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Two populations mean-field monomer-dimer model

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Abstract

A two populations mean-field monomer-dimer model including both hard-core and attractive interactions between dimers is considered. The pressure density in the thermodynamic limit is proved to satisfy a variational principle. A detailed analysis is made in the limit of one population is much smaller than the other and a ferromagnetic mean-field phase transition is found.

1 Introduction

Monomer-dimer models have been introduced in theoretical physics during the '70s to explain the absorption of diatomic molecules on a two-dimensional layer [21]. Fundamental results were obtained by Heilmann and Lieb, who proved the absence of phase transitions [15] when only the hard-core interaction is taken into account, while the presence of an additional interaction coupling dimers can generate critical behaviours [16]. Monomer-dimers models have been source of a renewed interest in the last years in mathematical physics [1, 2, 11, 13], condensed matter physics [19] and in the applications to computer science [17, 22] and social sciences [7, 10]. The presence of an interaction beyond the hard-core one that couples different dimers is fundamental for the applications where phase transitions are observed [7, 10]. Indeed in [3–5] the authors proved that a mean-field monomer-dimer model exhibits a ferromagnetic phase transition when a sufficiently strong interaction is introduced between pairs of dimers.

In this paper the investigation is extended to the case of a mean-field monomer-dimer model defined over two populations. The methods presented here can be extended to a higher number of populations. This multi-species framework has been already introduced in the context of spin models [8, 9, 18, 20] revealing interesting mathematical features. Multi-species monomer-dimer models are suitable to describe the experimental situation treated in [7, 10], where a mean-field type phase transition has been observed in the percentage of mixed marriages between native people and immigrants. The hard-core interaction between dimers naturally represents the monogamy constraint in marriages, while, as pointed out by the authors of [7], an additional imitative interaction between individuals can be at the origin of the observed critical behaviour.

In this work we consider a mean-field model built on two populations A and B (e.g., the immigrants population and the local one) which takes into account both the imitative and the hard-core interactions. Dimers can be divided into three classes: type A if they link two individuals in A , type B if they link two individuals in B and type AB if they link a mixed couple. When the total size of the system $N = N_A + N_B$ increases, we assume that the relative sizes of the two populations N_A/N , N_B/N take fixed values α , $1-\alpha$. The energy contribution of dimers is tuned by a three dimensional vector $h = (h_A, h_B, h_{AB}) \in \mathbb{R}^3$ where h_A tunes the activity of a dimer of type A and so on. Individuals have also a certain propensity to imitate or counter-imitate the behaviour of the other individuals; this feature is encoded in an additional contribution to the energy tuned by a 3×3 real matrix J . For example, the entry J_{AB}^{AB} couples dimers of type AB with other dimers of the same type. The main result we obtain is a representation of the pressure density in the thermodynamic limit in terms of a variational problem in \mathbb{R}^3 for all the values of the parameters h and J (see Theorem 1 in section 2 for the precise statement). This result is then applied to the case where the only non-zero parameters contributing to the energy are h_{AB} and J_{AB}^{AB} . As a consequence, the only relevant degree of freedom is the density of mixed dimers d_{AB} and the above variational problem leads to a consistency equation of the type

$$f_\alpha(d_{AB}) = h_{AB} + J_{AB}^{AB} d_{AB}.$$

Its analytical properties are investigated in detail for small α : the mean-field critical exponent $1/2$ is rigorously found, consistently with the experimental situation analyzed in [7, 10].

The paper is structured as follows. In section 2 we introduce the statistical mechanics model with the basic definitions and we prove the main result: the thermodynamic limit of the pressure density is expressed as a three-dimensional variational problem, where the order parameters are the dimer densities d_A , d_B internal to each population and the mixed dimer density d_{AB} .

In section 3 we focus on three non-zero parameters, α , h_{AB} , J_{AB}^{AB} , and we study in detail the critical behaviour of the system when one population is much larger than the other ($\alpha \rightarrow 0$), finding a phase transition with standard mean-field exponents.

Finally, in the Appendix we give an alternative proof for the existence of thermodynamic limit of the pressure density in the case $J = 0$, $h_A + h_B \geq 2h_{AB}$. This proof, which easily applies also to the standard single population case, uses a convexity inequality and is based on the Gaussian representation for the partition function [6].

2 Model and main result

Consider a system composed by N sites divided into two populations of sizes N_A and N_B respectively, $N_A + N_B = N$. We assume that the ratios $\alpha = N_A/N$ and $1 - \alpha = N_B/N$ are fixed when the total size N of the system varies. A *monomer-dimer configuration* can be identified with

a set Δ of edges that satisfies a hard-core condition:

$$e = \{i, j\} \in \Delta, e' = \{i', j'\} \in \Delta \Rightarrow e \cap e' = \emptyset \quad (1)$$

Given the configuration Δ (see Figure 1), the edges in Δ are called dimers and they can be partitioned into three families: denote by D_A the number of dimers having both endpoints in A , by D_B the number of dimers having both endpoints in B and by D_{AB} the number of dimers having one endpoint in A and the other one in B . Monomers, namely sites free of dimers, can be partitioned into two families: denote by M_A, M_B the number of monomers in A, B respectively. Observe that

$$2D_A + D_{AB} + M_A = N_A, \quad 2D_B + D_{AB} + M_B = N_B. \quad (2)$$

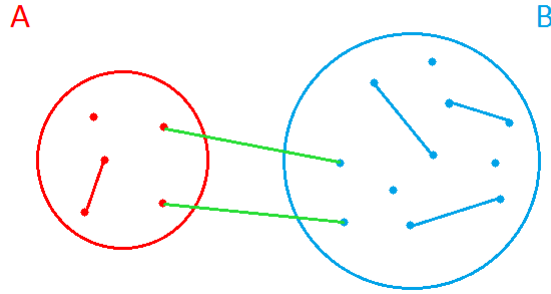


Figure 1: A monomer-dimer configuration on two populations of sizes $N_A = 5$, $N_B = 11$. In this example there are $D_A = 1$ dimers internal to population A , $D_B = 3$ dimers internal to population B and $D_{AB} = 2$ mixed dimers.

We denote by \mathcal{D}_N the set of all possible monomer-dimer configurations on N sites. For a given configuration $\Delta \in \mathcal{D}_N$, D denotes the vector of the cardinalities of the three families of dimers

$$D := \begin{pmatrix} D_A \\ D_B \\ D_{AB} \end{pmatrix}, \quad (3)$$

while

$$|D| := D_A + D_B + D_{AB} \quad (4)$$

represents the total number of dimers. The Hamiltonian function is defined as

$$H_N(D) = -h \cdot D - \frac{1}{2N} J D \cdot D \quad (5)$$

where \cdot denotes the standard scalar product in \mathbb{R}^3 , the dimer vector field h tunes the activity of dimers while the coupling matrix J tunes the interaction between sites according to the types of dimers they host:

$$h = \begin{pmatrix} h_A \\ h_B \\ h_{AB} \end{pmatrix} \quad J = \begin{pmatrix} J_A^A & J_A^B & J_A^{AB} \\ J_B^A & J_B^B & J_B^{AB} \\ J_{AB}^A & J_{AB}^B & J_{AB}^{AB} \end{pmatrix}. \quad (6)$$

The partition function of the model is

$$Z_N \equiv Z_N(h, J, \alpha) = \sum_{\Delta \in \mathcal{D}_N} N^{-|\Delta|} e^{-H_N(\Delta)}. \quad (7)$$

Since the number of dimers $|\Delta|$ is at most N , the Hamiltonian is of order N . On the other hand the term $N^{-|\Delta|}$ guarantees that the entropy, namely the logarithm of $Z_N(\Delta)$ (defined in equation (21)), is of order N . As we will see in Theorem 1 and in the Appendix, these two facts ensure a well defined thermodynamic limit of the model. Without loss of generality we assume the inverse temperature $\beta = 1$, since this parameter can be absorbed in h and J . Given $f: \mathcal{D}_N \rightarrow \mathbb{R}$ we call expected value of f with respect to the Gibbs measure the quantity

$$\langle f \rangle_N := \frac{1}{Z_N} \sum_{\Delta \in \mathcal{D}_N} N^{-|\Delta|} e^{-H_N(\Delta)} f(\Delta). \quad (8)$$

Let us introduce the definitions needed to state our main result. Denote by Ω_α the set of $d = (d_A, d_B, d_{AB})^T \in (\mathbb{R}_+)^3$ such that

$$2d_A + d_{AB} \leq \alpha, \quad 2d_B + d_{AB} \leq 1 - \alpha. \quad (9)$$

The above constraints on the vector d reflect the hard-core relations (2). Set

$$\gamma(x) := \exp(x \log x - x), \quad x \geq 0 \quad (10)$$

and define the following functions

$$\begin{aligned} s(d; \alpha) := & \log \gamma(\alpha) + \log \gamma(1 - \alpha) - \log \gamma(\alpha - 2d_A - d_{AB}) + \\ & - \log \gamma(1 - \alpha - 2d_B - d_{AB}) - \log \gamma(d_A) - \log \gamma(d_B) + \\ & - \log \gamma(d_{AB}) - d_A \log 2 - d_B \log 2 \end{aligned} \quad (11)$$

$$\epsilon(d; h, J) := -h \cdot d - \frac{1}{2} Jd \cdot d \quad (12)$$

$$\psi(d; h, J, \alpha) := s(d; \alpha) - \epsilon(d; h, J). \quad (13)$$

The functions ψ, s, ϵ represent respectively the variational pressure, entropy and energy densities.

Theorem 1. *For all $\alpha \in (0, 1)$, $h \in \mathbb{R}^3$ and $J \in \mathbb{R}^{3 \times 3}$, there exists*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log Z_N(h, J, \alpha) = \max_{d \in \Omega_\alpha} \psi(d; h, J, \alpha) =: p(h, J, \alpha) \quad (14)$$

The function $\psi(d; h, J, \alpha)$ attains its maximum in at least one point $d^* = d^*(h, J, \alpha) \in \Omega_\alpha$ which solves the following fixed point system:

$$\begin{cases} d_A = \frac{w_A}{2} m_A^2 \\ d_B = \frac{w_B}{2} m_B^2 \\ d_{AB} = w_{AB} m_A m_B \end{cases} \quad (15)$$

where we denote

$$m_A = \alpha - 2d_A - d_{AB}, \quad m_B = 1 - \alpha - 2d_B - d_{AB}, \quad (16)$$

$$w_A = e^{h_A + J_A d}, \quad w_B = e^{h_B + J_B d}, \quad w_{AB} = e^{h_{AB} + J_{AB} d}. \quad (17)$$

At $J = 0$ the system (15) has a unique solution $d^* = g(h, \alpha) \in \Omega_\alpha$ which is an analytic function of the parameters h, α . Clearly at any J the system (15) rewrites as

$$d = g(h + Jd, \alpha). \quad (18)$$

Provided that d^* is differentiable, $\nabla_h p = d^*$ and there exists

$$\lim_{N \rightarrow \infty} \frac{1}{N} \langle D \rangle_N = d^*. \quad (19)$$

Proof. The number of configurations $\Delta \in \mathcal{D}_N$ with given cardinalities D_A, D_B, D_{AB} can be computed by a standard combinatorial argument. Therefore the partition function rewrites as

$$Z_N = \sum_{D_A=0}^{N_A/2} \sum_{D_B=0}^{N_B/2} \sum_{D_{AB}=0}^{(N_A - 2D_A) \wedge (N_B - 2D_B)} \phi_N(D) e^{-H_N(D)} \quad (20)$$

with

$$\phi_N(D) := \frac{N_A! N_B! N - |D|}{(N_A - 2D_A - D_{AB})! (N_B - 2D_B - D_{AB})! D_A! D_B! D_{AB}! 2^{D_A} 2^{D_B}} \quad (21)$$

As we are interested in the limit $N_A, N_B \rightarrow \infty$ (while keeping fixed the ratio), in order to simplify the computations, we approximate the factorial by the continuous function γ defined in (10). We denote by $\tilde{\phi}_N$ the function obtained from ϕ_N by substituting any factorial $n!$ with $\gamma(n)$, then we denote by \tilde{Z}_N the partition function obtained from Z_N by substituting ϕ_N with $\tilde{\phi}_N$. The Stirling approximation and elementary computations give the following properties of γ :

- i. $1 \vee \sqrt{2\pi n} \leq n!/\gamma(n) \leq 1 \vee e^{\frac{1}{12}} \sqrt{2\pi n} \quad \forall n \in \mathbb{N}$
- ii. $\frac{d}{dx} \log \gamma(x) = \log x, \quad \log \gamma(x)$ is convex
- iii. $\frac{1}{N} \log \gamma(Nx) = \log \gamma(x) + x \log N$

By i. it follows that

$$\frac{1}{N} \log Z_N = \frac{1}{N} \log \tilde{Z}_N + \mathcal{O}\left(\frac{\log N}{N}\right), \quad (22)$$

by a standard argument

$$\frac{1}{N} \log \tilde{Z}_N = \max_{D \in \mathcal{N}\Omega_\alpha} \frac{1}{N} \left(\log \tilde{\phi}_N(D) - H_N(D) \right) + \mathcal{O}\left(\frac{\log N}{N}\right) \quad (23)$$

and using iii. a direct computation shows that for every $N \in \mathbb{N}$

$$\frac{1}{N} \left(\log \tilde{\phi}_N(Nd) - H_N(Nd) \right) = \psi(d; h, J, \alpha), \quad d \in \Omega_\alpha. \quad (24)$$

Therefore there exists

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log Z_N = \max_{d \in \Omega_\alpha} \psi(d; h, J, \alpha).$$

Using ii. one can easily compute

$$\nabla_d s = \left(\log \frac{m_A^2}{2d_A}, \log \frac{m_B^2}{2d_B}, \log \frac{m_A m_B}{d_{AB}} \right) \quad (25)$$

$$-\nabla_d \epsilon = (h_A + J_A \cdot d, h_B + J_B \cdot d, h_{AB} + J_{AB} \cdot d) \quad (26)$$

therefore

$$\nabla_d \psi(d; h, J, \alpha) = 0 \Leftrightarrow d \text{ is a solution of (15) .}$$

The first derivatives of $p(h, J, \alpha) = \psi(d^*(h, J, \alpha); h, J, \alpha)$ can be easily computed since $\nabla_d \psi(d^*; h, J, \alpha) = 0$.

3 The limit $\alpha \rightarrow 0$

In this section we choose a particular framework that simplifies the mathematical treatment of the problem and allows a detailed analysis of the thermodynamic properties of the system. The most peculiar parameters of the model are h_{AB} and J_{AB}^{AB} , describing respectively the AB -dimer field and the interaction between pairs of AB -dimers, indeed they have no correspondence in a bipopulated Ising model [18]. Moreover we focus on the case where one population is much smaller than the other ($\alpha \rightarrow 0$). Thus in this section we set $h_A = h_B = 0$, $J_A^A = J_B^B = J_A^B = J_B^A = J_{AB}^A = J_{AB}^B = J_{AB}^B = 0$ and we consider only the remaining coefficients h_{AB} and J_{AB}^{AB} . From now on, with a slight abuse of notation, we will denote

$$h := h_{AB}, \quad J := J_{AB}^{AB} > 0$$

and the mixed dimer density

$$d := d_{AB} = \frac{D_{AB}}{N} \in [0, \alpha]$$

In this framework the degrees of freedom of the variational problem (14) reduces from three to one, since d_A, d_B are explicit functions of $d_{AB} \equiv d$ as can be easily observed by looking to the consistency equation (15). Precisely, by setting $x_\alpha(d) := m_A = \sqrt{2d_A}$, $y_\alpha(d) := m_B = \sqrt{2d_B}$ one can easily see that $x_\alpha(d), y_\alpha(d)$ are the positive solutions of the following quadratic equations respectively

$$x^2 + x - (\alpha - d) = 0, \quad y^2 + y - (1 - \alpha - d) = 0 \quad (27)$$

namely

$$x_\alpha(d) = \frac{-1 + \sqrt{1 + 4(\alpha - d)}}{2}, \quad y_\alpha(d) = \frac{-1 + \sqrt{1 + 4(1 - \alpha - d)}}{2}. \quad (28)$$

Then one can easily prove from Theorem 1 that

$$p(h, J, \alpha) = \max_{d \in (0, \alpha)} \psi_1(d; h, J, \alpha) \quad (29)$$

where ψ_1 coincides with the function ψ defined by equation (13) evaluated at

$$\begin{pmatrix} d_A \\ d_B \\ d_{AB} \end{pmatrix} = \begin{pmatrix} x_\alpha(d)^2/2 \\ y_\alpha(d)^2/2 \\ d \end{pmatrix}. \quad (30)$$

Any solution $d^* = d^*(h, J, \alpha)$ of the one-dimensional variational problem (29) satisfies the fixed point equation

$$d = \exp(h + Jd) x_\alpha(d) y_\alpha(d) \quad (31)$$

It is convenient to set $f_\alpha(d) := \log d - \log x_\alpha(d) - \log y_\alpha(d)$ and rewrite equation (31) as $f_\alpha(d) = h + Jd$. Fix $\alpha \in (0, 1)$. f_α is the inverse function of a sigmoid function¹. Therefore the point (d_c, h_c, J_c) such that $f''_\alpha(d_c) = 0$, $f'_\alpha(d_c) = J_c$, $f_\alpha(d_c) = h_c + J_c d_c$ is the critical point of the system, where the density d^* branches from one to two values (see Figure 2).

For small values of α , the following estimates for the critical point can be obtained by expanding $f_\alpha(d)$ as $\alpha \rightarrow 0$:

$$d_c(\alpha) = \frac{\alpha}{2} + \mathcal{O}(\alpha^3) \quad (32)$$

$$J_c(\alpha) = \frac{4}{\alpha} + \mathcal{O}(\alpha) \quad (33)$$

$$h_c(\alpha) = -2 - \log \frac{\sqrt{5} - 1}{2} + \mathcal{O}(\alpha) \quad (34)$$

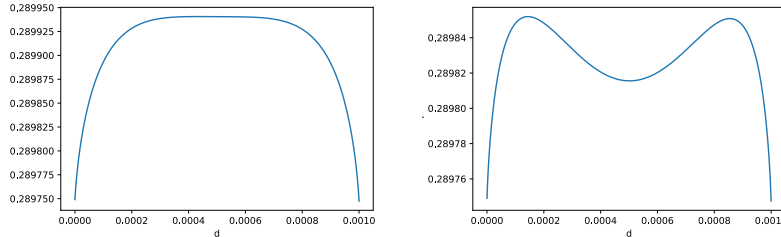


Figure 2: Plots of the variational pressure ψ_1 versus d , for $\alpha = 10^{-3}$ and different values of the parameters: critical parameters $J = J_c$, $h = h_c$ on the left-hand side; parameters $J = J_c + 10^3$, $h = h_c - d_c(J - J_c)$ on the right-hand side. The number of global maximum points of ψ_1 , that identify the phases of the system (see eq. (29)), passes from one to two when we move the parameters (J, h) away from the critical point along a suitable curve.

Fixing α close to zero and moving the parameters (h, J) towards their critical values, along the half line $h - h_c(\alpha) = -d_c(\alpha)(J - J_c(\alpha))$,

¹It is easy to check that $f_\alpha(d) \rightarrow -\infty$ as $d \searrow 0$, $f_\alpha(d) \rightarrow \infty$ as $d \nearrow \alpha$, $f'_\alpha > 0$, f''_α vanishes exactly once.

$J \geq J_c$, the mixed dimer density $d^*(h, J, \alpha)$ exhibits the following critical behaviour:

$$d^*(h, J, \alpha) - d_c(\alpha) = C(\alpha) \sqrt{J - J_c(\alpha)} + \mathcal{O}\left((J - J_c(\alpha))^{3/2}\right) \quad (35)$$

with $C(\alpha) = \sqrt{\frac{3}{16}}\alpha^3 + \mathcal{O}(\alpha^6)$. This fact can be proven using the Taylor expansion of $f_\alpha(d)$ around $d = d_c(\alpha)$ up to the third order.

Remark 1. The expansion (35) describes the mean-field critical behaviour with respect to the coupling J for fixed α . However one can also fix J and move α around the critical point. For example let's take $J = \alpha(1 - \alpha)J'$ with $J' \gg 1$. In this case we obtain

$$d - d_c = C(J') \sqrt{\alpha - \alpha_c} + \mathcal{O}\left((\alpha - \alpha_c)^{3/2}\right) \quad (36)$$

as $\alpha \rightarrow \alpha_c$, $h = h_c - d_c(\alpha - \alpha_c)$ and

$$\alpha_c = \frac{2}{\sqrt{J'}} + \mathcal{O}\left(\frac{1}{J'}\right) \quad (37)$$

$$h_c = -2 - \log \frac{\sqrt{5} - 1}{2} + \mathcal{O}\left(\frac{1}{\sqrt{J'}}\right) \quad (38)$$

$$d_c = \frac{1}{\sqrt{J'}} + \mathcal{O}\left(\frac{1}{J'^{3/2}}\right). \quad (39)$$

The critical behaviour (36) clearly has no counterpart in the single population case. This behaviour has been observed in the experimental situation [7], where the authors find that relation (36), with suitable parameters, fits well the data.

Remark 2. Equation (36) is a consequence of the fact that at the critical point the lowest order non vanishing derivative of the variational pressure ψ_1 in (29) is the fourth one. This fact suggests that the fluctuations of the order parameter at the critical point follows the standard mean field theory [3, 12]. From the above considerations we expect the fluctuations to scale as $N^{3/4}$ and to converge to a quartic exponential distribution.

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Appendix

Here we give a directed proof of the existence of the thermodynamic limit for the pressure density in the particular case

$$J = 0, \quad W = \begin{pmatrix} w_A & w_{AB} \\ w_{AB} & w_B \end{pmatrix} = \begin{pmatrix} e^{h_A} & e^{h_{AB}} \\ e^{h_{AB}} & e^{h_B} \end{pmatrix} > 0. \quad (40)$$

where $W > 0$ means that the matrix W is positive definite. This proof is independent from Theorem 1 and the strategy follows a basic idea introduced in [14] in the context of Spin Glass Theory. In this case the partition

function (7) admits a representation in terms of Gaussian moments:

$$Z_N = \sum_{\Delta \in \mathcal{D}_N} \left(\frac{w_A}{N}\right)^{D_A} \left(\frac{w_B}{N}\right)^{D_B} \left(\frac{w_{AB}}{N}\right)^{D_{AB}} = \mathbb{E} \left[(1 + \xi_A)^{N_A} (1 + \xi_B)^{N_B} \right], \quad (41)$$

where $\xi = (\xi_A, \xi_B)$ is a centred Gaussian vector of covariance matrix $\frac{1}{N}W$ (the hypothesis of positive definiteness is crucial) and \mathbb{E} denotes the expectation operator. The representation (41) is based on the Isserlis-Wick formula, see [6] (Proposition 2.2) for the proof.

Now consider the set $Q = \{\xi \in \mathbb{R}^2 : 1 + \xi_A > 0, 1 + \xi_B > 0\}$ and define a modified partition function

$$Z_N^* = \mathbb{E} \left[(1 + \xi_A)^{N_A} (1 + \xi_B)^{N_B} \mathbf{1}_Q(\xi) \right]. \quad (42)$$

Z_N^* can be rewritten as an integral over $\xi \in Q$, with integrand function proportional to $\exp(Nf(\xi))$ and

$$f(\xi) = -\frac{1}{2} \langle W^{-1} \xi, \xi \rangle + \alpha \log |1 + \xi_A| + (1 - \alpha) \log |1 + \xi_B|.$$

Since f approaches its global maximum on \mathbb{R}^2 only for $\xi_A \geq 0, \xi_B \geq 0$, standard Laplace type estimates implies that

$$\frac{Z_N}{Z_N^*} \rightarrow 1 \quad \text{as } N \rightarrow \infty. \quad (43)$$

Hence we can restrict our attention to the sequence $\log Z_N^*$, $N \in \mathbb{N}$. We claim that

Proposition 1. *For every $N_1, N_2, N \in \mathbb{N}$ such that $N = N_1 + N_2$, it holds that*

$$Z_{N_1}^* Z_{N_2}^* \leq Z_N^*. \quad (44)$$

Then the sequence $\log Z_N^*$ is super-additive and the ‘‘monotonic’’ convergence of the pressure density will follow immediately by Fekete’s lemma and equation (43):

Corollary 1. *Under the hypothesis (40), there exists*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log Z_N = \sup_N \frac{1}{N} \log Z_N^* \quad (45)$$

Only the proposition 1 remains to be proven.

Proof of the proposition 1. The strategy for the proof follows the basic ideas introduced in [14] for mean field spin models. For a fixed N consider two integers N_1, N_2 , such that $N = N_1 + N_2$ and set

$$\gamma = N_1/N, \quad 1 - \gamma = N_2/N,$$

We decompose each of the two parts of the system N_1, N_2 in two populations A, B according to the fixed ratio α , namely according to the relation

$$N_i = \alpha N_i + (1 - \alpha) N_i =: N_{iA} + N_{iB}, \quad i = 1, 2$$

Now we introduce two *independent* centred Gaussian vectors:

$$\xi_i = (\xi_{iA}, \xi_{iB}) \text{ with covariance matrix } \frac{1}{N_i} W, \quad i = 1, 2$$

and we prove the following lemmas.

Lemma 1.

$$\gamma \xi_1 + (1 - \gamma) \xi_2 \stackrel{d}{=} \xi$$

Proof. Since ξ_1, ξ_2 are independent centred Gaussian vectors, $\xi' := \gamma \xi_1 + (1 - \gamma) \xi_2$ is a centred Gaussian vector. Its covariance matrix is:

$$\gamma^2 \frac{W}{N_1} + (1 - \gamma)^2 \frac{W}{N_2} = \gamma \frac{W}{N} + (1 - \gamma) \frac{W}{N} = \frac{W}{N},$$

the same of ξ . □

Lemma 2.

$$(1 + x)^\gamma (1 + y)^{1-\gamma} \leq 1 + \gamma x + (1 - \gamma)y \quad \forall x > -1, y > -1, \gamma \in (0, 1)$$

Proof. Consider the function $f(x, y) = (1 + x)^\gamma (1 + y)^{1-\gamma}$ and its Taylor polynomial of first order at $(0, 0)$, $P(x, y) = 1 + \gamma x + (1 - \gamma)y$. The Hessian matrix of f is negative defined for $x > -1, y > -1$ (it has zero determinant and negative trace), hence $f(x, y) \leq P(x, y)$. □

Finally the proof of proposition 1 follows easily using the independence of ξ_1, ξ_2 , lemma 2 and lemma 1. □

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