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Limits on Sparse Data Acquisition: RIC Analysis of Finite Gaussian Matrices

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***Abstract*—One of the key issues in the acquisition of sparse data by means of compressed sensing (CS) is the design of the measurement matrix. Gaussian matrices have been proven to be information-theoretically optimal in terms of minimizing the required number of measurements for sparse recovery. In this paper we provide a new approach for the analysis of the restricted isometry constant (RIC) of finite dimensional Gaussian measurement matrices. The proposed method relies on the exact distributions of the extreme eigenvalues for Wishart matrices. First, we derive the probability that the restricted isometry property is satisfied for a given sufficient recovery condition on the RIC, and propose a probabilistic framework to study both the symmetric and asymmetric RICs. Then, we analyze the recovery of compressible signals in noise through the statistical characterization of stability and robustness. The presented framework determines limits on various sparse recovery algorithms for finite size problems. In particular, it provides a tight lower bound on the maximum sparsity order of the acquired data allowing signal recovery with a given target probability. Also, we derive simple approximations for the RICs based on the Tracy-Widom distribution.**

***Index Terms*—Data acquisition, compressed sensing, restricted isometry property, Wishart matrices, Gaussian measurement matrices, sparse reconstruction, robust recovery.**

I. Introduction

Compressed sensing (CS) is an acquisition technique for efficiently recovering a signal from a small set of linear measurements, provided that the sensed data is sparse, i.e., the number of its non-zero elements, s, is much less than its dimension n. If properly chosen, the number of measurements, m, can be much smaller than the signal dimension [1]–[7].

CS based techniques have been exploited to provide efficient solutions for several problems in signal processing and communication, e.g., source and channel coding, cryptography, random access, radar, channel estimation, and sub-Nyquist data acquisition [8]–[16]. The usability of such applications depends on the maximum sparsity order s such that recovery is guaranteed with high probability for given m and n.

The three main possible approaches to find the maximum sparsity order s guaranteeing recovery of all sparse vectors are based on the restricted isometry property (RIP) analysis, geometric methods, and coherence analysis. The RIP tells how well a linear transformation preserves distances between sparse vectors, and is quantified by the so-called restricted isometry constant (RIC) [1]. In general, the smaller the RIC, the closer the transformation to an isometry (a precise definition of the RIC is given later). Geometric based methods are useful for the recovery analysis of exactly sparse signals via ℓ_1-minimization in the noiseless case [17]–[20]. Sparse reconstruction can also be studied looking at the coherence of the measurement matrix. However, the resulting bounds are too pessimistic compared to RIP-based bounds [21, eq. (6.9) and eq. (6.14)]. This significant gap justifies preferring the RIP based analysis, whenever bounding the RIC is feasible. Furthermore, non-uniform recovery guarantees, like those based on Gaussian widths, provide tight bounds for the reconstruction of a fixed sparse vector, in contrast to the RIP method, which considers the recovery of all sparse vectors (uniform recovery) [21], [22].

Moreover, the RIP theory is more general compared to the geometric approach, as it also considers the stability for compressible signals and the robustness to noise, under different measurement matrices, for a wider range of sparse recovery algorithms. In fact, sufficient conditions for exact recovery have been obtained for several algorithms in terms of the RIC (see, e.g., [1], [23]–[28] for ℓ_1-minimization, [29], [30] for iterative hard thresholding (IHT), and [31]–[34] for greedy algorithms).

It has been shown by using information-theoretic methods that Gaussian random matrices with independent, identically distributed (i.i.d.) entries are optimal in terms of minimizing the number of measurements required for recovery [35]. Hence, precisely analyzing the RIP of such matrices is important. In fact, Gaussian matrices have been proved to satisfy the RIP with overwhelming probability [1], [3]. The two main tools adopted for the proof are the concentration of measure inequality for the distribution of the extreme eigenvalues of a Wishart matrix, and the union bound which accounts for all possible signal supports. However, if the aim is to quantify the maximal allowable sparsity order s for a given number of measurements, the use of the concentration inequalities leads to overly pessimistic results. In this regard, in [36]–[38] an improved analysis was presented, by bounding the asymptotic behavior of the distributions given in [39] for the extreme eigenvalues of a Wishart matrix instead of the concentration inequalities. Explicit bounds for the RIC have been obtained in some specific asymptotic regions [38], but no bounds are known in the general non-asymptotic setting. In fact, for finite measurement matrices the asymptotic analysis of the eigenvalues in [36]–[38] gives approximations of the

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true distributions; therefore, they cannot provide guaranteed bounds for a particular problem dimension (s, m, n).

This paper provides an accurate statistical analysis of the RIC for finite dimensional Gaussian measurement matrices, supporting the design of real CS applications (involving always finite size problems), with guaranteed recovery probability. In particular, we calculate the tightest, to our knowledge, lower bound on the probability of satisfying the RIP for an arbitrary condition on the RIC. For a specified number of measurements, the maximal sparsity order can then be found such that perfect recovery is feasible for all s-sparse vectors, i.e., the matrix satisfies the RIP, considering, on a random draw of the measurement matrix, a target probability $1-\epsilon$ of successful recovery. Differently, the usually adopted asymptotic setting considers that this probability tends to 1 (overwhelming probability).

To get better estimates on the maximal sparsity order, tight lower bounds on the cumulative distribution functions (CDFs) of the asymmetric RICs (ARICs) are derived, based on the exact probability that the extreme singular values of a Gaussian submatrix are within a range. Hence, starting from the derived CDFs, we can find thresholds, below which the ARICs lie with a predefined probability. These percentiles allow to calculate a lower bound on the maximal recoverable signal sparsity order, using several reconstruction methods, such as ℓ_1-minimization, greedy, and IHT algorithms. The new analysis is used in conjunction with the recovery conditions relaxed to asymmetric boundaries, as suggested in [36], to prove exact recovery for signals with larger sparsity orders. In this regard, we relax the symmetric RIC based condition in [28] to a weaker asymmetric one. Additionally, we provide approximations for the RIC CDFs based on the Tracy-Widom (TW) distribution, along with convergence investigation. In comparison with previous literature, the proposed analysis gives, for finite dimensional problems, a better estimation of the signal sparsity allowing guaranteed recovery.

The contributions of this paper can be summarized as follows:

- Accurate symmetric and asymmetric RIC analysis for finite dimensional problems, accounting for the exact distribution of finite Gaussian matrices (differently from previous methods based on asymptotic behavior of the distributions or loose concentration of measure bounds).
- Limits on compressive data acquisition in terms of the maximum achievable sparsity order guaranteeing arbitrary target reconstruction probability (instead of the common overwhelming probability approach) via various recovery algorithms.
- Accurate study for stable and robust recovery of compressible signals with tight bounds on the reconstruction error.
- Simple approximations for the RICs based on the TW laws.

Throughout this paper, we indicate with $\det(\cdot)$ the determinant of a matrix, with $\mathrm{card}(\cdot)$ the cardinality of a set, with $\|\cdot\|_q = (\sum_{i=1}^n |x_i|^q)^{\frac{1}{q}}$ the ℓ_q norm of an n-dimensional vector, with $\|\cdot\|$ the ℓ_2 norm, with $\Gamma(\cdot)$ the gamma function, with $\gamma(a;x,y) = \int_x^y t^{a-1}e^{-t}dt$ the generalized incomplete gamma function, with $P(a,x) = \frac{1}{\Gamma(a)}\gamma(a;0,x)$ the regularized lower incomplete gamma function, with $P(a;x,y) = \frac{1}{\Gamma(a)}\int_x^y t^{a-1}e^{-t}dt = P(a,y) - P(a,x)$ the generalized regularized incomplete gamma function, with $\mathcal{N}(\mu,\sigma^2)$ the Gaussian distribution with mean μ and variance σ^2.

II. Mathematical Background

Compressed sensing allows recovering a signal from a small number of linear measurements, under some constraints on both the sensed signal and the sensing system. More precisely, assume that we have

$$\mathbf{y} = \mathbf{A}\mathbf{x} \tag{1}$$

where $\mathbf{y} \in \mathbb{R}^m$ and $\mathbf{A} \in \mathbb{R}^{m\times n}$ are known, the number of equations is $m < n$, and $\mathbf{x} \in \mathbb{R}^n$ is the unknown. Since $m < n$, we can think of $\mathbf{y}$ as a compressed version of $\mathbf{x}$. Without other constraints, the system is underdetermined, and there are infinitely many distinct solutions of (1). If we assume that at most $s < m$ elements of $\mathbf{x}$ are non-zero (i.e., the vector is s-sparse), then there is a unique solution (the right one) to (1), provided that all possible submatrices consisting of $2s$ columns of $\mathbf{A}$ are maximum rank. The solution can be found by solving the following ℓ_0-minimization [1]

$$\hat{\mathbf{x}} = \arg\min \|\mathbf{x}\|_0 \text{ subject to } \mathbf{y} = \mathbf{A}\mathbf{x} \tag{2}$$

where $\|\mathbf{x}\|_0$ is the number of the non-zero elements of $\mathbf{x}$. However, even when the maximum rank condition is satisfied, the solution of (2) is computationally prohibitive for dimensions of practical interest. A much easier problem is to find the ℓ_1-minimization solution. It is proved in [1], under some conditions on $\mathbf{A}$, that the solution provided by the ℓ_1-minimization

$$\hat{\mathbf{x}} = \arg\min \|\mathbf{x}\|_1 \text{ subject to } \mathbf{y} = \mathbf{A}\mathbf{x} \tag{3}$$

is the same as that of (2). The conditions on $\mathbf{A}$ are given in term of the RIC.

Definition 1 (The RIC [1])**.** The RIC of order s of $\mathbf{A}$, $\delta_s(\mathbf{A})$, is the smallest constant, larger than zero, such that the inequalities

$$1-\delta_s(\mathbf{A}) \le \frac{\|\mathbf{A}_S\,\mathbf{c}\|^2}{\|\mathbf{c}\|^2} \le 1+\delta_s(\mathbf{A}) \tag{4}$$

are simultaneously satisfied for every $\mathbf{c} \in \mathbb{R}^s$ and every $m \times s$ submatrix $\mathbf{A}_S$ of $\mathbf{A}$ with columns indexed by $S \subset \Omega \triangleq \{1,2,...,n\}$ with $\mathrm{card}(S) = s$. Under this condition, the matrix $\mathbf{A}$ is said to satisfy the RIP of order s with constant $\delta_s(\mathbf{A})$.

Specifically, the importance of the RIP in CS comes from the possibility to use the computationally feasible ℓ_1-minimization instead of the impractical ℓ_0 one, under some constraints on the RIC. For example, it was shown that the ℓ_1 and the ℓ_0 solutions are coincident for every s-sparse vectors $\mathbf{x}$ if $\delta_s(\mathbf{A}) < \delta$ with $\delta = 1/3$ [28].

The next question is how to design a matrix $\mathbf{A}$ with a prescribed RIC. One possible way to design $\mathbf{A}$ consists simply

in randomly generating its entries according to some statistical distribution. In this case, for a given n, s and δ, the target is to find a way to generate $\mathbf{A}$ such that the probability $\mathbb{P}\{\delta_s(\mathbf{A}) < \delta\}$ is close to one. An optimal choice is to build the measurement matrix $\mathbf{A}$ with i.i.d. entries $a_{i,j} \sim \mathcal{N}(0, 1/m)$ [1], [35]. Then, in order to find the number of measurements m needed, we start by using the Rayleigh quotient inequality for a fixed S

$$\lambda_{\min}(\mathbf{W}) \leq \frac{\|\mathbf{A}_S\,\mathbf{c}\|^2}{\|\mathbf{c}\|^2} \leq \lambda_{\max}(\mathbf{W}) \tag{5}$$

where $\mathbf{W} = \mathbf{A}_S^T\mathbf{A}_S$, and $\lambda_{\min}(\mathbf{W})$ and $\lambda_{\max}(\mathbf{W})$ are its minimum and maximum eigenvalues, respectively. Considering that the inequalities in (4) should be satisfied for all the s-column submatrices of $\mathbf{A}$, the RIC constant can be written as

$$\delta_s(\mathbf{A}) = \max\left\{1 - \min_{\substack{S\subset\Omega\\ \mathrm{card}(S)=s}} \lambda_{\min}(\mathbf{W}),\ \max_{\substack{S\subset\Omega\\ \mathrm{card}(S)=s}} \lambda_{\max}(\mathbf{W}) - 1\right\}. \tag{6}$$

Hence, the probability that the measurement matrix satisfies the RIP with a RIC at most δ, denoted as $\beta(\delta) \triangleq \mathbb{P}\{\delta_s(\mathbf{A}) \leq \delta\}$, is represented by

$$\beta(\delta) = \mathbb{P}\left\{\min_{\substack{S\subset\Omega\\ \mathrm{card}(S)=s}} \lambda_{\min}(\mathbf{W}) \geq 1-\delta, \max_{\substack{S\subset\Omega\\ \mathrm{card}(S)=s}} \lambda_{\max}(\mathbf{W}) \leq 1+\delta\right\}. \tag{7}$$

The union bound gives a lower bound for the probability of satisfying the RIP as

$$\beta(\delta) \geq 1 - \binom{n}{s}\left[1 - P_{sw}(\delta)\right] \tag{8}$$

where $\binom{n}{s}$ is the binomial coefficient and $P_{sw}(\delta)$ is the probability that $\mathbf{A}_S$ is well conditioned defined as:

$$P_{sw}(\delta) \triangleq \mathbb{P}\{1 - \delta \leq \lambda_{\min}(\mathbf{W}), \lambda_{\max}(\mathbf{W}) \leq 1 + \delta\}. \tag{9}$$

The probability $P_{sw}(\delta)$ is of fundamental importance, since it determines the performance of CS. In the next section, an approach for exactly calculating (9) for Gaussian matrices is proposed.

III. Eigenvalues Statistics

In this section, we start by recalling the known concentration inequality based bound on $1 - P_{sw}(\delta)$, which is the approach used in [1], [2]. Then, an alternative method to find $P_{sw}(\delta)$ for Gaussian measurement matrices are provided. The proposed technique relies on the exact probability that the eigenvalues of $\mathbf{W}$ are within a predefined interval.

A. Eigenvalues Statistics Based on the Concentration Inequality

Deviation bounds for the largest and the smallest eigenvalues of the Wishart matrix $\mathbf{W}$ are obtained using the concentration of measure inequality [1], [2], as

$$\mathbb{P}\left\{\sqrt{\lambda_{\max}(\mathbf{W})} \geq 1 + \sqrt{s/m} + o(1) + t\right\} \leq e^{-mt^2/2} \tag{10}$$

and

$$\mathbb{P}\left\{\sqrt{\lambda_{\min}(\mathbf{W})} \leq 1 - \sqrt{s/m} + o(1) - t\right\} \leq e^{-mt^2/2} \tag{11}$$

where $t > 0$ and $o(1)$ is a small term tending to zero as m increases, which will be neglected in the following. Using the inequality $\mathbb{P}\{A^cB^c\} \geq 1 - \mathbb{P}\{A\} - \mathbb{P}\{B\}$ where A, B are arbitrary events, and A^c, B^c are their complements, i.e., the union bound, we get

$$P_{sw}(\delta) \geq 1 - e^{-\frac{1}{2}m\left[\left(-1-\sqrt{s/m}+\sqrt{1+\delta}\right)^+\right]^2} - e^{-\frac{1}{2}m\left[\left(1-\sqrt{s/m}-\sqrt{1-\delta}\right)^+\right]^2} \tag{12}$$

where $(x)^+ = \max\{0, x\}$. We will see later that this bound, which we use as a benchmark, is far from the exact probability.

B. Exact Eigenvalues Statistics

We propose a method to compute exactly the probability that a Wishart matrix is well conditioned, i.e, its eigenvalues are within a predefined limit. The method is based on the following recent result [40].

Theorem 1. *The probability that all non-zero eigenvalues of the real Wishart matrix $\mathbf{M} = \mathbf{G}_S^T\mathbf{G}_S$, where $\mathbf{G}_S$ is $m \times s$ matrix with entries $g_{i,j} \sim \mathcal{N}(0, 1)$, are within the interval $[a, b] \subset [0, \infty)$ is*

$$\begin{aligned}\psi_{ms}(a, b) &= \mathbb{P}\{a \leq \lambda_{\min}(\mathbf{M}), \lambda_{\max}(\mathbf{M}) \leq b\}\\ &= K'\sqrt{\det(\mathbf{Q}(a, b))}\end{aligned} \tag{13}$$

with the constant

$$K' = \frac{\pi^{s^2}/2}{2^{s\,m/2}\Gamma_s(m/2)\Gamma_s(s/2)}\,2^{\alpha s + s(s+1)/2}\prod_{\ell=1}^{s}\Gamma(\alpha + \ell)$$

where $\Gamma_s(a) \triangleq \pi^{s(s-1)/4}\prod_{i=1}^{s}\Gamma(a - (i-1)/2)$, and $\alpha = \frac{m-s-1}{2}$. In (13), when s is even the elements of the $s \times s$ skew-symmetric matrix $\mathbf{Q}(a, b)$ are

$$\begin{aligned}q_{i,j} = &\left[P\left(\alpha_j, \frac{b}{2}\right) + P\left(\alpha_j, \frac{a}{2}\right)\right]P\left(\alpha_i; \frac{a}{2}, \frac{b}{2}\right)\\ &- \frac{2}{\Gamma(\alpha_i)}\int_{a/2}^{b/2} x^{\alpha+i-1}e^{-x}P(\alpha_j, x)\,dx\end{aligned} \tag{14}$$

for $i, j = 1, \ldots, s$, where $\alpha_\ell = \alpha + \ell$. When s is odd, the elements of the $(s+1)\times(s+1)$ skew-symmetric matrix $\mathbf{Q}(a, b)$ are as in (14), with the additional elements

$$\begin{aligned}q_{i,s+1} &= P\left(\alpha_i; \frac{a}{2}, \frac{b}{2}\right) && i = 1, \ldots, s\\ q_{s+1,j} &= -q_{j,s+1} && j = 1, \ldots, s\\ q_{s+1,s+1} &= 0\,.\end{aligned} \tag{15}$$

Moreover, the elements $q_{i,j}$ can be computed iteratively, without numerical integration or series expansion [40, Algorithm 1].

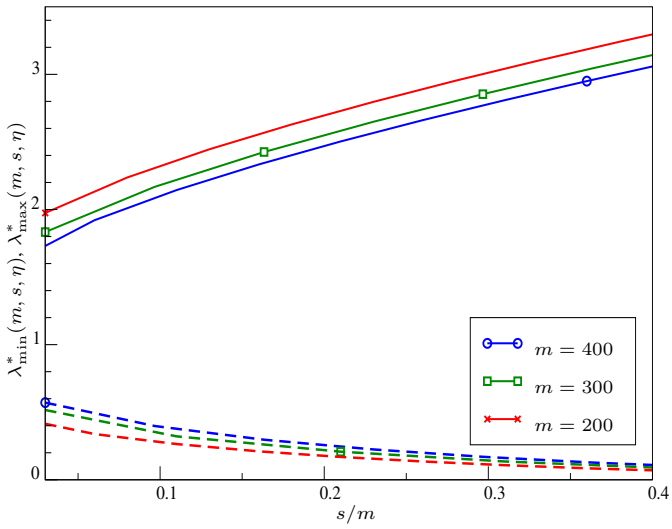

Fig. 1. The asymmetric extreme eigenvalues thresholds of the Wishart matrix $\mathbf{W}$ as a function of s/m, for $\eta = 10^{-10}$. The lower threshold $\lambda^*_{\min}(m,s,\eta)$ and the upper threshold $\lambda^*_{\max}(m,s,\eta)$ are represented by dashed and solid lines, respectively.

Considering that in our case the entries of $\mathbf{A}_S$ are distributed as $\mathcal{N}(0, 1/m)$, the exact probability that $\mathbf{A}_S$ is well conditioned is calculated from Theorem 1 as

$$\begin{aligned} P_{sw}(\delta) &= \mathbb{P}\left\{\lambda_{\min}(\mathbf{W}) \geq 1-\delta, \lambda_{\max}(\mathbf{W}) \leq 1+\delta\right\} \\ &= \psi_{ms}\big(m[1-\delta], m[1+\delta]\big) \end{aligned} \tag{16}$$

where $\psi_{ms}(a,b)$ can now be computed exactly. The exact expression (16) is computationally easy for moderate matrix dimensions (we used it up to $m = 1 \cdot 10^5$ and $s = 150$).

C. Asymmetric Nature of the Extreme Eigenvalues

Clearly, the RIC in (6) depends on the deviation of the extreme eigenvalues from unity. It has been shown that the smallest and the largest eigenvalues of Wishart matrices asymptotically deviate from 1 [36]. Hence, the symmetric RIC can not efficiently describe the RIP of Gaussian matrices. Now, it is essential to illustrate whether such asymmetric behavior is still valid for finite measurement matrices. In this regard, we proposed to find the two percentiles $\lambda^*_{\min}(m,s,\eta)$ and $\lambda^*_{\max}(m,s,\eta)$ for the extreme eigenvalues of $\mathbf{W}$, such that

$$\begin{aligned} &\mathbb{P}\left\{\lambda_{\min}(\mathbf{W}) \leq \lambda^*_{\min}(m,s,\eta)\right\} \\ &\qquad = \mathbb{P}\left\{\lambda_{\max}(\mathbf{W}) \geq \lambda^*_{\max}(m,s,\eta)\right\} = \eta\,. \end{aligned}$$

In fact, such percentiles can be calculated form the exact eigenvalues distribution in Theorem 1 as

$$\lambda^*_{\min}(m,s,\eta) = \psi^{-1}_{\min}(1-\eta), \quad \lambda^*_{\max}(m,s,\eta) = \psi^{-1}_{\max}(1-\eta) \tag{17}$$

where $\psi^{-1}_{\min}(y)$ and $\psi^{-1}_{\max}(y)$ are the inverse of $\psi_{ms}(m\,x, \infty)$ and $\psi_{ms}(0, m\,x)$, respectively.

In Fig. 1 we report the thresholds $\lambda^*_{\min}(m,s,\eta)$ and $\lambda^*_{\max}(m,s,\eta)$ as a function of s/m, for some finite values of m and a fixed exceeding probability $\eta = 10^{-10}$. We can see that they asymmetrically deviate from unity, as already observed for asymptotic large matrices in [36]. Additionally, since for small values of m the deviation of the extreme eigenvalues from unity is more significant, i.e., the RIC should be larger, the asymptotic tail behavior of the eigenvalues distributions in [36]–[38] cannot be used for upper bounding the RICs in the finite case.

Definition 2 (ARIC [24], [36])**.** The lower RIC (LRIC) of order s of $\mathbf{A}$, $\underline{\delta}_s(\mathbf{A})$, is defined as the smallest constant larger than zero that satisfies

$$1-\underline{\delta}_s(\mathbf{A}) \leq \frac{\|\mathbf{A}_S\,\mathbf{c}\|^2}{\|\mathbf{c}\|^2} \quad \forall\,\mathbf{c} \in \mathbb{R}^s, \forall\, S \subset \Omega : \mathrm{card}(S) = s \tag{18}$$

and the upper RIC (URIC) of order s of $\mathbf{A}$, $\overline{\delta}_s(\mathbf{A})$, is defined as the smallest constant larger than zero that satisfies

$$\frac{\|\mathbf{A}_S\,\mathbf{c}\|^2}{\|\mathbf{c}\|^2} \leq 1+\overline{\delta}_s(\mathbf{A}) \quad \forall\,\mathbf{c} \in \mathbb{R}^s, \forall\, S \subset \Omega : \mathrm{card}(S) = s. \tag{19}$$

Clearly, the relation with the symmetric RIC is $\delta_s(\mathbf{A}) = \max\{\underline{\delta}_s(\mathbf{A}), \overline{\delta}_s(\mathbf{A})\}$. Moreover, from Definition 2 and (5), we can represent the ARICs as

$$\underline{\delta}_s(\mathbf{A}) = 1 - \min_{\substack{S\subset\Omega \\ \mathrm{card}(S)=s}} \lambda_{\min}(\mathbf{W}) \tag{20}$$

$$\overline{\delta}_s(\mathbf{A}) = \max_{\substack{S\subset\Omega \\ \mathrm{card}(S)=s}} \lambda_{\max}(\mathbf{W}) - 1. \tag{21}$$

IV. Symmetric and Asymmetric RICs

The symmetric and asymmetric RICs of a Gaussian matrix can be seen as functions of the extreme eigenvalues of Wishart matrices as in (20) and (21), and hence are themselves random variables (r.v.s). In this section, we derive at first lower bounds on the probability of satisfying RIP for finite dimensional Gaussian random matrices using the exact eigenvalues distribution, and then a lower bound on the RIC. Additionally, the CDFs of the ARICs are lower bounded using the CDFs of the extreme eigenvalues. Finally, thresholds for ARICs that are not exceeded with a target probability are deduced.

In the following, the analysis derived starting from the exact eigenvalues statistic (16) will be referred as the exact eigenvalues distribution (EED) based approach.

A. RIP Analysis for Gaussian Matrices

A Gaussian matrix is said to satisfy the RIP of order s if its RIC, $\delta_s(\mathbf{A})$, is less than a constant δ with high probability on a random draw of $\mathbf{A}$. In other words, if a sufficient condition for perfect reconstruction using a sparse recovery algorithm is satisfied with high probability. This probability can be lower bounded from (8) and (16) as

$$\beta(\delta, m, n, s) \geq 1 - \binom{n}{s}\Big[1 - \psi_{ms}\big(m[1-\delta], m[1+\delta]\big)\Big]\,. \tag{22}$$

The expression (22) gives, to the best of our knowledge, the tightest lower bound on the probability of satisfying the RIP, $\beta(\delta)$, for finite dimensional Gaussian matrices. This is attributed to employing the exact joint distribution of the extreme eigenvalues of Wishart matrices, providing a quantitatively sharper estimates compared to the concentration bound and the asymptotic approaches.

When applying CS, it is important to estimate the RIC to assess the recovery property of the measurement matrix. Let us define $\delta^*_{s,\min}(m,n,\epsilon)$ as the RIC which is exceeded with probability ϵ, such that

$$\mathbb{P}\{\delta_s(\mathbf{A}) \leq \delta^*_{s,\min}(m,n,\epsilon)\} = 1-\epsilon\,. \tag{23}$$

Using (22) we can upper bound this value as

$$\delta^*_{s,\min}(m,n,\epsilon) \leq \delta^*_s(m,n,\epsilon) \triangleq \psi^{-1}_{ms}\left(1-\epsilon/\binom{n}{s}\right) \tag{24}$$

where $\psi^{-1}_{ms}(y)$ is the inverse of $\psi_{ms}\big(m(1-x), m[1+x]\big)$. In the following we will refer to $\delta^*_s(m,n,\epsilon)$ in (24) as the RIC threshold (RICt), where from (23) and (24) we have

$$\mathbb{P}\{\delta_s(\mathbf{A}) \leq \delta^*_s(m,n,\epsilon)\} \geq 1-\epsilon\,. \tag{25}$$

B. Asymmetric RIP Analysis for Gaussian Matrices

Let $\underline{\delta}_s(\mathbf{A})$ be the LRIC as defined in (20). The CDF of the LRIC, $F_{\ell\text{RIC}}(x)$, is lower bounded as

$$\mathbb{P}\left\{\underline{\delta}_s(\mathbf{A}) \leq x\right\} \geq 1-\binom{n}{s}\left[1-\psi_{ms}\big(m[1-x],\infty\big)\right] \tag{26}$$

In fact, from (20) the CDF of the LRIC $\underline{\delta}_s(\mathbf{A})$ is

$$\begin{aligned} F_{\ell\text{RIC}}(x) &= \mathbb{P}\left\{1-\min_{\substack{S\subset\Omega\\ \text{card}(S)=s}} \lambda_{\min}(\mathbf{W}) \;\leq x\right\} \\ &\geq 1-\binom{n}{s}\mathbb{P}\left\{\lambda_{\min}(\mathbf{W}) \leq 1-x\right\} \qquad (27)\\ &= 1-\binom{n}{s}\left[1-\psi_{ms}\big(m\,[1-x],\infty\big)\right]. \end{aligned}$$

Let us define $\underline{\delta}^*_{s,\min}(m,n,\epsilon)$ as the LRIC which is exceeded with probability ϵ, such that

$$\mathbb{P}\{\underline{\delta}_s(\mathbf{A}) \leq \underline{\delta}^*_{s,\min}(m,n,\epsilon)\} = 1-\epsilon\,.$$

This quantity is upper bounded as follows

$$\underline{\delta}^*_{s,\min}(m,n,\epsilon) \leq \underline{\delta}^*_s(m,n,\epsilon) = \psi^{-1}_{ms,\text{lower}}\left(1-\epsilon/\binom{n}{s}\right) \tag{28}$$

where $\psi^{-1}_{ms,\text{lower}}(y)$ is the inverse of $\psi_{ms}\big(m[1-x],\infty\big)$. In the following we will refer to $\underline{\delta}^*_s(m,n,\epsilon)$ as the LRIC threshold (LRICt).

Similarly, for the CDF of the URIC, $F_{\text{uRIC}}(x)$, we have

$$\begin{aligned} \mathbb{P}\left\{\overline{\delta}_s(\mathbf{A}) \leq x\right\} &\geq 1-\binom{n}{s}\mathbb{P}\left\{\lambda_{\max}(\mathbf{W}) \geq 1+x\right\} \qquad (29)\\ &= 1-\binom{n}{s}\left[1-\psi_{ms}\big(0,m\,[1+x]\big)\right]. \end{aligned}$$

Then, we can compute a threshold such that $\mathbb{P}\{\overline{\delta}_s(\mathbf{A}) \leq \overline{\delta}^*_{s,\min}(m,n,\epsilon)\} = 1-\epsilon$, which leads to

$$\overline{\delta}^*_{s,\min}(m,n,\epsilon) \leq \overline{\delta}^*_s(m,n,\epsilon) = \psi^{-1}_{ms,\text{upper}}\left(1-\frac{\epsilon}{\binom{n}{s}}\right) \tag{30}$$

where $\psi^{-1}_{ms,\text{upper}}(y)$ is the inverse of $\psi_{ms}\big(0,m[1+x]\big)$. In the following we will refer to $\overline{\delta}^*_s(m,n,\epsilon)$ as the URIC threshold (URICt).

Note that, while previously known approaches refer to infinite dimensional matrices, our analysis accounts for the (always finite) true dimensions of the problem.

V. Conditions For Perfect Recovery

In this section, the estimated thresholds for the RICs (both symmetric and asymmetric) of finite matrices are used to quantify the maximum allowed signal sparsity order for various recovery algorithms.

Definition 3 (The maximum sparsity order)**.** Let $\mathbf{A}$ be a random $m\times n$ measurement matrix, s be the signal sparsity order, and $0<\epsilon<1$ be an arbitrary constant. The maximum sparsity order, s^*, is the value such that every s-sparse vector with $s<s^*$ can be recovered perfectly with probability P_{PR} at least $1-\epsilon$ on a random draw of $\mathbf{A}$. Then the maximum oversampling ratio, a finite regime version of the asymptotic phase transition function, is defined as s^*/m.

The maximum sparsity order is used to compare the performance of different recovery algorithms and their associated sufficient conditions. As mentioned before, the perfect reconstruction conditions for many sparse recovery algorithms are stated in terms of the RICs [1], [23], [25]–[34]. We now exploit these conditions to provide a probabilistic framework for the recovery problem.

A. Symmetric RIC Based Sparse Recovery

About the symmetric RIC, the sufficient condition for perfect signal recovery via ℓ_1-minimization can be represented in a generic form as $\delta_{ks}(\mathbf{A})<\delta$, where k is a positive integer and δ is a constant. As a consequence, the probability of perfect recovery can be bounded as

$$P_{\text{PR}} \geq \mathbb{P}\left\{\delta_{ks}(\mathbf{A})<\delta\right\} = \beta(\delta,m,n,ks) \tag{31}$$

with the proposed (22). Sufficient recovery condition of this class are, e.g., $\delta_s(\mathbf{A})<1/3$ [28], $\delta_{2s}(\mathbf{A})<0.6246$ [21], etc.

The inverse problem is the calculation of the maximum sparsity order, for a given m and a given n, such that the P_{PR} is at least $1-\epsilon$. For this target we have

$$s^* = \max\left\{s : \beta(\delta,m,n,ks) \geq 1-\epsilon\right\}. \tag{32}$$

B. Asymmetric RIC Based Sparse Recovery

Although the asymmetric RICs are less investigated, it is known that the conditions stated in terms of them lead to tighter bounds for the maximum sparsity order [36]. This is attributed to the asymmetric behavior of the extreme eigenvalues for Wishart matrices as analyzed in section III-C.

A general class of sufficient recovery conditions based on the ARICs has the form

$$\mu(s,\mathbf{A}) \triangleq f\left(\underline{\delta}_{k_1 s}(\mathbf{A}), \overline{\delta}_{k_2 s}(\mathbf{A})\right) < 1 \tag{33}$$

where k_1 and k_2 are arbitrary positive integers and $f\left(\underline{\delta}_{k_1 s}(\mathbf{A}), \overline{\delta}_{k_2 s}(\mathbf{A})\right)$ is a non-decreasing function in both $\underline{\delta}_{k_1 s}(\mathbf{A})$ and $\overline{\delta}_{k_2 s}(\mathbf{A})$. In this regard, we propose a generalization of the symmetric RIC based condition, $\delta_s(\mathbf{A})<\frac{1}{3}$, to an asymmetric one. In particular, it is possible to prove that if the following condition is satisfied

$$\mu_{\text{ECG}}(s,\mathbf{A}) \triangleq 2\,\underline{\delta}_s(\mathbf{A}) + \overline{\delta}_s(\mathbf{A}) < 1 \tag{34}$$

then all s-sparse vectors can be recovered perfectly using ℓ_1-minimization.[1] Other sufficient conditions in the form of (33) are found in [24], [41]. For example, it is shown in [24] that if

$$\mu_{\text{FL}}(s,\mathbf{A}) \triangleq \frac{1}{4}\left(1+\sqrt{2}\right)\left(\frac{1+\overline{\delta}_{2s}(\mathbf{A})}{1-\underline{\delta}_{2s}(\mathbf{A})}-1\right)<1 \tag{35}$$

and in [41] that if

$$\mu_{\text{BT}}(s,\mathbf{A}) \triangleq \underline{\delta}_{2s}(\mathbf{A})+\left[\underline{\delta}_{6s}(\mathbf{A})+\overline{\delta}_{6s}(\mathbf{A})\right]/4<1$$

then perfect reconstruction is also guaranteed.

Therefore, for random measurement matrices, the probability of perfect recovery by incorporating the ARICs can be bounded as

$$P_{\text{PR}} \geq \mathbb{P}\left\{\mu(s,\mathbf{A})<1\right\}. \tag{36}$$

For the design problem of calculating the maximum sparsity order, by exploiting the monotonicity of the function $f(\cdot,\cdot)$, we have

$$\begin{aligned}\mathbb{P}\left\{\mu(s,\mathbf{A})\leq 1\right\} &\geq \mathbb{P}\left\{\underline{\delta}_{k_1 s}(\mathbf{A})\leq \underline{\delta}^*_{k_1 s}, \overline{\delta}_{k_2 s}(\mathbf{A})\leq \overline{\delta}^*_{k_2 s}\right\}\\ &\geq 1-\mathbb{P}\left\{\underline{\delta}_{k_1 s}(\mathbf{A})\geq \underline{\delta}^*_{k_1 s}\right\}-\mathbb{P}\left\{\overline{\delta}_{k_2 s}(\mathbf{A})\geq \overline{\delta}^*_{k_2 s}\right\}\end{aligned} \tag{37}$$

for any $\underline{\delta}^*_{k_1 s}$ and $\overline{\delta}^*_{k_2 s}$ such that $f\left(\underline{\delta}^*_{k_1 s},\overline{\delta}^*_{k_2 s}\right)<1$. Equation (37) is due to the union bound, (20), and (21). Setting the bound (37) to $1-\eta$ and distributing equally the probability on the lower and upper RICs, we get

$$\mathbb{P}\left\{\overline{\delta}_{k_2 s}(\mathbf{A})\leq \overline{\delta}^*_{k_2 s}\right\}=\mathbb{P}\left\{\underline{\delta}_{k_1 s}(\mathbf{A})\leq \underline{\delta}^*_{k_1 s}\right\}=1-\frac{\eta}{2}. \tag{38}$$

Finally, the maximum sparsity order s^* is the maximum s compatible with $f\left(\underline{\delta}^*_{k_1 s},\overline{\delta}^*_{k_2 s}\right)<1$, where $\underline{\delta}^*_{k_1 s},\overline{\delta}^*_{k_2 s}$ are calculated from (28) and (30) with $\epsilon=\eta/2$ to satisfy (38). Then, every sparse vector with $s<s^*$ can be perfectly recovered with probability at least $1-\eta$ on a random draw of $\mathbf{A}$.

Although we focused on ℓ_1-minimization based recovery, the same approach can be used to estimate the maximum sparsity order using greedy or thresholding algorithms. For example, sufficient conditions on the RIC for perfect recovery using compressive sampling matching pursuit (CoSaMP), orthogonal matching pursuit (OMP), and IHT are $\delta_{4s}(\mathbf{A})<0.4782$ [21], $\delta_{13s}(\mathbf{A})<0.1666$ [21], [34], and $\delta_{3s}(\mathbf{A})<0.5773$ [42], respectively. Additionally, asymmetric RIC based conditions have been obtained in [43] for CoSaMP, IHT, and subspace pursuit (SP). For example,

$$\mu_{\text{BCTT}}(s,\mathbf{A}) \triangleq 2\sqrt{2}\left(\frac{\overline{\delta}_{3s}(\mathbf{A})+\underline{\delta}_{3s}(\mathbf{A})}{2+\overline{\delta}_{3s}(\mathbf{A})-\underline{\delta}_{3s}(\mathbf{A})}\right)<1$$

is a sufficient condition for perfect recovery using IHT [43].

[1]The proof is obtained by reformulating equations (33) and (34) in [28] to account for the asymmetric RICs.

VI. Robust Recovery of Compressible Signals

Up to now, we have studied the case of perfect recovery of sparse data in noiseless setting. However, in practice signals can also be not exactly sparse, but rather compressible, i.e., the data is well approximated by a sparse signal. Moreover, noise can be present during the acquisition process.

A measure of the discrepancy between a compressible signal and its sparse representation is the ℓ_1-error of best s-term approximation $\sigma_s(\mathbf{x})_1$, defined as

$$\sigma_s(\mathbf{x})_1 \triangleq \inf\{\|\mathbf{x}-\mathbf{x}_s\|_1, \quad \mathbf{x}_s\in\mathbb{R}^n \text{ is } s\text{-sparse}\}. \tag{39}$$

Hence, a signal is well approximated by an s-sparse vector if $\sigma_s(\mathbf{x})_1$ is small [21]. Besides considering compressible signals, we can also include the measurement noise in the model, so that the measured vector can be written as

$$\mathbf{y}=\mathbf{A}\mathbf{x}+\mathbf{z} \tag{40}$$

where $\mathbf{z}$ is a bounded noise with $\|\mathbf{z}\|\leq\kappa$. Assuming κ is known, we can account for the noise term by modifying the constraint in the ℓ_1-minimization problem (3) as

$$\hat{\mathbf{x}}=\arg\min\|\mathbf{x}\|_1 \text{ subject to } \|\mathbf{y}-\mathbf{A}\mathbf{x}\|\leq\kappa. \tag{41}$$

This algorithm is called quadratically constrained ℓ_1-minimization [44]. There are also other algorithms for sparse recovery in noisy cases, e.g., Dantzig selector [45], basis pursuit denoising [46], denoising-orthogonal approximate message passing [47], etc.

For the model illustrated in (40), we cannot guarantee perfect signal recovery, but rather an approximate reconstruction can be assured with bounded error. For example, it was shown in [28] that if $\delta_s(\mathbf{A})<1/3$, the error after recovery can be bounded by a weighted combination of κ and $\sigma_s(\mathbf{x})_1$, i.e.,

$$\|\hat{\mathbf{x}}-\mathbf{x}\|\leq C_1\kappa+C_2\frac{\sigma_s(\mathbf{x})_1}{\sqrt{s}} \tag{42}$$

where

$$C_1\left(\delta_s(\mathbf{A})\right)=\frac{\sqrt{8\left[1+\delta_s(\mathbf{A})\right]}}{1-3\,\delta_s(\mathbf{A})} \tag{43}$$

$$C_2\left(\delta_s(\mathbf{A})\right)=\frac{\sqrt{8}\left[2\,\delta_s(\mathbf{A})+\sqrt{\left[1-3\,\delta_s(\mathbf{A})\right]\delta_s(\mathbf{A})}\right]}{1-3\,\delta_s(\mathbf{A})}+2. \tag{44}$$

The constants C_1 and C_2 give an insight about both the robustness (ability to handle noise) and the stability (ability to handle compressible signals) of the recovery algorithm, respectively.

When $\mathbf{A}$ is a random matrix, both C_1 and C_2 are random variables. To characterize their statistical distribution, we propose to find a bound on the threshold $C^*_{i,\min}$, with $i=1,2$, which is not exceeded with a predefined probability ϵ_i, i.e.,

$$\mathbb{P}\left\{C_i\left(\delta_s(\mathbf{A})\right)\leq C^*_{i,\min}\right\}=1-\epsilon_i. \tag{45}$$

Noting that $C_i\left(\delta_s(\mathbf{A})\right)$ is monotonically increasing in $\delta_s(\mathbf{A})$, we have

$$\begin{aligned}&\mathbb{P}\left\{C_i\left(\delta_s(\mathbf{A})\right)\leq C_i\left(\delta^*_s(m,n,\epsilon_i)\right)\right\}\\ &\qquad=\mathbb{P}\left\{\delta_s(\mathbf{A})\leq\delta^*_s(m,n,\epsilon_i)\right\}\geq 1-\epsilon_i\end{aligned} \tag{46}$$

where the RICt $\delta_s^*(m,n,\epsilon_i)$ can be calculated from (24). Consequently, from (45) and (46) we upper bound $C_{i,\min}^*$ as

$$C_{i,\min}^* \leq C_i^* \triangleq C_i\left(\delta_s^*(m,n,\epsilon_i)\right). \tag{47}$$

The inverse problem is finding the maximum sparsity order, for a given m and a given n, such that the r.v. C_i, with $i=1,2$, is less than a targeted constant c_i with probability at least $1-\epsilon_i$. For this aim we have

$$s^* = \max\left\{s : C_i\left(\delta_s^*(m,n,\epsilon_i)\right) \leq c_i\right\}.$$

Analogous results relating the recovery error with $\sigma_s(\mathbf{x})_1$ and κ have been obtained for different algorithms under suitable symmetric and asymmetric RIC based sufficient conditions [24], [43], [48]–[50]. By following the same approach, the proposed methodology can be applied to describe the statistics of the stability and robustness constants also for these cases.

VII. Tracy-Widom Based RIC Analysis

Although the proposed framework based on the exact distribution of the eigenvalues (16) provides tight bounds on the RICs, it could be computationally expensive for large matrices, for which easier approaches are preferred.

In this section, we derive approximations for the RICs of finite matrices based on the TW distribution, much tighter than those obtained from concentration of measure inequalities. Also, we study the convergence rate of the distribution of extreme eigenvalues to those based on the TW by exploiting the small deviation analysis of the extreme eigenvalues around their mean. In particular, we prove that TW based distributions approximate the eigenvalues statistics of finite Gaussian matrices with exponentially small error in m, leading to accurate estimation of the RICs.

In fact, it is well known that the distribution of the smallest and largest eigenvalues of Wishart matrices tend, under some conditions, to a properly scaled and shifted TW distributions [51]–[57]. Specifically, it has been shown that for the real Wishart matrix $\mathbf{M}$ when $m,s\to\infty$ and $m/s\to\gamma\in(0,\infty)$

$$\frac{\lambda_{\max}(\mathbf{M})-\mu_{ms}}{\sigma_{ms}} \xrightarrow{\mathcal{D}} \mathcal{TW}_1 \tag{48}$$

where $\mathcal{TW}_1$ is a Tracy-Widom r.v. of order 1 with complementary CDF (CCDF) $\Psi_{\text{TW1}}(t)$, $\mu_{ms}=\left(\sqrt{m}+\sqrt{s}\right)^2$, and $\sigma_{ms}=\sqrt{\mu_{ms}}(1/\sqrt{s}+1/\sqrt{m})^{1/3}$ [53]. More precisely, from the convergence in distribution definition and letting $\rho\triangleq s/m$ we have

$$\begin{aligned}&\lim_{m\to\infty}\mathbb{P}\left\{\lambda_{\max}(\mathbf{M})\geq\mu_{ms}+t\,\sigma_{ms}\right\}=\\&\lim_{m\to\infty}\mathbb{P}\left\{\lambda_{\max}(\mathbf{W})\geq(1+\sqrt{\rho})^2+t\,m^{-\frac{2}{3}}\,\rho^{-\frac{1}{6}}\,(1+\sqrt{\rho})^{\frac{4}{3}}\right\}\\&\quad=\Psi_{\text{TW1}}(t).\end{aligned} \tag{49}$$

Similarly, for the smallest eigenvalue, when $m,s\to\infty$ and $m/s\to\gamma\in(1,\infty)$ [56]

$$-\frac{\ln\lambda_{\min}(\mathbf{M})-v_{ms}}{\tau_{ms}} \xrightarrow{\mathcal{D}} \mathcal{TW}_1 \tag{50}$$

with scaling and centering parameters

$$\tau_{ms}=\frac{\left[(s-1/2)^{-1/2}-(m-1/2)^{-1/2}\right]^{1/3}}{\sqrt{m-1/2}-\sqrt{s-1/2}}$$

$$v_{ms}=2\ln\left(\sqrt{m-1/2}-\sqrt{s-1/2}\right)+\frac{1}{8}\tau_{ms}^2.$$

Regarding the RIC analysis for finite Gaussian matrices, let $\overline{\delta}_s^*(m,n,\epsilon)$, $\underline{\delta}_s^*(m,n,\epsilon)$, and $\delta_s^*(m,n,\epsilon)$ be the RIC thresholds as defined in (30), (28), and (24), respectively. We will show that they can be approximated as

$$\begin{aligned}\overline{\delta}_s^*(m,n,\epsilon)\simeq\overline{\delta}_{\text{TW}}^*\triangleq\;&m^{-\frac{2}{3}}\rho^{-\frac{1}{6}}\left(1+\sqrt{\rho}\right)^{\frac{4}{3}}\Psi_{\text{TW1}}^{-1}\left(\epsilon/\binom{n}{s}\right)\\&+\rho+2\sqrt{\rho}\end{aligned} \tag{51}$$

$$\underline{\delta}_s^*(m,n,\epsilon)\simeq\underline{\delta}_{\text{TW}}^*\triangleq 1-\frac{1}{m}\exp\left(v_{ms}-\tau_{ms}\Psi_{\text{TW1}}^{-1}\left(\epsilon/\binom{n}{s}\right)\right) \tag{52}$$

$$\delta_s^*(m,n,\epsilon)\simeq\delta_{\text{TW}}^*\triangleq\widetilde{P}_{sw}^{-1}\left(1-\epsilon/\binom{n}{s}\right) \tag{53}$$

for $\overline{\delta}_{\text{TW}}^*$, $\underline{\delta}_{\text{TW}}^*$, and δ_{TW}^* less than one, where $\Psi_{\text{TW1}}^{-1}(y)$ is the inverse of the TW's CCDF and $\widetilde{P}_{sw}^{-1}(y)$ is the inverse of

$$\begin{aligned}\widetilde{P}_{sw}(x)\triangleq 1-\Psi_{\text{TW1}}&\left(\frac{v_{ms}-\ln\left(m[1-x]\right)}{\tau_{ms}}\right)\\&-\Psi_{\text{TW1}}\left(\frac{m[1+x]-\mu_{ms}}{\sigma_{ms}}\right).\end{aligned} \tag{54}$$

In order to prove these formulas, at first the convergence rate of the extreme eigenvalue distributions to those based on the TW is provided. For the URIC, it has been shown in [58, Theorem 2] that there exists a constant $c>0$, depending only on ρ, such that

$$\mathbb{P}\Big\{\lambda_{\max}(\mathbf{M})\geq\mu_{ms}[1+z]\Big\}\leq c\exp\left(-\frac{1}{c}\,s\,z^{\frac{3}{2}}\right) \tag{55}$$

for all $m>s\geq 1$ and $0<z\leq 1$. This small deviation analysis provides tighter bounds compared to the concentration inequality (10) and Edelman bound [39, Lemma 4.2] used for large m in [36]–[38]. From (55), the L.H.S. of (49) can be tightly bounded for finite m and for $t\leq m^{2/3}\rho^{1/6}\left(1+\sqrt{\rho}\right)^{2/3}$ as

$$\begin{aligned}\mathbb{P}\left\{\lambda_{\max}(\mathbf{W})\geq(1+\sqrt{\rho})^2+t\,m^{-\frac{2}{3}}\,\rho^{-\frac{1}{6}}\,(1+\sqrt{\rho})^{\frac{4}{3}}\right\}&\\\leq c\exp\left(-c_1\,t^{\frac{3}{2}}\right)&\end{aligned} \tag{56}$$

where $c_1\triangleq c^{-1}\,\rho^{3/4}\left(1+\sqrt{\rho}\right)^{-1}$. Regarding the R.H.S, for sufficiently large t we have

$$\Psi_{\text{TW1}}(t)\leq c_2\exp\left(-c_3\,t^{\frac{3}{2}}\right) \tag{57}$$

where $c_2>0$ and $c_3>0$ are constants [59, eq. (2)], [60]. Now the error in using the TW can be bounded as

$$\begin{aligned}&\Big|\mathbb{P}\left\{\lambda_{\max}(\mathbf{W})\geq(1+\sqrt{\rho})^2+t\,m^{-\frac{2}{3}}\,\rho^{-\frac{1}{6}}\,(1+\sqrt{\rho})^{\frac{4}{3}}\right\}\\&-\Psi_{\text{TW1}}(t)\Big|\leq c_4\exp\left(-c_5\,t^{\frac{3}{2}}\right)\end{aligned} \tag{58}$$

where $c_4 = \max\{c, c_2\}$ and $c_5 = \min\{c_1, c_3\}$. Therefore, the error due to approximating $\mathbb{P}\{\lambda_{\max}(\mathbf{W}) \geq 1 + x\}$ in (29) by that of the TW can be bounded from (58) as

$$\Big|\mathbb{P}\{\lambda_{\max}(\mathbf{W}) \geq 1 + x\} - \Psi_{\text{TW1}}\Big((x - 2\sqrt{\rho} - \rho) \times m^{\frac{2}{3}} \rho^{\frac{1}{6}} (1+\sqrt{\rho})^{-\frac{4}{3}} \Big)\Big| \leq c_4 \exp\Big(-m\,(x - 2\sqrt{\rho} - \rho)^{\frac{3}{2}} \times c_5\, \rho^{\frac{1}{4}} (1 + \sqrt{\rho})^{-2}\Big) \tag{59}$$

for $x \leq 2\left(1+\sqrt{\rho}\right)^2 - 1$.[2] Hence, the absolute error in approximating the exact probability with that based on the TW distribution is exponentially small in m and the URICt can be approximated by (51).

A similar reasoning can be used to derive the thresholds for the lower and symmetric RICs (the proof is not reported here for the sake of conciseness).

Finally, we would like to remark that Tracy-Widom based approaches could be used not only for Wishart ensembles, but also for a wider class of matrices like those drawn from some sub-Gaussian distributions, e.g., Rademacher and Bernoulli measurement matrices. This is motivated by the universality of the TW laws for the extreme eigenvalues of large random matrices [61], [62], although further research is required to investigate such extensions.

VIII. Numerical Results

In this section, numerical results are presented to compare the proposed exact and TW approaches with the concentration inequalities, for analyzing the probability that the RIP is satisfied. Moreover, the statistics of the RICs, the probability of perfect reconstruction, the maximum sparsity order for various recovery algorithms, and the robustness and stability constants are also investigated.

Fig. 2 shows upper bounds on the probability of not satisfying the RIP, $\mathbb{P}\{\delta_s(\mathbf{A}) \geq 1/3\}$, using the EED based approach (22), the TW approximation (8), (54), and the concentration bound (8), (12). Note that when the sparsity level is beyond some threshold value, the probability of not satisfying the RIP rapidly increases from zero to one. This figure also illustrates the limit on the maximum sparsity ratio that still permits satisfying the RIP with a targeted probability. We can see that the EED based approach indicates higher sparsity ratios (less sparse vectors) compared to those estimated by the well-known concentration bound (more than 220% increase in s/n when the probability is 10^{-14} and $m/n = 0.4$). In fact, the concentration inequality is quite loose in bounding the probability that a submatrix is ill conditioned, $1 - P_{sw}(\delta)$, and consequently in analyzing the RIP.

Regarding the ARICs, the upper RIC thresholds, $\overline{\delta}_s^*(m, n, \epsilon)$, computed by means of (30) and (51), are plotted in Fig. 3 for an excess probability $\epsilon = 10^{-3}$, as a function of the compression ratio, m/n, and the oversampling ratio, s/m. In this figure, we set $m = 4000$ and vary n from $2 \cdot 10^5$ to 4000. As can be noticed the TW approximation is quite accurate.

[2]Note that $x \leq 1$ is a stronger condition than $x \leq 2\left(1+\sqrt{\rho}\right)^2 - 1$.

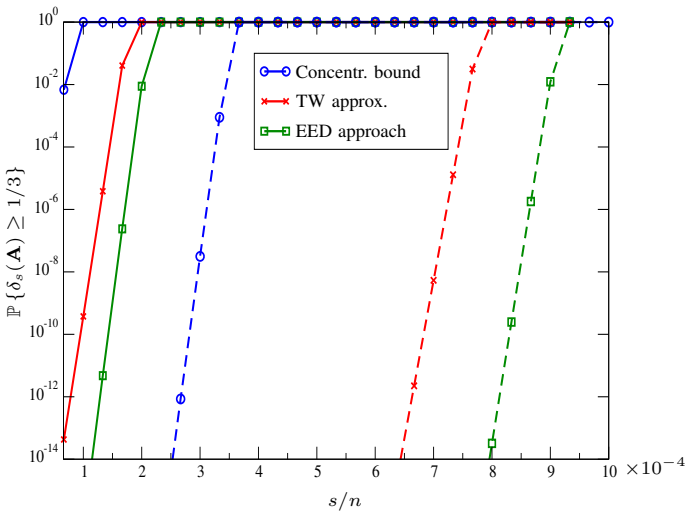

Fig. 2. Symmetric RIP: upper bounds on the probability of not satisfying the RIP, $\mathbb{P}\{\delta_s(\mathbf{A}) \geq 1/3\}$, for $m/n = 0.1$ (solid) and $m/n = 0.4$ (dashed). The signal dimension is $n = 3 \cdot 10^4$. Curves obtained through the concentration bound, (8) and (12), the EED, (22), and the TW approximation, (8) and (54).

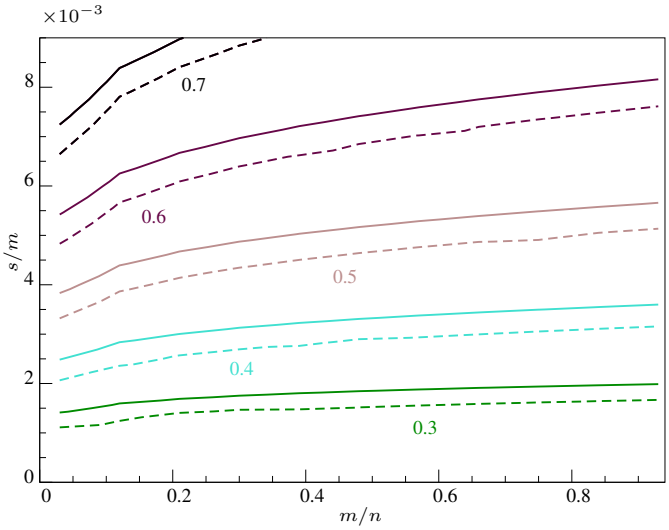

Fig. 3. Level sets of the upper RIC threshold $\overline{\delta}_s^*(m, n, \epsilon) \in \{0.3, 0.4, 0.5, 0.6, 0.7\}$ such that $\mathbb{P}\{\overline{\delta}_s(\mathbf{A}) \geq \overline{\delta}_s^*(m, n, \epsilon)\} \leq \epsilon$, using the EED (solid) and TW (dashed), for $m = 4000$ and $\epsilon = 10^{-3}$.

To further investigate the RIC bounds, we report in Table I both the LRIC and URIC thresholds for different m/n using various approaches: the proposed EED (28), (30), the TW approximation (52), (51), the empirical lower bounds in [63], and the asymptotic bounds in [36], [37]. We can see that the upper bounds on the RICs obtained from the EED approach is sharp, with small differences from the empirical lower bounds (averaged over 100 different realizations) indicated by [63].

With the aim of comparing different sufficient recovery conditions via ℓ_1-minimization, IHT, and CoSaMP algorithms, in Fig. 4 we report the maximum oversampling ratio, s^*/m, such that $P_{\text{PR}} \geq 0.999$. All curves have been obtained by using the EED based approach. Specifically, for ℓ_1-minimization we consider the symmetric RIC condition $\delta_s(\mathbf{A}) \leq 1/3$ [28], its relaxed asymmetric extension $\mu_{\text{ECG}}(s, \mathbf{A}) < 1$ proposed

TABLE I
THE RIC THRESHOLDS USING THE EED BOUND AND TW APPROXIMATION FOR $\epsilon = 10^{-2}$, EMPIRICAL AVERAGED LOWER BOUNDS [63], BCT [36], AND BT [37] APPROACHES, FOR $m = 2000$ AND $s = 4$. FOR EACH m/n, THE TWO ROWS GIVE THE UPPER AND LOWER RIC.

$m/n \downarrow$	Finite			Asymptotic	
	EED upper bounds (30), (28)	TW approximation (51), (52)	Empirical lower bounds [63]	BCT [36]	BT [37]
0.4	0.3071	0.3395	0.2703	0.3408	0.3402
	0.2561	0.2846	0.2322	0.2777	0.2772
0.6	0.3000	0.3304	0.2626	0.3344	0.3337
	0.2512	0.2778	0.2268	0.2734	0.2729
0.8	0.2949	0.3239	0.2580	0.3297	0.3291
	0.2477	0.2729	0.2214	0.2703	0.2698

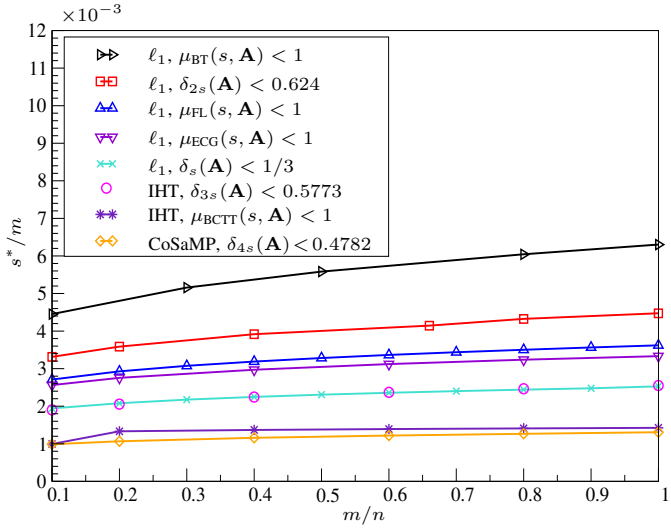

Fig. 4. The maximum oversampling ratio, s^*/m, for various recovery algorithms and their associated sufficient conditions using the proposed EED based approach, for $m = 4000$ and $P_{\text{PR}} \geq 0.999$ ($\eta = 10^{-3}$).

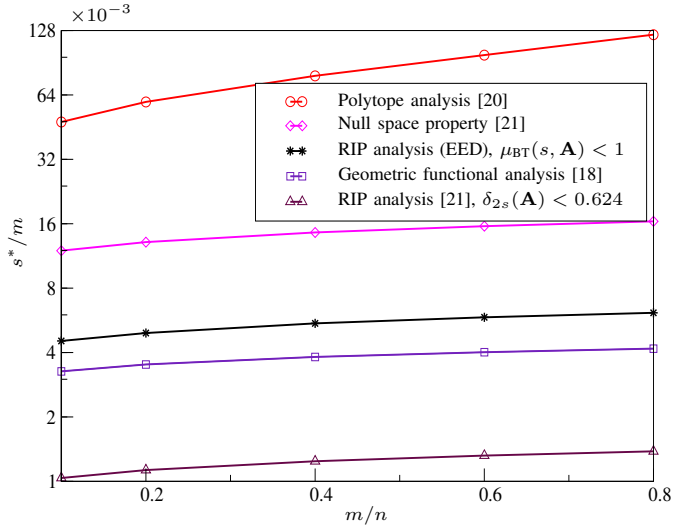

Fig. 5. The maximum oversampling ratio, s^*/m, for perfect recovery via ℓ_1-minimization, estimated by the proposed RIP based approach (EED) along with the RIP [21], polytope [20], Null space [21], and geometric functional [18] analyses, for $m = 4000$ and $P_{\text{PR}} \geq 0.5$.

in Section V-B, $\delta_{2s}(\mathbf{A}) < 0.624$ [21], $\mu_{\text{FL}}(s, \mathbf{A}) < 1$ [24], and $\mu_{\text{BT}}(s, \mathbf{A}) < 1$ [41]. For IHT we used the conditions $\delta_{3s}(\mathbf{A}) < 0.5773$ [42] and $\mu_{\text{BCTT}}(s, \mathbf{A}) < 1$ [43], while for the CoSaMP we considered $\delta_{4s}(\mathbf{A}) < 0.4782$ [21]. We can see that the asymmetric conditions provide higher estimates of the sparsity which can be handled by compressed sensing, compared to the symmetric conditions (more than 40% increase in s). As known, the ℓ_l-minimization and IHT algorithms allow higher oversampling ratios than the CoSaMP algorithm.

Moreover, we provide in Fig. 5 the maximum oversampling ratio, for uniform recovery, indicated by our proposed approach along with those obtained from the polytope [20], Null space [21, Theorem 9.29], geometric functional [18, Theorem 4.1], and RIP [21, Theorem 9.27] analyses for finite matrices with $m = 4000$ and $P_{\text{PR}} \geq 0.5$. However, we would like to note that the polytope based approach suggests tighter bounds on the maximum sparsity order, as it fully exploits the geometry of the ℓ_1-minimization for signal recovery from Gaussian measurements. On the other hand, the RIP is suitable for analyzing the robust and stable reconstruction with several sparse recovery algorithms, such as optimization, greedy, and thresholding.

Finally, regarding the analysis for compressible signals in noise, the contours for robustness and stability thresholds C_1^* and C_2^* are shown in Fig. 6. As can be seen for small s/m the thresholds are small, indicating that the more sparse is the signal, the more robust and stable is the reconstruction process. Therefore, a compromise between sparsity and robustness/stability should be considered when designing the acquisition system. This figure also gives the maximum oversampling ratio for a given m and n, such that the minimization program (41) can approximately recover the measured signal with a predefined discrepancy.

IX. CONCLUSION

For sparse data acquisition we have found that the concentration of measure inequality provides a loose upper bound on the probability that a measurement submatrix is ill conditioned. For example, in some cases it overestimate the maximum sparsity ratio by over 220% with respect to the proposed exact eigenvalues based approach. For finite matrices, by tightly

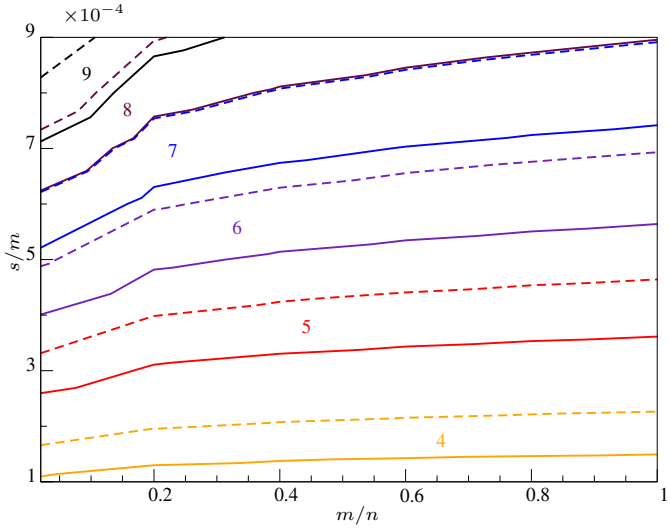

Fig. 6. Level sets of robustness and stability thresholds in Section VI, C_1^* (solid) and C_2^* (dashed), with $C_1^*, C_2^* \in \{4, 5, 6, 7, 8, 9\}$, for $m = 2 \cdot 10^4$ and $\epsilon_1 = \epsilon_2 = 10^{-3}$, using the EED based approach.

bounding the symmetric and asymmetric RICs, the best current lower bound on the maximum sparsity order guaranteeing successful recovery has been provided, for various sparse reconstruction algorithms. For stable and robust recovery of compressible data, we have noticed that when the sparsity order decreases the discrepancy between the recovered and original signals reduces. Finally, we have shown that simple approximations for the RICs can be obtained based on TW distributions.

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