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A SIMPLE CHARACTERIZATION OF SPECIAL MATCHINGS IN LOWER BRUHAT INTERVALS

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ABSTRACT. We give a simple characterization of special matchings in lower Bruhat intervals (that is, intervals starting from the identity element) of a Coxeter group. As a byproduct, we obtain some results on the action of special matchings.

1. INTRODUCTION

Over the last few years, special matchings have shown many applications in Coxeter group theory in general (see [11], [12]) and in the computation of Kazhdan–Lusztig polynomials in particular (see [2], [3], [4], [13], [14], [16]).

The main achievements obtained by using special matchings are in the problem of the combinatorial invariance of Kazhdan–Lusztig polynomials. These famous polynomials, introduced by Kazhdan and Lusztig in [10], are polynomials indexed by pairs of elements u, v in a Coxeter group, with $u \leq v$ under Bruhat order. In the 80's, Lusztig in private and, independently, Dyer [7] have conjectured that the Kazhdan–Lusztig polynomial $P_{u,v}(q)$ only depends on the combinatorial structure of the interval $[u, v]$ (that is, on the isomorphism class of $[u, v]$ as a poset). This conjecture is often referred to as the Combinatorial Invariance Conjecture of Kazhdan–Lusztig polynomials.

Over the last 15 years, new results about the Combinatorial Invariance Conjecture (and its generalization to the parabolic setting) for lower intervals, that is, intervals starting from the identity element, were obtained by proving a recursive formula for Kazhdan–Lusztig polynomials which depends on special matchings. This result was first obtained for the ordinary Kazhdan–Lusztig polynomials of the symmetric group [2], then for the ordinary Kazhdan–Lusztig polynomials of an arbitrary Coxeter group [3] (see also [6]), then for the parabolic Kazhdan–Lusztig polynomials of a doubly-laced Coxeter group [13], and very recently for the parabolic Kazhdan–Lusztig polynomials of an arbitrary Coxeter group [14]. Each time, these improvements were made possible by a better control on the special matchings of Coxeter groups. The impression is that a further understanding of special matchings of Coxeter groups might bring to other results on the combinatorial invariance of Kazhdan–Lusztig polynomials.

The purpose of this paper is to give a simple characterization of special matchings of lower intervals in any arbitrary Coxeter group.

With respect to the classification given in [5], the present classification has the advantages of being more compact and simpler. It is also aesthetically pleasing to have only one self-dual type of system instead of two types of systems, one dual to the other.

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Furthermore, we provide a simpler description of the action of a special matching, which is easy to work with. More precisely, given a special matching M of an element $w \in W$, we show that, for all $u \in W$ with $u \leq w$, the element $M(u)$ can be computed from any factorization of u satisfying certain hypotheses, while in [5] we could compute $M(u)$ only starting from one specific factorization.

As an application of the classification in this paper, we give a new proof of the Combinatorial Invariance Conjecture for lower Bruhat intervals in any Coxeter group, which is shorter than the original one in [3].

2. STATEMENTS OF MAIN RESULTS

We fix an arbitrary Coxeter system (W, S) .

For $w \in W$, the *support* of w , denoted $\text{Supp}(w)$, is $\{s \in S : s \leq w\}$. For $H \subseteq S$, we let $\text{Supp}_H(w)$ denote the intersection $\text{Supp}(w) \cap H$. For $s \in S$, we denote by C_s the set of generators commuting with s , that is $\{c \in S : sc = cs\}$. For $w \in W$ and $H \subseteq S$, we denote by $w_0(H)$ the longest element of $[e, w] \cap W_H$.

Definition 2.1. Let $s \in S$ and $J, K \subseteq S$. We say that J and K are *s-complementary* if $J \cup K = S$ and $J \cap K = C_s$. We also say that K is the *s-complement* of J , and vice versa.

Definition 2.2. A *system* \mathcal{S} for $w \in W$ is a triple (J, H, M) with $H \subseteq S$, $|H| \in \{1, 2\}$, M a matching of the Hasse diagram of $[e, w_0(H)]$, and $C_{M(e)} \subseteq J \subseteq S$, such that

- [S0] M is a multiplication matching (if and) only if $|H| = 1$;
- [S1] $w^J \in W_K$ (equivalently ${}^K w \in W_J$), where K is the $M(e)$ -complement of J ;
- [S2]
 - $|\text{Supp}_H((w^J)^H)| \leq 1$ and if $\alpha \in \text{Supp}_H((w^J)^H)$ then M commutes with λ_α ,
 - $|\text{Supp}_H({}^H({}^K w))| \leq 1$ and if $\beta \in \text{Supp}_H({}^H({}^K w))$ then M commutes with ρ_β .

Definition 2.3. Let \mathcal{S} be a system (J, H, M) for w , and $u \in W$. An *\mathcal{S} -factorization* of u is a factorization of $u = a \cdot b \cdot c$, for some $a, b, c \in W$ satisfying the following properties:

- $\ell(u) = \ell(a) + \ell(b) + \ell(c)$;
- $a \in W_K \cap W^H$, $|\text{Supp}_H(a)| \leq 1$, and if $\alpha \in \text{Supp}_H(a)$ then M commutes with λ_α ;
- $b \in W_H$;
- $c \in W_J \cap {}^H W$, $|\text{Supp}_H(c)| \leq 1$, and if $\beta \in \text{Supp}_H(c)$ then M commutes with ρ_β .

In the sequel, we prove that, to each system $\mathcal{S} = (J, H, M)$, we can attach a special matching $M_{\mathcal{S}}$ of w by letting, for all $u \in W$ with $u \leq w$

$$M_{\mathcal{S}}(u) = a \cdot M(b) \cdot c$$

where $u = a \cdot b \cdot c$ is an arbitrary \mathcal{S} -factorization. While it is easy to see that \mathcal{S} -factorizations exist for all $u \in W$ with $u \leq w$, the fact that $M_{\mathcal{S}}$ does not depend on the chosen \mathcal{S} -factorization and is indeed a matching of w (when $M_{\mathcal{S}}(w) \triangleleft w$) is a key point in the sequel.

The main results of this work are collected in the following.

Theorem. *Let (W, S) be an arbitrary Coxeter system and w be an arbitrary element of W .*

- *If M is a special matching of w , then there exists a system \mathcal{S} for w such that $M = M_{\mathcal{S}}$.*
- *Vice versa, if \mathcal{S} is a system for w such that $M_{\mathcal{S}}(w) \triangleleft w$, then $M_{\mathcal{S}}$ is a special matching for w .*
- *If M is a special matching of w associated with a system \mathcal{S} , $w = a \cdot b \cdot c$ is an \mathcal{S} -factorization of w , and u is an element smaller than w , then*

$$M(u) = a' M_{\mathcal{S}}(b') c',$$

for all factorizations $u = a' \cdot b' \cdot c'$ of u such that $a' \leq a$, $b' \leq b$, $c' \leq c$, and $\ell(u) = \ell(a') + \ell(b') + \ell(c')$.

- *Moreover, let SM_w be the set of systems $\mathcal{S} = (J, H, M)$ for w such that*
 - $M_{\mathcal{S}}(w) \triangleleft w$,
 - for all $r \in S$,

$$M_{\mathcal{S}}(r) = sr \text{ if and only if } r \in J.$$

Then the set

$$\{M_{\mathcal{S}} : \mathcal{S} \in SM_w\}$$

is a complete list of all distinct special matchings of w .

3. NOTATION, DEFINITIONS AND BACKGROUND

In this section, we collect some notation, definitions, and results that are used in the rest of this work.

We follow [15, Chapter 3] for undefined notation and terminology concerning partially ordered sets. In particular, given x and y in a partially ordered set P , we say that y covers x and we write $x \triangleleft y$ if the interval $\{z \in P : x \leq z \leq y\}$, denoted $[x, y]$, has two elements, x and y . We say that a poset P is *graded* if P has a minimum and there is a function $\rho : P \rightarrow \mathbb{N}$ such that $\rho(\hat{0}) = 0$ and $\rho(y) = \rho(x) + 1$ for all $x, y \in P$ with $x \triangleleft y$. (This definition is slightly different from the one given in [15], but is more convenient for our purposes.) We call ρ the *rank function* of P . The *Hasse diagram* of P is any drawing of the graph having P as vertex set and $\{\{x, y\} \in \binom{P}{2} : \text{either } x \triangleleft y \text{ or } y \triangleleft x\}$ as edge set, with the convention that, if $x \triangleleft y$, then the edge $\{x, y\}$ goes upward from x to y . When no confusion arises, we make no distinction between the Hasse diagram and its underlying graph.

A *matching* of a poset P is an involution $M : P \rightarrow P$ such that $\{v, M(v)\}$ is an edge in the Hasse diagram of P , for all $v \in P$. A matching M of P is *special* if

$$u \triangleleft v \implies M(u) \leq M(v),$$

for all $u, v \in P$ such that $M(u) \neq v$.

The following result (see [11, Lemma 2.6]) is an immediate generalization of the well-known Lifting Property for Coxeter groups (see, for example, [1, Proposition 2.2.7] or [9, Proposition 5.9]).

Lemma 3.1. *(Lifting Property for special matchings) Let M be a special matching of a locally finite ranked poset P , and let $u, v \in P$, with $u \leq v$. Then*

- (1) if $M(v) \triangleleft v$ and $M(u) \triangleleft u$ then $M(u) \leq M(v)$,
- (2) if $M(v) \triangleright v$ and $M(u) \triangleright u$ then $M(u) \leq M(v)$,
- (3) if $M(v) \triangleleft v$ and $M(u) \triangleright u$ then $M(u) \leq v$ and $u \leq M(v)$.

We follow [1] for undefined Coxeter groups notation and terminology.

Given a Coxeter system (W, S) and $s, r \in S$, we denote by $m_{s,r}$ the order of the product sr . Given $w \in W$, we denote by $\ell(w)$ the length of w with respect to S , and by $D_R(w)$ and $D_L(w)$ the sets $\{s \in S: \ell(ws) < \ell(w)\}$ and $\{s \in S: \ell(sw) < \ell(w)\}$, respectively. We call the elements of $D_R(w)$ and $D_L(w)$, respectively, the *right descents* and the *left descents* of w . We denote by e the identity of W , and we let $T = \{wsw^{-1}: w \in W, s \in S\}$ be the set of *reflections* of W .

The Coxeter group W is partially ordered by *Bruhat order* (see, for example, [1, §2.1] or [9, §5.9]), which is denoted by \leq . The Bruhat order is the partial order whose covering relation \triangleleft is as follows: if $u, v \in W$, then $u \triangleleft v$ if and only if $u^{-1}v \in T$ and $\ell(u) = \ell(v) - 1$. There is a well-known characterization of Bruhat order on a Coxeter group (usually referred to as the *Subword Property*) that we use repeatedly in this work, often without explicit mention. We recall it here for the reader's convenience.

By a *subword* of a word $s_1s_2\cdots s_q$ (where we use the symbol “-” to separate letters in a word in the alphabet S) we mean a word of the form $s_{i_1}s_{i_2}\cdots s_{i_k}$, where $1 \leq i_1 < \cdots < i_k \leq q$. If $w \in W$ then a *reduced expression* for w is a word $s_1s_2\cdots s_q$ such that $w = s_1s_2\cdots s_q$ and $\ell(w) = q$. When no confusion arises, we also say in this case that $s_1s_2\cdots s_q$ is a reduced expression for w .

Theorem 3.2 (Subword Property). *Let $u, w \in W$. Then the following are equivalent:*

- $u \leq w$ in the Bruhat order,
- every reduced expression for w has a subword that is a reduced expression for u ,
- there exists a reduced expression for w having a subword that is a reduced expression for u .

A proof of the preceding result can be found in [1, §2.2] or [9, §5.10]. It is well known that W , partially ordered by Bruhat order, is a graded poset having ℓ as its rank function.

We recall that two reduced expressions of an element are always linked by a sequence of *braid moves*, where a braid move consists in substituting a factor $s-t-s\cdots$ ($m_{s,t}$ letters) with a factor $t-s-t\cdots$ ($m_{s,t}$ letters), for some $s, t \in S$. We also recall that, if $w \in W$ and $s, t \in D_R(w)$, then there exists a reduced expression for w of the form

$$s_1\cdots s_k \underbrace{s-t-s\cdots}_{m_{s,t} \text{ letters}}$$

Given $u, v \in W$, we write $u \cdot v$ instead of simply uv when $\ell(uv) = \ell(u) + \ell(v)$ and we want to stress this additivity. On the other hand, when we write uv , we mean that $\ell(uv)$ can be either $\ell(u) + \ell(v)$ or smaller.

For each subset $J \subseteq S$, we denote by W_J the parabolic subgroup of W generated by J , and by W^J the set of minimal coset representatives:

$$W^J = \{w \in W: D_R(w) \subseteq S \setminus J\}.$$

The following is a useful factorization of W (see, for example, [1, §2.4] or [9, §1.10]).

Proposition 3.3. *If $J \subseteq S$, then*

- (i) *every $w \in W$ has a unique factorization $w = w^J w_J$ with $w^J \in W^J$ and $w_J \in W_J$;*
- (ii) *for this factorization, $\ell(w) = \ell(w^J) + \ell(w_J)$.*

There are, of course, left versions of the above definition and result. Namely, if we let

$${}^J W = \{w \in W : D_L(w) \subseteq S \setminus J\} = (W^J)^{-1},$$

then every $w \in W$ can be uniquely factorized $w = {}_J w \cdot {}^J w$, where ${}_J w \in W_J$, ${}^J w \in {}^J W$, and $\ell(w) = \ell({}_J w) + \ell({}^J w)$.

We also need the two following well-known results (a proof of the first can be found in [8, Lemma 7], while the second is easy to prove).

Proposition 3.4. *Let $J \subseteq S$ and $w \in W$. The set $W_J \cap [e, w]$ has a unique maximal element $w_0(J)$, so $W_J \cap [e, w]$ is the interval $[e, w_0(J)]$.*

We note that the term $w_0(J)$ denotes something different in [1] and that, if $J = \{s, t\}$, we adopt the lighter notation $w_0(s, t)$ to mean $w_0(\{s, t\})$.

Proposition 3.5. *If $J \subseteq S$ and $v, w \in W$, with $v \leq w$, then $v^J \leq w^J$ and ${}^J v \leq {}^J w$.*

The following elementary results are needed later (see [5, Lemmas 4.6, 4.7, and 4.8] for a proof).

Lemma 3.6. *Fix $H \subseteq S$ and $u = u^H \cdot u_H \in W$. If $j \in D_R(u) \setminus H$, then $j \in D_R(u^H)$.*

Lemma 3.7. *Fix $u \in W$ and $t, j \in S$, with $t \leq u$ and $j \in D_R(u)$. If there exists a reduced expression X for u such that t - j is not a subword of X , then t and j commute.*

Lemma 3.8. *Let $u \in W$, $s, t \in S$ with $m_{s,t} \geq 3$, $t \not\leq u^{\{s,t\}}$ and $ts \leq u_{\{s,t\}}$. If $j \in D_R(u) \setminus \{s, t\}$, then j commutes with s and t .*

We are interested in the special matchings of a Coxeter group W (to be precise, of intervals in W) partially ordered by Bruhat order. Given $w \in W$, we say that M is a matching of w if M is a matching of the lower Bruhat interval $[e, w]$. If $s \in D_R(w)$ (respectively, $s \in D_L(w)$), we define a matching ρ_s (respectively, λ_s) of w by $\rho_s(u) = us$ (respectively, $\lambda_s(u) = su$) for all $u \leq w$. From the ‘‘Lifting Property’’ (see, for example, [1, Proposition 2.2.7] or [9, Proposition 5.9]), it easily follows that ρ_s (respectively, λ_s) is a special matching of w . We call a matching M of w a *left multiplication matching* if there exists $s \in S$ such that $M = \lambda_s$ on $[e, w]$, and we call it a *right multiplication matching* if there exists $s \in S$ such that $M = \rho_s$ on $[e, w]$.

Definition 3.9. A right system for $w \in W$ is a quadruple $\mathcal{R} = (J, s, t, M_{st})$ such that:

- R1. $J \subseteq S$, $s \in J$, $t \in S \setminus J$, and M_{st} is a special matching of $w_0(s, t)$ such that $M_{st}(e) = s$ and $M_{st}(t) = ts$;
- R2. $(u^J)^{\{s,t\}} \cdot M_{st}\left((u^J)_{\{s,t\}} \cdot {}_{\{s\}}(u_J)\right) \cdot {}^{\{s\}}(u_J) \leq w$, for all $u \leq w$;
- R3. if $r \in J$ and $r \leq w^J$, then r and s commute;
- R4. (a) if $s \leq (w^J)^{\{s,t\}}$ and $t \leq (w^J)^{\{s,t\}}$, then $M_{st} = \rho_s$,
 (b) if $s \leq (w^J)^{\{s,t\}}$ and $t \not\leq (w^J)^{\{s,t\}}$, then M_{st} commutes with λ_s ,
 (c) if $s \not\leq (w^J)^{\{s,t\}}$ and $t \leq (w^J)^{\{s,t\}}$, then M_{st} commutes with λ_t ;

R5. if $s \leq \{s\}(w_J)$, then M_{st} commutes with ρ_s on $[e, w_0(s, t)]$.

Definition 3.10. A left system for $w \in W$ is a quadruple $\mathcal{L} = (J, s, t, M_{st})$ such that:

- L1. $J \subseteq S$, $s \in J$, $t \in S \setminus J$, and M_{st} is a special matching of $w_0(s, t)$ such that $M_{st}(e) = s$ and $M_{st}(t) = st$;
- L2. $(Ju)^{\{s\}} \cdot M_{st} \left((Ju)_{\{s\}} \cdot \{s, t\}(Ju) \right) \cdot \{s, t\}(Ju) \leq w$, for all $u \leq w$;
- L3. if $r \in J$ and $r \leq {}^J w$, then r and s commute;
- L4. (a) if $s \leq \{s, t\}({}^J w)$ and $t \leq \{s, t\}({}^J w)$, then $M_{st} = \lambda_s$,
 (b) if $s \leq \{s, t\}({}^J w)$ and $t \not\leq \{s, t\}({}^J w)$, then M_{st} commutes with ρ_s ,
 (c) if $s \not\leq \{s, t\}({}^J w)$ and $t \leq \{s, t\}({}^J w)$, then M_{st} commutes with ρ_t ;
- L5. if $s \leq ({}^J w)^{\{s\}}$, then M_{st} commutes with λ_s on $[e, w_0(s, t)]$.

As shown in [5, Lemma 4.3], Properties R5 and L5 are equivalent to the, a priori, more restrictive Properties R5 and L5 appearing in [13]. We also note that Properties R4 (a) and L4 (a) do not appear in the definitions of right and left systems in [5] since they appear there in Lemma 4.2 as a consequence of the other properties. Nevertheless we realized that there are a few (irrelevant) exceptions to the statement of [5, Lemma 4.2], so it is more correct to add Properties R4 (a) and L4 (a) in the present definition of right and left systems.

With a right system $\mathcal{R} = (J, s, t, M_{st})$ for w , we associate the matching $M_{\mathcal{R}}$ on $[e, w]$ sending $u \in [e, w]$ to

$$M_{\mathcal{R}}(u) = (u^J)^{\{s, t\}} \cdot M_{st} \left((u^J)_{\{s, t\}} \cdot \{s\}(u_J) \right) \cdot \{s\}(u_J).$$

Symmetrically, we associate with any left system \mathcal{L} for w the matching ${}_{\mathcal{L}}M$ sending $u \in [e, w]$ to ${}_{\mathcal{L}}M(u) = (M_{\mathcal{L}}(u^{-1}))^{-1}$, where $M_{\mathcal{L}}$ is the matching on $[e, w^{-1}]$ associated to \mathcal{L} as a right system for w^{-1} .

The maps $M_{\mathcal{R}}$ and ${}_{\mathcal{L}}M$ are actually matchings, as shown in [5, Corollary 4.10].

The following is the main result of [5] (see [5, Theorem 5.9]).

Theorem 3.11. *Let (W, S) be a Coxeter system. If $w \in W$ then*

- (1) *the matching associated with a right or left system of w is special;*
- (2) *a special matching of w is the matching associated with a right or left system of w ;*
- (3) *if $\mathcal{R} = (J, s, t, M_{st})$ and $\mathcal{R}' = (J', s', t', M_{s't'})$ are right systems, then $M_{\mathcal{R}} = M_{\mathcal{R}'}$ if and only if $s = s'$, $J \cup C_s = J' \cup C_{s'}$ and one of the following conditions is satisfied:*
 - $M_{st}(u) = us$ for all $u \leq w_0(s, t)$ and $M_{s't'}(u) = us$ for all $u \leq w_0(s', t')$,
 - $t = t'$ and $M_{st} = M_{s't'}$;
- (4) *if $\mathcal{R} = (J, s, t, M_{st})$ is a right system and $\mathcal{L} = (K, s', t', M_{s't'})$ is a left system, then $M_{\mathcal{R}} = {}_{\mathcal{L}}M$ if and only if $s = s'$, $J \cap K \subseteq C_s$, $J \cup K \subseteq S \setminus C_s$, $M_{st} = \rho_s$, $M_{s't'} = \lambda_s$.*

4. PRELIMINARY RESULTS

We fix an arbitrary Coxeter system (W, S) .

For $u, v \in W$, we say that u is a *prefix* of v if $\ell(u) + \ell(u^{-1}v) = \ell(v)$. We similarly define a *suffix*. The proof of the next result is easy and is left to the reader.

Lemma 4.1. *Let $u, w \in W$ be such that u is a prefix of w . If $u \in W_J$, then u is a prefix of ${}_J w$ and ${}_J w$ is a suffix of $u^{-1}w$.*

Lemma 4.2. *Let $w \in W$ and $J, K \subset S$. Then $w^J \in W_K$ if and only if ${}^K w \in W_J$, and in this case $w = w^J \cdot v \cdot {}^K w$ with $\ell(w) = \ell(w^J) + \ell(v) + \ell({}^K w)$ and $v \in W_{J \cap K}$.*

Proof. Assume $w^J \in W_K$. By Lemma 4.1, w^J is a prefix of ${}^K w$ and ${}^K w$ is a suffix of w_J , and in particular ${}^K w \in W_J$. Symmetrically, Lemma 4.1 also implies that, if ${}^K w \in W_J$, then $w^J \in W_K$. The second part is now straightforward. \square

The reader is invited to recall the definitions in Section 2. Given a system $\mathcal{S} = (J, H, M)$ for w , we always denote $M(e)$ by s , the s -complement of J by K and, if $|H| = 2$, we let $t \in S$ be such that $H = \{s, t\}$.

Our target is to show that special matchings can be described in terms of systems in a very precise way.

The first goal is to identify a class of elements admitting \mathcal{S} -factorizations. This class contains the lower interval $[e, w]$. In order to do so, we separate the proof in two cases. Although this seems unavoidable (see Example 4.6), the resulting classification is uniform and self-dual.

Definition 4.3. We say that a system $\mathcal{S} = (J, H, M)$ for w is of the *first* (respectively, *second*) *kind* if $H \subseteq K$ (respectively, $H \subseteq J$).

Note that a system $\mathcal{S} = (J, H, M)$ is of both the first and the second kind if and only if $|H| = 1$.

We now concentrate on systems of the first kind. Given a system $\mathcal{S} = (J, H, M)$ of the first kind for some $w \in W$, we let

$$\begin{aligned} a_{\mathcal{S}}(u) &= (u^J)^H \\ b_{\mathcal{S}}(u) &= (u^J)_H \cdot {}_H(u_J) \\ c_{\mathcal{S}}(u) &= {}^H(u_J) \end{aligned}$$

for every $u \in W$, and

$$W_{\mathcal{S}} = \{u \in W : u_0(H) \leq w_0(H), a_{\mathcal{S}}(u) \leq a_{\mathcal{S}}(w)\}.$$

Observe that $W_{\mathcal{S}}$ is an order ideal which contains the lower Bruhat interval $[e, w]$.

We show next that elements in $W_{\mathcal{S}}$ admit \mathcal{S} -factorizations. We need the following result.

Lemma 4.4. *Let $\mathcal{S} = (J, H, M)$ be a system for w of the first kind. If $u \in W_{\mathcal{S}}$, then*

- (1) $c_{\mathcal{S}}(u) \geq {}^H({}^K u)$,
- (2) $s \leq c_{\mathcal{S}}(u)$ if and only if $s \leq {}^H({}^K u)$.

Proof. (1). Since $a_{\mathcal{S}}(u)b_{\mathcal{S}}(u) \in W_K$, Lemma 4.1 implies that ${}^K u$ is a suffix of $c_{\mathcal{S}}(u)$. Thus $c_{\mathcal{S}}(u) \geq {}^K u = {}^H({}^K u)$.

(2). The “if part” follows by (1).

Suppose $s \leq c_{\mathcal{S}}(u)$. We already know that ${}^K c_{\mathcal{S}}(u) = {}^K u$ (by Lemma 3.5 together with the fact that ${}^K u$ is a suffix of $c_{\mathcal{S}}(u)$). Hence $c_{\mathcal{S}}(u) = {}_K(c_{\mathcal{S}}(u)) \cdot {}^K u$ and ${}_K(c_{\mathcal{S}}(u)) \in W_J \cap W_K = W_{C_s}$ since $c_{\mathcal{S}}(u) \in W_J$. Thus $s \not\leq {}_K(c_{\mathcal{S}}(u))$ otherwise s would be a left descent of ${}_K(c_{\mathcal{S}}(u))$ (since all letters in ${}_K(c_{\mathcal{S}}(u))$ commute with s), so also a left descent of $c_{\mathcal{S}}(u)$, which is impossible since $c_{\mathcal{S}}(u) \in {}^H W$. Hence $s \leq {}^K u$ and we get the assertion since ${}^K u = {}^H({}^K u)$ (by hypothesis, \mathcal{S} is of the first kind, so $H \subseteq K$). \square

Lemma 4.5. *Let $\mathcal{S} = (J, H, M)$ be a system for w of the first kind. If $u \in W_{\mathcal{S}}$, then*

$$u = a_{\mathcal{S}}(u) \cdot b_{\mathcal{S}}(u) \cdot c_{\mathcal{S}}(u)$$

is an \mathcal{S} -factorization for u .

Proof. We show that the 4 properties of the definition of an \mathcal{S} -factorization are satisfied.

- The additivity of lengths in the factorization $u = a_{\mathcal{S}}(u) \cdot b_{\mathcal{S}}(u) \cdot c_{\mathcal{S}}(u)$ is clear by Proposition 3.3.
- Since $a_{\mathcal{S}}(u) = (u^J)^H \leq (w^J)^H \leq w^J$ and $w^J \in W_K$ by Property [S1], we have $a_{\mathcal{S}}(u) \in (W_K)^H$, and $a_{\mathcal{S}}(u)$ satisfies also the other desired properties since $(w^J)^H$ does by Property [S2].
- Clearly $b_{\mathcal{S}}(u) \in W_H$.
- Since $c_{\mathcal{S}}(u) = {}^H(u_J) \leq u_J$, we have $c_{\mathcal{S}}(u) \in {}^H(W_J)$. Since \mathcal{S} is of the first kind we have $H \cap J = \{s\}$. By Lemma 4.4, if $s \leq c_{\mathcal{S}}(u)$ then $s \leq {}^H({}^K u)$, and the desired property follows by Property [S2].

\square

Example 4.6. We show in this example that the assumption that \mathcal{S} is of the first kind is necessary for Lemma 4.5. We let $J = S = \{r, s, t\}$, $H = \{s, t\}$, $m_{s,r} = 2$, $m_{r,t} \geq 3$, $m_{s,t} \geq 5$ and $w = rtsts$. We let M be any matching of $[e, tsts]$ which is not a multiplication matching. One can check that (J, H, M) is a system for w of the second kind. Nevertheless one can also check that $c_{\mathcal{S}}(w) = rtsts$, so $\text{Supp}_H(c_{\mathcal{S}}(w)) = \{s, t\}$.

For a system $\mathcal{S} = (J, H, M)$ of the first kind and for all $u \in W_{\mathcal{S}}$, we let

$$M_{\mathcal{S}}(u) = a_{\mathcal{S}}(u)M(b_{\mathcal{S}}(u))c_{\mathcal{S}}(u).$$

We observe that the restriction of $M_{\mathcal{S}}$ to $[e, w_0(H)]$ agrees with M . Our next target is to show that $M_{\mathcal{S}}$ is well behaved with respect to the factorization $u = a_{\mathcal{S}}(u) \cdot b_{\mathcal{S}}(u) \cdot c_{\mathcal{S}}(u)$: we need the following preliminary result where, if $A, B \subset W$, we let

$$A \cdot B = \{ab : a \in A, b \in B\}.$$

Lemma 4.7. *Let $\mathcal{S} = (J, H, M)$ be a system for w of the first kind. If $u \in W_{\mathcal{S}}$, then*

$$a_{\mathcal{S}}(u) M(b_{\mathcal{S}}(u)) \in W^J \cdot W_s.$$

where $s = M(e)$.

Proof. For short, we let $a = a_{\mathcal{S}}(u)$, $b = b_{\mathcal{S}}(u)$, and $c = c_{\mathcal{S}}(u)$. By definition, $a \cdot b = u^J \cdot {}_s(u_J) \in W^J \cdot W_s$. The result holds if $M(b) = bs$. This happens, in particular, if $|H| = 1$ (that is, $H = \{s\}$).

We can therefore assume $M(b) \neq bs$ and $|H| = 2$, $H = \{s, t\}$. Note that, in particular, $t \leq M(b)$. We let $\varepsilon \in \{e, s\}$ be such that $s \notin D_R(a \cdot M(b)\varepsilon)$, and we show that

$a \cdot M(b)\varepsilon \in W^J$. Otherwise, there exists $j \in J \setminus \{s\}$ such that $j \in D_R(a \cdot M(b)\varepsilon)$. We claim that

- (1) $j \in D_R(a)$,
- (2) j commutes with s and t .

Lemma 3.6 immediately implies that (1) holds. In particular, $j \leq a = (u^J)^H \leq (w^J)^H \leq w^J \in W_K$, so $j \in K \cap J = C_s$, and hence j commutes with s . We also need to show that j commutes with t . If $t \not\leq a$, we may conclude using Lemma 3.7 applied to $aM(b)\varepsilon$. Assume $t \leq a$, so M commutes with λ_t by the definition of \mathcal{S} -factorization. If $M(b)\varepsilon = t$, then

$$e = \lambda_t(M(b)\varepsilon) = \lambda_t(M(b))\varepsilon = M(\lambda_t(b))\varepsilon = M(tb)\varepsilon,$$

so $\varepsilon = M(tb)$, $s\varepsilon = M(\varepsilon) = tb$ and $b = ts\varepsilon = t\varepsilon s = M(b)s$; thus $M(b) = bs$, which contradicts our assumption. Therefore, $M(b)\varepsilon \neq t$; note also that $M(b)\varepsilon \neq e$ since $M(b) \neq bs$. Since $s \notin D_R(M(b)\varepsilon)$, we have $st \leq M(b)\varepsilon$ and $st \neq ts$. Moreover, since $t \leq a$, we have $s \not\leq a$, since $a \cdot b \cdot c$ is an \mathcal{S} -factorization, and we can conclude using Lemma 3.8 applied to $aM(b)\varepsilon$.

The claim is proved: so we have that $j \neq s$ satisfies (1) and (2) above. Since $ab \in W^J \cdot W_s$, we have that there exists $\varepsilon' \in W_s$ such that $x = a \cdot b\varepsilon' \in W^J$ (we have $\ell(x) = \ell(a) + \ell(b\varepsilon')$ since $x^H = a$ and $x_H = b\varepsilon'$). But $\ell(xj) = \ell(ajb\varepsilon') = \ell(aj) + \ell(b\varepsilon') = \ell(a) - 1 + \ell(b\varepsilon') = \ell(x) - 1$. In particular $j \in D_R(x)$, which is a contradiction since $x \in W^J$. \square

Proposition 4.8. *Let $\mathcal{S} = (J, H, M)$ be a system for w of the first kind. If $u \in W_{\mathcal{S}}$, then*

- $a_{\mathcal{S}}(M_{\mathcal{S}}(u)) = a_{\mathcal{S}}(u)$;
- $b_{\mathcal{S}}(M_{\mathcal{S}}(u)) = M(b_{\mathcal{S}}(u))$;
- $c_{\mathcal{S}}(M_{\mathcal{S}}(u)) = c_{\mathcal{S}}(u)$.

Furthermore, $M_{\mathcal{S}}(u) \in W_{\mathcal{S}}$.

Proof. For short, we let $a = a_{\mathcal{S}}(u)$, $b = b_{\mathcal{S}}(u)$, and $c = c_{\mathcal{S}}(u)$.

Let $\varepsilon \in W_s$ be such that $aM(b)\varepsilon \in W^J$ (see Lemma 4.7). Thus $M_{\mathcal{S}}(u)^J = aM(b)\varepsilon$ and $M_{\mathcal{S}}(u)_J = \varepsilon c$. From this parabolic decomposition, it follows that $(M_{\mathcal{S}}(u)^J)^H = a$, $(M_{\mathcal{S}}(u)^J)_H = M(b)\varepsilon$, ${}_H(M_{\mathcal{S}}(u)_J) = \varepsilon$ and ${}^H(M_{\mathcal{S}}(u)_J) = c$, and the three assertions are proved.

To show that $M_{\mathcal{S}}(u) \in W_{\mathcal{S}}$ we only have to verify that $M_{\mathcal{S}}(u)_0(H) \leq w_0(H)$. This follows from the observation that $u_0(H) = \varepsilon \cdot b_{\mathcal{S}}(u) \cdot \varepsilon'$ with $\varepsilon, \varepsilon' \in H \cup \{e\}$. Indeed we have $M_{\mathcal{S}}(u)_0(H) \in \{\varepsilon \cdot M(b_{\mathcal{S}}(u)) \cdot \varepsilon', M(b_{\mathcal{S}}(u)) \cdot \varepsilon', \varepsilon \cdot M(b_{\mathcal{S}}(u)), M(b_{\mathcal{S}}(u))\}$, so the result follows since in this situation the Bruhat interval $[e, w_0(H)]$ is closed under M, λ_{ε} , and $\rho_{\varepsilon'}$. \square

Corollary 4.9. *Let $\mathcal{S} = (J, H, M)$ be a system for w of the first kind. Then $M_{\mathcal{S}}$ defines a matching on $W_{\mathcal{S}}$. Moreover, for all $u \in W_{\mathcal{S}}$, we have $u \triangleleft M_{\mathcal{S}}(u)$ if and only if $b_{\mathcal{S}}(u) \triangleleft M(b_{\mathcal{S}}(u))$.*

Proof. This follows by Proposition 4.8. \square

Some comments for systems of the second kind are in order. The situation here is right-to-left symmetrical with respect to systems of the first kind. If $\mathcal{S} = (J, H, M)$ is a system for w of the second kind, one can define the decomposition

$$\begin{aligned} a'_\mathcal{S}(u) &= ({}_K u)^H \\ b'_\mathcal{S}(u) &= ({}_K u)^H \cdot^H ({}^K u) \\ c'_\mathcal{S}(u) &= {}^H ({}^K u) \end{aligned}$$

for all $u \in W$. We let

$$W'_\mathcal{S} = \{u \in W : u_0(H) \leq w_0(H), c'_\mathcal{S}(u) \leq c'_\mathcal{S}(w)\}.$$

Left versions of Lemmas 4.4, 4.5, and 4.7, Proposition 4.8, and Corollary 4.9 hold. In particular, for all $u \in W'_\mathcal{S}$, the factorization $u = a'_\mathcal{S}(u) \cdot b'_\mathcal{S}(u) \cdot c'_\mathcal{S}(u)$ is an \mathcal{S} -factorization. Moreover, the map

$$M_\mathcal{S}(u) = a'_\mathcal{S}(u) \cdot M(b'_\mathcal{S}(u)) \cdot c'_\mathcal{S}(u)$$

defines a matching on $W'_\mathcal{S}$.

The problem of consistency of the two definitions of $M_\mathcal{S}$ if \mathcal{S} is both of the first and second kind disappears after the following fundamental result.

Theorem 4.10. *Let $\mathcal{S} = (J, H, M)$ be a system for w , and $u \leq w$ (or, more generally, $u \in W_\mathcal{S}$ if \mathcal{S} is of the first kind, $u \in W'_\mathcal{S}$ if \mathcal{S} is of the second kind). If $u = a \cdot b \cdot c$ is any \mathcal{S} -factorization for u , then*

$$M_\mathcal{S}(u) = a \cdot M(b) \cdot c,$$

(so $a \cdot M(b) \cdot c$ does not depend on the chosen factorization), and $u \triangleleft M_\mathcal{S}(u)$ if and only if $b \triangleleft M(b)$.

Consequently, $M_\mathcal{S}(u) = a \cdot M(b) \cdot c$ is an \mathcal{S} -factorization for $M_\mathcal{S}(u)$.

Proof. The main idea of this proof is already present in the proof of [5, Theorem 5.7], and we show here that it is still valid in this more general context with weaker hypotheses. We assume that \mathcal{S} is a system of the first kind, the argument for a system of the second kind being entirely similar. Let

$$\varepsilon = \begin{cases} e & \text{if } bs > b; \\ s & \text{if } bs < b. \end{cases}$$

We prove the two statements $M_\mathcal{S}(u) = a \cdot M(b) \cdot c$ and $u \triangleleft M_\mathcal{S}(u)$ if and only if $b \triangleleft M(b)$ by induction on $\ell(a)$. We first observe that if $a = a_\mathcal{S}(u)$, $b = b_\mathcal{S}(u)$ and $c = c_\mathcal{S}(u)$, then the results follow by the definition of $M_\mathcal{S}$ and Proposition 4.8. If $\ell(a) = 0$, then $u^J = b\varepsilon$ and $u_J = \varepsilon c$, and therefore $a_\mathcal{S}(u) = e = a$, $b_\mathcal{S}(u) = b$ and $c_\mathcal{S}(u) = c$, and we are done.

Assume $\ell(a) \geq 1$. If $ab\varepsilon \in W^J$, that is, $ab\varepsilon = u^J$, then $a_\mathcal{S}(u) = a$, $b_\mathcal{S}(u) = b$ and $c_\mathcal{S}(u) = c$, and the results again follow.

If $ab\varepsilon \notin W^J$, there exists $r \in J \cap K = C_s$ such that $r \in D_R(ab\varepsilon)$, and we first claim that $r \in D_R(a)$. If $b\varepsilon = e$ this is trivial, otherwise we have $|H| = 2$, $H = \{s, t\}$ and $t \in D_R(ab\varepsilon)$. Note that $s \notin D_R(ab\varepsilon)$ by Proposition 3.3. Hence $r \neq s$. By Lemma 3.6, we have $r \in D_R(a)$.

Now, if r commutes with both s and t , then $abc = (ar)b(rc)$; this triplet is an \mathcal{S} -factorization and the result clearly follows by induction.

Therefore, we can assume that r does not commute with t . If $b\varepsilon = e$, that is, $b \in \{e, s\}$, then $abc = (ar)(b)(rc)$; we observe that $s \notin D_R(ar)$ since otherwise $s \in D_R(a)$, and similarly $s \notin D_L(rc)$. If $t \notin D_R(ar)$, this triplet is an \mathcal{S} -factorization and the result follows by induction. If $t \in D_R(ar)$ then $s \not\leq a$ and M commutes with λ_t by definition of an \mathcal{S} -factorization, and again the result follows by induction by considering the \mathcal{S} -factorization $(art)(tb)(rc)$ since

$$(art)M(tb)(rc) = (art)tM(b)(rc) = (ar)M(b)(rc) = aM(b)c.$$

The second statement follows since $M(b) \neq tb$ in this case and so $M(b) \triangleleft b$ if and only if $M(tb) \triangleleft tb$.

We are therefore reduced to the case $b\varepsilon \neq e$, so t is a right descent of $b\varepsilon$; hence both t and r are right descents of $ab\varepsilon$. In particular, we have a reduced expression for $ab\varepsilon$ which ends with $t-r-t$ and so $tr \leq ab\varepsilon$. This forces $t \leq a$, so, by definition of an \mathcal{S} -factorization, M commutes with λ_t and $s \not\leq a$: in particular, $M(t) = ts$ otherwise $M \circ \lambda_t(e)$ would not be equal to $\lambda_t \circ M(e)$.

Now, if $b\varepsilon = t$ we let $m = trt \cdots$ ($m_{t,r}$ factors): so $ab\varepsilon = x \cdot m$ with $\ell(ab\varepsilon) = \ell(x) + \ell(m)$ and therefore, since $b\varepsilon = t$, we have $a = x \cdot mt$ with $\ell(a) = \ell(x) + \ell(m) - 1$. Moreover, since r and t are both right descents of $xm = ab\varepsilon$, we have that $\ell(xmrt) = \ell(xm) - 2 = \ell(a) - 1$. Thus

$$u = xm\varepsilon c = xmrt t\varepsilon c = (xmrt)(t\varepsilon)(rc) = (xmrt)b(rc).$$

This factorization is an \mathcal{S} -factorization of u (recall that $s \not\leq a$, so $s \not\leq xmrt$) and hence we can conclude by induction (as we have already observed, $\ell(xmrt) = \ell(a) - 1$) that

$$\begin{aligned} M_{\mathcal{S}}(u) &= xmrtM(b)rc = xmrt t\varepsilon s rc \\ &= xm\varepsilon sc = absc \\ &= aM(b)c, \end{aligned}$$

as, clearly, since $b\varepsilon = t$ we have either $b = ts$ or $b = t$ and in both cases $M(b) = bs$.

We are left with the case $st \leq b\varepsilon$. We observe that $ab\varepsilon$ has a reduced expression which ends with $t-r-t$ and a reduced expression which ends with $s-t$; therefore $ab\varepsilon t$ has a reduced expression which ends with $t-r$ and a reduced expression which ends with s . In order to transform one of these two reduced expressions to the other using braid moves, we necessarily have to perform a braid relation between s and t . Therefore, $tst \cdots$ ($m_{s,t}$ factors) is smaller or equal than $ab\varepsilon t$. As we already know $s \not\leq a$, we deduce $b\varepsilon t \geq sts \cdots$ ($m_{s,t} - 1$ factors). But $b\varepsilon t \in W_H$ and $t \notin D_R(b\varepsilon t)$ by construction, so $b\varepsilon t = sts \cdots$ ($m_{s,t} - 1$ factors). Therefore, $b\varepsilon = sts \cdots$ ($m_{s,t}$ factors). This is a contradiction since $s \notin D_R(b\varepsilon)$.

In order to show that $M_{\mathcal{S}}(u) = a \cdot M(b) \cdot c$ is an \mathcal{S} -factorization for $M_{\mathcal{S}}(u)$, observe that all axioms of the definition of an \mathcal{S} -factorization follows immediately by the fact that $a \cdot b \cdot c$ is an \mathcal{S} -factorization, except the first one, $\ell(M_{\mathcal{S}}(u)) = \ell(a) + \ell(M(b)) + \ell(c)$, which follows by the fact that

$$u \triangleleft M_{\mathcal{S}}(u) \iff b \triangleleft M(b).$$

The proof is complete. \square

5. MAIN RESULTS

This section contains the main results of this work. In particular, given a system \mathcal{S} for w , we present a simple way to compute $M_{\mathcal{S}}(u)$ for every $u \in [e, w]$, and use this to show that $M_{\mathcal{S}}$ is a special matching. We finally observe, using known results, that all special matchings are of this form and show the desired classification of special matchings.

The next result shows that the action of a matching $M_{\mathcal{S}}$ on an element u can sometimes be calculated also if we do not have an \mathcal{S} -factorization of u . The first part is used in this section to prove that $M_{\mathcal{S}}$ is a special matching, while the second part is needed only in Section 6.

Theorem 5.1. *Let $\mathcal{S} = (J, H, M)$ be a system for w and $w = a \cdot b \cdot c$ be an \mathcal{S} -factorization of w . Let $u \leq w$ and $u = a' \cdot b' \cdot c'$ with $a' \leq a$, $b' \leq b$ and $c' \leq c$. Assume that*

$$u = a' \cdot b' \cdot c' \quad (\text{that is, } \ell(u) = \ell(a) + \ell(b) + \ell(c)),$$

or $M_{\mathcal{S}}$ is a special matching of w and there exists $r \in D_L(a)$ with

$$ru = (ra') \cdot b' \cdot c' \quad (\text{that is, } \ell(ru) = \ell(ra') + \ell(b') + \ell(c')).$$

Then

$$M_{\mathcal{S}}(u) = a' M(b') c'.$$

Proof. We first assume that $u = a' \cdot b' \cdot c'$. We recall that by definition of \mathcal{S} -factorization we have $|\text{Supp}_H(a)| \leq 1$ and $|\text{Supp}_H(c)| \leq 1$. We let

$$\eta = \begin{cases} e & \text{if } H \cap D_R(a') = \emptyset; \\ \alpha & \text{if } H \cap D_R(a') = \{\alpha\}. \end{cases}$$

and we let

$$\varepsilon = \begin{cases} e & \text{if } H \cap D_L(c') = \emptyset; \\ \beta & \text{if } H \cap D_L(c') = \{\beta\}. \end{cases}$$

Now we observe that the factorization

$$u = (a'\eta) \cdot (\eta b'\varepsilon) \cdot (\varepsilon c')$$

is an \mathcal{S} -factorization and that M commutes with the multiplication on the left by η (since $\eta \leq a' \leq a$) and the multiplication on the right by ε (since $\varepsilon \leq c' \leq c$). Therefore, by Theorem 4.10

$$M_{\mathcal{S}}(u) = a'\eta \cdot M(\eta b'\varepsilon) \cdot \varepsilon c' = a'\eta \cdot \eta M(b')\varepsilon \cdot \varepsilon c' = a' \cdot M(b') \cdot c'.$$

Now we assume that $ru = (ra') \cdot b' \cdot c'$ and that $M_{\mathcal{S}}$ is a special matching of w . By the first part, since $ra' \leq a$ by the Lifting Property, we have $M_{\mathcal{S}}(ru) = ra' M(b') c'$, so if the result fails we have $M_{\mathcal{S}}(ru) \neq r M_{\mathcal{S}}(u)$. Applying if necessary $M_{\mathcal{S}}$ and λ_r repeatedly to u we can find an element $x \leq w$ such that $M_{\mathcal{S}}(x) \triangleleft x$, $rx \triangleleft x$ and $M_{\mathcal{S}}(rx) \neq r M_{\mathcal{S}}(x)$. We consider an element x of smallest length satisfying these conditions. If $M_{\mathcal{S}}(x)$ and rx are the only coatoms of x , then $x \in W_{\{r, M(e)\}}$, and this is a contradiction since $r \leq a$ and thus $M_{\mathcal{S}}$ commutes with λ_r on $W_{\{r, M(e)\}}$ (by the definition of an \mathcal{S} -factorization

if $r \in H$, by the definition of $M_{\mathcal{S}}$ otherwise). Otherwise there exists an element y such that $y \triangleleft x$, with $y \notin \{M_{\mathcal{S}}(x), rx\}$. By the definition of special matchings we have $M_{\mathcal{S}}(y) \triangleleft y$ and $ry \triangleleft y$. By the minimality of x we also have $M_{\mathcal{S}}(ry) = rM_{\mathcal{S}}(y)$. Two cases occur and the reader is invited to draw a picture of the following such cases:

- $M_{\mathcal{S}}(y) = ry$: in this case we have $ry \triangleleft rx$ with $M_{\mathcal{S}}(ry) \neq rx$ but $M_{\mathcal{S}}(ry) \not\leq M_{\mathcal{S}}(rx)$ contradicting the definition of special matching;
- $M_{\mathcal{S}}(y) \neq ry$: in this case we can similarly show that $M_{\mathcal{S}}(ry) = rM_{\mathcal{S}}(y) \triangleleft rM_{\mathcal{S}}(x)$ with $M_{\mathcal{S}}(rM_{\mathcal{S}}(y)) \neq rM_{\mathcal{S}}(x)$ but $M_{\mathcal{S}}(rM_{\mathcal{S}}(y)) \not\leq M_{\mathcal{S}}(rM_{\mathcal{S}}(x))$ contradicting again the definition of special matching.

□

A right version of the previous result holds (with a similar proof).

Theorem 5.1 implies the following result.

Corollary 5.2. *Let $\mathcal{S} = (J, H, M)$ be a system for w such that $M_{\mathcal{S}}(w) \triangleleft w$. Then $M_{\mathcal{S}}(u) \leq w$ for all $u \in [e, w]$.*

Proof. Let $w = a \cdot b \cdot c$ and $u = a' \cdot b' \cdot c'$ with $a' \leq a$, $b' \leq b$, and $c' \leq c$. Theorem 5.1 implies $M_{\mathcal{S}}(u) = a'M(b')c'$. If $M(b') \triangleleft b'$ then the result is trivial. Otherwise, we have

$$M(b') \leq b$$

by the Lifting Property for the special matching M (see Lemma 3.1), since $b' \leq b$ and $M(b) \triangleleft b$ (being $M_{\mathcal{S}}(w) < w$). We conclude that $a' \cdot M(b') \cdot c' \leq w$ by the subword property. □

The next result tells us that if $H = \{s, t\}$ then we can always assume that $M(t) = ts$ if and only if $t \in K$.

Proposition 5.3. *Let $\mathcal{S} = (J, H, M)$ be a system for w , and $H = \{s, t\}$. If $t \notin J$ then one of the following applies:*

- $M(t) = ts$;
- $t \not\leq (w^J)^H$, $\mathcal{S}' = (J \cup \{t\}, H, M)$ is a system for w , and $M_{\mathcal{S}} = M_{\mathcal{S}'}$.

Symmetrically, if $t \in J$ then one of the following applies:

- $M(t) = st$;
- $t \not\leq {}^H(Kw)$, $\mathcal{S}' = (J \setminus \{t\}, H, M)$ is a system for w , and $M_{\mathcal{S}} = M_{\mathcal{S}'}$.

Proof. We only prove the first statement.

We recall that s and t do not commute. For short, let $J' = J \cup \{t\}$. Clearly \mathcal{S}' satisfies Property [S0] since \mathcal{S} does.

If $t \leq (w^J)^H$ then M commutes with λ_t and thus $M(t) = M \circ \lambda_t(e) = \lambda_t \circ M(e) = ts$. Hence we can assume that $t \not\leq (w^J)^H$. Since

$$w = (w^J)^H \cdot (w^J)_H \cdot w_J$$

with $(w^J)_H \cdot w_J \in W_{J'}$, we deduce that $w^{J'} \leq (w^J)^H$, so $t \not\leq w^{J'} = (w^{J'})^H$, and if $s \leq (w^{J'})^H$ then $s \leq (w^J)^H$; thus \mathcal{S}' satisfies the first part of Property [S2]. Moreover, we have $w^{J'} \in W_{K' \setminus \{t\}} = W_{K'}$, where K' is the s -complement of J' ; thus \mathcal{S}' satisfies Property [S1].

In order to show that \mathcal{S}' satisfies the second part of Property [S2], we prove that $t \not\leq H(K'w)$ and that $s \leq H(K'w)$ implies $s \leq H(Kw)$ (this is enough since \mathcal{S} satisfies Property [S2]).

Since \mathcal{S} is a system for w , Lemma 4.2 implies

$$w = w^J \cdot v \cdot {}^K w = (w^J)^H \cdot (w^J)_H \cdot v \cdot {}^K w,$$

with $v \in W_{C_s}$. We set, for simplicity, $y = (w^J)_H \cdot v \cdot {}^K w$, so $w = (w^J)^H \cdot y$. Let

$$\varepsilon = \begin{cases} s & \text{if } s \in D_L(v) \\ e & \text{otherwise.} \end{cases}$$

Note $t \not\leq {}^K w$ since ${}^K w \in W_J$. The parabolic components of y are as follows: ${}_H y = (w^J)_H \cdot \varepsilon$ and ${}^H y = (\varepsilon v) \cdot {}^K w$ since $t \not\leq (\varepsilon v) \cdot {}^K w$ and $s \notin D_L({}^K w)$ being $s \in K$ (notice that $\varepsilon v \in W_{C_s \setminus \{s\}}$). Finally, $tst \leq {}_H y$ since $tst \leq w$ (because M is not a multiplication matching on $[e, w_0(H)]$) and t does not appear in the other factors of $w = (w^J)^H \cdot {}_H y \cdot {}^H y$.

Therefore, we have that the factorization $w = (w^J)^H \cdot y$ is a factorization of the form $w = x \cdot a$ which satisfies the following properties

- $\ell(w) = \ell(x) + \ell(a)$,
- $x \in W_{K'}$,
- ${}^H a \in W_J$,
- $s \leq {}^H a$ if and only if $s \leq {}^K w$.

We claim that if $w = x \cdot a$ is a factorization satisfying these properties with $\ell(x)$ as big as possible, then $x = {}^{K'} w$ and $a = {}^{K'} w$. From the claim, it follows that $t \not\leq H(K'w) = {}^H a$ and $s \leq H(K'w)$ if and only if $s \leq H(Kw)$, since ${}^H(Kw) = {}^K w$ (because $H \subseteq K$).

Let $x \cdot a$ satisfy the conditions above with $\ell(x)$ as big as possible: we have to show that a does not have left descents in K' . By contradiction, let $r \in K' \cap D_L(a)$.

If $r = s$, then $s \in D_L({}^H a)$ and the factorization $w = (xs) \cdot (sa)$ still satisfies the conditions above (since ${}_H(sa) = s({}^H a)$ and ${}^H(sa) = {}^H a$) and contradicts the maximality of $\ell(x)$.

If $r \neq s$ and $s \notin D_L(a)$, then ${}_H a$ has a prefix t - s - t and by (the left version of) Lemma 3.8 we deduce that r commutes with s and t . Therefore, we have

$$w = (xr) \cdot ({}_H a) \cdot (r {}^H a).$$

We observe that $r \in D_L({}^H a)$, by Lemma 3.6, and that $r {}^H a \in {}^H W$, since r commutes with s and t . In particular the factorization

$$w = (xr) \cdot (ra)$$

still contradicts the maximality of $\ell(x)$ since ${}_H(ra) = {}_H a$ and ${}^H(ra) = r {}^H a$.

We need to show that $M_{\mathcal{S}} = M_{\mathcal{S}'}$. Let $u \leq w$: since \mathcal{S} is a system for w , Lemma 4.2 implies

$$u = u^J \cdot \bar{v} \cdot {}^K u = (u^J)^H \cdot (u^J)_H \cdot \bar{v} \cdot {}^K u$$

with $\bar{v} \in W_{C_s}$. Let

$$\bar{\varepsilon} = \begin{cases} s & \text{if } s \in D_L(\bar{v}) \\ e & \text{otherwise.} \end{cases}$$

Then the factorization

$$u = (u^J)^H \cdot \left((u^J)_H \cdot \bar{\varepsilon} \right) \cdot (\bar{\varepsilon}v \cdot {}^K u)$$

is both an \mathcal{S} -factorization and an \mathcal{S}' -factorization: the unique non immediate property follows by the fact that $t \not\leq \bar{\varepsilon}v \cdot {}^K u$ and the fact that $s \leq \bar{\varepsilon}v \cdot {}^K u$ implies $s \leq {}^K u = {}^H({}^K u) \leq {}^H({}^K w)$. Thus $M_{\mathcal{S}}(u) = M_{\mathcal{S}'}(u)$ by Theorem 4.10. \square

Corollary 5.4. *Let $w \in W$. If M is a special matching of w , then there exists a system \mathcal{S} for w such that $M = M_{\mathcal{S}}$. Vice versa, if \mathcal{S} is a system for w such that $M_{\mathcal{S}}(w) \triangleleft w$, then $M_{\mathcal{S}}$ is a special matching of w .*

Proof. We use Theorem 3.11. Let M be a special matching of w and suppose that M is associated with a right system $\mathcal{R} = (J, s, t, M)$ for w . If M is a multiplication matching on $[e, w_0(s, t)]$, then $\mathcal{S} = (J, \{s\}, M)$ is a system for w and $M_{\mathcal{R}} = M_{\mathcal{S}}$. If M is not a multiplication matching on $[e, w_0(s, t)]$, then $\mathcal{S} = (J, \{s, t\}, M)$ is a system and $M_{\mathcal{R}} = M_{\mathcal{S}}$. The situation is similar in the case when M is associated with a left systems.

Now let $\mathcal{S} = (J, H, M)$ be a system for w such that $M_{\mathcal{S}}(w) \triangleleft w$. If $|H| = 1$ (that is, $H = \{s\}$), then $M_{\mathcal{S}} = M_{\mathcal{R}}$ where $\mathcal{R} = (J, s, t, \rho_s)$ and t is any element in $(S \setminus J) \cup (C_s \setminus \{s\})$. If $|H| = 2$, $H = \{s, t\}$, then it follows by Proposition 5.3 that $M_{\mathcal{S}} = M_{\mathcal{R}}$ with

$$\mathcal{R} = \begin{cases} \text{the right system } (J, s, t, M) & \text{if } t \notin J \text{ and } M(t) = ts \\ \text{the right system } (J \setminus \{t\}, s, t, M) & \text{if } t \in J \text{ and } M(t) = ts \\ \text{the left system } (K \setminus \{t\}, s, t, M) & \text{if } t \notin J \text{ and } M(t) = st \\ \text{the left system } (K, s, t, M) & \text{if } t \in J \text{ and } M(t) = st \end{cases}$$

where K is the s -complement of J . \square

For $w \in W$, let SM_w be the set of systems $\mathcal{S} = (J, H, M)$ for w such that $M_{\mathcal{S}}(w) \triangleleft w$ and such that, for all $\alpha \in H$, $M_{\mathcal{S}}(\alpha) = s\alpha$ if and only if $\alpha \in J$.

Theorem 5.5. *The set*

$$\{M_{\mathcal{S}} : \mathcal{S} \in SM_w\}$$

is a complete list of all distinct special matchings of w .

Proof. Straightforward by Theorem 3.11 and Corollary 5.4. \square

Remark 5.6. We observe that one of the main achievements in Definition 2.2 is the elimination of Properties R2 and L2 that we have in Definitions 3.9 and 3.10: we no longer need to know in advance that the associated map is a matching on the lower Bruhat interval (which was also the most difficult and technical part to verify in the definition of right and left systems): it is now a consequence of the other axioms (Corollary 5.2).

6. THE COMBINATORIAL INVARIANCE

In this section we show how the main result of [3] (that is, the combinatorial invariance of Kazhdan-Lusztig polynomials of lower Bruhat intervals in any arbitrary Coxeter group) can be easily deduced from results in the previous sections.

We fix a system $\mathcal{S} = (J, H, M)$ for $w \in W$ and we simply denote $M_{\mathcal{S}}$ by M .

Proposition 6.1. *Suppose that $[e, w]$ is not a dihedral interval. There exists a multiplication matching N of w such that $M(w) \neq N(w)$ and $MN(u) = NM(u)$ for all $u \leq w$.*

Proof. Let $w = a \cdot b \cdot c$ be an \mathcal{S} -factorization. If $a = c = e$ then $[e, w]$ is a dihedral interval contradicting our hypothesis. We may therefore assume $a \neq e$, the case $c \neq e$ being entirely similar. Let $r \in D_L(a)$ and we prove the result with $N = \lambda_r$. It is clear that $\lambda_r(w) \neq M(w)$ since they are obtained from a reduced expression of w (namely a concatenation of reduced expressions of a , b and c) by deleting two distinct letters.

Now let $u \leq w$ and $u = a' \cdot b' \cdot c'$ with $a' \leq a$, $b' \leq b$ and $c' \leq c$.

The first part of Theorem 5.1 implies $M(u) = a'M(b')c'$. Moreover, the second part of Theorem 5.1 implies $M(ru) = ra'M(b')c'$ since $ra' \leq a$ (by the Lifting Property) and $r(ru) = (r(ra')) \cdot b' \cdot c'$. The result follows. \square

Let M be a special matching of w and N be a multiplication matching of w such that $M(w) \neq N(w)$ and $MN(u) = NM(u)$ for all $u \leq w$ (N exists by Proposition 6.1). Fix $u \in [e, w]$. We let \mathcal{O}_u be the orbit of u under the action of the group generated by the special matchings M and N , so

$$\mathcal{O}_u = \begin{cases} \{u, M(u)\} & \text{if } M(u) = N(u) \\ \{u, M(u), N(u), MN(u)\} & \text{if } M(u) \neq N(u). \end{cases}$$

We let \mathcal{U} be the free $\mathbb{Z}[q]$ -module with basis \mathcal{O}_u .

Lemma 6.2. *Suppose $|\mathcal{O}_u| = 4$ and let, for all $x \in \mathcal{O}_u$, the polynomials $f_x, \tilde{f}_x, g_x, \tilde{g}_x \in \mathbb{Z}[q]$ be such that $f_x = f_{N(x)}$, $\tilde{f}_x = \tilde{f}_{N(x)}$, $g_x = g_{M(x)}$ and $\tilde{g}_x = \tilde{g}_{N(x)}$. Consider the endomorphisms F, G of \mathcal{U} uniquely determined by*

$$F(x) = f_x x + \tilde{f}_x M(x) \quad G(x) = g_x x + \tilde{g}_x N(x)$$

for all $x \in \mathcal{O}_u$. Then $F \circ G = G \circ F$.

Proof. Given $x \in \mathcal{O}_u$, we have

$$\begin{aligned} F \circ G(x) &= F((g_x x + \tilde{g}_x N(x))) \\ &= f_x g_x x + f_{N(x)} \tilde{g}_x N(x) + \tilde{f}_x g_x M(x) + \tilde{f}_{N(x)} \tilde{g}_x MN(x). \end{aligned}$$

and similarly

$$G \circ F(x) = g_x f_x x + g_{M(x)} \tilde{f}_x M(x) + \tilde{g}_x f_x N(x) + \tilde{g}_{M(x)} \tilde{f}_x NM(x).$$

The result follows. \square

The combinatorial invariance of Kazhdan-Lusztig polynomials is equivalent to the analogous result for R -polynomials (see [1, Chapter 5]). R -polynomials are polynomials in $\mathbb{Z}[q]$ indexed by pairs of elements of W and it is well known that they can be computed with a recursive algorithm based on the following facts. If N is a multiplication matching of w , then

$$R_{u,w} = (q^{c(N,u)} - 1)R_{u,N(w)} + q^{c(N,u)}R_{N(u),N(w)},$$

where

$$c(N, u) = \begin{cases} 1 & \text{if } N(u) \triangleright u \\ 0 & \text{if } N(u) \triangleleft u, \end{cases}$$

for all $u < w$. Moreover, $R_{u,u} = 1$ for all $u \in W$, and $R_{u,v} = 0$ if $u \not\leq v$.

The next result, which was originally proved in [3], shows that one can substitute N in the previous formula with any special matching M , and that in particular R -polynomials can be computed from the knowledge of the lower Bruhat interval as an abstract poset.

Theorem 6.3. *Let M be a special matching of w . If $u \in [e, w]$, then*

$$R_{u,w} = (q^{c(M,u)} - 1)R_{u,M(w)} + q^{c(M,u)}R_{M(u),M(w)},$$

where

$$c(M, u) = \begin{cases} 1 & \text{if } M(u) \triangleright u \\ 0 & \text{if } M(u) \triangleleft u. \end{cases}$$

Proof. The result is known if M is a multiplication matching and is easy to prove if $[e, w]$ is a dihedral interval. Otherwise, let N be a multiplication matching of w such that $M(w) \neq N(w)$ and $MN(u) = NM(u)$ for all $u \leq w$, which exists by Proposition 6.1. We consider the endomorphisms F and G of \mathcal{U} such that, for $x \in \mathcal{O}_u$,

$$F(x) = (q^{c(M,x)} - 1)x + q^{c(M,x)}M(x)$$

and

$$G(x) = (q^{c(N,x)} - 1)x + q^{c(N,x)}N(x).$$

If $|\mathcal{O}_u| = 2$ then $F(x) = G(x)$, and if $|\mathcal{O}_u| = 4$ then $FG = GF$ by Lemma 6.2. In particular, in both cases, we have $F \circ G = G \circ F$.

For all $z \in W$, we consider the $\mathbb{Z}[q]$ -morphism $v_z: \mathcal{U} \rightarrow \mathbb{Z}[q]$ given by

$$v_z(x) = R_{x,z}$$

for all $x \in \mathcal{O}_u$ and extended by linearity. The statement is equivalent to

$$(6.1) \quad v_w(x) = v_{M(w)}(F(x))$$

with $x = u$. We prove Eq. (6.1) for all $x \in \mathcal{U}$ by induction on $\ell(w)$. We have

$$\begin{aligned} v_w(x) &= v_{N(w)}(G(x)) \\ &= v_{MN(w)}(FG(x)) \end{aligned}$$

where we use in the first line that Eq. (6.1) holds for N and G since N is a multiplication matching, and in the second line our induction hypothesis, since M restricts to a special matching of $N(w)$. Now, since $M \circ N = N \circ M$ and $F \circ G = G \circ F$, we can conclude

$$\begin{aligned} v_w(x) &= v_{NM(w)}(GF(x)) \\ &= v_{M(w)}(F(x)) \end{aligned}$$

where we use again Eq. (6.1) for N and G . □

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