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Published Version:
Topological dynamics of Nondeterministic Cellular Automata / Pietro Di Lena. - In: INFORMATION AND COMPUTATION. - ISSN 1090-2651. - ELETTRONICO. - 274:(2020), pp. 104532.1-104532.30. [10.1016/j.ic.2020.104532]

This version is available at: https://hdl.handle.net/11585/653252 since: 2021-06-03

Published:

DOI: http://doi.org/10.1016/j.ic.2020.104532

Topological dynamics of Nondeterministic Cellular Automata

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Abstract

Cellular Automata (CA) are discrete dynamical systems and an abstract model of parallel computation. Nondeterministic Cellular Automata (NCA) are the class of multi-valued functions obtained by allowing nondeterminism in CA. In this study we extend to multi-valued functions the definition of some important topological properties and investigate the differences between the dynamical behaviour of one-dimensional NCA and one-dimensional CA in such classes.

Keywords: nondeterministic cellular automata, equicontinuity, sensitivity, transitivity, expansivity, positive expansivity

1. Introduction

Cellular Automata (CA) are discrete dynamical systems and one of the simplest abstract models for parallel computation. The dynamical [4, 6, 10, 8, 14, 23, 24] and computational [5, 11, 13, 15, 16, 22, 26, 29] properties of the CA formalism, as well as those of its asynchronous and non-uniform variants [7, 9, 12], have been well studied in literature.

A cellular automaton is defined by a fixed infinite grid of cells, each one in one of a finite number of possible states. The simplest grid is a *one-dimensional* (1D) infinite line of cells, although it can be in any finite number of dimensions. The value of each cell is updated synchronously at discrete temporal instants. The update rule, called *local rule*, uses a finite amount of information and depends on the current state of the cell and that of its neighbours. CA can be easily extended to nondeterminism by simply allowing a nondeterministic update local rule.

From the mathematical point of view, Nondeterministic Cellular Automata (NCA) are a special class of multi-valued functions, i.e. a function that is allowed to map a point in its domain to more than one point in its range. In particular, NCA multi-valued dynamical systems can be seen as the parallel composition of an uncountable collection of continuous self maps, i.e. the uncountable set of maps that can be defined by selecting cell-specific local rules from a finite set of local rules. The mathematical background on multi-valued functions is not well-developed as for the single-valued counterpart. In particular, standard notions of dynamical system theory, such as equicontinuity and sensitivity to initial conditions, cannot be immediately extended to multi-valued dynamical systems. To the best of our knowledge, the only mention in literature of such topological properties for multi-valued dynamical systems is in recent works on induced dynamics on the hyperspace with Hausdorff metric topology [25, 30, 31, 32], where the multi-valued dynamical systems are, however, defined by the parallel composition of a finite collection of continuous self maps and, thus, do not include the NCA class. The lack of formal mathematical background maybe the reason why, despite its attractiveness and simplicity, the NCA model received so far very little attention in literature [3, 17, 18, 19, 20, 27, 28, 33].

In [17] we started to study the most basic properties of the 1D NCA mappings. In particular, we proved necessary and sufficient conditions that characterize the class of NCA, proved that the NCA multifunction is continuous, and that the image of a compact set under the NCA continuous multifunction is compact. Such properties simplify the effort to study the NCA formalism from the point of view of dynamical system theory. In this study we continue our investigation on the 1D NCA dynamical system starting from some important topological properties that have been well characterized for 1D CA. Part of these studies have already appeared in [18].

In detail, we extend to NCA the definition of well-studied topological properties related to the degree of chaoticity and complex behaviour of CA. The considered properties are equicontinuity, almost equicontinuity, sensitivity, topological transitivity, expansivity and positive expansivity. Our definitions are quite intuitive and are equivalent to their single-valued counterparts when applied to CA. With the help of simple examples we show how, in comparison to CA, the dynamical behaviour of NCA is much more complex and less constrained. The most immediate differences are related to the equicontinuous behaviour. Surjective equicontinuous CA have a strongly periodic behaviour, hence they are bijective and reversible. On the contrary, there are surjective equicontinuous NCA that are transitive and not reversible. Furthermore, equicontinuity points of CA are characterized by the occurrence of blocking words, i.e. finite words that block the information coming from the left and right under the iteration of the map. As a consequence of the existence of blocking words it is possible to prove that for every CA the set of equicontinuous points is either empty or dense, and thus that every CA can be either almost equicontinuous or sensitive. There is no immediate generalization of the concept of blocking word for NCA, which leaves open the question whether there are NCA with a non-empty and non-dense set of equicontinuous points. A further property, not preserved in NCA, is that the set of equicontinuous point of NCA is not inversely invariant. This property is used in the single-valued setting to prove that transitive dynamical systems can be either almost equicontinuous or sensitive. Although it is well known that in the CA dynamical system transitivity implies sensitivity, we can show examples of transitive almost equicontinuous NCA but remains open the question whether there are non-sensitive and non-almost equicontinuous transitive NCA.

The paper is organized as follows. In Section 2 we introduce the basic notation and background on symbolic dynamical systems, CA and NCA. Sections 3-6 are devoted to equicontinuity, sensitivity, transitivity and expansivity, respectively. Section 7 contains the final remarks.

2. Preliminaries

2.1. Symbolic Dynamics

We introduce the basic notation and terminology we will use throughout the rest of the paper. We assume that the reader is familiar with the elementary notions from Symbolic Dynamics and Topology Theory [21, 24].

Let $\mathbb{Z}$ denote the set of integers and $\mathbb{N}$ the set of non-negative integers. Let A be a finite set with at least two elements. We denote with A^k, the set of words over A of length $k > 0$ and with $A^+ = \cup_{k>0} A^k$ the set of finite words on A. The set $A^{\mathbb{Z}}$ denotes the set of doubly infinite sequences $(x_i)_{i\in\mathbb{Z}}$ of symbols $x_i \in A$. Given $x \in A^{\mathbb{Z}}$ we use the shortcut $x_{[i,j]}$ for the sub-word $x_i x_{i+1}..x_j \in A^{j-i+1}$. A sequence containing a periodic repetition of the word $w \in A^+$ is denoted with ${}^\infty w^\infty$, i.e $x = {}^\infty w^\infty$ if $\forall i \in \mathbb{Z}, x_{[i\cdot|w|,(i+1)|w|-1]} = w$.

The mapping $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$, defined by $\sigma(x)_i = x_{i+1}$, is called *shift map*. The pair $(A^{\mathbb{Z}}, \sigma)$ is a dynamical system, called a *full shift*.

Consider the metric $d(x,y) = 2^{-n}$ on $A^{\mathbb{Z}}$, where $n = \min\{|i| \mid x_i \neq y_i\}$. The full shift $A^{\mathbb{Z}}$ endowed with metric d is a Cantor space, i.e. a compact, totally disconnected, metric space. For every word $u \in A^+$ and $i \in \mathbb{Z}$, the set $[u]_i = \{x \in A^{\mathbb{Z}} \mid x_{[i,|u|-1]} = u\}$ is called *cylinder set*. We can extend the cylinder set notation to set of words $U \subseteq A^+$, $[U]_i = \cup_{u \in U}[u]_i$. A cylinder set is a clopen (closed and open) set in $A^{\mathbb{Z}}$. Given $x \in A^{\mathbb{Z}}$ and $\epsilon = 2^{-r} > 0$, the open ball $\mathcal{B}_\epsilon(x) = \{y \in A^{\mathbb{Z}} \mid d(x,y) < \epsilon\}$ coincides with the cylinder set $[x_{[-r,r]}]_{-r}$. Every open set $U \subseteq A^{\mathbb{Z}}$ is defined by a countable union of cylinders.

A configuration $x \in A^{\mathbb{Z}}$ is said to be *bitransitive* if every word in A^+ occurs in x infinitely often on the left and on the right, i.e. $\forall w \in A^+$, $\forall i \in \mathbb{N}, \exists i' \leq -i, \exists i'' \geq i$ such that $x_{[i', i'+|w|-1]} = x_{[i'', i''+|w|-1]} = w$. The orbit of a bitransitive point under σ is dense in $A^{\mathbb{Z}}$. That is, the closure of the orbit of a bitransitive point $x \in A^{\mathbb{Z}}$ is the entire space $A^{\mathbb{Z}} = \overline{\{\sigma^i(x) \mid i \in z\}}$.

A map $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is a *sliding block code* if there exists a *block map* $f : A^{2r+1} \to A$, for some *radius* $r \geq 0$, such that for every point $x \in A^{\mathbb{Z}}$, $F(x)_i = f(x_{[i-r,i+r]})$. We call f the *local rule* of F. Local rules of radius zero are usually called *one-block maps*.

The fundamental theorem of symbolic dynamics [21], states that a mapping $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is a sliding block code if and only if F is *continuous* and *commutes with the shift*, i.e. $F(\sigma(x)) = \sigma(F(x))$. The shift map σ itself is a sliding block code.

2.2. Cellular Automata

The continuous and σ-commuting mappings $(A^{\mathbb{Z}}, F)$ are usually known as *Cellular Automata* (CA). Cellular Automata are a class of discrete time-and-space dynamical systems, which exhibit a rich dynamical behaviour. We review here only the most basic properties of CA dynamical systems that are relevant to this study.

Injective and surjective CA are well characterized. It is well known that every injective CA is surjective, hence bijective, and that its inverse F^{-1} is again a sliding block code [21, 28]. Furthermore, every point in the configuration space of a surjective CA has a bounded number of pre-images [21].

The following topological dynamical properties and their correlations have been widely studied in literature for the class of CA dynamical systems [24].

Definition 2.1. Let $(A^{\mathbb{Z}}, F)$ be a CA.

- The set $\mathcal{E} \subseteq A^{\mathbb{Z}}$ of **equicontinuous points** of $(A^{\mathbb{Z}}, F)$ is defined by

$$x \in \mathcal{E} \iff \forall \epsilon > 0, \exists \delta > 0, \forall y \in B_\delta(x), \forall n \in \mathbb{N}, d(F^n(x), F^n(y)) < \epsilon$$

- $(A^{\mathbb{Z}}, F)$ is **equicontinuous** if $\mathcal{E} = A^{\mathbb{Z}}$
- $(A^{\mathbb{Z}}, F)$ is **almost equicontinuous** if $\mathcal{E}$ is dense in $A^{\mathbb{Z}}$.
- $(A^{\mathbb{Z}}, F)$ is **sensitive to initial conditions** if

  $$\exists \epsilon > 0, \forall x \in A^{\mathbb{Z}}, \forall \delta > 0, \exists y \in B_\delta(x), \exists n \in \mathbb{N}, d(F^n(x), F^n(y)) \geq \epsilon$$

  The constant ϵ is called *sensitivity constant.*
- $(A^{\mathbb{Z}}, F)$ is **topologically transitive** if

  $$\forall U, V \subseteq A^{\mathbb{Z}}, \text{ open and non-empty}, \exists n \in \mathbb{N}, F^n(U) \cap V \neq \emptyset.$$
- $(A^{\mathbb{Z}}, F)$ is **expansive** if it is bijective and

  $$\exists \epsilon > 0, \forall x \neq y \in A^{\mathbb{Z}}, \exists n \in \mathbb{Z}, d(F^n(x), F^n(y)) \geq \epsilon$$

  The constant ϵ is called *expansivity constant.*
- $(A^{\mathbb{Z}}, F)$ is **positively expansive** if

  $$\exists \epsilon > 0, \forall x \neq y \in A^{\mathbb{Z}}, \exists n \in \mathbb{N}, d(F^n(x), F^n(y)) \geq \epsilon$$

  The constant ϵ is called *expansivity constant.*

$\square$

By definition, a sensitive dynamical system cannot have equicontinuous points but the converse is not generally true. However, it is well known that every CA is either sensitive or almost equicontinuous [23]. This dichotomy is a consequence of the presence of *blocking words* on configurations that are equicontinuity points.

Definition 2.2. Let $(A^{\mathbb{Z}}, F)$ be a CA. A word $w \in A^{2d+1}, d \geq 0$ is a *blocking word* if there is some $0 \leq k \leq d$ such that

$$\forall x, y \in [w]_{-d}, \forall n \geq 0, F^n(x)_{[-k,k]} = F^n(y)_{[-k,k]}.$$

By definition, a blocking word is a word that *blocks* the information coming from the right and left under the iteration of the map. In CA, an equicontinuous point is characterised by the occurrence of infinitely many blocking words in both directions. This implies that every bitransitive point of a CA that has equicontinuity points is equicontinuous and, thus, that every CA can have a either dense or empty set of equicontinuoity points. In equicontinuous CA every sufficiently large word is blocking. In fact, equicontinuous CA have a strongly periodic behaviour. In particular, surjective equicontinuous CA are injective and behave like the identity under some power of the map [4], while non-surjective equicontinuous CA are *eventually periodic*, i.e. there exists a preperiod $k > 0$ and a period $n > 0$ such $F^{k+n} = F^k$ [23]. The most simple class of equicontinuous CA is the class of mappings with radius zero, i.e. the class of one-block maps.

It is well known that transitive dynamical systems can be either sensitive or almost equicontinuous [1]. This is basically a (not immediate) consequence of the fact that the set of equicontinuous points of a dynamical system is inversely invariant. On the other end, the dynamics of transitive CA is more constrained, since transitive CA are known to be sensitive [23]. Finally, note that the definitions of positively expansivity and expansivity imply sensitivity. We further have that positively expansive and expansive CA are transitive [23]. Since transitivity implies surjectivity, expansive and positively expansive CA are surjective. The most simple example of expansive CA is the shift map, while the most simple example of positively expansive CA is the so-called *rule 90* on binary alphabet, whose local rule is defined by $\forall a, b, c \in \{0, 1\}, f(a, b, c) = (a + c) \mod 2$.

2.3. Nondeterministic Cellular Automata

Nondeterministic Cellular Automata (NCA) are the class of *multi-valued functions* (or *multimaps*) definable by *nondeterministic or multi-valued block maps*. By definition, CA are a special subclass of NCA.

Definition 2.3. Let A be some alphabet A with at least two elements.

- A (multi-valued) block map $f : A^{2r+1} \rightrightarrows A$ of radius $r \geq 0$ is a *nondeterministic block map* if,

$$\forall w \in A^{2r+1}, \emptyset \neq f(w) \subseteq A$$

- A (multi-valued) mapping $F : A^{\mathbb{Z}} \rightrightarrows A^{\mathbb{Z}}$ is a *nondeterministic cellular automaton* if there is some nondeterministic block map $f : A^{2r+1} \rightrightarrows A$ such that:

$$\forall x \in A^{\mathbb{Z}}, F(x) = \{y \in A^{\mathbb{Z}} \mid \forall i \in \mathbb{Z}, y_i \in f(x_{[i-r,i+r]})\}$$

Orbits and trajectories of multi-valued dynamical systems are defined in terms of sets of points.

Definition 2.4. Let $F : A^{\mathbb{Z}} \rightrightarrows A^{\mathbb{Z}}$ be a multimap and let $x \in A^{\mathbb{Z}}$.

- The *orbit* of x under F is the set:

$$O(x) = \{F^n(x) \mid n \geq 0\}$$

- The *trajectory* of x under F is the sequence of sets:

$$(X_n)_{n\geq 0} \text{ such that } X_n = F^n(x)$$

Continuity notion for (single-valued) functions can be extended to multi-valued functions by means of the dual concepts of *upper* and *lower semicontinuity* (also referred to as *upper* and *lower hemicontinuity*), which collapse to the ordinary notion of continuity in the single-valued setting. The upper and lower semicontinuity properties have a simple characterization in terms of preimages of closed and open sets (see [2]).

Definition 2.5. Let $F : A^{\mathbb{Z}} \rightrightarrows A^{\mathbb{Z}}$ be a multimap.

- F is said *upper semicontinuous* at $x \in A^{\mathbb{Z}}$ if for any open subset $V \subseteq A^{\mathbb{Z}}$ such that $F(x) \subseteq V$,

$$\exists \delta > 0 \text{ such that } \forall x' \in \mathcal{B}_\delta(x),\ F(x') \subseteq V$$

- F is said *lower semicontinuous* at $x \in A^{\mathbb{Z}}$ if for any open subset $V \subseteq A^{\mathbb{Z}}$ such that $F(x) \cap V \neq \emptyset$,

$$\exists \delta > 0 \text{ such that } \forall x' \in \mathcal{B}_\delta(x),\ F(x') \cap V \neq \emptyset$$

- F is said *continuous* if it is both lower and upper semicontinuous at every $x \in A^{\mathbb{Z}}$.

Proposition 2.1. *Let $F : A^{\mathbb{Z}} \rightrightarrows A^{\mathbb{Z}}$ be a multimap.*

1. *F is upper semicontinuous if and only if for any closed set $V \subseteq A^{\mathbb{Z}}$, $F^{-1}(V) = \{x \in A^{\mathbb{Z}} \mid V \cap F(x) \neq \emptyset\}$ is closed in $A^{\mathbb{Z}}$.*
2. *F is lower semicontinuous if and only if for any open set $V \subseteq A^{\mathbb{Z}}$, $F^{-1}(V) = \{x \in A^{\mathbb{Z}} \mid V \cap F(x) \neq \emptyset\}$ is open in $A^{\mathbb{Z}}$.*

In the single-valued setting, upper and lower semicontinuity are equivalent and collapse to the ordinary notion of continuity.

Proposition 2.2. *Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a map. The following conditions are equivalent.*

1. *F is continuous.*
2. *For every open set $U \subseteq A^{\mathbb{Z}}$, $F^{-1}(U)$ is open in $A^{\mathbb{Z}}$.*
3. *For every closed set $V \subseteq A^{\mathbb{Z}}$, $F^{-1}(V)$ is closed in $A^{\mathbb{Z}}$.*

It is easy to prove that multi-valued block mappings are σ-commuting and continuous. However, these two properties alone are not sufficient to characterize the class of multi-valued functions definable by multi-valued block maps.

Definition 2.6. Let $F : A^{\mathbb{Z}} \rightrightarrows A^{\mathbb{Z}}$ be a multimap.

- We say that F is *locally independent at $x \in A^{\mathbb{Z}}$* if

  $$y \notin F(x) \text{ if and only if } \exists i \in \mathbb{Z} \text{ such that } y_i \notin F(x)_i$$

  or, equivalently,

  $$y \in F(x) \text{ if and only if } \forall i \in \mathbb{Z} \text{ we have } y_i \in F(x)_i$$

  where

  $$F(x)_i = \{a \in A \mid \exists z \in F(x), z_i = a\}$$

- We say that F is *locally independent* if it is locally independent at every $x \in A^{\mathbb{Z}}$.

Theorem 2.1. *[17] A multimap $F : A^{\mathbb{Z}} \rightrightarrows A^{\mathbb{Z}}$ is a NCA if and only if it is continuous, σ-commuting and locally independent.*

It is not generally true for multi-valued functions that the continuous image of a compact set is compact. It is possible to prove that this property holds for multi-valued block mappings.

Theorem 2.2. *[17] Let $(A^{\mathbb{Z}}, F)$ be a NCA. Then $F(U)$ is compact (closed) for every compact (closed) subset $U \subseteq A^{\mathbb{Z}}$.*

Two interesting classes of NCA are the class of surjective and reversible NCA. In CA, surjectivity imposes strong constrains to the number of preimages of configurations. In particular, every configuration of a surjective CA has a bounded number of preimages. On the contrary, surjective strictly non-deterministic CA are characterized as the class of NCA that have a dense set of configurations with uncountable number of both images and preimages [17]. This is basically a consequence of the following property.

Definition 2.7. Let $(A^{\mathbb{Z}}, F)$ a NCA of radius r. We say that $(A^{\mathbb{Z}}, F')$ of radius r is a **deterministic sub-NCA** of F if the block map $f' : A^{2r+1} \to A$ of F' has the following property

$$\forall w \in A^{2r+1}, \emptyset \neq f'(w) \in f(w)$$

Definition 2.8. Let $(A^{\mathbb{Z}}, F)$ a NCA. We say that F is surjective if

$$\forall y \in A^{\mathbb{Z}}, \exists x \in A^{\mathbb{Z}}, y \in F(x)$$

Proposition 2.3. *[17] Let $(A^{\mathbb{Z}}, F)$ be a strictly NCA. Then there is a non-surjective deterministic sub-NCA F' of F.*

The class of deterministic sub-maps of a NCA gives us also a sufficient but not necessary condition for surjectivity.

Proposition 2.4. *[17] Let $(A^{\mathbb{Z}}, F)$ be a NCA. If there is some surjective deterministic sub-NCA F' of F then $(A^{\mathbb{Z}}, F)$ is surjective.*

In the single-valued CA setting reversibility coincides with the injectivity, hence bijectivity, property. In the multi-valued setting, the scenario is more complex.

Definition 2.9. Let $(A^{\mathbb{Z}}, F)$ be a NCA.

- The *reversed* map $F^{-1} : F(A^{\mathbb{Z}}) \rightrightarrows A^{\mathbb{Z}}$ is defined by

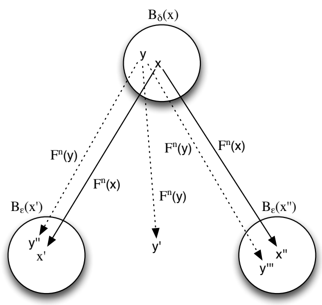

Figure 1: lower equicontinuous point - upper sensitive

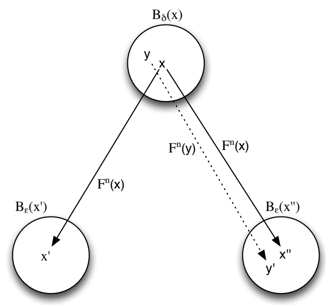

Figure 2: upper equicontinuous point - lower sensitive

$$F^{-1}(x) = \{y \in A^{\mathbb{Z}} \mid x \in F(y)\}$$

- We say that F is a *reversible* NCA if F^{-1} is a nondeterministic sliding block code.

Note that, by definition F^{-1} is a NCA only if F is both reversible and surjective. If F is reversible but not surjective, then its inverse F^{-1} is a nondeterministic sliding block code from the subshift $F(A^{\mathbb{Z}}) \subset A^{\mathbb{Z}}$ to $A^{\mathbb{Z}}$.

Definition 2.10. Let $(A^{\mathbb{Z}}, F)$ be a NCA. We say that F is injective if

$$\forall x, y \in A^{\mathbb{Z}}, x \neq y, F(x) \cap F(y) = \emptyset.$$

In [17] we showed that there are no injective NCA, with the exception of the class of injective CA. Moreover, we showed that if a surjective CA $(A^{\mathbb{Z}}, F)$ is reversible, then $(A^{\mathbb{Z}}, F^{-1})$ is an injective CA, i.e. it is not multi-valued. This implies that if $(A^{\mathbb{Z}}, F)$ is a strictly multi-valued, surjective and reversible NCA, then $(A^{\mathbb{Z}}, F^{-1})$ is again a strictly multi-valued, surjective and reversible NCA. A further characteristic of reversible NCA is that they don't need to be surjective. This property is in contrast with the scenario in the single-valued setting in which reversibility (i.e. injectivity) implies surjectivity. The simplest, non trivial example of (surjective or not surjective) reversible NCA is the class of multi-valued one-block maps.

3. Equicontinuity

In this section we extend the equicontinuity notion to multi-valued functions and point out the main differences between equicontinuous CA and equicontinuous NCA. In particular, we show here an example of NCA whose set of equicontinuous points is not inversely invariant. We also show that there is no immediate generalization of the blocking word definition for NCA. In fact, we show an example of an almost equicontinuous NCA that has a point which is not equicontinuous although it contains infinitely many occurrences of blocking words. Furthermore, we show an example of surjective and equicontinuous NCA that is not reversible.

A point x is called equicontinuous if the family of iterations $(F^n)_{n\geq 0}$ is equicontinuous at x. As for the continuity notion for multi-valued functions, the dual properties of *lower equicontinuity* and *upper equicontinuity* facilitate the extension of equicontinuity to iterations of the multimap.

Definition 3.1. Let $F : A^{\mathbb{Z}} \rightrightarrows A^{\mathbb{Z}}$ be a NCA.

- We say that $x \in A^{\mathbb{Z}}$ is an **upper equicontinuous** point (Fig. 2) if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall y \in B_\delta(x), \forall n \geq 0, F^n(y) \subseteq B_\epsilon(F^n(x))$$

- We say that $x \in A^{\mathbb{Z}}$ is a **lower equicontinuous** point (Fig. 1) if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall y \in B_\delta(x), \forall n \geq 0, F^n(x) \subseteq B_\epsilon(F^n(y))$$

- We say that $x \in A^{\mathbb{Z}}$ is an **equicontinuous** point if it is both upper and lower equicontinuous:

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall y \in B_\delta(x), \forall n \geq 0, B_\epsilon(F^n(x)) = B_\epsilon(F^n(y))$$

Definition 3.2. Let $\mathcal{E} \subseteq A^{\mathbb{Z}}$ be the set of **equicontinuous points** of $(A^{\mathbb{Z}}, F)$.

- We say that $(A^{\mathbb{Z}}, F)$ is **equicontinuous** if $\mathcal{E} = A^{\mathbb{Z}}$.
- We say that $(A^{\mathbb{Z}}, F)$ is **almost equicontinuous** if $\mathcal{E}$ dense in $A^{\mathbb{Z}}$.

The most simple class of equicontinuous CA is the class of mappings with local rules of radius zero. Such class can be easily characterized also for NCA.

Proposition 3.1. *Any NCA with radius zero is equicontinuous.*

Proof. If the local rule f has radius zero, we have that $\forall x \in A^{\mathbb{Z}}$, $\forall n \geq 0$, $F^n(x)_i = f^n(x_i)$. This implies that $\forall \epsilon = 2^{-k} > 0$, if $y \in B_\epsilon(x) = [x_{[-k,k]}]_{-k}$ then $\forall n \geq 0$

$$\begin{aligned} B_\epsilon(F^n(x)) &= [F^n(x)_{[-k,k]}]_{-k} = [f^n(x_{[-k,k]})]_{-k} = [f^n(y_{[-k,k]})]_{-k} = \\ &= [F^n(y)_{[-k,k]}]_{-k} = B_\epsilon(F^n(y)) \end{aligned}$$

□

It is well known that every surjective equicontinuous CA is injective, hence reversible. We have already shown in [17] that every NCA with radius zero is reversible. Thus, multi-valued local rules of radius zero give rise to a non trivial class of (either surjective or not) equicontinuous and reversible NCA. However, we can easily show that not every surjective and equicontinuous NCA is reversible.

Example 3.1. (**Irreversible and equicontinuous NCA**) *Consider the NCA $(A^{\mathbb{Z}}, F)$ on the alphabet $A = \{0,1\}$, defined by the following local rule of radius 1:*

$$\forall a,b,c \in A, f(a,b,c) = \begin{cases} \{0,1\} & \text{if } a=1, b=0, c=1 \\ \{b\} & \text{otherwise} \end{cases}$$

The mapping F is essentially the identity on A, except for the word 101, which is mapped nondeterministically by the local rule to $\{0,1\}$. F is clearly surjective, since $\forall x \in A^{\mathbb{Z}}, x \in F(x)$. In order to see that $(A^{\mathbb{Z}}, F)$ is equicontinuous, note that 1 is a quiescent symbol, i.e. $\forall x \in A^{\mathbb{Z}}, \forall n \geq 0$ if $x_i = 1$ then $F^n(x)_i = \{1\}$. The symbol 0 is quiescent everywhere except when it is immediately surrounded by two (quiescent) 1s. Then,

$$\forall w \in A^3, \forall x,y \in [w]_{-1}, \forall n \geq 0, F^n(x)_0 = F^n(y)_0$$

and, generalizing,

$$\forall w \in A^{2k+1}, \forall x,y \in [w]_{-k}, \forall n \geq 0, F^n(x)_{[-k+1,k-1]} = F^n(y)_{[-k+1,k-1]}$$

which implies equicontinuity. We conclude by showing that F is not reversible. Consider the configuration $\tilde{x} = {}^\infty 1^\infty$, and note that $\tilde{x} \in F({}^\infty(01)^\infty)$ and $\tilde{x} \notin F({}^\infty 0^\infty)$. Now, if F^{-1} is defined by some multi-valued block map $f^{-1} : A^{2k+1} \rightrightarrows A$, the only possibility is that $f^{-1}(1^{2k+1}) = \{0,1\}$. But in this way, $F^{-1}(\tilde{x}) = A^{\mathbb{Z}}$, while ${}^\infty 0^\infty \notin F^{-1}(\tilde{x})$. □

In topological (single-valued) dynamical systems, the set of equicontinuous points is inversely invariant. The (not immediate) consequence of such property is that transitive dynamical systems can be either sensitive or almost equicontinuous. As shown by the following example, this property does not hold for multi-valued mappings.

Example 3.2. (**Almost equicontinuous NCA with not inversely invariant set of equicontinuous points**) *Consider the NCA* $(A^{\mathbb{Z}}, F)$ *on alphabet* $A = \{0, 1, 2\}$*, defined by the following multi-valued local rule of radius* 1*:*

$$\forall a, b, c \in A, f(a, b, c) = \begin{cases} \{2\} & \textit{if } a = 2 \textit{ or } b = 2 \textit{ or } c = 2 \\ \{0, 2\} & \textit{if } a = b = c = 0 \\ \{c\} & \textit{otherwise} \end{cases}$$

Note that, the symbol 2 *is a quiescent symbol that spreads to the left and to the right. The point* ${}^{\infty}2^{\infty}$ *is thus an equicontinuous point of* $(A^{\mathbb{Z}}, F)$*. Consider the set of sequences that contain infinitely many occurrences of the symbol* 2 *to the left and to the right.*

$$U = \{x \in A^{\mathbb{Z}} \mid \forall i \in \mathbb{N}, \exists k' \geq i, k'' \leq -i, \textit{ such that } x_{k'} = 2, x_{k''} = 2\}$$

Clearly U *is dense and, since the quiescent symbol* 2 *spreads to the left and to the right, every element of* U *is an equicontinuity point, i.e.* $U \subseteq \mathcal{E}$*. We show that* $\mathcal{E} \neq A^{\mathbb{Z}}$*, since* $\tilde{x} = {}^{\infty}0^{\infty}$ *is not an equicontinuous point. Let* $\delta = 2^{-k}, k \geq 0$ *and consider the point* $y \in B_{\delta} = [0^{2k+1}]_{-k}$ *such that*

$$y_i = \begin{cases} \tilde{x}_i & \textit{if } i \neq k+1 \\ 1 & \textit{if } i = k+1 \end{cases}$$

Then $F^{k+1}(\tilde{x})_0 = \{0, 2\} \neq \{1, 2\} = F^{k+1}(y)_0$*, which implies that* $\tilde{x}$ *is not an equicontinuous point and that* $(A^{\mathbb{Z}}, F)$ *is almost equicontinuous but not equicontinuous. To conclude, note that* $\tilde{x} \in F^{-1}(U)$*, since* ${}^{\infty}2^{\infty} \in F(\tilde{x})$*. But, since* $\tilde{x} \notin U$*, we conclude that* $F^{-1}(\mathcal{E}) \not\subset \mathcal{E}$*.* □

Recall that, in CA dynamical systems equicontinuity is strictly related to the presence of blocking words, i.e. words that block the information coming from the left and from the right under the iteration of the map. In particular, equicontinuity points of CA are characterized by the presence of infinitely many occurrences of blocking words. This strong characterization implies that the set of equicontinuous points of a cellular automaton is either empty or dense. There is no immediate generalization of such property for NCA, since the formal property of blocking word (Definition 2.2), when applied in the non-deterministic setting, gives rise to words that do not necessarily block the information flow from the right to the left (and conversely). This leaves open the question whether there are NCA whose set of equicontinuous points is non-empty and non-dense. In the following example we build an almost equicontinuous NCA that has a not equicontinuous point with infinitely many occurrences of blocking words.

Example 3.3. (**Almost equicontinuous NCA that has a point not equicontinuous containing infinitely many blocking words**) *Consider the NCA $(A^{\mathbb{Z}}, F)$ on the alphabet $A = \{0, 1, 2\}$, defined by the following multi-valued local rule of radius 1:*

$$\forall a, b, c \in A, f(a, b, c) = \begin{cases} \{0, 1, 2\} & \text{if } b = 2 \\ \{0\} & \text{if } b \neq 2 \text{ and } (a = 2 \text{ or } c = 2) \\ \{c\} & \text{if } a, b, c \in \{0, 1\} \end{cases}$$

Note that the function F behaves like the shift map on $\{0, 1\}^{\mathbb{Z}}$ and that for every $x \in A^{\mathbb{Z}}, F(x) \cap \{0, 1\}^{\mathbb{Z}} \neq \emptyset$. On the other end, the symbol 2 does not move and generates all the other symbols. We first show that $(A^{\mathbb{Z}}, F)$ has a dense set of equicontinuous points. It is easy to see that $\tilde{x} = {}^{\infty}2^{\infty}$ is an equicontinuity point, since $\forall \delta = 2^{-d}, d \geq 0$ we have that

$$\forall y \in B_\delta(\tilde{x}), \forall n > 0, F^n(y)_{[-d,d]} = F^n(\tilde{x})_{[-d,d]} = A^{2d+1}$$

In the same way, for every $w \in A^+$, all the points in

$$U_w = \{x \in A^{\mathbb{Z}} \mid \exists i \in \mathbb{Z}, x_{[i,i+|w|-1]} = w \wedge \forall j \notin [i, i + |w| - 1], x_j = 2\}$$

are equicontinuous. In fact, note that

$$\forall w \in A^+, \forall x \in [2^{|w|}w2^{|w|}]_{-w}, \forall n > |w|, F^n(x)_{[0,|w|-1]} = \{0, 1\}^{|w|}$$

Then the dense set $U = \cup_{w \in A^+} U_w$ is contained in $\mathcal{E}$.

Now, fix some $\epsilon = 2^{-k}, k \geq 0$. For simplicity we consider $k = 0$, but what follows can be generalized to larger k. By definition 2.2, the word $w = 2$ is a blocking word. In fact, for all $x, y \in [2]_0$, and for all $n \in \mathbb{N}$

$$F^n(x)_0 = F^n(y)_0 = A$$

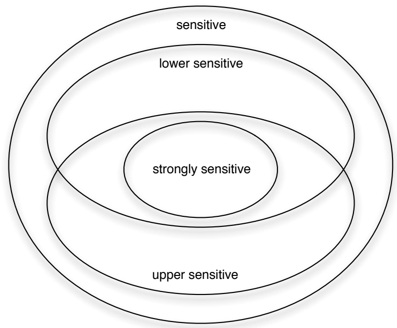

Figure 3: Sensitivity classes

On the other end, we show that w does not always block the information coming from the left and right by building a not equicontinuous configuration that contains infinitely many occurrences of w. Consider, the periodic sequence $\tilde{x} = {}^{\infty}(020)^{\infty}$. Note that, in $\tilde{x}$ only the 2 symbol can generate the 1 symbol and that, by definition of the local rule, every symbol generated by 2 is surrounded by a 0 symbol immediately to the left and to the right. This implies that, under the iteration of the map, it is not possible that two 1s get closer together in the trajectories of $\tilde{x}$. In particular, for every $n \in \mathbb{N}$ and for every $z \in F^n(\tilde{x})$, the 3-length word 111 cannot appear in z. Then, fix $\epsilon = 2^{-1}$. For every $\delta = 2^{-d}, d \geq 1$ we can build the configuration $y \in B_\delta(\tilde{x})$ such that

$$y_i = \begin{cases} 1 & \text{if } i > d \\ \tilde{x}_i & \text{otherwise} \end{cases}$$

By definition of F, the configuration ${}^{\infty}(0)(1)^{\infty} \in F(y)$ and, since F is the shift map on $\{0,1\}^{\mathbb{Z}}$, the 111 word will travel infinitely far to the left under the iteration of the map. Then

$$\exists n > 0, 111 \in F^n(y)_{[-1,1]}, \text{ while } \forall n \geq 0, 111 \notin F^n(\tilde{x})_{[-1,1]}$$

which implies that $\tilde{x}$ is not a point of equicontinuity for F. □

4. Sensitivity

In sensitive dynamical systems small perturbations of the initial configuration may lead to significantly different trajectories. In some sense, sensitivity is the opposite of equicontinuity and, in fact, the two notions are strictly related: a sensitive dynamical system cannot have points of equicontinuity. The converse is not generally true, although it is for the CA dynamical systems.

There is no standard definition of sensitivity for multimaps. We extend the usual definition of sensitivity to multimaps by introducing the notion of upper and lower sensitivity. We get different classes of sensitivity that coincide with the classical definition when the mapping is single-valued.

Definition 4.1. Let $F : A^{\mathbb{Z}} \rightrightarrows A^{\mathbb{Z}}$ be a NCA.

- We say that $(A^{\mathbb{Z}}, F)$ is **upper sensitive** (Fig. 1) if

$$\exists \epsilon > 0,\ \forall x \in A^{\mathbb{Z}}, \forall \delta > 0, \exists y \in B_\delta(x), \exists n \geq 0, F^n(y) \not\subset B_\epsilon(F^n(x))$$

- We say that $(A^{\mathbb{Z}}, F)$ is **lower sensitive** (Fig. 2) if

$$\exists \epsilon > 0,\ \forall x \in A^{\mathbb{Z}}, \forall \delta > 0, \exists y \in B_\delta(x), \exists n \geq 0, F^n(x) \not\subset B_\epsilon(F^n(y))$$

- We say that $(A^{\mathbb{Z}}, F)$ is **sensitive** if

$$\exists \epsilon > 0, \forall x \in A^{\mathbb{Z}}, \forall \delta > 0, \exists y \in B_\delta(x), \exists n \geq 0, B_\epsilon(F^n(x)) \neq B_\epsilon(F^n(y))$$

- We say that $(A^{\mathbb{Z}}, F)$ is **strongly sensitive** if

$$\exists \epsilon > 0, \forall x \in A^{\mathbb{Z}}, \forall \delta > 0, \exists y \in B_\delta(x), \exists n \geq 0, B_\epsilon(F^n(y)) \cap B_\epsilon(F^n(x)) = \emptyset$$

The ϵ constant is called *sensitivity constant* of the map. All four sensitivity classes imply no equicontinuous points. Note that upper and lower sensitivity imply sensitivity and that strong sensitivity immediately implies lower and upper sensitivity. While in the single-valued setting all four definitions are equivalent, in the multi-valued setting, the four definitions give rise to different classes of sensitivity. We show that all such classes are non empty and distinct (see Fig. 3).

In the following two lemmas we prove two useful properties:

1. if there is one point whose trajectory (i.e. sequence of sets generated by iterating the map) appears in the trajectory of every other point, then the NCA is not lower sensitive,

2. if there is a point that is mapped to the entire space, then the NCA is not upper sensitive.

We will use these two properties to build examples of NCA that are not lower-or-upper sensitive.

Lemma 4.1. *Let $(A^{\mathbb{Z}}, F)$ be a NCA. Assume that there is some point $x \in A^{\mathbb{Z}}$ such that*

$$\forall n > 0, \forall y \in A^{\mathbb{Z}}, F^n(x) \subseteq F^n(y)$$

then $(A^{\mathbb{Z}}, F)$ is not lower sensitive.

Proof. Let $(A^{\mathbb{Z}}, F)$ be of radius $r \geq 0$ and assume there is one point $x \in A^{\mathbb{Z}}$ as defined in the statement. Consider some $\epsilon > 0$. Then

$$\forall \delta > 0, \forall y \in B_\delta(x), \forall n > 0, F^n(x) \subseteq F^n(y) \subseteq B_\epsilon(F^n(y)),$$

which implies that F is not lower sensitive. □

Lemma 4.2. *Let $(A^{\mathbb{Z}}, F)$ be a NCA. Assume that there is some point $x \in A^{\mathbb{Z}}$ such that*

$$F(x) = A^{\mathbb{Z}}.$$

Then F in not upper sensitive.

Proof. First of all, note that if $F(x) = A^{\mathbb{Z}}$, then $\forall n > 0, F^n(x) = A^{\mathbb{Z}}$. Consider some $\epsilon > 0$, then

$$\forall \delta > 0, \forall y \in B_\delta(x), \forall n > 0, F^n(y) \subseteq A^{\mathbb{Z}} \subseteq B_\epsilon(F^n(x)),$$

which implies that F is not upper sensitive. □

All the following examples are based on the shift map. We first show that sensitivity does not imply lower and upper sensitivity.

Example 4.1. (**Sensitive but not lower/upper sensitive NCA**) *Consider the NCA $(A^{\mathbb{Z}}, F)$ on alphabet $A = \{0, 1\}$ defined by the following multi-valued local rule of radius 1:*

$$\forall a, b, c \in A, f(a, b, c) = \begin{cases} \{0\} & \text{if } c = 0 \\ \{0, 1\} & \text{if } c = 1 \end{cases}$$

This multi-valued map contains both the shift map and the constant map, which sends every configuration to the uniform configuration $^\infty 0^\infty$.

We first show that $(A^{\mathbb{Z}}, F)$ *is sensitive. Consider some configuration* $x \in A^{\mathbb{Z}}$ *and note that, by definition of the local rule* f, *for every* $i > 0$

$$F^i(x)_0 = \begin{cases} \{0\} & \text{if } x_i = 0 \\ \{0,1\} & \text{if } x_i = 1 \end{cases}$$

Let $x \in A^{\mathbb{Z}}$ *and let* $k > 0$. *Let* $y \in [x_{[-k,k]}]_{-k}$ *be such that*

$$y_i = \begin{cases} x_i & \text{if } i \neq k+1 \\ 1 - x_i & \text{if } i = k+1 \end{cases}$$

Then $F^{k+1}(x)_0 \neq F^{k+1}(y)_0$, *which implies that* F *is sensitive with sensitivity constant* $\epsilon = 2^0$. *We now show that* F *is neither lower nor upper sensitive.*

1. F *is not upper sensitive. Consider the configuration* $\tilde{x} = {}^\infty 1^\infty \in A^{\mathbb{Z}}$. *We have that,* $\tilde{x}$ *is mapped to the entire configuration space, i.e.*

   $$\forall n > 0, F^n(\tilde{x}) = A^{\mathbb{Z}}.$$

   then, by Lemma 4.2, F *is not upper sensitive.*
2. F *is not lower sensitive. Consider the configuration* $\tilde{x} = {}^\infty 0^\infty \in A^{\mathbb{Z}}$. *We have that* $\tilde{x}$ *is a quiescent configuration that appears in every trajectory, i.e.*

   $$\forall n > 0, \forall y \in A^{\mathbb{Z}}, F^n(\tilde{x}) = \{\tilde{x}\} \subseteq F^n(y).$$

   then, by Lemma 4.1, F *is not lower sensitive.*

□

The following two examples show that upper sensitivity does not imply lower sensitivity, and conversely.

Example 4.2. (**Upper sensitive and not lower sensitive NCA**) *Consider the NCA* $(A^{\mathbb{Z}}, F)$ *on alphabet* $A = \{0, 1, 2\}$ *defined by the following multi-valued local rule of radius* 1*:*

$$\forall a, b, c \in A, f(a,b,c) = \begin{cases} \{0\} & \text{if } c = 0 \\ \{0,1\} & \text{if } c = 1 \\ \{0,2\} & \text{if } c = 2 \end{cases}$$

Consider some configuration $x \in A^{\mathbb{Z}}$ and note that, by definition of the local rule f, for every $i > 0$

- *if $x_i = 0$, then $F^i(x)_0 = \{0\}$,*
- *if $x_i = 1$, then $F^i(x)_0 = \{0, 1\}$,*
- *if $x_i = 2$, then $F^i(x)_0 = \{0, 2\}$,*

Then, for every $x \in A^{\mathbb{Z}}$ and $\delta = 2^{-k}, k \geq 0$ we can build the configuration $y \in B_\delta(x) = [x_{[-k,k]}]_{-k}$ such that

$$y_i = \begin{cases} x_i & \text{if } i \neq k+1 \\ 2 & \text{if } i = k+1 \text{ and } x_i \in \{0,1\} \\ 1 & \text{if } i = k+1 \text{ and } x_i = 2 \end{cases}$$

It is clear that $F^{k+1}(y)_0 \not\subset F^{k+1}(x)_0$, which proves that F is upper sensitive with sensitivity constant $\epsilon = 2^0$. In order to see that F is not lower sensitive, consider the uniform configuration $\tilde{x} = {}^{\infty}0^{\infty} \in A^{\mathbb{Z}}$, which is mapped to itself, i.e. $\forall n \geq 0, F^n(\tilde{x}) = \{\tilde{x}\}$. Note that $\forall y \in A^{\mathbb{Z}}$ and $\forall n > 0, F^n(\tilde{x}) \subseteq F^n(y)$, then by Lemma 4.1, F is not lower sensitive. □

Example 4.3. (**Lower sensitive and not upper sensitive NCA**) *Consider the NCA on alphabet $A = \{0, 1, 2\}$ defined by the following multi-valued local rule of radius 1:*

$$\forall a, b, c \in A, f(a, b, c) = \begin{cases} \{0, 1, 2\} & \text{if } a = b = c = 0 \\ \{c\} & \text{otherwise} \end{cases}$$

Note that, for every $x \in A^{\mathbb{Z}}$ and $i > 0$

$$F^i(x)_0 = \begin{cases} \{0, 1, 2\} & \text{if } x_{[i-2,i]} = 000 \\ \{x_i\} & \text{otherwise} \end{cases}$$

For every $x \in A^{\mathbb{Z}}$ and for every $\delta = 2^{-k}, k \geq 0$ we can build the configuration $y \in B_\delta(x) = [x_{[-k,k]}]_{-k}$ such that

$$y_i = \begin{cases} x_i & \text{if } i \neq k+1 \\ 2 & \text{if } i = k+1 \text{ and } x_i \in \{0,1\} \\ 1 & \text{if } i = k+1 \text{ and } x_i = 2 \end{cases}$$

By construction, we have that, if $F^{k+1}(x)_0 = \{0,1,2\}$ or $F^{k+1}(x)_0 = \{1\}$, then $F^{k+1}(y)_0 = \{2\}$, while if $F^{k+1}(x)_0 = \{1\}$ then $F^{k+1}(y)_0 = \{2\}$. In both cases, $F^{k+1}(x)_0 \not\subset F^{k+1}(y)_0$, which implies that F is lower sensitive with sensitivity constant $\epsilon = 2^0$. In order to see that F is not upper sensitive, note that the configuration $\tilde{x} =^\infty 0^\infty \in A^{\mathbb{Z}}$ is mapped to the entire space, i.e. $F^n(\tilde{x}) = A^{\mathbb{Z}}$, $\forall n > 0$. Then, by Lemma 4.2, F is not upper sensitive. $\square$

Since CA are a subset of NCA, all sensitive CA belong to the strongly sensitive class. We show that such class contains also strictly NCA. The simplest example is the multi-valued reformulation of the shift map.

Example 4.4. (**Strongly sensitive NCA**) *Consider the NCA $(A^{\mathbb{Z}}, F)$ on alphabet $A = \{0,1,2,3\}$ defined by the following multi-valued local rule of radius 1:*

$$\forall a,b,c \in A, f(a,b,c) = \begin{cases} \{0,2\} & \text{if } c \in \{0,2\} \\ \{1,3\} & \text{if } c \in \{1,3\} \end{cases}$$

Note that, F is essentially a nondeterministic shift map on the two sets $\{0,2\}$ and $\{1,3\}$:

$$\forall i > 0, F^i(x)_0 = \begin{cases} \{0,2\} & \text{if } x_i \in \{0,2\} \\ \{1,3\} & \text{if } x_i \in \{1,3\} \end{cases}$$

For every $x \in A^{\mathbb{Z}}$ and $\delta = 2^{-k}, k \geq 0$ there is the configuration $y \in B_\delta(x)$ such that

$$y_i = \begin{cases} x_i & \text{if } i \neq k+1 \\ (x_i + 1) \mod 4 & \text{if } i = k+1 \end{cases}$$

It is easy to see that $F^{k+1}(x)_0 \cap F^{k+1}(y)_0 = \emptyset$, which implies that F is strongly sensitive with sensitivity constant $\epsilon = 2^0$.

We leave open the question whether there are NCA, both upper and lower sensitive, that are not strongly sensitive.

5. Transitivity

In a *topologically transitive* dynamical system every non-empty open set has points whose orbit intersects any other non-empty open set. The topological definition of transitivity does not need to be re-defined for multi-valued mappings. While, in general, a transitive dynamical system can be either sensitive or almost equicontinuous, it is well known that topologically transitive CA are sensitive. We show here some examples of equicontinuous and transitive NCA. It is open the question whether a transitive NCA needs to be either sensitive or almost equicontinuous.

The following general property holds for any continuous endomorphism of a compact space.

Proposition 5.1. *Any transitive NCA is surjective.*

Proof. Since F is topologically transitive, for every non-empty open set $U \in A^{\mathbb{Z}}$, $F(A^{\mathbb{Z}}) \cap U \neq \emptyset$, which implies that $F(A^{\mathbb{Z}})$ is dense in $A^{\mathbb{Z}}$. Since F is continuous and $A^{\mathbb{Z}}$ compact, $F(A^{\mathbb{Z}})$ is closed, then $F(A^{\mathbb{Z}}) = A^{\mathbb{Z}}$. □

The following sufficient condition is useful to build examples of transitive NCA.

Lemma 5.1. *If there is a deterministic sub-NCA F' such that $(A^{\mathbb{Z}}, F')$ is transitive, then $(A^{\mathbb{Z}}, F)$ is transitive.*

Proof. If $(A^{\mathbb{Z}}, F')$ is transitive then, for every non-empty open sets $U, V \subseteq A^{\mathbb{Z}}$ there is $n \geq 0$ such that $F^n(U) \cap V \supseteq F'^n(U) \cap V \neq \emptyset$. □

By Lemma 5.1, all the examples of sensitive NCA in Section 4 are transitive, since all of them contain the shift map as deterministic sub-NCA. We conclude this section by showing examples of reversible and non-reversible transitive equicontinuous NCA. We also show an example of transitive almost equicontinuous NCA which is not reversible.

The most simple example of transitive NCA is the map that sends every point into the entire configuration space. Such map is equicontinuous.

Example 5.1. (**Transitive, reversible and equicontinuous NCA**). *Consider the NCA on alphabet $A = \{0, 1\}$ defined by the following multi-valued local rule of radius zero:*

$$\forall a \in A, f(a) = A$$

By Proposition 3.1, $(A^{\mathbb{Z}}, F)$ is equicontinuous. It is clearly transitive, since $\forall n > 0, \forall x \in A^{\mathbb{Z}}$, and for every open set $U \subseteq A^{\mathbb{Z}}$, $F^n(x) \cap U = A^{\mathbb{Z}} \cap U = U$. This example is also easily reversible and the inverse is the map itself. □

With a small modification of the previous example, we can get a non-reversible, equicontinuous and transitive NCA.

Example 5.2. (**Transitive, irreversible and equicontinuous NCA**). *Consider the NCA on alphabet $A = \{0,1\}$ defined by the following multi-valued local rule of radius 1:*

$$\forall a,b,c \in A, f(a,b,c) = \begin{cases} \{0\} & \text{if } a = c = 1, b = 0 \\ \{0,1\} & \text{otherwise} \end{cases}$$

By Lemma 5.1, $(A^{\mathbb{Z}}, F)$ is transitive, since it contains the sensitive elementary rule 90. It is easy to see that it is equicontinuous, since $\forall x \in A^{\mathbb{Z}}, \forall n \geq 2, F^n(x) = A^{\mathbb{Z}}$. In order to see that it is not reversible, consider the configurations $\tilde{x} = {}^{\infty}1^{\infty}$ and $\tilde{y} = {}^{\infty}0^{\infty}$. Note that $\tilde{x} \in F(\tilde{x})$ and $\tilde{x} \in F(\tilde{y})$, thus, if F^{-1} is a multi-valued block map, the only possibility is that for some $r \geq 0, f^{-1}(1^{2r+1}) = \{0,1\}$, which implies that $F^{-1}(\tilde{x}) = A^{\mathbb{Z}}$. This is not possible, since $\tilde{x} \notin F({}^{\infty}(01)^{\infty})$. □

Example 5.3. (**Transitive, irreversible and almost equicontinuous NCA**). *Consider the NCA on alphabet $A = \{0,1\}$ defined by the following multi-valued local rule of radius 1:*

$$\forall a,b,c \in A, f(a,b,c) = \begin{cases} \{0,1\} & \text{if } b = 0 \\ \{c\} & \text{otherwise} \end{cases}$$

By Lemma 5.1, $(A^{\mathbb{Z}}, F)$ is transitive, since it contains the shift map.

We first show that $(A^{\mathbb{Z}}, F)$ is not reversible. Consider the two configurations $\tilde{x} = {}^{\infty}1^{\infty}$ and $\tilde{y} = {}^{\infty}0^{\infty}$. Note that $\tilde{x} \in F(\tilde{y})$ and $\tilde{x} \in F(\tilde{x})$. If F is reversible the only possibility is that for some $k \in \mathbb{N}, f^{-1}(1^{2k+1}) = \{0,1\}$ and then $F^{-1}(\tilde{x}) = A^{\mathbb{Z}}$, but this is not possible since $\tilde{x} \notin F({}^{\infty}(10)^{\infty})$.

Now we show that $\tilde{x}$ is not an equicontinuous point, while $\tilde{y}$ it is. For every $k \geq 0$, we can build the configuration x^k, defined by

$$x_i^k = \begin{cases} 1 & \text{if } i \leq k \\ 0 & \text{otherwise} \end{cases}$$

Fix $\epsilon = 2^0$, then for all $k \geq 0$, we have that

$$F^{k+1}(\tilde{x})_0 = 1 \neq \{0,1\} = F^{k+1}(x^k)_0.$$

On the contrary, it is easy to see that $\tilde{y}$ is an equicontinuous point, since

$$\forall k \in \mathbb{N}, \forall y \in [\tilde{y}_{[-k,k]}]_{-k}, \forall n > 0,\ F^n(y)_{[-k,k]} = A^{2k+1}.$$

We just need to show that the set of equicontinuous points of $(A^{\mathbb{Z}}, F)$ is dense. For any word $w \in A^+, |w| = k$, consider the cylinder set $[0w0^k]_i, i \in \mathbb{Z}$ and note that

$$\forall x, y \in [0w0^k]_i, \forall n \in \mathbb{N}, F^n(x)_{[i,i+2k]} = F^n(y)_{[i,i+2k]}$$

In particular, note that

$$\forall x, y \in [0w0^k]_i, \forall n \geq k+1, F^n(x)_{[i,i+2k]} = F^n(y)_{[i,i+2k]} = A^{2k+1}$$

Now, let $(w_i)_{i \geq 0}$ be an enumeration of all finite words on A^+ such that $|w_i| \leq |w_{i+1}|$and consider the sequence

$$\tilde{z} = ..w_2 0^{|w_2|} w_1 0^{|w_1|} w_0 0^{|w_0|} w_1 0^{|w_1|} w_2 0^{|w_2|}..$$

The sequence $\tilde{z}$ is bitransitive under the shift map and, by construction, a point of equicontinuity. Then, the set of equicontinuous points $\{\sigma^i(\tilde{z}) \mid i \in \mathbb{Z}\} \subset \mathcal{E}$ is dense in $A^{\mathbb{Z}}$. □

6. Expansivity

In expansive dynamical systems every point of the space has a distinctive trajectory. The classes of expansive and positively expansive CA are well understood and characterized. For instance, both positively expansive and expansive CA are transitive, hence surjective.There is no standard definition of expansivity and positive expansivity for the class of multimaps. In this section we propose a definition of expansivity and positive expansivity for NCA and discuss some basic questions on such classes of automata.

Definition 6.1. Let $(A^{\mathbb{Z}}, F)$ be a NCA. We say that:

- $(A^{\mathbb{Z}}, F)$ is **expansive** if it is surjective, reversible and

$$\exists \epsilon > 0, \forall x \neq y, \exists n \in \mathbb{Z}, B_\epsilon(F^n(x)) \neq B_\epsilon(F^n(y))$$

- $(A^{\mathbb{Z}}, F)$ is **positively expansive** if

$$\exists \epsilon > 0, \forall x \neq y, \exists n \in \mathbb{N}, B_\epsilon(F^n(x)) \neq B_\epsilon(F^n(y))$$

- $(A^{\mathbb{Z}}, F)$ is **strongly expansive** if it is surjective reversible and

$$\exists \epsilon > 0, \forall x \neq y, \exists n \in \mathbb{Z}, B_\epsilon(F^n(x)) \cap B_\epsilon(F^n(y)) = \emptyset$$

- $(A^\mathbb{Z}, F)$ is **strongly positively expansive** if

$$\exists \epsilon > 0, \forall x \neq y, \exists n \in \mathbb{N}, B_\epsilon(F^n(x)) \cap B_\epsilon(F^n(y)) = \emptyset$$

□

As for the previous properties, in the single-valued setting, the proposed definitions of expansivity and strong expansivity are equivalent and collapse to the usual definition of expansivity. The same holds for the two definitions of positive expansivity. In the multi-valued setting, it is clear that strong expansivity implies expansivity (resp. strong positive expansivity implies positive expansivity) but the question whether these definitions are equivalent is not immediate. By definition, expansive and positively expansive NCA are sensitive, while strongly expansive and strongly positively expansive NCA are strongly sensitive. Furthermore, note that in the expansivity definition we require reversibility since if F is reversible $F^{-}1$ is a NCA (i.e. it is defined on the entire configuration space $A^\mathbb{Z}$) only if F is also surjective.

We show that there cannot be proper multi-valued strongly expansive and positive expansive NCA.

Proposition 6.1. *Let $(A^\mathbb{Z}, F)$ be a NCA. If it is strongly expansive (resp. positively expansive) then $(A^\mathbb{Z}, F)$ is single-valued.*

Proof. Let $(A^\mathbb{Z}, F)$ be a strongly expansive (resp. positively expansive) NCA with expansivity constant $\epsilon > 0$. Assume by absurd that $(A^\mathbb{Z}, F)$ is strictly multi-valued. Then, by Proposition 2.3, it contains a non-surjective deterministic sub-NCA $(A^\mathbb{Z}, F_k)$, which is not expansive (resp. positively expansive). Then, there are $x, y \in A^\mathbb{Z}$ such that $x \neq y$ and such that $d(F_k^n(x), F_k^n(y)) < \epsilon$ for every $n \in \mathbb{Z}$ (resp. $n \geq 0$). This implies that for every $n \in \mathbb{Z}$ (resp. $n \geq 0$), $B_\epsilon(F^n(x)) \cap B_\epsilon(F^n(y)) \neq \emptyset$, which contradicts the hypothesis that $(A^\mathbb{Z}, F)$ is multi-valued. Then, the only possibility is that $(A^\mathbb{Z}, F)$ is single-valued. □

We show an example of strictly multi-valued expansive NCA.

Example 6.1. (**Expansive NCA**) *Consider the surjective NCA $(A^\mathbb{Z}, F)$ on alphabet $A = \{0, 1, 2\}$ defined by the following multi-valued local rule of radius 1:*

$$\forall a,b,c \in A, f(a,b,c) = \begin{cases} \{0,2\} & \text{if } c=2 \\ \{c\} & \text{otherwise} \end{cases}$$

The NCA $(A^{\mathbb{Z}}, F)$ *contains the shift map as deterministic sub-NCA and it is easy to see that it is reversible with local rule:*

$$\forall a,b,c \in A, f^{-1}(a,b,c) = \begin{cases} \{0,2\} & \text{if } a=0 \\ \{a\} & \text{otherwise} \end{cases}$$

We show that $(A^{\mathbb{Z}}, F)$ *is expansive with expansivity constant* $\epsilon = 2^0$*. Consider the one-block mapping* $\phi : A \to \{0,1\}$ *defined by* $\phi(0) = \phi(2) = 0$ *and* $\phi(1) = 1$ *and consider two configurations* $x, y \in A^{\mathbb{Z}}$ *such that* $x \neq y$*. We have two possibile cases:*

- $\exists i \in \mathbb{Z}$ *such that* $\phi(x_i) \neq \phi(y_i)$*. In this case the only possibility is that either* $x_i = 1, y_i \in \{0,2\}$ *or* $x_i \in \{0,2\}, y_i = 1$*. Without loss of generality, assume that* $x_i = 1$ *and* $y_i \in \{0,2\}$*. Then, by definition of the local rule,* $1 = F^i(x)_0 \neq F^i(y)_0 \in \{0,2\}$*.*

- $\forall i \in \mathbb{Z}, \phi(x_i) = \phi(y_i)$*. Since* $x \neq y$*, the only possibility is that there is* $i \in \mathbb{Z}$ *such that either* $x_i = 0, y_i = 2$ *or* $x_i = 2, y_i = 0$*. Without loss of generality assume that* $x_i = 0, y_i = 2$ *and* $i \neq 0$*. Then, by definition of the local rule and its inverse, if* $i > 0$*,* $F^i(x)_0 = 0$ *and* $F^i(y)_0 = \{0,2\}$*. Otherwise, if* $i < 0$*,* $F^i(x)_0 = \{0,2\}$ *and* $F^i(y)_0 = 2$*.*

□

We do not have an example of strictly multi-valued positively expansive NCA. One such example is not completely trivial on bi-infinite full shifts. However, consider that the NCA in Example 6.1 is positively expansive if we restrict the configuration space to a one-sided full shift. Furthermore, the only example we can provide, either one-sided or two-sided, is based on the shift map and thus, by Lemma 5.1, it is transitive. This does not give any interesting clue about the differences between the expansivity behaviour in the multi-valued and single-valued settings. In conclusion, the characteristics of expansive and positively expansive multi-valued maps cannot be observed by considering simple examples built starting from expansive and positively expansive CA. As a final remark, we note that most of the properties of expansive and positively expansive CA are derived from the fact that they are conjugated to two-sided and one-sided subshifts, respectively. In the multi-valued setting such conjugacy cannot exist, hence different techniques need to be developed in other to study such classes of NCA.

7. Conclusions

We extended to NCA the definition of some relevant topological properties, well characterized for CA, and studied the differences between the dynamical behaviour of NCA and CA in such classes. The considered properties are equicontinuity, almost equicontinuity, sensitivity, transitivity, expansivity and positive expansivity. The intersection classes between these properties for CA and NCA are shown in Figure 4 and 5, respectively. In the two figures we did not include the expansive and positively expansive classes since we do not have a strong enough characterization of these classe in the NCA setting. In the paper we showed NCA examples for almost all intersection classes that are empty for CA. We do not have an example of a transitive, reversible and almost equicontinuous NCA that is not equicontinuous, however we conjecture that one such example exists. The most interesting open question is whether there are (transitive) NCA which are neither sensitive nor almost equicontinuous.

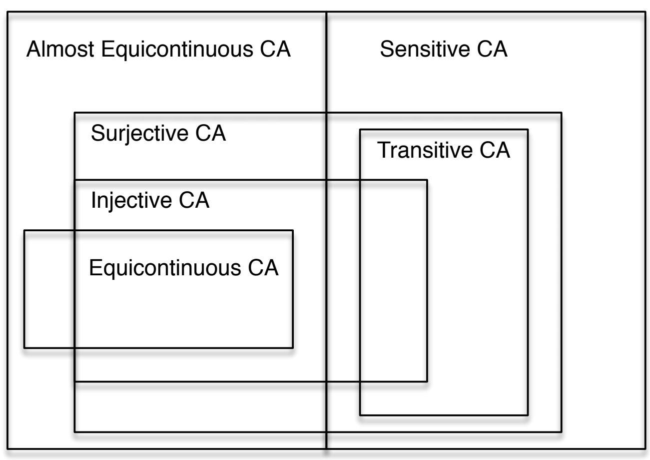

Figure 4: CA classes

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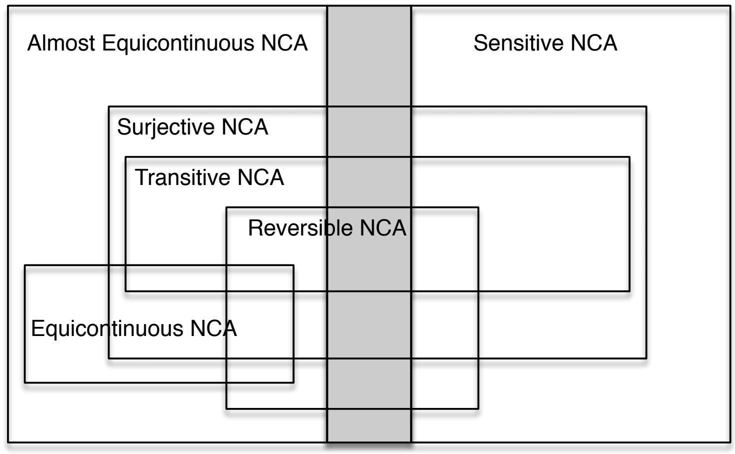

Figure 5: NCA classes

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