Robust Control of an Aerial Manipulator
Interacting with the Environment *

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Abstract: This paper deals with the problem of modelling and controlling a vertical take-off and landing aircraft equipped with a lightweight robotic arm. This system is able to perform complex operations that require the physical interaction with the environment while remaining airborne. Once the dynamical model in the 3D case is provided, a control law is able to let the degrees of freedom of the system to track a desired trajectory is derived. The proposed controller takes into account the interaction between the robotic arm and the aerial platform both during free-flight and in the presence of unknown contact forces deriving from the interaction with the environment. The effectiveness and main properties of the proposed control algorithm have been analytically investigated and then demonstrated with the help of an experiment.

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Keywords: control of UAVs, robust control, robotics, interaction control, nested saturations

1. INTRODUCTION

Applications in which Unmanned Aerial Vehicles (UAVs) are required to physically interact with the surrounding environment have become a popular research topic in the control and robotic community, see e.g. Mellinger et al. (2013); Marconi et al. (2011); Fumagalli et al. (2012); Lippiello and Ruggiero (2012). In this work, the control of an aerial manipulator that consists of a Vertical Take-Off and Landing (VTOL) airframe, more specifically a ducted-fan configuration (Naldi et al. (2010); Marconi and Naldi (2012)), and a miniature robotic arm is presented.

The control design has been specifically tailored to address the applicative scenario described hereafter. A human operator commands the UAV to reach a constant desired position close to the infrastructure to be inspected, and then performs some inspection-by-contact tasks with the manipulator. The environment is modelled as a vertical compliant surface, which means that reaction forces are applied back to the end-effector and, in turn, to the aerial vehicle during the inspection. To succeed in the task, the control law has to guarantee that the vehicle remains closed to the desired position in presence of the disturbances introduced by the interaction with the vertical surface. To investigate the main dynamical properties of the system, a 3D model is proposed. The model takes into account the 3D dynamics of the UAV, and the manipulator is modeled as an n = 3 degrees-of-freedom (d.o.f.) robotic arm with actuated joints. Results have been experimentally validated on a real setup that has been obtained by attaching a lightweight parallel manipulator to a miniature ducted-fan prototype.

Since the UAV is under-actuated, the main challenge is to compensate the coupling forces between the manipulator and the floating base that appear either when the robot moves in free-space, and when the end-effector is in contact situations. With respect to existing contributions in the field, see e.g. Fumagalli et al. (2012); Kobilarov (2014); Lippiello and Ruggiero (2012), uncertainties in the knowledge of the contact forces motivate the adoption of robust control techniques. Then, inspired by classical vectored-thrust control paradigms (Hua et al. (2009); Abdessameud and Tayebi (2010)), a novel control strategy is proposed: the coupling forces are directly taken into account in the definition of the desired force control vector, now obtained by tilting the vehicle in the desired direction and then by applying a certain thrust. This approach naturally leads to a cascade control strategy as in Isidori et al. (2003); Marconi and Naldi (2007) in which the position controller (outer-loop) generates the attitude reference for the inner attitude loop by taking into account, throughout the knowledge of the coupling forces, the interaction with the manipulator and the environment. The stability of the closed-loop system is then investigated using total stability tools for nonlinear control systems (Isidori (1999)). The analysis takes into account the control and mechanical parameters of the system to establish conditions leading to asymptotic or practical tracking of the desired forces both in free flight and in the presence of unknown contact forces.

2. NOTATION AND DEFINITIONS

In this paper, let \( \mathbb{R} \), \( \mathbb{R}_{>0} \), and \( \mathbb{R}_{>0} \) denote the set of real, positive real and non-negative real numbers, respectively. Given \( x \in \mathbb{R}^n \), \( |x| \) denotes the Euclidean norm, while, for a function \( f : [0, +\infty) \to \mathbb{R}^k \), \( k > 0 \), define \( |f|_{\infty} = \sup_{t \in [0, +\infty)} |f(t)| \), and \( |f|_{l} = \lim sup_{t \to +\infty} |f(t)| \). Given

\[ \ast \] This work has been partially supported by the European project AirBorne (ICT 780960).

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Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2018.07.335
Here, the notion of Input-to-State Stability (ISS) with restrictions given in (Isidori et al., 2003, Appendix B) is used, and reported below for sake of completeness. Consider a nonlinear system
\[ \dot{x}(t) = f(x(t), u(t)) \]  
with state \( x \in \mathbb{R}^n \), input \( u \in \mathbb{R}^m \), in which \( f(0, 0) = 0 \) and \( f(x, u) \) is locally Lipschitz on \( \mathbb{R}^n \times \mathbb{R}^m \). Let \( X \) be an open subset of \( \mathbb{R}^n \) containing the origin, and let \( U \) be a positive number. System (1) is said to be ISS with restriction \( X \) on the initial state \( x(0) \) and restriction \( U \) on the input \( u(\cdot) \) if there exist class-\( \mathcal{K} \) functions \( \gamma_0 \) and \( \gamma_u \) such that, for any \( x(0) \in X \) and any input \( u \in L^\infty_0 \) satisfying \( |u|_\infty \leq U \), the solution \( x(t) \) satisfies
\[ |x|_\infty \leq \max \{ \gamma_0 (|x(0)|), \gamma_u (|u|_\infty) \}, \]
\[ |x|_a \leq \gamma_u (|u|_a). \]
Finally, a saturation function is a mapping \( \sigma : \mathbb{R}^n \rightarrow \mathbb{R}^n \) such that, for \( n = 1 \)
\[ |\sigma'(s)| = |dx(s)/ds| \leq 2, \text{ for all } s, \]
\[ s \sigma(s) > 0, \text{ for all } s \neq 0, \sigma(0) = 0, \]
\[ \sigma(s) = \text{sign}(s), \text{ for } |s| \geq 1, \]
\[ |s| < \sigma(s) < 1, \text{ for } |s| < 1. \]
For \( n > 1 \), the properties listed above are intended to hold component-wise.

3. DYNAMICAL MODEL OF THE AERIAL MANIPULATOR

The prototype considered in this work is the ducted-fan aircraft presented in Naldi and Marconi (2014) equipped with a manipulator (see Fig. 1). The ducted-fan aerial vehicle is a particular configuration of VTOL aircraft in which the propeller is protected by an annular fuselage, denoted as the duct. The airframe is composed of two main subsystems. The former consists of a propeller driven by an electric motor and is responsible for producing the thrust force \( T \) required to counteract the gravity force. The second subsystem consists of a set of actuated aerodynamic surfaces, denoted as control flaps, acting on the airflow produced by the propeller so as to produce an aerodynamic lift force \( F \) and, then, a torque contribution \( \tau \) that can be employed to govern the attitude of the vehicle. The robotic arm employed in the experiments consists of a parallel Delta configuration, see e.g. Merlet (2000). On the other hand, the theory is developed under the assumption that the manipulator is a generic device with \( n = 3 \) d.o.f., so that the end-effector can reach a desired position in space with respect to the base of the manipulator itself.

By considering the mechanical layout of the prototype employed in the experiments of Section 5, a number of approximations can be introduced to obtain a model suitable for control design. As far as the manipulator is concerned, by assuming that the mass of the links is negligible compared to the one of the end-effector, denoted by \( m \), the following approximated model is obtained:
\[ m \ddot{p}_e = R(\theta)J^{-T}(q)\tau_q + f_c - mg\hat{e}_3. \]

The second subsystem is the dynamical model of the UAV that is driven by the thrust \( T \) and the torque \( \tau \). By assuming that the mass of the manipulator is negligible compared to the one of the vehicle, namely \( m \ll M \) in the experimental set-up, we have that \( M \approx 1.8 \text{ kg}, \) and \( m \approx 0.1 \text{ kg} \), the lateral and longitudinal dynamics can be approximated as
\[ M \ddot{p}_e = -M g\hat{e}_3 + R(\theta)T\hat{e}_3 - R(\theta)J^{-T}(q)\tau_q \]
while, under the hypothesis that the effect of the aerodynamic force can be neglected, the attitude dynamics are governed by
\[ \Omega(\theta)\dot{\theta} = \omega, \]
\[ J_{\text{nav}} \dot{\omega} = -\omega \times J_{\text{nav}} \omega + \tau_p - p'_p \times [J^{-T}(q)\tau_q] \]
where \( p \in \mathbb{R}^3 \) is the position of the gravity center with respect to \((\hat{e}_1, \hat{e}_2, \hat{e}_3), \omega \in \mathbb{R}^3 \) the angular velocity, \( \Omega(\theta) \) a linear mapping that depends on the specific representation \( \theta \) of the UAV attitude, \( J_{\text{nav}} \) the moment of inertia tensor of the UAV, and \( p'_p \in \mathbb{R}^3 \) the position of the base of the manipulator in the \((\hat{e}_1', \hat{e}_2', \hat{e}_3')\) coordinates. Note that the \( \hat{e}_3' \) axis is chosen in the same direction of the thrust \( T \).

The dynamics of the aerial vehicle are affected by the joint generalised torques \( \tau_q \) governing the manipulator. Moreover, the inertial position of the manipulator is related to the inertial position and orientation of the UAV, and to the joint position of the manipulator via the direct kinematic:
\[ p_e = p + R(\theta) [p'_p + f_{\text{kin}}(q)] \]
Due to this kinematic constraint, the next assumption on the trajectories of the system is necessary.

Assumption 3.1. Denote by \( Q \subset \mathbb{R}^3 \) the joint workspace of the robotic manipulator. Then, we assume that for the complete system the trajectories evolve in the set.
\[ \Theta = \{(q, p_c, p, \theta) \mid q \in Q, p_c, p, \theta \in \mathbb{R}^3 \text{ s.t. (5) holds}\} \]

### 4. CONTROL OF THE AERIAL MANIPULATOR

The inspiring applicative scenario is that the system has to perform inspection-by-contact tasks by means of the manipulator, so the control problem tackled in this paper is to let the end-effector inertial position to track a desired reference trajectory, while the aerial vehicle is maintained at a constant position. As summarised in Assumption 3.1, the reference position of the UAV has to be selected in such a way that the inertial position of the end-effector is within the manipulator operative space. As a main challenge, the control law has to be robust with respect to the presence of possible contacts with the environment, namely when the unknown contact force \( f_c \) is applied to the end-effector.

#### 4.1 Robust control of the robotic arm

Given the reference trajectory \( p_c^*(t) \) for the inertial position of the end-effector, consider the following control law

\[
\tau_q = J^T(q)R^T(\theta) [m(p_c^* + \dot{g}e_3) - \kappa (\ddot{p}_c, \dot{p}_c)]
\]

in which \( \ddot{p}_c = p_c - p_c^* \) is the position error of the end-effector in the inertial coordinates, and where \( \kappa : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) is an error feedback controller that is designed by means of the following nested saturation control law:

\[
\ddot{\hat{p}}_c = \lambda_2 \sigma_k \left( \frac{k_2}{\lambda_2} \left( \ddot{p}_c + \lambda_1 \sigma \left( \frac{k_1}{\lambda_1} \dot{p}_c \right) \right) \right)
\]

in which, by following (Isidori et al., 2003, Appendix B), the parameters \( k_1, k_2, \lambda_1, \) and \( \lambda_2 \) are selected as

\[
\lambda_i = \epsilon^{(i-1)} \lambda_i^*, \quad \ddot{\hat{p}}_c = \epsilon \ddot{p}_c
\]

with \( i = 1, 2 \), and where \( k_i^* \) and \( \lambda_i^* \) are such that

\[
\lambda_i^* > 0, \quad \lambda_i^* > \lambda_i^* > \frac{1}{4} m \frac{\lambda_1^*}{m} \frac{6}{24} \frac{\lambda_1^*}{m} < \frac{1}{24} m
\]

\[ \] (9)

Finally, a mapping \( \phi : \mathbb{R} \rightarrow \mathbb{R}^3 \) can be employed to govern the attitude of the manipulator, being

\[
\phi(p) = Mg\dot{e}_3 - \kappa(\ddot{p}, \dot{p})
\]

in which \( \ddot{p} = p - p^* \) is the position error, and \( \kappa : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) is a feedback control law.

The control vector \( v_c \) is applied to the vehicle position dynamics (3) by properly vectorizing the thrust produced by the propeller. By taking advantage of the knowledge of the manipulator control inputs \( \tau_q \), a control thrust \( T_c \) and a control attitude \( \theta_c \) are computed to have

\[
R(\theta_c) [T_c \dot{e}_3' - J^{-1}(q) \tau_q] = v_c
\]

To compute a solution to (14), let us assume that, for all time \( t \geq 0 \), we have that \( |v_c| > 0 \), and

\[
|T_c - J^{-1}(q) \dot{e}_3'|^T \tau_q > 0
\]

(15)

The above assumptions are satisfied by properly tuning the position control laws of the manipulator and the UAV, see Prop. 3. From (14), we get that \( T_c \) is positive solution of

\[
T_c^2 - 2 J^{-1}(q) \dot{e}_3'|^T \tau_q + \tau_q J^{-1}(q) J^{-T}(q) \tau_q = v_c v_c^T
\]

provided that \( v_c \) is such that

\[
v_c^T v_c + \tau_q J^{-1}(q) \left[ \dot{e}_3' (\ddot{e}_3')^T - I \right] J^{-T}(q) \tau_q > 0
\]

(16)

As far as the attitude \( \theta_c \) is concerned, if \( \hat{v} \) and \( \hat{\theta} \) are the unitary vectors aligned with \( v_c \) and \( T_c \dot{e}_3' - J^{-1}(q) \tau_q \), respectively, we have that

\[
R(\theta_c) = I + S(w) + \frac{1 - c}{s^2} S^2(w)
\]

(17)

where \( w = \hat{v} \times \hat{\theta}, \ s = |w| \) and \( c = \hat{v} \cdot \hat{\theta} \) the rotation axis, and the sine and cosine of the angle, respectively, that allow to align \( \hat{v} \) with \( \hat{\theta} \), and where \( S(\cdot) \) is the skew-symmetric cross-product matrix of a vector. While the control thrust \( T_c \) can be directly applied to the vehicle by choosing \( T = T_c \), the control attitude \( \theta_c \) is employed as a reference for the attitude stabilising control law.

To stabilize the position dynamics of the aerial vehicle, we focus on the following nested saturation control law

\[
\kappa(\ddot{p}, \dot{p}) = \lambda_3 \sigma_k \left( \frac{k_2}{\lambda_2} \left( \ddot{p} + \lambda_1 \sigma \left( \frac{k_1}{\lambda_1} \dot{p} \right) \right) \right)
\]

(18)

in which \( k_1, k_2, \lambda_1, \) and \( \lambda_2 \) are selected as

\[
\lambda_i = \epsilon^{(i-1)} \lambda_i^*, \quad k_i = \epsilon k_i^*
\]

(19)

with \( i = 1, 2, \epsilon > 0 \), and where \( k_i^*, \lambda_i^* \) are the same of Prop. 1, and then such that (10) holds.

#### 4.2 Modified thrust-vectoring control of the UAV

To stabilise the position of aerial platform to the constant references \( p^* \), the following control vector is defined

\[
v_c = Mg\dot{e}_3 - \kappa(\ddot{p}, \dot{p}) \]

in which \( \ddot{p} = p - p^* \) is the position error, and \( \kappa : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) is a feedback control law.
Let the control torque $\tau_c$ be designed as:

$$\tau_c = \tau_{FF}(\tau_q, q) + \tau_{FB}(\theta, \dot{\theta}, \ddot{\theta})$$

in which

$$\tau_{FF}(\tau_q, q) = p_c^* \times \left[ J^{-1}(q) \tau_q \right]$$

is the feed-forward control action compensating for the reaction torque produced by the manipulator, and

$$\tau_{FB}(\theta, \dot{\theta}, \ddot{\theta}) = -k_p \left[ \theta - \theta_c \right] + k_d \dot{\theta}$$

is the feedback stabilising control law. The stability of the system resulting from the interaction with the robotic arm both in free-flight and in contact with the environment is discussed in the next proposition.

**Proposition 3.** Let us consider the system (3)-(4), in which the control inputs $T$ and $\tau$ are selected as $T = T_c$ and $\tau = \tau_c$. Let the trajectory of the complete system be such that $(q(t), p_c(t), p(t), \theta(t)) \in \Theta$, with $\Theta$ defined in (6), for all $t \geq 0$ (see Assumption 3.1), the references $p_c^*$ satisfy Assumption 4.1, and $\epsilon > 0$ be chosen such that

$$\sqrt{3} \Delta \dot{\theta} \leq M g - \ddot{\nu}$$

for some $m g < \ddot{\nu} < M g$, and let $|f_c|_{\infty} \leq \bar{F}$ for some $F > 0$. Then, there exists $k_D > 0$ and, for all $k_D < k_D^*$, there exists positive $k_p^*(k_D)$, $m^*$ and $\dot{\epsilon}^*$ with

$$\sqrt{6} \left[ \dot{m}^* + m^* k_D + \lambda^2 \dot{\epsilon}^* \right] > \ddot{\nu}$$

such that for all $k_D > k_D^*$, $m < m^*$, and $0 < \dot{\epsilon} < \dot{\epsilon}^*$, there exists a $\Delta_0 > 0$ and a class-$K$ function $\gamma_p$ such that the closed-loop system is ISS with restriction $\Delta_0$ on the initial conditions, restriction $\bar{F}$ on the exogenous input $f_c$, and

$$|\ddot{p}|_a \leq \gamma_p \left( k_D \frac{k_P}{k_F} \left( |p_c^*|_a + |f_c|_a \right) \right)$$

**Proof.** The proof is in Appendix A.

The purpose of this section is to validate the theory presented in Section 4 by showing some experimental results. The aerial manipulator (see Fig. 1) consists of the ducted-fan prototype presented in Naldi and Marconi (2014) rigidly connected to the base plate of a parallel Delta robot, described in Keemink et al. (2012). The end-effector is an ultrasonic non-destructive testing sensor (see e.g. Hayward et al. (2006)) usually employed for inspection-by-contact of infrastructures. It is driven by 3 electric motors and it is characterized by a total weight lower than 150 gr. The workspace of the manipulator is approximately a sphere of 10 cm radius.

The prototype has been equipped with suitable avionics hardware. In particular, an Inertial Measurement Unit (IMU) is employed to obtain a high bandwidth (500 Hz) attitude information of the vehicle. On the other hand, the position of the ducted-fan is obtained by means of an OptiTrack motion tracking system capable of millimeter accuracy. Then, the position of the end-effector is obtained by the knowledge of the inertial position of the vehicle, and of the joint position of the manipulator thanks to the direct kinematic (5). The mechanical parameters of the system and the control parameters employed in the control laws (8), (18) and (23) are listed in Table 1.

The goal of the experiment is to show how the aerial vehicle can be stabilised to a constant position while the manipulator enters in contact with the environment. In Fig. 1, it is possible to observe the aerial robot performing a “docking manoeuvre,” i.e. the robot enters in contact with a vertical surface, parallel to the $\dot{e}_3$ axis of the inertial reference frame, and at a certain position $\bar{x}$ along its path in the $\dot{e}_1$ direction. Due to space limitations, the experimental data only show the planar dynamics, i.e. the motion of the system in the $\dot{e}_1$ and $\dot{e}_3$ direction (here denoted by $x$ and $y$), and the rotation $\theta$ around the $\dot{e}_1$ axis. In Fig. 2, the behaviour of the UAV during the two different phases that compose the experiment are presented. The first one, approximately from 70 s to 105 s, is denoted as the Free Flight phase since the vehicle is not in contact with the surface. The task is

### Table 1. Parameters of the setup.

| $M$ | 1.8 Kg |
| $J_{uu}$ | diag(1.9, 1.9, 0.8) Kg · m² |
| $m$ | 0.1 Kg |
| $(k_1^c, k_2^c)$ | (1, 150) |
| $(\lambda_1^c, \lambda_2^c)$ | (5, 150) |
| $(\epsilon, \eta)$ | (0.1, 0.05) |
| $(k_P, k_F)$ | (30, 9) |
Fig. 2. Reference and trajectories in the $x$, $y$, and $\theta$ directions for the ducted fan UAV.

Fig. 3. The control input of the UAV: the thrust $T$ and the torque $\tau$ applied around the $\hat{e}_1 \equiv \hat{e}_z$ axis.

to track a longitudinal trajectory so as to stabilise a constant final desired position. The second phase, i.e. the Docking / Contact phase, starts after 105 s. Note, in fact, that the lateral position $x$ that does not follow the “right” trajectory because of the obstacle. Furthermore, at the same time, the UAV tilts since the $\theta$ dynamics is directly influenced by the lateral error. The contact force also affects the vertical dynamics of the vehicle: in fact, a small tracking error can be also observed for the vertical position $y$. In summary, the aerial vehicle remains stable during the entire maneuver and, in the presence of the unknown contact forces, practical stability of the desired reference position is obtained.

In Fig. 3, the control inputs of the UAV, i.e. the thrust $T$ and the torque $\tau$ have been reported. Note that, during the contact, the torque $\tau$ reaches higher values in order to stabilise the attitude dynamics. Finally, in the last picture (Fig. 4) the two contact forces $f_y$ and $f_z$ applied to the end effector of the robot manipulator are reported. It is interesting to observe that they are close to zero during the Free Flight maneuver (the mass of the end-effector is relatively small), and that they assume values different from zero during the docking phase.

Fig. 4. The forces $f_y$ and $f_z$ applied to the end effector of the Delta manipulator.

6. CONCLUSIONS AND FUTURE ACTIVITIES

In this work, the control of a ducted-fan aerial robot equipped with a lightweight robotic arm and able to accomplish operations requiring the physical interaction with the surrounding environment is presented. Stability of the system both in the free flight and in the presence of contacts is achieved. The idea is to take into account explicitly the reaction forces applied by the manipulator to the aerial platform so as to stabilize the vehicle to a constant desired position both in free-flight and during contacts. Interestingly enough, the proposed approach does not require an exact knowledge of the external force applied by the environment during interaction. Experiments, obtained with a ducted-fan aerial robot endowed with a Delta robotic arm, show how docking to a surface can be robustly achieved.

REFERENCES


**Appendix A. PROOF OF PROPOSITION 3**

First of all, note that (24), (25) and $\epsilon < \tilde{\epsilon}^*$ ensure that (15) and (16) are satisfied. Given the position dynamics of the UAV (3), the control law (18), and the change of coordinates $\zeta_1 = \tilde{p}$ and $\zeta_2 = \tilde{p} + \lambda_2 \sigma \left( \frac{k_2}{\lambda_2} \xi_1 \right)$, the position error dynamics can be written as

$$
\dot{\zeta}_1 = -\lambda_1 \sigma \left( \frac{k_1}{\lambda_1} \xi_1 \right) + \xi_2
$$

$$
M \dot{\zeta}_2 = -\lambda_2 \sigma \left( \frac{k_2}{\lambda_2} \xi_2 \right) + M k_1 \sigma' \left( \frac{k_1}{\lambda_1} \xi_1 \right) + \Gamma_{\eta}(\eta_1, \tau_q)
$$

(A.1)

where $\Gamma_{\eta}(\eta_1, \tau_q) = \left[ R(\theta) - R(\theta_t) \right] T_c \dot{e}_q^c - J^{-T}(q)\sigma_1$ is the auxiliary input, being $\eta_1 = \theta - \theta_t$. Note that $\Gamma_{\eta}(0, \tau_q) = 0$ for all $\tau_q \in \mathbb{R}^3$, and for all $\eta_1 \in \mathbb{R}^3$ we have that $|\Gamma_{\eta}(\eta_1, \tau_q)| \leq \Gamma_{\eta,1} |\tau_q| + \Gamma_{\eta,2}$ for some positive constants $\Gamma_{\eta,1}$ and $\Gamma_{\eta,2}$. In the new coordinates, we have that $\kappa(\xi_1, \xi_2) = \lambda_2 \sigma \left( \frac{k_2}{\lambda_2} \xi_2 \right)$, which implies that

$$
\frac{d \kappa}{dt} (\zeta_1(t), \zeta_2(t)) = \frac{k_2}{M} \sigma' \left( \frac{k_2}{\lambda_2} \xi_2 \right) \left[ \lambda_2 \sigma \left( \frac{k_2}{\lambda_2} \xi_2 \right) + M k_1 \sigma' \left( \frac{k_1}{\lambda_1} \xi_1 \right) \zeta_1 + \Gamma_{\eta}(\eta_1, \tau_q) \right]
$$

(A.2)

From (Isidori et al., 2003, Appendix C), it is possible to prove that (A.1) is ISS with non-zero restrictions on the input $\Gamma_{\eta}$. Moreover, thanks to the change of coordinates $\eta_1 = \theta - \theta_t$ and $\eta_2 = \tilde{\epsilon} + \frac{\bar{m}}{k_D} \xi_1$, the closed-loop attitude error dynamics can be written as

$$
\dot{\eta}_1 = -\frac{\eta_1}{k_D} + \eta_2 - \dot{\theta}_t
$$

$$
J_{nau} \Omega(\eta_1 + \theta_c) \dot{\eta}_2 = -\omega \times J_{nau} \omega - k_p k_D \eta_2 +
$$

$$
+ J_{nau} \left[ \frac{1}{k_D} + \Omega(\eta_1 + \theta_c) \right] \cdot \left( \eta_2 - \eta_1 \right) + J_{nau} \dot{\theta}_t
$$

(A.3)

in which we have that $\omega = \Omega(\theta) \dot{\theta} = \Omega(\eta_1 + \theta_c) \left( \eta_2 - \frac{\bar{m}}{k_D} \xi_1 \right)$, with $\Omega(\theta)$ introduced in (4). Note that (17) implies that both $\dot{\theta}_t$ and $\dot{\theta}$ are some bounded functions of $v_c, \tilde{v}_c, \tau_q$ and $\dot{\tau}_q$. From (7) and (13) we get that $\tau_q$ and $v_c$ are bounded by construction, while from (13), we have that $\dot{v}_c = -\kappa(\tilde{p}, \tilde{p})$, with $\kappa$ computed in (A.2). From (7), we can also write that

$$
\dot{\tau}_q = J^T(q) R^T(\theta) \left[ m p_c^{*}(3) - \frac{d}{dt} \kappa(\tilde{p}_c, \dot{\tilde{p}}_c) \right] +
$$

$$
+ \left\{ \frac{d}{dt} \left[ J^T(q) R^T(\theta) \right] \right\} \left[ m (\tilde{p}_c + g \xi_3) - \hat{\kappa}(\tilde{p}_c, \dot{\tilde{p}}_c) \right]
$$

and then by Prop. 2 that

$$
|\dot{\tau}_q| \leq \lambda_2^* \left[ m (D_2^* + g) + \epsilon \right] \left| \eta_2 \right| + \frac{|\eta_1|}{k_D} +
$$

$$
+ J \left| p_c^{*}(3) \right| + |\dot{\kappa}(\tilde{p}_c, \dot{\tilde{p}}_c)|
$$

for some positive $\lambda_2^*$.

The closed-loop dynamics of the UAV results from the feedback interconnection of (A.1) and (A.3). Due to space limitations this part of the proof is only sketched. By considering an ISS-Lyapunov function $V(\eta_1, \eta_2) = \frac{1}{2} \eta_1^T \eta_1 + \frac{\bar{m}}{k_D} \Omega^T(\eta_1) J_{nau} \Omega(\eta_1) \eta_2$, and choosing $k_D$ and $k_P$ sufficiently small and large, respectively, and for $m$ and $\bar{m}$ sufficiently small, system (A.3) can be shown to be ISS with respect to the exogenous input $\dot{v}_c = -\kappa(\tilde{p}, \tilde{p})$, $p_c^{*}(3)$, and $\dot{\kappa}(\tilde{p}_c, \dot{\tilde{p}}_c)$, with an arbitrary asymptotic gain. From Propositions 1 and 2, and from the fact that $|f_{\epsilon}|_{\epsilon} \equiv \infty$ is bounded, it is clear that $\tilde{\epsilon}$ and $\tilde{\epsilon}$ are also bounded. Moreover, according to Assumption 4.1, also $p_c^{*}(3)$ is bounded. Then, it is possible to choose $k_P$ and $k_P$ to satisfy the restrictions on the input (see Prop. 2) on the position error subsystem (A.1) in finite time, and then to enforce the small gain condition. The final result is a consequence of standard ISS arguments, as in (Isidori et al., 2003, Appendix C).