ARCHIVIO ISTITUZIONALE
DELLA RICERCA

Alma Mater Studiorum Università di Bologna
Archivio istituzionale della ricerca

Two Populations Mean-Field Monomer–Dimer Model

This is the final peer-reviewed author's accepted manuscript (postprint) of the following publication:

Published Version:
Two Populations Mean-Field Monomer–Dimer Model / Alberici, Diego; Mingione, Emanuele*. - In: JOURNAL OF STATISTICAL PHYSICS. - ISSN 0022-4715. - STAMPA. - 171:1(2018), pp. 96-105. [10.1007/s10955-018-1989-x]

Availability:
This version is available at: https://hdl.handle.net/11585/657640 since: 2019-01-25

Published:
DOI: http://doi.org/10.1007/s10955-018-1989-x

Terms of use:

Some rights reserved. The terms and conditions for the reuse of this version of the manuscript are specified in the publishing policy. For all terms of use and more information see the publisher's website.

This item was downloaded from IRIS Università di Bologna (https://cris.unibo.it/).
When citing, please refer to the published version.

(Article begins on next page)

25 April 2024

This is the final peer-reviewed accepted manuscript of:

Alberici, D., Mingione, E. Two Populations Mean-Field Monomer–Dimer Model. *J Stat Phys* 171, 96–105 (2018).

The final published version is available online at **: https://doi.org/10.1007/s10955-018-1989-x**

Rights / License:

The terms and conditions for the reuse of this version of the manuscript are specified in the publishing policy. For all terms of use and more information see the publisher's website.

This item was downloaded from IRIS Università di Bologna (https://cris.unibo.it/)

When citing, please refer to the published version.

Two populations mean-field monomer-dimer model

Diego Alberici, Emanuele Mingione

Abstract

A two populations mean-field monomer-dimer model including both hard-core and attractive interactions between dimers is considered. The pressure density in the thermodynamic limit is proved to satisfy a variational principle. A detailed analysis is made in the limit of one population is much smaller than the other and a ferromagnetic mean-field phase transition is found.

1 Introduction

Monomer-dimer models have been introduced in theoretical physics during the '70s to explain the absorption of diatomic molecules on a two-dimensional layer [21]. Fundamental results were obtained by Heilmann and Lieb, who proved the absence of phase transitions [15] when only the hard-core interaction is taken into account, while the presence of an additional interaction coupling dimers can generate critical behaviours [16]. Monomer-dimers models have been source of a renewed interest in the last years in mathematical physics [1, 2, 11, 13], condensed matter physics [19] and in the applications to computer science [17, 22] and social sciences [7, 10]. The presence of an interaction beyond the hard-core one that couples different dimers is fundamental for the applications where phase transitions are observed [7, 10]. Indeed in [3–5] the authors proved that a mean-field monomer-dimer model exhibits a ferromagnetic phase transition when a sufficiently strong interaction is introduced between pairs of dimers.

In this paper the investigation is extended to the case of a mean-field monomer-dimer model defined over two populations. The methods presented here can be extended to a higher number of populations. This multi-species framework has been already introduced in the context of spin models [8,9,18,20] reveling interesting mathematical features. Multi-species monomer-dimer models are suitable to describe the experimental situation treated in [7, 10], where a mean-field type phase transition has been observed in the percentage of mixed marriages between native people and immigrants. The hard-core interaction between dimers naturally represents the monogamy constraint in marriages, while, as pointed out by the authors of [7], an additional imitative interaction between individuals can be at the origin of the observed critical behaviour.

In this work we consider a mean-field model built on two populations A and B (e.g., the immigrants population and the local one) which takes into account both the imitative and the hard-core interactions. Dimers can be divided into three classes: type A if they link two individuals in A, type B if they link two individuals in B and type AB if they link a mixed couple. When the total size of the system $N = N_A + N_B$ increases, we assume that the relative sizes of the two populations N_A/N, N_B/N take fixed values α, $1-\alpha$. The energy contribution of dimers is tuned by a three dimensional vector $h = (h_A, h_B, h_{AB}) \in \mathbb{R}^3$ where h_A tunes the activity of a dimer of type A and so on. Individuals have also a certain propensity to imitate or counter-imitate the behaviour of the other individuals; this feature is is encoded in an additional contribution to the energy tuned by a 3×3 real matrix J. For example, the entry J_{AB}^{AB} couples dimers of type AB with other dimers of the same type. The main result we obtain is a representation of the pressure density in the thermodynamic limit in terms of a variational problem in $\mathbb{R}^3$ for all the values of the parameters h and J (see Theorem 1 in section 2 for the precise statement). This result is then applied to the case where the only non-zero parameters contributing to the energy are h_{AB} and J_{AB}^{AB}. As a consequence, the only relevant degree of freedom is the density of mixed dimers d_{AB} and the above variational problem leads to a consistency equation of the type

$$f_\alpha(d_{AB}) \,=\, h_{AB} + J_{AB}^{AB}\, d_{AB}\,.$$

Its analytical properties are investigated in detail for small α: the mean-field critical exponent $1/2$ is rigorously found, consistently with the experimental situation analyzed in [7, 10].

The paper is structured as follows. In section 2 we introduce the statistical mechanics model with the basic definitions and we prove the main result: the thermodynamic limit of the pressure density is expressed as a three-dimensional variational problem, where the order parameters are the dimer densities d_A, d_B internal to each population and the mixed dimer density d_{AB}.

In section 3 we focus on three non-zero parameters, α, h_{AB}, J_{AB}^{AB}, and we study in detail the critical behaviour of the system when one population is much larger than the other ($\alpha \to 0$), finding a phase transition with standard mean-field exponents.

Finally, in the Appendix we give an alternative proof for the existence of thermodynamic limit of the pressure density in the case $J = 0$, $h_A + h_B \geq 2h_{AB}$. This proof, which easily applies also to the standard single population case, uses a convexity inequality and is based on the Gaussian representation for the partition function [6].

2 Model and main result

Consider a system composed by N sites divided into two populations of sizes N_A and N_B respectively, $N_A + N_B = N$. We assume that the ratios $\alpha = N_A/N$ and $1 - \alpha = N_B/N$ are fixed when the total size N of the system varies. A *monomer-dimer configuration* can be identified with a set Δ of edges that satisfies a hard-core condition:

$$e = \{i, j\} \in \Delta \ , \ e' = \{i', j'\} \in \Delta \quad \Rightarrow \quad e \cap e' = \emptyset \tag{1}$$

Given the configuration Δ (see Figure 1), the edges in Δ are called dimers and they can be partitioned into three families: denote by D_A the number of dimers having both endpoints in A, by D_B the number of dimers having both endpoints in B and by D_{AB} the number of dimers having one endpoint in A and the other one in B. Monomers, namely sites free of dimers, can be partitioned into two families: denote by M_A, M_B the number of monomers in A, B respectively. Observe that

$$2\, D_A + D_{AB} + M_A = N_A \quad , \quad 2\, D_B + D_{AB} + M_B = N_B \ . \tag{2}$$

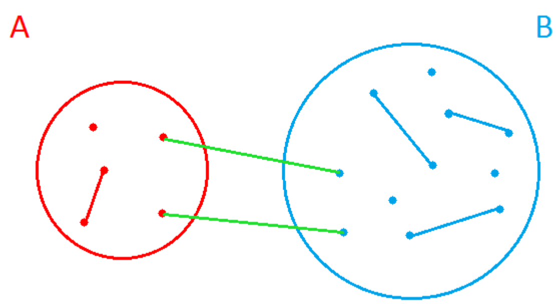

Figure 1: A monomer-dimer configuration on two populations of sizes $N_A = 5$, $N_B = 11$. In this example there are $D_A = 1$ dimers internal to population A, $D_B = 3$ dimers internal to population B and $D_{AB} = 2$ mixed dimers.

We denote by $\mathscr{D}_N$ the set of all possible monomer-dimer configurations on N sites. For a given configuration $\Delta \in \mathscr{D}_N$, D denotes the vector of the cardinalities of the three families of dimers

$$D \,:=\, \begin{pmatrix} D_A \\ D_B \\ D_{AB} \end{pmatrix} , \tag{3}$$

while

$$|D| \,:=\, D_A + D_B + D_{AB} \tag{4}$$

represents the total number of dimers. The Hamiltonian function is defined as

$$H_N(D) \,=\, -h \cdot D - \frac{1}{2N} JD \cdot D \tag{5}$$

where $\cdot$ denotes the standard scalar product in $\mathbb{R}^3$, the dimer vector field h tunes the activity of dimers while the coupling matrix J tunes the interaction between sites according to the types of dimers they host:

$$h \,=\, \begin{pmatrix} h_A \\ h_B \\ h_{AB} \end{pmatrix} \quad J \,=\, \begin{pmatrix} J_A^A & J_A^B & J_A^{AB} \\ J_B^A & J_B^B & J_B^{AB} \\ J_{AB}^A & J_{AB}^B & J_{AB}^{AB} \end{pmatrix} . \tag{6}$$

The partition function of the model is

$$Z_N \equiv Z_N(h,J,\alpha) = \sum_{\Delta\in\mathscr{D}_N} N^{-|D|}\, e^{-H_N(D)} . \tag{7}$$

Since the number of dimers $|D|$ is at most N, the Hamiltonian is of order N. On the other hand the term $N^{-|D|}$ guarantees that the entropy, namely the logarithm of $\Phi_N(D)$ (defined in equation (21)), is of order N. As we will see in Theorem 1 and in the Appendix, these two facts ensure a well defined thermodynamic limit of the model. Without loss of generality we assume the inverse temperature $\beta = 1$, since this parameter can be absorbed in h and J. Given $f\colon \mathscr{D}_N \to \mathbb{R}$ we call expected value of f with respect to the Gibbs measure the quantity

$$\langle\, f\,\rangle_N := \frac{1}{Z_N}\sum_{\Delta\in\mathscr{D}_N} N^{-|D|} e^{-H_N(D)}\, f(\Delta) . \tag{8}$$

Let us introduce the definitions needed to state our main result. Denote by Ω_α the set of $d = (d_A, d_B, d_{AB})^T \in (\mathbb{R}_+)^3$ such that

$$2d_A + d_{AB} \le \alpha\,,\;\; 2d_B + d_{AB} \le 1-\alpha\,. \tag{9}$$

The above constraints on the vector d reflect the hard-core relations (2). Set

$$\gamma(x) := \exp(x\log x - x)\;, \quad x \ge 0 \tag{10}$$

and define the following functions

$$\begin{aligned} s(d;\alpha) :=\; & \log\gamma(\alpha) + \log\gamma(1-\alpha) - \log\gamma(\alpha - 2d_A - d_{AB}) + \\ & -\log\gamma(1-\alpha-2d_B-d_{AB}) - \log\gamma(d_A) - \log\gamma(d_B) + \\ & -\log\gamma(d_{AB}) - d_A\log 2 - d_B\log 2 \end{aligned} \tag{11}$$

$$\epsilon(d;h,J) := -h\cdot d - \frac{1}{2}\,Jd\cdot d \tag{12}$$

$$\psi(d;h,J,\alpha) := s(d;\alpha) - \epsilon(d;h,J)\,. \tag{13}$$

The functions ψ, s, ϵ represent respectively the variational pressure, entropy and energy densities.

Theorem 1. *For all $\alpha\in(0,1)$, $h\in\mathbb{R}^3$ and $J\in\mathbb{R}^{3\times3}$, there exists*

$$\lim_{N\to\infty}\frac{1}{N}\log Z_N(h,J,\alpha) = \max_{d\in\Omega_\alpha}\psi(d;h,J,\alpha) =: p(h,J,\alpha) \tag{14}$$

The function $\psi(d;h,J,\alpha)$ attains its maximum in at least one point $d^* = d^*(h,J,\alpha)\in\Omega_\alpha$ which solves the following fixed point system:

$$\begin{cases} d_A = \frac{w_A}{2}\, m_A^2 \\ d_B = \frac{w_B}{2}\, m_B^2 \\ d_{AB} = w_{AB}\, m_A\, m_B \end{cases} \tag{15}$$

where we denote

$$m_A = \alpha - 2d_A - d_{AB}\;, \quad m_B = 1-\alpha-2d_B-d_{AB}\;, \tag{16}$$

$$w_A = e^{h_A + J_A d} \ , \quad w_B = e^{h_B + J_B d} \ , \quad w_{AB} = e^{h_{AB} + J_{AB} d} \ . \tag{17}$$

At $J = 0$ the system (15) has a unique solution $d^* = g(h, \alpha) \in \Omega_\alpha$ which is an analytic function of the parameters h, α. Clearly at any J the system (15) rewrites as

$$d \ = \ g(h + Jd \, , \, \alpha) \ . \tag{18}$$

Provided that d^* is differentiable, $\nabla_h \, p = d^*$ and there exists

$$\lim_{N \to \infty} \frac{1}{N} \left\langle \, D \, \right\rangle_N \ = \ d^* \ . \tag{19}$$

Proof. The number of configurations $\Delta \in \mathscr{D}_N$ with given cardinalities D_A, D_B, D_{AB} can be computed by a standard combinatorial argument. Therefore the partition function rewrites as

$$Z_N \ = \sum_{D_A = 0}^{N_A/2} \sum_{D_B = 0}^{N_B/2} \sum_{D_{AB} = 0}^{(N_A - 2D_A) \wedge (N_B - 2D_B)} \phi_N(D) \, e^{-H_N(D)} \tag{20}$$

with

$$\phi_N(D) := \frac{N_A! \, N_B! \, N{-}|D|}{(N_A - 2D_A - D_{AB})! \, (N_B - 2D_B - D_{AB})! \, D_A! \, D_B! \, D_{AB}! \, 2^{D_A} \, 2^{D_B}} \tag{21}$$

As we are interested in the limit $N_A, N_B \to \infty$ (while keeping fixed the ratio), in order to simplify the computations, we approximate the factorial by the continuous function γ defined in (10). We denote by $\tilde{\phi}_N$ the function obtained from ϕ_N by substituting any factorial $n!$ with $\gamma(n)$, then we denote by $\tilde{Z}_N$ the partition function obtained from Z_N by substituting ϕ_N with $\tilde{\phi}_N$. The Stirling approximation and elementary computations give the following properties of γ:

i. $1 \vee \sqrt{2\pi n} \ \leq \ n!/\gamma(n) \ \leq \ 1 \vee e^{\frac{1}{12}} \sqrt{2\pi n} \quad \forall n \in \mathbb{N}$

ii. $\frac{d}{dx} \log \gamma(x) = \log x \, , \quad \log \gamma(x)$ is convex

iii. $\frac{1}{N} \log \gamma(Nx) = \log \gamma(x) + x \, \log N$

By i. it follows that

$$\frac{1}{N} \log Z_N \ = \ \frac{1}{N} \log \tilde{Z}_N \ + \ \mathcal{O}\left(\frac{\log N}{N} \right) \ , \tag{22}$$

by a standard argument

$$\frac{1}{N} \log \tilde{Z}_N \ = \ \max_{D \in N\Omega_\alpha} \frac{1}{N} \left(\log \tilde{\phi}_N(D) - H_N(D) \right) \ + \ \mathcal{O}\left(\frac{\log N}{N} \right) \tag{23}$$

and using iii. a direct computation shows that for every $N \in \mathbb{N}$

$$\frac{1}{N} \left(\log \tilde{\phi}_N(Nd) - H_N(Nd) \right) \ = \ \psi(d; h, J, \alpha) \, , \quad d \in \Omega_\alpha \ . \tag{24}$$

Therefore there exists

$$\lim_{N \to \infty} \frac{1}{N} \log Z_N = \max_{d \in \Omega_\alpha} \psi(d; h, J, \alpha) \ .$$

Using ii. one can easily compute

$$\nabla_d\, s \;=\; \left(\log \frac{m_A^2}{2d_A} \;,\; \log \frac{m_B^2}{2d_B} \;,\; \log \frac{m_A m_B}{d_{AB}} \right) \tag{25}$$

$$-\nabla_d\, \epsilon \;=\; (\, h_A + J_A \cdot d \;,\; h_B + J_B \cdot d \;,\; h_{AB} + J_{AB} \cdot d\,) \tag{26}$$

therefore

$$\nabla_d\, \psi(d; h, J, \alpha) \;=\; 0 \;\Leftrightarrow\; d \text{ is a solution of } (15)\;.$$

The first derivatives of $p(h, J, \alpha) \;=\; \psi(d^*(h, J, \alpha); h, J, \alpha)$ can be easily computed since $\nabla_d\, \psi(d^*; h, J, \alpha) = 0$.

3 The limit $\alpha \to 0$

In this section we choose a particular framework that simplifies the mathematical treatment of the problem and allows a detailed analysis of the thermodynamic properties of the system. The most peculiar parameters of the model are h_{AB} and J_{AB}^{AB}, describing respectively the AB-dimer field and the interaction between pairs of AB-dimers, indeed they have no correspondence in a bipopulated Ising model [18]. Moreover we focus on the case where one population is much smaller than the other ($\alpha \to 0$). Thus in this section we set $h_A = h_B = 0$, $J_A^A = J_B^B = J_A^B = J_B^A = J_A^{AB} = J_{AB}^A = J_B^{AB} = J_{AB}^B = 0$ and we consider only the remaining coefficients h_{AB} and J_{AB}^{AB}. From now on, with a slight abuse of notation, we will denote

$$h := h_{AB}\;, \quad J := J_{AB}^{AB} > 0$$

and the mixed dimer density

$$d := d_{AB} = \frac{D_{AB}}{N} \in [0, \alpha]$$

In this framework the degrees of freedom of the variational problem (14) reduces from three to one, since d_A, d_B are explicit functions of $d_{AB} \equiv d$ as can be easily observed by looking to the consistency equation (15). Precisely, by setting $x_\alpha(d) := m_A = \sqrt{2d_A}\,$, $y_\alpha(d) := m_B = \sqrt{2d_B}$ one can easily see that $x_\alpha(d)$, $y_\alpha(d)$ are the positive solutions of the following quadratic equations respectively

$$x^2 + x - (\alpha - d) = 0 \quad , \quad y^2 + y - (1 - \alpha - d) = 0 \tag{27}$$

namely

$$x_\alpha(d) = \frac{-1 + \sqrt{1 + 4(\alpha - d)}}{2} \quad , \quad y_\alpha(d) = \frac{-1 + \sqrt{1 + 4(1 - \alpha - d)}}{2}\;. \tag{28}$$

Then one can easily prove from Theorem 1 that

$$p(h, J, \alpha) \;=\; \max_{d\, \in\, (0,\alpha)} \psi_1(d; h, J, \alpha) \tag{29}$$

where ψ_1 coincides with the function ψ defined by equation (13) evaluated at

$$\begin{pmatrix} d_A \\ d_B \\ d_{AB} \end{pmatrix} = \begin{pmatrix} x_\alpha(d)^2/2 \\ y_\alpha(d)^2/2 \\ d \end{pmatrix} . \tag{30}$$

Any solution $d^* = d^*(h, J, \alpha)$ of the one-dimensional variational problem (29) satisfies the fixed point equation

$$d \,=\, \exp(h + Jd)\, x_\alpha(d)\, y_\alpha(d) \tag{31}$$

It is convenient to set $f_\alpha(d) := \log d - \log x_\alpha(d) - \log y_\alpha(d)$ and rewrite equation (31) as $f_\alpha(d) = h + Jd$. Fix $\alpha \in (0,1)$. f_α is the inverse function of a sigmoid function[1]. Therefore the point (d_c, h_c, J_c) such that $f''_\alpha(d_c) = 0$, $f'_\alpha(d_c) = J_c$, $f_\alpha(d_c) = h_c + J_c\, d_c$ is the critical point of the system, where the density d^* branches from one to two values (see Figure 2).

For small values of α, the following estimates for the critical point can be obtained by expanding $f_\alpha(d)$ as $\alpha \to 0$:

$$d_c(\alpha) = \frac{\alpha}{2} + \mathcal{O}(\alpha^3) \tag{32}$$

$$J_c(\alpha) = \frac{4}{\alpha} + \mathcal{O}(\alpha) \tag{33}$$

$$h_c(\alpha) = -2 - \log \frac{\sqrt{5}-1}{2} + \mathcal{O}(\alpha) \tag{34}$$

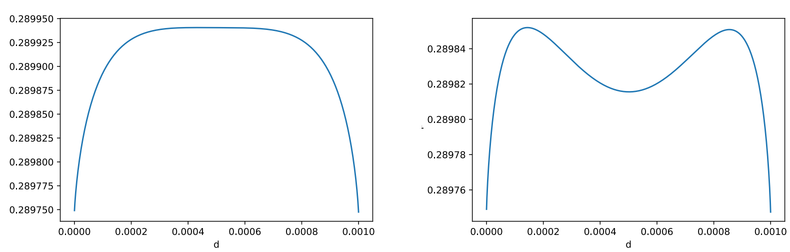

Figure 2: Plots of the variational pressure ψ_1 versus d, for $\alpha = 10^{-3}$ and different values of the parameters: critical parameters $J = J_c$, $h = h_c$ on the left-hand side; parameters $J = J_c + 10^3$, $h = h_c - d_c\,(J - J_c)$ on the right-hand side. The number of global maximum points of ψ_1, that identify the phases of the system (see eq. (29)), passes from one to two when we move the parameters (J, h) away from the critical point along a suitable curve.

Fixing α close to zero and moving the parameters (h, J) towards their critical values, along the half line $h - h_c(\alpha) = -d_c(\alpha)\,\big(J - J_c(\alpha)\big)$,

[1] It is easy to check that $f_\alpha(d) \to -\infty$ as $d \searrow 0$, $f_\alpha(d) \to \infty$ as $d \nearrow \alpha$, $f'_\alpha > 0$, f''_α vanishes exactly once.

$J \geq J_c$, the mixed dimer density $d^*(h, J, \alpha)$ exhibits the following critical behaviour:

$$d^*(h, J, \alpha) - d_c(\alpha) \; = \; C(\alpha)\, \sqrt{J - J_c(\alpha)} \; + \; \mathcal{O}\Big((J - J_c(\alpha))^{3/2} \Big) \qquad (35)$$

with $C(\alpha) = \sqrt{\frac{3}{16}\alpha^3 + \mathcal{O}(\alpha^6)}$. This fact can be proven using the Taylor expansion of $f_\alpha(d)$ around $d = d_c(\alpha)$ up to the third order.

Remark 1. The expansion (35) describes the mean-field critical behaviour with respect to the coupling J for fixed α. However one can also fix J and move α around the critical point. For example let's take $J = \alpha\,(1-\alpha)\,J'$ with $J' \gg 1$. In this case we obtain

$$d - d_c \; = \; C(J')\, \sqrt{\alpha - \alpha_c} \; + \; \mathcal{O}\big((\alpha - \alpha_c)^{3/2} \big) \qquad (36)$$

as $\alpha \to \alpha_c$, $h = h_c - d_c\,(\alpha - \alpha_c)$ and

$$\alpha_c = \frac{2}{\sqrt{J'}} \; + \mathcal{O}\left(\frac{1}{J'}\right) \qquad (37)$$

$$h_c = -2 - \log\frac{\sqrt{5}-1}{2} \; + \mathcal{O}\left(\frac{1}{\sqrt{J'}}\right) \qquad (38)$$

$$d_c = \frac{1}{\sqrt{J'}} \; + \mathcal{O}\left(\frac{1}{J'^{\,3/2}}\right) . \qquad (39)$$

The critical behaviour (36) clearly has no counterpart in the single population case. This behaviour has been observed in the experimental situation [7], where the authors find that relation (36), with suitable parameters, fits well the data.

Remark 2. Equation (36) is a consequence of the fact that at the critical point the lowest order non vanishing derivative of the variational pressure ψ_1 in (29) is the fourth one. This fact suggests that the fluctuations of the order parameter at the critical point follows the standard mean field theory [3, 12]. From the above considerations we expect the fluctuations to scale as $N^{3/4}$ and to converge to a quartic exponential distribution.

Acknowledgment: We thank Pierluigi Contucci for bringing the problem to our attention and we acknowledge financial support by GNFM-INdAM Progetto Giovani 2017.

Appendix

Here we give a directed proof of the existence of the thermodynamic limit for the pressure density in the particular case

$$J = 0 \;, \quad W = \begin{pmatrix} w_A & w_{AB} \\ w_{AB} & w_B \end{pmatrix} = \begin{pmatrix} e^{h_A} & e^{h_{AB}} \\ e^{h_{AB}} & e^{h_B} \end{pmatrix} > 0 \; . \qquad (40)$$

where $W > 0$ means that the matrix W is positive definite. This proof is independent from Theorem 1 and the strategy follows a basic idea introduced in [14] in the context of Spin Glass Theory. In this case the partition function (7) admits a representation in terms of Gaussian moments:

$$Z_N = \sum_{\Delta\in\mathscr{D}_N} \left(\frac{w_A}{N}\right)^{D_A} \left(\frac{w_B}{N}\right)^{D_B} \left(\frac{w_{AB}}{N}\right)^{D_{AB}} = \mathbb{E}\left[(1+\xi_A)^{N_A}(1+\xi_B)^{N_B}\right] , \tag{41}$$

where $\xi = (\xi_A, \xi_B)$ is a centred Gaussian vector of covariance matrix $\frac{1}{N}W$ (the hypothesis of positive definiteness is crucial) and $\mathbb{E}$ denotes the expectation operator. The representation (41) is based on the Isserlis-Wick formula, see [6] (Proposition 2.2) for the proof.

Now consider the set $Q = \{\xi \in \mathbb{R}^2 \,:\, 1+\xi_A > 0,\, 1+\xi_B > 0\}$ and define a modified partition function

$$Z^*_N = \mathbb{E}\left[(1+\xi_A)^{N_A}(1+\xi_B)^{N_B}\, \mathbb{1}_Q(\xi)\right] . \tag{42}$$

Z^*_N can be rewritten as an integral over $\xi \in Q$, with integrand function proportional to $\exp(N\, f(\xi))$ and

$$f(\xi) = -\frac{1}{2}\langle W^{-1}\xi, \xi\rangle + \alpha \log|1+\xi_A| + (1-\alpha)\log|1+\xi_B| \,.$$

Since f approaches its global maximum on $\mathbb{R}^2$ only for $\xi_A \geq 0$, $\xi_B \geq 0$, standard Laplace type estimates implies that

$$\frac{Z_N}{Z^*_N} \to 1 \quad \text{as } N \to \infty \,. \tag{43}$$

Hence we can restrict our attention to the sequence $\log Z^*_N$, $N \in \mathbb{N}$. We claim that

Proposition 1. *For every $N_1, N_2, N \in \mathbb{N}$ such that $N = N_1 + N_2$, it holds that*

$$Z^*_{N_1}\, Z^*_{N_2} \,\leq\, Z^*_N \,. \tag{44}$$

Then the sequence $\log Z^*_N$ is super-additive and the "monotonic" convergence of the pressure density will follow immediately by Fekete's lemma and equation (43):

Corollary 1. *Under the hypothesis* (40)*, there exists*

$$\lim_{N\to\infty} \frac{1}{N}\log Z_N \,=\, \sup_N \frac{1}{N}\log Z^*_N \tag{45}$$

Only the proposition 1 remains to be proven.

Proof of the proposition 1. The strategy for the proof follows the basic ideas introduced in [14] for mean field spin models. For a fixed N consider two integers N_1, N_2, such that $N = N_1 + N_2$ and set

$$\gamma = N_1/N\,,\;\; 1-\gamma = N_2/N\,,$$

We decompose each of the two parts of the system N_1, N_2 in two populations A, B according to the fixed ratio α, namely according to the relation

$$N_i = \alpha N_i + (1-\alpha)N_i =: N_{iA} + N_{iB}\,, \quad i = 1,2$$

Now we introduce two *independent* centred Gaussian vectors:

$$\xi_i = (\xi_{iA}\,,\,\xi_{iB}) \text{ with covariance matrix } \frac{1}{N_i}\,W\,, \quad i=1,2$$

and we prove the following lemmas.

Lemma 1.

$$\gamma\,\xi_1 + (1-\gamma)\,\xi_2 \overset{d}{=} \xi$$

Proof. Since ξ_1, ξ_2 are independent centred Gaussian vectors, $\xi' := \gamma\,\xi_1 + (1-\gamma)\,\xi_2$ is a centred Gaussian vector. Its covariance matrix is:

$$\gamma^2\,\frac{W}{N_1} + (1-\gamma)^2\,\frac{W}{N_2} \;=\; \gamma\,\frac{W}{N} + (1-\gamma)\,\frac{W}{N} \;=\; \frac{W}{N}\;,$$

the same of ξ. □

Lemma 2.

$$(1+x)^{\gamma}\,(1+y)^{1-\gamma} \leq 1+\gamma x + (1-\gamma)y \quad \forall\, x>-1,\, y>-1, \gamma\in(0,1)$$

Proof. Consider the function $f(x,y) = (1+x)^{\gamma}\,(1+y)^{1-\gamma}$ and its Taylor polynomial of first order at $(0,0)$, $P(x,y) = 1+\gamma x+(1-\gamma)y$. The Hessian matrix of f is negative defined for $x>-1$, $y>-1$ (it has zero determinant and negative trace), hence $f(x,y) \leq P(x,y)$. □

Finally the proof of proposition 1 follows easily using the independence of ξ_1, ξ_2, lemma 2 and lemma 1 . □

References

[1] D. Alberici, "A cluster expansion approach to the Heilmann-Lieb liquid crystal model", *Journal of Statistical Physics* 162(3), 761-791 (2016)

[2] D. Alberici, P. Contucci, "Solution of the monomer-dimer model on locally tree-like graphs. Rigorous results", *Communications in Mathematical Physics* 331, 975-1003 (2014)

[3] D. Alberici, P. Contucci, M. Fedele, E. Mingione, "Limit theorems for monomer-dimer mean-field models with attractive potential", *Communications in Mathematical Physic* 346(3), 781-799 (2016)

[4] D. Alberici, P. Contucci, E. Mingione, "A mean-field monomer-dimer model with attractive interaction. Exact solution and rigorous results", *Journal of Mathematical Physics* 55, 1-27 (2014)

[5] D. Alberici, P. Contucci, E. Mingione, "The exact solution of a mean-field monomer-dimer model with attractive potential", *Europhysics Letters* 106, 1-5 (2014)

[6] D. Alberici, P. Contucci, E. Mingione, "A mean-field monomer-dimer model with randomness. Exact solution and rigorous results", *Journal of Statistical Physics* 160, 1721-1732 (2015)

[7] A. Barra, P. Contucci, R. Sandell, C. Vernia, "An analysis of a large dataset on immigrant integration in Spain. The statistical mechanics perspective on social action", *Scientific Reports* 4, 4174 (2014)

[8] A. Barra, P. Contucci, E. Mingione, D. Tantari, "Multi-species mean-field spin-glasses. Rigorous results", *Annales Henri Poincaré* 16(3), 691-708 (2015)

[9] A. Barra, G. Genovese, F. Guerra, "Equilibrium statistical mechanics of bipartite spin systems",*Journal of Physics A: Mathematical and Theoretical* 44, 24 (2011)

[10] P. Contucci, C. Vernia, "Alienation in italian cities. Social network fragmentation from collective data", preprint: https://arxiv.org/pdf/1410.0501.pdf

[11] M. Disertori, A. Giuliani, "The nematic phase of a system of long hard rods", *Communications in Mathematical Physics* **323**(1), 143-175 (2013)

[12] R.S. Ellis, C.M. Newman, "Limit theorems for sums of dependent random variables occurring in statistical mechanics", *Probability Theory and Related Fields* 44, 117-139 (1978)

[13] A. Giuliani, I. Jauslin, E.H. Lieb, "A Pfaffian formula for monomer-dimer partition functions", *Journal of Statistical Physics* 163(2), 211-238 (2016)

[14] F. Guerra, F.L. Toninelli, "The thermodynamic limit in mean field spin glass models", *Communications in Mathematical Physics* 230(1), 71-79 (2002)

[15] O.J. Heilmann, E.H. Lieb, "Theory of monomer-dimer systems", *Communications in Mathematical Physics* 25, 190-232 (1972)

[16] O.J. Heilmann, E.H. Lieb, "Lattice models for liquid crystals", *Journal of Statistical Physics* 20, 680-693 (1979)

[17] R. Karp, M. Sipser, "Maximum matchings in sparse random graphs", *Proceedings of the 22nd Annual Symposium on Foundations of Computer Science, IEEE Computer Society Press*, 364-375 (1981)

[18] M. Fedele, P. Contucci, "Scaling Limits for Multi-species Statistical Mechanics Mean-Field Models", *Journal of Statistical Physics* 144, 1186 (2011)

[19] R. Moessner, K. S. Raman, "Quantum dimer models", *Introduction to frustrated magnetism, Springer Series in Solid-State Sciences* 164, 437-479 (2011)

[20] D. Panchenko, "The free energy in a multi-species Sherrington-Kirkpatrick model", *Annals of Probability* 43(6), (2015)

[21] J.K. Roberts, "Some properties of mobile and immobile adsorbed films", *Proceedings of the Cambridge Philosophical Society* 34, 399-411 (1938)

[22] L. Zdeborová, M. Mézard, "The number of matchings in random graphs", *Journal of Statistical Mechanics* 5, P05003 (2006)