and only if β^P is a safe element with respect to α^J in $\mathbb{K}$.

Since the revision operator defined here is prioritised (the new information has priority), the first postulate establishes that the revision should be successful. That is, the result of revising a belief base $\mathbb{K}$ by a sentence α^J should be a new belief base in which α is effective during the interval determined by the time interval J. The *Inclusion* postulate says that the result of applying the temporal change operator over an arbitrary base and an effective sentence in the time determined by the interval J is included in the (unrestricted) union of them. *Consistency* determines that the changed base is temporally consistent whenever the input is consistent. *Uniformity* determines that if two sentences are temporally inconsistent in the same time with the same subsets of the original belief base $\mathbb{K}$ then the respective sentences erased from $\mathbb{K}$ should be identical. *Safe Retainment* expresses that the prevailing elements after revision will be the elements that are safe before the revision, similarly to [1]. That is, every element that is retained after revision is there because it was not involved in any conflict with respect to the epistemic input, or it was involved but there was another element also involved in the conflict that was considered to be less safe.

In the following section we define the construction of the temporalised belief revision operator.

5. Legal Belief Revision

From a rational point of view, as was mentioned in Remark 1, a legal system should be temporally consistent, i.e., it cannot contain contradictory norms at any time. Hence, we propose a **prioritised legal revision operator** that allows to consistently add a temporalised sentence $\alpha^{[t_i,t_j]}$ to a consistent legal system $\mathbb{K}$.

Our special revision operator is inspired by the rule semantics explained above in Section 3 (an adapted version from the one proposed in [19]). Thus, following the concept of consistency of Section 3, the revision operator may remove temporalised sentences or, in some cases, may only modify the intervals to maintain consistency.

To incorporate a norm $\neg\beta^J$ into a legal system, it is necessary to consider all possible contradictions that may arise if the norm is added without checking for consistency. For this reason, it is necessary to compute all proofs of β considering only those temporalised sentences β^P whose effectiveness time is overlapped with the time interval J, that is, $J \top P$. Note that it is optimal to compute all minimal proofs of a temporal sentence considering only those in which the time interval is overlapped with the time interval of the input sentence. Next, a set of minimal proofs for a sentence is defined.

Definition 6 (Minimal proof)**.** *Let $\mathbb{K}$ be a temporalised belief base and α^J a temporalised sentence. Then, $\mathbb{H}$ is a minimal proof of α^J if and only if*

1. $\mathbb{H} \subseteq \mathbb{K}$,
2. $\alpha^P \in Cn^t(\mathbb{H})$ *with* $J \top P$, *and*
3. *if* $\mathbb{H}' \subset \mathbb{H}$, *then* $\alpha^P \notin Cn^t(\mathbb{H}')$ *with* $J \top P$.

Given a temporalised sentence α^J, the function $\Pi(\alpha^J, \mathbb{K})$ returns the set of all the minimal proofs for α^J from $\mathbb{K}$.

REMARK 2. *Each set of $\Pi(\alpha^J, \mathbb{K})$ derives α in at least one time instant of J.*

EXAMPLE 6. *Consider the temporalised belief base of Example 4. Then $\Pi(\beta^{[t_5,t_6]}, \mathbb{K})$ = $\{\mathbb{H}_1, \mathbb{H}_2, \mathbb{H}_3, \mathbb{H}_4\}$ where:*

- $\mathbb{H}_1 = \{\alpha^{[t_1,t_3]}, \alpha^{[t_4]}, \alpha^{[t_1,t_4]} \to \beta^{[t_4,t_6]}\}$,
- $\mathbb{H}_2 = \{\beta^{[t_5,t_6]}\}$,
- $\mathbb{H}_3 = \{\beta^{[t_6,t_8]}\}$,
- $\mathbb{H}_4 = \{\omega^{[t_2,t_8]}, \omega^{[t_4]} \to \beta^{[t_6,\infty]}\}$

Note that $\mathbb{H}_1$ is minimal: α should be derived from t_1 to t_4 to use the rule $\alpha^{[t_1,t_4]} \to \beta^{[t_4,t_6]}$ hence, $\alpha^{[t_1,t_3]}$ and $\alpha^{[t_4]}$ should be in $\mathbb{H}_1$.

Now, we will define a type of legal revision operator. The construction of prioritised legal revision by a temporalised sentence is based on the concept of a minimal proof; to complete the construction, we must define an incision function which selects in every minimal proof the sentence to be erased later and which can produce legal effects in favour of a possible contradiction with the new norm. So, in what follows we say that a temporalised sentence β^P of $\mathbb{K}$ is not safe with respect to revision by α^J in $\mathbb{K}$ if and only if β^P belongs to some minimal subset of $\mathbb{K}$ that proves $\neg\alpha^J$, $J \top P$ and $\beta = \neg\alpha$ or $\beta = \delta \to \neg\alpha$ for any δ in $\mathbb{L}$.

Our operator is based on a selection of sentences in the knowledge base that are relevant to derive the sentence to be retracted or modified. In order to perform a revision, following kernel contractions [26], this approach uses *incision functions*, which select from the minimal subsets entailing the piece of information to be revoked or modified. We adapt this notion of incision function from [26] and import in our epistemic model. An incision function only selects sentences that can be relevant for α and at least one element from each $\Pi(\alpha^J, \mathbb{K})$:

Definition 7 (Incision function)**.** *Let $\mathbb{K}$ be a temporalised belief base. An incision function σ for $\mathbb{K}$ is a function such that for all $\alpha^J \in Cn^t(\mathbb{K})$:*

- $\sigma(\Pi(\alpha^J, \mathbb{K})) \subseteq \bigcup(\Pi(\alpha^J, \mathbb{K}))$.
- *For each $\mathbb{H} \in \Pi(\alpha^J, \mathbb{K})$, $\mathbb{H} \cap \sigma(\Pi(\alpha^J, \mathbb{K})) \neq \emptyset$.*

In Hansson's approach it is not specified how the incision function selects the sentences that will be discarded of each minimal proof. In our approach, this will be solved by considering those sentences that can produce legal effects in favour of a possible contradiction with the new norm. Thus, if the new norm is $\neg\beta^J$ then the incision function will select the temporalised sentences β^P or $\alpha^Q \rightarrow \beta^F$ of each $\Pi(\beta^J, \mathbb{K})$.

Definition 8 (Search consequence function)**.** *Sc: $\mathbb{L} \times \mathbb{K} \mapsto \mathbb{K}$, is a function such that for a given sentence α and a given temporalised base $\mathbb{K}$ with $\mathbb{H} \subseteq \mathbb{K}$,*

$$Sc(\alpha, \mathbb{H}) = \{\alpha^J : \alpha^J \in \mathbb{H}\} \cup \{\beta^P \rightarrow \alpha^Q : \beta^P \rightarrow \alpha^Q \in \mathbb{H} \text{ and } \beta \in \mathbb{L}\}.$$

Definition 9 (Consequence incision function)**.** *Given a set of minimal proofs $\Pi(\alpha^J, \mathbb{K})$, σ^c is a consequence incision function if it is a incision function for $\mathbb{K}$ such that*

$$\sigma^c(\Pi(\alpha^J, \mathbb{K})) = \bigcup_{\mathbb{H} \in \Pi(\alpha^J, \mathbb{K})} Sc(\alpha, \mathbb{H}).$$

Example 7. *Consider Examples 4 and 6. Then, $Sc(\beta, \mathbb{H}_1) = \{\alpha^{[t_1,t_4]} \rightarrow \beta^{[t_4,t_6]}\}$, $Sc(\beta, \mathbb{H}_2) = \{\beta^{[t_5,t_6]}\}$, $Sc(\beta, \mathbb{H}_3) = \{\beta^{[t_6,t_8]}\}$, and $Sc(\beta, \mathbb{H}_4) = \{\omega^{[t_4]} \rightarrow \beta^{[t_6,\infty]}\}$. Thus, $\sigma^c(\Pi(\beta^{[t_5,t_6]}, \mathbb{K})) = \bigcup_{\mathbb{H} \in \Pi(\beta^{[t_5,t_6]}, \mathbb{K})} Sc(\beta, \mathbb{H}) = \{\alpha^{[t_1,t_4]} \rightarrow \beta^{[t_4,t_6]}, \beta^{[t_5,t_6]}, \beta^{[t_6,t_8]}, \omega^{[t_4]} \rightarrow \beta^{[t_6,\infty]}\}$.*

As mentioned before, the revision operator may remove temporalised sentences or, in some cases, may modify the intervals to maintain consistency. Next, a temporal projection will be defined based on a given time interval. The idea here is, given a temporalised belief base $\mathbb{K}$ and given a time interval $[t_i, t_j]$, to return a temporalised belief base $\mathbb{K}'$ containing those sentences from $\mathbb{K}$ whose time intervals be out of $[t_i, t_j]$.

Definition 10 (Excluding temporal projection)**.** *Let $\mathbb{K}$ be a temporalised belief base and let $[t_i, t_j]$ be a time interval where $t_i, t_j \in \mathbb{T}$. A excluding temporal projection of $\mathbb{K}$ from t_i to t_j, denoted $out(\mathbb{K}, [t_i, t_j])$, is a subset of $\mathbb{K}$ where for all $\alpha^{[t_p,t_q]} \in \mathbb{K}$, $out(\mathbb{K}, [t_i, t_j])$ will contain:*

- $\alpha^{[t_p,t_i-1]}$ *if $t_p < t_i$, $t_q \geq t_i$ and $t_q \leq t_j$,*
- $\alpha^{[t_j+1,t_q]}$ *if $t_p \geq t_i$, $t_q > t_j$ and $t_p \leq t_j$,*

- $\alpha^{[t_p,t_i-1]}$ *and* $\alpha^{[t_j+1,t_q]}$ *if* $t_p < t_i$, $t_q > t_j$,
- $\alpha^{[t_p,t_q]}$ *if* $t_q < t_i$ *or* $t_p > t_j$.

REMARK 3. *Note that when* $t_p \geq t_i$ *and* $t_q \leq t_j$*, the temporal sentence is not considered. In this case, this sentence is erased.*

REMARK 4. *Note that if* $\delta^{[t_h,t_k]} \in out(\mathbb{K},[t_i,t_j])$ *and the interval* $[t_h,t_k]$ *is generated through excluding temporal projection of* $\mathbb{K}$ *from* t_i *to* t_j *then there exists a temporal sentence* $\delta^{[t_p,t_q]}$ *in* $\mathbb{K}$ *such that* $[t_h,t_k] \subseteq [t_p,t_q]$.

EXAMPLE 8. *Consider Example 7 and suppose that S is a temporalised belief base and* $S = \sigma^c(\Pi(\beta^{[t_5,t_6]},\mathbb{K}))$. *Then,* $out(S,[t_5,t_6]) = \{\alpha^{[t_1,t_4]} \rightarrow \beta^{[t_4]},\ \beta^{[t_7,t_8]},\ \omega^{[t_4]} \rightarrow \beta^{[t_7,\infty]}\}$.

Following the notion of excluding temporal projection (Definition 10) a norm prioritised revision operator can be defined. That is, an operator that allows to *consistently* add temporalised sentences in a temporalised belief base. If a contradiction arises, then the revision operator may remove temporalised sentences or modify the corresponding intervals in order to maintain consistency.

Definition 11. *Let* $\mathbb{K}$ *be a temporalised belief base and* α^J *be a temporalised sentence. The operator "* $\otimes$*", called prioritised legal revision operator, is defined as follow:*

$$\mathbb{K} \otimes \alpha^J = (\mathbb{K} \setminus \sigma^c(\Pi(\neg\alpha^J,\mathbb{K}))) \cup out(\sigma^c(\Pi(\neg\alpha^J,\mathbb{K})),J) \cup \{\alpha^J\}.$$

EXAMPLE 9. *Consider Example 4 and suppose that a new norm* $\neg\beta^{[t_5,t_6]}$ *it is wished to add. To do this, it is necessary to do* $\mathbb{K} \otimes \neg\beta^{[t_5,t_6]}$. *Consider Examples 6 and 7. Then,* $\mathbb{K} \otimes \neg\beta^{[t_5,t_6]} = \{\alpha^{[t_1,t_3]},\ \alpha^{[t_4]},\ \alpha^{[t_1,t_4]} \rightarrow \beta^{[t_4]},\ \beta^{[t_7,t_8]},\ \beta^{[t_{10}]},\ \delta^{[t_{11}]},\ \delta^{[t_{11}]} \rightarrow \beta^{[t_{15},t_{20}]},\ \omega^{[t_2,t_8]},\ \omega^{[t_4]} \rightarrow \beta^{[t_7,\infty]},\ \varepsilon^{[t_1,\infty]},\ \neg\beta^{[t_5,t_6]}\}$. *Note that, this new temporalised base is temporally consistent.*

The following example shows how our operator works in a particular situation when a legal system undergoes many changes and has rules that complement each other.

EXAMPLE 10. *Consider following temporalised belief base* $\mathbb{K} = \{\beta^{[t_1,t_{10}]}, \beta^{[t_1,t_5]} \rightarrow \alpha^{[t_1,t_5]}, \beta^{[t_6,t_{10}]} \rightarrow \alpha^{[t_6,t_{10}]}, \delta^{[t_4]}\}$. *Note that,* $\mathbb{K} \vdash^t \alpha^{[t_1,t_{10}]}$ *because* $\mathbb{K} \vdash^t \alpha^{[t_i]}$ *for all* $t_i \in [t_1,t_{10}]$. *Suppose that it is necessary to adopt* $\neg\alpha^{[t_1,t_{10}]}$. *To do this, it is necessary to compute all the minimal proofs of* $\alpha^{[t_1,t_{10}]}$ *in* $\mathbb{K}$. *In this case,* $\Pi(\alpha^{[t_1,t_{10}]},\mathbb{K}) = \{\{\beta^{[t_1,t_{10}]}, \beta^{[t_1,t_5]} \rightarrow \alpha^{[t_1,t_5]}, \beta^{[t_6,t_{10}]} \rightarrow \alpha^{[t_6,t_{10}]}\}\}$. *Then,* $S = \sigma^c(\Pi(\alpha^{[t_1,t_{10}]},\mathbb{K})) = \{\beta^{[t_1,t_5]} \rightarrow \alpha^{[t_1,t_5]}, \beta^{[t_6,t_{10}]} \rightarrow \alpha^{[t_6,t_{10}]}\}$. *Thus,* $out(S,[t_1,t_{10}]) = \emptyset$. *Therefore,* $\mathbb{K} \otimes \neg\alpha^{[t_1,t_10]} = \{\beta^{[t_1,t_{10}]}, \delta^{[t_4]}, \neg\alpha^{[t_1,t_{10}]}\}$.

After introducing the legal revision operator $\otimes$, we will now complete its presentation with a proper characterisation of its behaviour with respect to the proposed postulates. To that end, below we will offer a Representation Theorem for the operator that establishes the correspondence between the postulates and the construction that formalises it.

THEOREM 1. *An operator* $\otimes$ *is a prioritised legal revision for* $\mathbb{K}$ *if and only if it satisfies the postulates of (TBR-1) Success, (TBR-2) Inclusion, (TBR-3) Consistency, (TBR-4) Uniformity, and (TBR-5) Safe Retainment.*

Proof. See Appendix. □

The Representation Theorem above properly shows that a temporalised legal operator is defined in terms of our timed postulates.

6. Related work

Alchourrón and Makinson were the first to logically study the changes of a legal code [3,4,2]. The addition of a new norm n causes an enlargement of the code, consisting of the new norm plus all the regulations that can be derived from n. Alchourrón and Makinson distinguish two other types of change. When the new norm is incoherent with the existing ones, we have an *amendment* of the code: in order to coherently add the new regulation, we need to reject those norms that conflict with n. Finally, *derogation* is the elimination of a norm n together with whatever part of the legal code that implies n.

In [5], inspired by the works above, the so called AGM framework for belief revision is proposed. This area proved to a very fertile one and the phenomenon of revision of logical theories has been thoroughly investigated. It is then natural to ask if belief revision offers a satisfactory framework for the problem of norm revision. Some of the AGM axioms seem to be rational requirements in a legal context, whereas they have been criticised when imposed on belief change operators. An example is the *success* postulate, requiring that a new input must always be accepted in the belief set. It is reasonable to impose such a requirement when we wish to enforce a new norm or obligation. However, it gives rise to irrational behaviors when imposed to a belief set, as observed in [15].

The AGM operation of contraction is perhaps the most controversial one, due to some postulates such as recovery [19,34], and to elusive nature of legal changes such as derogations and repeals, which are all meant to contract legal effects but in remarkably different ways [19]. Standard AGM framework is of little help here: it has the advantage of being very abstract—it works with theories consisting of simple logical assertions—but precisely for this reason it is more suitable to capture the dynamics of obligations and permissions than the one of legal norms. In fact, it is hard in AGM to represent how the same set of legal effects can be contracted in many different ways, depending on how norms are changed. For this reason, previous works [16,17,19] proposed to combine a rule-based system with some forms of temporal reasoning.

Difficulties behind standard AGM have been considered and some research has been carried out to reframe AGM ideas within reasonably richer rule-based logical systems, combining AGM ideas with Defeasible Logic [30,21] or Input/Output Logic [9,32]. [34] suggested a different route, i.e., employing in the law existing techniques—such as iterated belief change, two-dimensional belief change, belief bases, and weakened contraction—that can obviate problems identified in [19] for standard AGM.

In this paper we showed to extend base revision with temporal reasoning, and, in particular, with time intervals. Our approach, like in [19], is able to deal with constituents holding in an interval of time, thus an expression $\Longrightarrow a^{[t_1,t_2]}$ meaning that a holds between t_1 and t_2 can be seen as a shorthand of the pair of rules from [19] (defeasible and defeater) $\Longrightarrow a^{[t_1,pers]}$ and $\rightsquigarrow \neg a$. We have taken this design decision because it simplifies the construction of the revision operator: following the semantics of the temporalised rule proposed in [19] and explained in Section 3 (an adapted version), the revision operator in many cases only consists in modifying the intervals to maintain the consistency.

Intervals as a model of time were applied in several contexts. In our work intervals are used as a structure denoting periods of time attached to propositional formulas. Since our focus is put on belief revision operations, we do not elaborate here on interval semantics. However, other formalisms make use of this model of time. First introduced by Allen (as in [7]), intervals were later applied to different scenarios, such as modal logics [24,13], defeasible logics [8,23] and abstract argumentation ([12]). [23] focuses on duration and periodicity and relationships with various forms of causality. [8] proposed a sophisticated interaction of defeasible reasoning and standard temporal reasoning (i.e., mutual relationships of intervals and constraints on the combination of intervals). In both cases it is not clear whether the techniques employed there are relevant to the application to norm modifications, and such works consider only a single temporal dimension.

In [33], similar to our approach, a change operator is proposed. There, the change model focuses on the knowledge of the agent and it proposes that the agent has a kind of short-term memory in which the recently computed results are stored. When a result is important, or frequently used, the agent can store it explicitly in its long-term memory. Changes in a belief state take place initially in a short-term memory, rather than in the long-term memory. Due to our focus on legal systems, in our proposal, instead, all sentences are stored in a long-term memory. In addition, in [33], time interval is not used to represent information, but introduces a structure to represent an agent's belief state that distinguishes different type of beliefs according to whether or not they are explicitly represented, whether they are currently active and whether they are fully accepted or provisional. Despite all these differences, in [33], similar to us, they suggest representing the knowledge base as a belief base, a set of formulas which is not closed under logical consequence. Also there, the author proposes a particular consequence operation that considers only the relevant parts of a belief base and shown that this consequence operation can be used to define a local version of the operator proposed in [26] (among others). Thus, in [33], postulates similar to those in [26] were proposed. In our approach, we have been inspired by these postulates to axiomatize our operator.

There some works in the literature that have discussed the relation between belief revision and temporal reasoning, though none of them addressed the issue in the normative domain. Two prominent lines of investigation are [10,11] and [31].

[10,11] address belief revision in a temporal logic setting. In these articles, consider sets of sentences closed under logical consequence. In contrast to this, our proposal is based on an adapted version of belief bases which have additional information (time intervals). The use of belief bases makes the representation of the legal system state more natural and computationally tractable. That is, following [27, page 24] and [33], we consider that legal systems sentences could be represented by a finite number of sentences that correspond to the explicit beliefs on the legal system. The main purpose of [10,11] is to represent the AGM postulates as axioms in a modal language. The assumption is that belief revision has to do with the interaction of belief and information over time, thus temporal logic seemed a natural starting point. The technical solution is to consider branching-time frames to represent different possible evolutions of beliefs. Hence, belief revision operators are interpreted over possible worlds. Unlike this we work with legal system in which each temporalised sentence defines a norm whose time interval determines the effectiveness time. Then, the revision process defined in this article may remove temporalised sentences or, in some cases, may only modify the intervals.

[31] is based on a well-developed theory of action in the situation calculus extended to deal with belief. The authors add this framework a notion of plausibility over situations, an show how to handle nested belief, belief introspection, mistaken belief, belief revision and belief update together with iterated belief change.

An interesting line of investigation is to study possible correlations with these two last research lines in literature as compared to our system. Such a comparison cannot be directly done from technical viewpoint for two reasons. First of all, our work is specifically focused in a propositional language following kernel contraction construction proposed in [26]. Second, our propositional language is equipped with explicit time-stamps and with temporal intervals, which allow us for expressing richer temporal specifications in the language.

7. Conclusions and Future Work

Law is, by nature, a *dynamic* system of rules. As times goes by, rules are introduced in the system, which may be either unexpectedly in conflict with existing rules or be intended to provide new, different norms for society. This demands a consistent revision of the rules of the system. In particular, some dynamic features of legal reasoning can be captured by considering a temporal dimension applied to normative elements such as rules. Since the normative system is revised as a consequence of new rules, two dynamic aspects of the law must be considered: the change of the set of rules to take into account new pieces of information, and the ability to reason about temporalised knowledge. Hence, formal models of belief revision under timed rules should be suitable to capture this intrinsic dynamism of law.

Following that idea, in this work we have introduced an interval-based model of temporalised knowledge, together with a time-based belief revision operator for legal systems. Intervals are used to model a period of time for a piece of knowledge to be effective or relevant, leading to the definition of a new kind of temporal rules. On these interval-decorated rules we have defined the corresponding temporalised derivation.

The consideration of time requires an adaptation of the notions of contradiction and inconsistency in the classical sense. We state that temporalised knowledge base is inconsistent only if contradictory information can be derived for the same moment of time. Hence, if a new rule is added to the legal system causing a and $\neg a$ to be both consequences of the theory at moment i, a revision of rules is required. We have defined a novel belief revision operator that allows the consistent addition of temporalised sentences in a temporalised belief base. If a contradiction arises, then the revision operator may either completely remove conflictive temporalised sentences or modify the intervals of some rules. This last action is made because a given consequence a at interval I may fall in contradiction during a sub-interval of I. Thus, a should be a consequence, after the revision, only for the rest of I. Then, intervals in rules should be taken into account for the revision process.

Change operators are presented following the AGM model [5] where the operators are defined through constructions and representation theorems. We then presented a new temporalised revision operator. This operator is based on the notion of kernel [26], and the selection of norms to remove are either contradictory norms or rules with a contradictory consequent, both with respect to the new information to be added to the legal system. As expected, intervals of existing rules may be affected towards consistency.

The operator is characterised through a set of rationality postulates that consider intervals of time. These new timed postulates provide a rational formalisation of the process of belief change. We have also introduced a Representation Theorem that proves the relation between our operator and the timed rationality postulates, as it is mandatory when defining belief revision operators.

Acknowledgments

A preliminary version of this work was published in the proceedings of JURIX 2017 [30]. We would like to thank the anonymous reviewers of JURIX 2017 and the conference audience for their useful comments. This work was partially supported by PGI-UNS (grants 24/ZN30, 24/ZN32) and EU H2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No. 690974 for the project MIREL: MIning and REasoning with Legal texts.

Appendix

THEOREM 1. An operator $\otimes$ is a prioritised legal revision for $\mathbb{K}$ if and only if it satisfies the postulates of (TBR-1) Success, (TBR-2) Inclusion, (TBR-3) Consistency, (TBR-4) Uniformity, and (TBR-5) Safe Retainment.
Proof
Proof has two parts. First, we start from the satisfaction of postulates to the construction as a legal revision operator. Second, we prove that an operator is a legal revision if previous postulates are satisfied.

$\Leftarrow$) Postulates to construction:

Let $*$ be an operator that satisfies *Success*, *Inclusion*, *Consistency*, *Uniformity* and *Safe Retainment*. We have to show that $*$ is a legal revision operator.

(1) Let σ^c be a function such that for every temporalised base $\mathbb{K}$ and for every temporalised sentence α^J holds $\sigma^c(\Pi(\neg\alpha^J,\mathbb{K})) = \mathbb{K} \setminus \mathbb{K} * \alpha^J$.

We first show that σ^c is an incision function. To do this we show that the conditions in Definition 9 are satisfied by σ^c; that is:

- σ^c is a well-defined function: if $\neg\alpha^J$ and $\neg\beta^J$ are such that $\Pi(\neg\alpha^J,\mathbb{K}) = \Pi(\neg\beta^J,\mathbb{K})$ then $\sigma^c(\Pi(\neg\alpha^J,\mathbb{K})) = \sigma^c(\Pi(\neg\beta^J,\mathbb{K}))$.
  Let $\neg\alpha^J$ and $\neg\beta^J$ be two temporalised sentences such that $\Pi(\neg\alpha^J,\mathbb{K}) = \Pi(\neg\beta^J,\mathbb{K})$. We need to show that $\sigma^c(\Pi(\neg\alpha^J,\mathbb{K})) = \sigma^c(\Pi(\neg\beta^J,\mathbb{K}))$. By Definition 6 and Definition 5, for all subset $\mathbb{K}'$ of $\mathbb{K}$, $\neg\alpha^J \in Cn^t(\mathbb{K}')$ if and only if $\neg\beta^J \in Cn^t(\mathbb{K}')$. Then $\neg\alpha^J \cup \mathbb{K}'$ is temporally inconsistent if and only if $\neg\beta^J \cup \mathbb{K}'$ is temporally inconsistent. Thus, by **uniformity**, $\mathbb{K} \cap (\mathbb{K} * \alpha^J) = \mathbb{K} \cap (\mathbb{K} * \beta^J)$. Then, $\mathbb{K} \setminus (\mathbb{K} * \alpha^J) = \mathbb{K} \setminus (\mathbb{K} * \beta^J)$. Therefore, by (1), $\sigma^c(\Pi(\neg\alpha^J,\mathbb{K})) = \sigma^c(\Pi(\neg\beta^J,\mathbb{K}))$.

- $\sigma^c(\Pi(\neg\alpha^J,\mathbb{K})) \subseteq \bigcup(\Pi(\neg\alpha^J,\mathbb{K}))$.
  Let $\beta^P \in \sigma(\Pi(\neg\alpha^J,\mathbb{K}))$. By (1), we have that $\beta^P \in \mathbb{K}\setminus\mathbb{K}*\alpha^J$. Then, it holds that $\beta^P \notin \mathbb{K}*\alpha^J$, and from **Safe Retainment** we have that β^P is not a safe element, otherwise it will be part of the revision. Since β^P is not a safe element then it holds that β^P is a sentence that can produce effects in favour of a possible temporal contradiction with α^J where J and P are overlapped time interval. Then, β^P is in a minimal subset (under set inclusion) $\mathbb{H}$ of $\mathbb{K}$ such that $\mathbb{H}\cup\alpha^J$ is temporally inconsistent. By Definition 6, if $\mathbb{H}$ is a minimal subset such that $\mathbb{H}\cup\alpha^J$ is temporally inconsistent then $\mathbb{H}\in\Pi(\neg\alpha^J,\mathbb{K})$, and therefore $\beta^P \in \bigcup(\Pi(\neg\alpha^J,\mathbb{K}))$. Since this holds for any arbitrary $\beta^P \in \sigma(\Pi(\neg\alpha^J,\mathbb{K}))$ we have that $\sigma^c(\Pi(\neg\alpha^J,\mathbb{K})) \subseteq \bigcup(\Pi(\neg\alpha^J,\mathbb{K}))$.
- If $\mathbb{H}\in\Pi(\alpha^J,\mathbb{K})$, $\mathbb{H}\cap\sigma^c(\Pi(\alpha^J,\mathbb{K}))\neq\emptyset$.
  Let $\mathbb{H}\in\Pi(\alpha^J,\mathbb{K})$. We need to show that $\mathbb{H}\cap\sigma^c(\Pi(\alpha^J,\mathbb{K}))\neq\emptyset$. We should prove that, there exists $\beta^P\in\mathbb{H}$ such that $\beta^P\in\sigma^c(\Pi(\alpha^J,\mathbb{K}))$. Suppose $\neg\alpha^J$ is consistent. Since we have assumed that $\mathbb{K}$ is temporally consistent, by **consistency**, $\mathbb{K}*\neg\alpha^J$ is temporally consistent. Since $\mathbb{H}$ is inconsistent with $\neg\alpha^J$ then $\mathbb{H}\not\subseteq\mathbb{K}*\neg\alpha^J$ by **success**. This means that there is some $\beta^P\in\mathbb{H}$ and $\beta^P\notin\mathbb{K}*\neg\alpha^J$. Since $\mathbb{H}\subseteq\mathbb{K}$ it follows that $\beta^P\in\mathbb{K}\setminus\mathbb{K}*\neg\alpha^J$; *i.e.*, by our definition of σ^c, $\beta^P\in\sigma^c(\Pi(\alpha^J,\mathbb{K}))$. Therefore, $\mathbb{H}\cap\sigma^c(\Pi(\alpha^J,\mathbb{K}))\neq\emptyset$.
- $\beta^P\in\sigma^c(\Pi(\alpha^J,\mathbb{K}))$ then for some $\mathbb{H}\in\Pi(\alpha^J,\mathbb{K})$ such that $\beta^P\in\mathbb{H}$ it holds that $\beta=\alpha$ or $\beta^P=\delta^Q\rightarrow\alpha^P$ and $\delta\in\mathbb{L}$.
  Let $\beta^P\in\sigma^c(\Pi(\alpha^J,\mathbb{K}))$. Then $\beta^P\in\mathbb{K}\setminus(\mathbb{K}*\neg\alpha^J)$, hence $\beta^P\notin\mathbb{K}*\neg\alpha^J$. Therefore, by **Safe Retainment** we have that β^P is not a safe element. Since β^P is not a safe element then it holds that β^P is a sentence that can produce effects in favour of a possible temporally contradiction with α^J where J and P are overlapped time interval. Then, β^P is in a minimal subset (under set inclusion) $\mathbb{H}$ of $\mathbb{K}$ such that $\mathbb{H}\cup\alpha^J$ is temporally inconsistent. By Definition 6, if $\mathbb{H}$ is a minimal subset such that $\mathbb{H}\cup\alpha^J$ is temporally inconsistent then $\mathbb{H}\in\Pi(\alpha^J,\mathbb{K})$. Then, since β^P is a sentence that can produce effects in favour of a possible temporally contradiction with α^J, $\beta=\alpha$ or $\beta^P=\delta^Q\rightarrow\alpha^P$ and $\delta\in\mathbb{L}$.

Once we have proven that σ^c is a proper incision function, to finalise the proof we must show that $\mathbb{K}*\alpha^J=\mathbb{K}\otimes\alpha^J$.

($\subseteq$) Let $\beta^{[t_i]}\in\mathbb{K}*\alpha^J$.
It follows by **inclusion** that $\beta^{[t_i]}\in\mathbb{K}\cup\{\alpha^J\}$.
Then, $\beta^{[t_i]}\in\mathbb{K}$.
It follows from $\beta^{[t_i]}\in\mathbb{K}*\alpha^J$ and $\beta^{[t_i]}\in\mathbb{K}$ that $\beta^{[t_i]}\notin\mathbb{K}\setminus\mathbb{K}*\alpha^J$.
Thus, by (1), $\beta^{[t_i]}\notin\sigma^c(\Pi(\neg\alpha^J,\mathbb{K}))$. Hence, $\beta^{[t_i]}\in\mathbb{K}\otimes\alpha^J$.

($\supseteq$) Let $\beta^{[t_i]}\in\mathbb{K}\otimes\alpha^J$.
By definition, $\beta^{[t_i]}\in(\mathbb{K}\setminus\sigma^c(\Pi(\neg\alpha^J,\mathbb{K})))\cup out(\sigma^c(\Pi(\neg\alpha^J,\mathbb{K})),J)\cup\{\alpha^J\}$.
From Remark 4, $out(\sigma^c(\Pi(\neg\alpha^J,\mathbb{K})),J)\subseteq\mathbb{K}$ and then, $\beta^{[t_i]}\in\mathbb{K}$ and $\beta^{[t_i]}\notin\sigma^c(\Pi(\neg\alpha^J,\mathbb{K}))$.
Thus, by (1), $\beta^{[t_i]}\notin\mathbb{K}\setminus\mathbb{K}*\alpha^J$.
Hence, $\beta^{[t_i]}\in\mathbb{K}*\alpha^J$.

The second part of the demonstration follows.

⇒) Construction to postulates: Let σ^c be a consequence incision function and $\otimes$ its associated operator and $\mathbb{K}$ a knowledge base. Then, for all α^J:

$\mathbb{K} \otimes \alpha^J = (\mathbb{K} \setminus S) \cup out(S,J) \cup \{\alpha^J\}$ where $S = \sigma^c(\Pi(\neg\alpha^J, \mathbb{K}))$.

We prove that the postulates hold for the given construction, as follows.

- **Success**: $\alpha^J \in \mathbb{K} \otimes \alpha^J$.
  Straightforward by Definition 11.
- **Inclusion**: If $\beta^{[t_i]} \in \mathbb{K} \otimes \alpha^J$ then $\beta^{[t_i]} \in \mathbb{K} \cup \{\alpha^J\}$.
  Let $\beta^{[t_i]} \in \mathbb{K} \otimes \alpha^J$. From Definition 11 we have that $\mathbb{K} \otimes \alpha^J = (\mathbb{K} \setminus S) \cup out(S,J) \cup \{\alpha^J\}$ where $S = \sigma^c(\Pi(\neg\alpha^J, \mathbb{K}))$. Following Remark 4, for all $\beta^P \in S$ there exists $\beta^Q \in out(S,J)$ such that $Q \subseteq P$. Then, $out(S,J) \subseteq \mathbb{K}$. Therefore, $\beta^{[t_i]} \in \mathbb{K} \cup \{\alpha^J\}$.
- **Consistence**: if α^J is consistent then $\mathbb{K} \otimes \alpha^J$ is temporally consistent.
  Suppose α is consistent. By Definition 8, $\sigma^c(\Pi(\neg\alpha^J, \mathbb{K}))$ returns a set of sentences which they are selected from every subset of $\mathbb{K}$ temporally inconsistent with α^J. Then, since $\mathbb{K}$ is temporally consistent (Remark 1), $\mathbb{K} \setminus \sigma^c(\Pi(\neg\alpha^J, \mathbb{K}))$ is temporally consistent. From Definition 10, if there exists $\neg\alpha^Q$ or $\beta^P \to \alpha^Q$ in $out(\sigma^c(\Pi(\neg\alpha^J, \mathbb{K})), J)$ then the intervals J and Q are not overlapped. Therefore, $\mathbb{K} \setminus \sigma^c(\Pi(\neg\alpha^J, \mathbb{K})) \cup out(\sigma^c(\Pi(\neg\alpha^J, \mathbb{K})), J)$ is temporally consistent. Then, following Definition 11, $\mathbb{K} \otimes \alpha^J$ is temporally consistent.
- **Uniformity**: if for all $\mathbb{K}' \subseteq \mathbb{K}$, $\{\alpha^J\} \cup \mathbb{K}'$ is temporally inconsistent if and only if $\{\beta^J\} \cup \mathbb{K}'$ is temporally inconsistent then $\mathbb{K} \cap (\mathbb{K} \otimes \alpha^J) = \mathbb{K} \cap (\mathbb{K} \otimes \beta^J)$.
  Let α and β be consistent sentences and J a time interval. Suppose that for all subset $\mathbb{K}$' of $\mathbb{K}$, $\{\alpha^J\} \cup \mathbb{K}'$ is temporally inconsistent if and only if $\{\beta^J\} \cup \mathbb{K}'$ is temporally inconsistent. Then $\Pi(\alpha^J, \mathbb{K}) = \Pi(\beta^J, \mathbb{K})$ and since σ^c is a well defined function then $\sigma^c(\Pi(\alpha^J, \mathbb{K})) = \sigma^c(\Pi(\beta^J, \mathbb{K}))$. In the same way $out(\sigma^c(\Pi(\alpha^J, \mathbb{K})), J) = out(\sigma^c(\Pi(\beta^J, \mathbb{K})), J)$. Therefore, $\mathbb{K} \cap (\mathbb{K} \otimes \alpha^J) = \mathbb{K} \cap (\mathbb{K} \otimes \beta^J)$.
- **Safe retaiment**: $\beta^P \in \mathbb{K} \otimes \alpha^J$ if and only if β^P is a safe element with respect to α^J in $\mathbb{K}$.
  - Proof that if $\beta^P \in \mathbb{K} \otimes \alpha^J$ then β^P is a safe element with respect to α^J in $\mathbb{K}$.
    Let $\beta^P \in \mathbb{K} \otimes \alpha^J$ then by Definition 11 we have two alternatives:
    * $\beta^P \notin \sigma^c(\Pi(\neg\alpha^J, \mathbb{K}))$. We can identify two different cases: either β^P does not belong to any minimal proof, or it does. Let us consider the two cases separately.
      If $\beta^P \notin X$ for every $X \in \Pi(\neg\alpha^{\mathbb{K}}, J)$ then $\neg\alpha^J$ does not belong to any minimal set (under set inclusion) B of $\mathbb{K}$ such that $\neg\alpha^Q \in Cn^t(B)$ with $J \top Q$ and then it is a safe element with respect to α^J in $\mathbb{K}$.
      Now consider the case where $\beta^P \in X$ for every $X \in \Pi(\neg\alpha^{\mathbb{K}}, J)$. Since $\beta^P \notin \sigma^c(\Pi(\neg\alpha^J, \mathbb{K}))$ then by Definition 9 and Definition 8, $\beta \neq \alpha$ and $\beta \neq \delta \to \alpha$. Then β^P is not a sentence that can produce effects in favour of a possible temporally contradiction with α^J. Hence, β^P is a safe element with respect to α^J in $\mathbb{K}$.

* $\beta^P \in \sigma^c(\Pi(\neg\alpha^J, \mathbb{K}))$. Since $\beta^P \in \mathbb{K} \otimes \alpha^J$ then, by Definition 11, $\beta^P \in out(\sigma^c(\Pi(\neg\alpha^J, \mathbb{K})), J)$. Then, by Definition 10, the time intervals P and J are not overlapped. Therefore, β^P is a safe element with respect to α^J in $\mathbb{K}$.

- Proof that if β^P is a safe element with respect to α^J in K then $\beta^P \in \mathbb{K} \otimes \alpha^J$. Let $\beta^P \in \mathbb{K}$ be a safe element with respect to the revision of $\mathbb{K}$ by α^J. Then, β^P is not a sentence that can produce effects in favour of a possible temporally contradiction with α^J where J and P are overlapped time interval. Then, β^P does not belong to any minimal subset under set inclusion X of $\mathbb{K}$ such that $\neg\alpha^J \in Cn^t(X)$ with $J \top P$. Thus, by Definition 6, $\beta^P \notin \bigcup(\Pi(\neg\alpha^J, \mathbb{K}))$. Following Definition 9, $\beta^P \notin \sigma^c(\Pi(\neg\alpha^J, \mathbb{K}))$. Therefore, by Definition 11, $\beta^P \in \mathbb{K} \otimes \alpha^J$.

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