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Product Differentiation with Multiple Qualities

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Abstract

We study subgame-perfect equilibria of the classical quality-price, multistage game of vertical product differentiation. Each of two firms can choose the levels of an arbitrary number of qualities. Consumers' valuations are drawn from independent and general distributions. The unit cost of production is increasing and convex in qualities. We characterize equilibrium prices, and the effects of qualities on the rival's equilibrium price in the general model. Equilibrium qualities depend on what we call the Spence and price-reaction effects. For any equilibrium, we characterize conditions for quality differentiation.

Keywords: multidimensional product differentiation, quality and price competition

JEL: D43, L13

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1 Introduction

Firms using differentiated products to soften intense Bertrand price competition is a basic principle in industrial organization. Following Hotelling (1929), D'Aspremont, Gabszewicz and Thisse (1979) clarify theoretical issues and solve the basic horizontal differentiation model. Gabszewicz and Thisse (1979), and Shaked and Sutton (1982, 1983) work out equilibria of the basic vertical differentiation model.

The standard model of horizontal-vertical product differentiation is the following multistage game between two firms: in Stage 1, firms choose product attributes or qualities; in Stage 2, firms choose prices, and then consumers pick a firm to purchase from. In the literature, models have seldom gone beyond two possible qualities, have assumed that consumers' quality valuations are uniformly distributed, and have let production or mismatch costs be nonexistent, linear, or quadratic. We make none of these assumptions. In this paper, each of two firms produces goods with an arbitrary number of quality attributes. Consumers' valuations on each quality follow a general distribution. A firm's unit production cost is an increasing and convex function of qualities. In this general environment, we characterize subgame-perfect equilibria of the standard differentiation model. By doing so, we show that results in the literature are less general than previously thought.

Using the uniform quality-valuation distribution and the separable cost assumptions, researchers have managed to solve for equilibrium prices explicitly as functions of qualities. Equilibrium qualities then can be characterized. What has emerged in the literature are a few classes of equilibria with largest or smallest differences in equilibrium qualities (see the literature review below, in particular the "Max-Min-...-Min" result by Irlen and Thisse (1998)). In equilibrium, firms may successfully differentiate their products only in some qualities. We show that uniform distributions and separable quality costs (or mismatch disutility) are drivers for nondifferentiation results. Indeed, when valuations are uniformly distributed and costs are separable, multidimensional models can be simplified to single-dimensional ones. However, how robust are maximal or minimal differentiation results? To what extent are they driven by these assumptions? What are the fundamental forces that determine product differentiation?

In this paper we solve the tractability-generalizability dilemma that has been posed by the literature. For the

quality-price, multistage game, we characterize subgame-perfect equilibria. First, we find out how qualities affect equilibrium prices—without solving for the equilibrium prices explicitly in terms of qualities. Second, we identify two separate effects for the characterization of equilibrium qualities. The first is what we call the *Spence effect* (because it is originally exposted in Spence (1975); see footnote 7). For maximum profit, a firm chooses a quality which is efficient for the consumer who is just indifferent between buying from the firm and its rival. Consumers must buy from one of the two firms, so firms share the same set of indifferent consumers. The Spence effect says that each firm should choose those qualities that are efficient for the equilibrium set of indifferent consumers. The Spence effect alone is a motivation for minimal product differentiation.

The second effect is what we call the *price-reaction effect*, which is how a firm’s quality in Stage 1 affects the rival firm’s price in Stage 2. Prices are strategic complements, so each firm would like to use its qualities in Stage 1 to raise the rival’s price in Stage 2. The two firms engage in a race, each trying to solicit a positive price reaction from the rival by increasing qualities. However, higher qualities raise the unit production cost. Equilibrium qualities reflect each firm balancing between the benefit from price-reaction effects and higher unit cost, against the rival’s strategy. The entire quality profile, not just one single quality, determines the overall effects on production costs and the rival’s price reactions.

In an environment where consumers’ quality valuations are uniformly distributed, firms’ equilibrium price-reaction effects are equal. In addition, when cost is separable in qualities, firms’ balancing between price reaction and higher quality cost pushes them to produce equal qualities except in one dimension. Nevertheless, when firms’ price-reaction effects are not proportional to the quality marginal cost difference, or when cost is nonseparable, firms likely choose different qualities in equilibrium. Thus, firms that produce “high-end” products will still differentiate—if only in small details of their product qualities. For example, *BMW* and *Lexus* are companies that differentiate even in the high qualities of their cars. All *BMW* and *Lexus* cars are high-quality automobiles, but the common consensus is that *BMW* has a higher “performance” quality than *Lexus*, but the opposite is true when it comes to the “comfort” quality. However, any car by *BMW* or *Lexus* will be a better performer and more comfortable than any car by *Yugo*. In fact, in the automobile and most other markets, it is impossible to find products that have identical quality attributes. These observations are consistent with the general tenet of product differentiation.

What is behind our solution to the tractability-generalizability dilemma? The key is to show that equilibrium prices and equilibrium demands can be decomposed into two systems. The equilibrium prices must satisfy the usual inverse demand elasticity rule, whereas the equilibrium set of indifferent consumers which determines demands, must satisfy an integral equation. Furthermore, the solution of the integral equation takes the form of a set of implicit and explicit functions of the model primitives. Then a firm's equilibrium price can be characterized in terms of qualities, through the solution of the integral equation. In other words, we dispense with the need for computation of equilibrium prices, which would require explicit specification of the model primitives.

We use a vertical differentiation model, but Cremer and Thisse (1991) show that for the usual model specification, the Hotelling, horizontal differentiation model is a special case of the vertical differentiation model. The intuition is simply that firms' demand functions in a Hotelling model can be directly translated to the demand functions in a quality model. Cremer and Thisse (1991) state the result for a single location or quality dimension, but their result extends straightforwardly to an arbitrary number of such dimensions. (A model with a combination of horizontal and vertical dimensions can also be translated to a model with only vertical dimensions.) Hence, our results in this paper apply to horizontal differentiation models. In particular, our method of solving for equilibrium prices is valid for Hotelling models.¹

Whereas we have here characterized subgame-perfect equilibria of a general quality-then-price duopoly, our analysis is incomplete. We take for granted the existence of subgame-perfect equilibria. Our methodology is to exploit equilibrium properties. To date, the only known equilibrium-existence results are for models with i) a single generally distributed dimension of consumer locations (or valuations) and quadratic cost, and ii) multiple uniformly distributed dimensions of consumer locations and quadratic costs (see Anderson, Goeree and Ramer (1997) and Irmen and Thisse (1998), discussed below). Our results can be regarded as necessary conditions, so can be used to obtain candidate equilibria. This already simplifies the search for equilibria. (In fact, in Subsection 4.2.1, we verify that one candidate equilibrium of an example is an equilibrium, whereas another candidate fails to be an equilibrium.) In any case, our characterization uses no

¹Differences between horizontal and vertical models may also be due to specification of the strategy sets. Here, we allow qualities to take any positive values; in location models, firms' positions may not vary as much.

assumptions except those for existence of equilibria in the price subgame, so does not present any impediment on existence research.

We continue with a subsection on the literature. In Section 2, we define consumers' preferences and firms' technology. Then we set up the quality-price, multistage game. Section 3 is divided into four subsections. In Subsection 3.1, we characterize subgame-perfect equilibrium prices. Lemma 1 presents the solution of the integral equation, the key step in expressing equilibrium prices as functions of qualities. In Subsection 3.2, we characterize how prices change with qualities. In Subsection 3.3, we characterize equilibrium qualities, and establish the price-reaction and Spence effects. Subsection 3.4 presents a number of implications. We specialize our model by adopting common assumptions (uniform quality-valuation distribution and separable cost function), and draw connections between earlier results and ours. A number of examples are studied in Section 4. These examples illustrate our general results and how they can be used. We also verify the existence of equilibria in some examples. The last section contains some remarks on open issues. Proofs of results and statements of some intermediate steps are in the Appendix. *Mathematica* files for computations in Subsections 4.1. and 4.2 are available online.

1.1 Literature review

The modern literature on product differentiation and competition begins with D'Aspremont, Gabszewicz and Thisse (1979), Gabszewicz and Thisse (1979) and Shaked and Sutton (1982, 1983). In the past few decades, the principle of product differentiation relaxing price competition has been stated in texts of industrial organization at all levels: Tirole (1988), Anderson, De Palma, and Thisse (1992), and Belleflamme and Peitz (2010) for graduate level, as well as Cabral (2000), Carlton and Perloff (2005), and Pepall, Richards, and Norman (2014). Many researchers use the basic horizontal and vertical differentiation models as their investigation workhorse.

The research here focuses on equilibrium differentiation. In both horizontal and vertical models, a common theme has been to solve for subgame-perfect equilibria in the quality-price, multistage game in various environments.² First, earlier papers have looked at single or multiple horizontal and vertical dimensions of

²As we have already mentioned, Cremer and Thisse (1991) (following on a suggestion by Champsaur and Rochet

consumer preferences. Second, most papers have adopted the assumption that these preferences are uniformly distributed. Third, most papers in the horizontal model have used a quadratic consumer mismatch disutility function, whereas those in the vertical model have assumed that the unit production cost is either independent of, or linear in, quality.

Single dimension models

Anderson, Goeree and Ramer (1997) study equilibrium existence and characterization in a single-dimension horizontal model. They use a general consumer preference distribution but quadratic mismatch disutility. Our multidimensional vertical model can be recast into the single-dimensional model in Anderson, Goeree and Ramer. A few other papers have adopted nonuniform distributions on consumer locations. Neven (1986) shows that firms tend to locate inside the market when consumers' densities are higher near the center. Tabuchi and Thisse (1995) assume a triangular distribution and find that there is no symmetric location equilibria but that asymmetric location equilibria exist. Yurko (2010) uses a vertical model for studying entry decisions, but her results are based on numerical simulations. Benassi, Chirco, and Colombo (2006) allow consumers the nonpurchase option. They relate various trapezoidal valuation distributions to degrees of equilibrium quality differentiation. Finally, Loertscher and Muehlheusser (2011) consider sequential location entry games without price competition. They study equilibria under the uniform and some nonuniform consumer-location distributions.

Multiple dimensions models

A few papers have studied vertical models with two dimensions. These are Vandebosch and Weinberg (1995), Lauga and Ofek (2011), and Garella and Lambertini (2014). All three papers use the uniform valuation distribution. In the end of Subsection 3.4, we will present the relationship between our results here to those in these papers. Here, we note that these papers have assumed zero production cost, or unit cost that is linear or discontinuous in quality. By contrast, we use a strictly convex quality cost function.

In a recent paper, Chen and Riordan (2015) have proposed using the copula to model consumers' correlated multidimensional preferences on product varieties. In their formulation, each variety is a distinct good,

(1989)) show that horizontal-location models are special cases of vertical models.

and a consumer considers buying some variety. Their analysis has assumed that average production cost is fixed, so a firm’s variety choice has no cost consequences. By contrast, we let consumers’ preferences on different qualities be independent, but all qualities are embedded in a good. We also let the unit production cost be increasing and convex in qualities.

For horizontal models with multiple dimensions, the key paper is Irmen and Thisse (1998), who set up an N dimensional model to derive what they call “Max-Min-...-Min” equilibria. We will relate our results to those in Irmen and Thisse in Subsection 3.4, right after Corollary 4. Tabuchi (1994) and Vendorp and Majeed (1995) are special cases of Irmen and Thisse (1998) at $N = 2$. Ansari, Economides, and Steckel (1998) study two and three dimensional Hotelling models, and derive similar results as in Irmen and Thisse (1998). All assume that consumers’ locations are uniformly distributed, and that the mismatch disutility is Euclidean and therefore separable. We are unaware of any paper in the multidimensional horizontal literature that adopts general consumer preferences distributions, or general, nonseparable mismatch disutility.

Finally, Degryse and Irmen (2001) use a model with both horizontal and vertical differentiation. For the horizontal dimension, consumer locations are uniformly distributed. For the vertical dimension, consumers have the same valuation (as in the model in Garella and Lambertini (2014)). However, the mismatch disutility depends also on quality, which corresponds to the case of a nonseparable mismatch disutility or quality cost function. This can be thought of as a special case of the model here.

2 The Model

We begin with describing consumers and their preferences. Then we present two identical firms. Finally, we define demands, profits, and the extensive form of quality-price competition.

2.1 Consumers and preferences for qualities

There is a set of consumers, with total mass normalized at 1. Each consumer would like to buy one unit of a good, which has $N \geq 2$ quality attributes. A good is defined by a vector of qualities $(q_1, q_2, \dots, q_N) \in \mathfrak{R}_+^N$, where q_i is the level of the i^{th} quality, $i = 1, 2, \dots, N$.

A consumer's preferences on goods are described by his quality valuations, represented by the vector $(v_1, \dots, v_i, \dots, v_N) \in \prod_{i=1}^N [\underline{v}_i, \bar{v}_i] \subset \mathfrak{R}_{++}^N$. The valuation on quality q_i is v_i , which varies in a bounded, and strictly positive interval. If a consumer with valuation vector $(v_1, \dots, v_i, \dots, v_N) \equiv v$ buys a good with qualities $(q_1, q_2, \dots, q_N) \equiv q$ at price p , his utility is $v_1 q_1 + v_2 q_2 + \dots + v_N q_N - p$. (We may sometimes call this consumer (v_1, \dots, v_N) or simply consumer v .) The quasi-linear utility function is commonly adopted in the literature (see such standard texts as Tirole (1988) and Belleflamme and Peitz (2010)). (Throughout the paper, a vector is a mathematical symbol without a subscript; components of a vector are distinguished by subscripts (either numerals or Roman letters). Besides, we use v_{-i} to denote the vector v with the i^{th} component omitted. Any exception should not create confusion.)

Consumers' heterogeneous preferences on qualities are modeled by letting the valuation vector be random. We use the standard independence distribution assumption: the valuation v_i follows the distribution function F_i with the corresponding density f_i , $i = 1, \dots, N$, and these distributions are all independent. Each density is assumed to be differentiable (almost everywhere) and logconcave. The logconcavity of f_i implies that the joint density of $(v_1, \dots, v_i, \dots, v_N) \equiv v$ is logconcave,³ and it guarantees that profit functions, to be defined below, are quasi-concave (see Proposition 4 in Caplin and Nalebuff 1991, p.39).

2.2 Firms and extensive form

There are two firms and they have access to the same technology. If a firm produces a good at quality vector $(q_1, q_2, \dots, q_N) \equiv q$, the per-unit production cost is $C(q)$. There is no fixed cost, so if a firm produces D units of the good at quality q , its total cost is D multiplied by $C(q)$. We assume that the per-unit quality cost function $C : \mathbb{R}_+^N \rightarrow \mathbb{R}_+$ is strictly increasing and strictly convex. We also assume that C is continuous and differentiable, and satisfies the usual conditions: $\lim_{q \rightarrow 0} C(q) = 0$, $\lim_{q \rightarrow 0} dC(q) = 0$, $\lim_{q \rightarrow \infty} C(q) = \infty$, and $\lim_{q \rightarrow \infty} dC(q) = \infty$ (where q is a quality vector, 0 stands for a zero or an N -vector of all zeros, and ∞ stands for an infinity or an N -vector of infinities, and dC is a vector of C 's first-order derivatives).⁴ If a firm

³Because of independence, the joint density of $(v_1, \dots, v_i, \dots, v_N)$ is $\prod_{i=1}^N f_i$. Hence, $\ln \prod f_i = \sum \ln f_i$. Because $\ln f_i$ is concave, so is $\sum \ln f_i$.

⁴The Inada conditions do not necessarily imply that all equilibrium qualities must be strictly positive; this is due to price-reaction effects to be derived below. However, they do eliminate nondifferentiation due to marginal costs being too high at very low quality levels.

sells D units of the good with quality vector q at price p , its profit is $D \cdot [p - C(q)]$.

The two firms are called Firm A and Firm B . We use the notation q for Firm A 's vector of qualities $(q_1, q_2, \dots, q_N) \equiv q$. We use the notation r for Firm B 's vector of qualities $(r_1, r_2, \dots, r_N) \equiv r$. Hence, when we say quality q_i , it indicates the level of Firm A 's i^{th} quality attribute, whereas when we say quality r_i , it indicates the level of Firm B 's i^{th} quality attribute. Let Firm A 's price be p_A , and Firm B 's price be p_B . We use the notation p for the price vector (p_A, p_B) .

Given the two firms' quality choices, consumer $v = (v_1, \dots, v_N)$ obtains utilities $v_1q_1 + v_2q_2 + \dots + v_Nq_N - p_A$ and $v_1r_1 + v_2r_2 + \dots + v_Nr_N - p_B$ from Firm A and Firm B , respectively. Consumer v purchases from Firm A if and only if $v \cdot q - p_A > v \cdot r - p_B$. If the consumer is indifferent because $v \cdot q - p_A = v \cdot r - p_B$, he picks a firm to buy from with probability 0.5. For given quality vectors and prices, the demands for Firm A and Firm B are, respectively,

$$\int_{v \cdot q - p_A \geq v \cdot r - p_B} \dots \int dF_1 dF_2 \dots dF_N \quad \text{and} \quad \int_{v \cdot q - p_A \leq v \cdot r - p_B} \dots \int dF_1 dF_2 \dots dF_N.$$

The two firms' profits are

$$\left\{ \int_{v \cdot q - p_A \geq v \cdot r - p_B} \dots \int dF_1 dF_2 \dots dF_N \right\} [p_A - C(q)] \equiv \pi_A(p_A, p_B; q, r) \quad (1)$$

$$\left\{ \int_{v \cdot q - p_A \leq v \cdot r - p_B} \dots \int dF_1 dF_2 \dots dF_N \right\} [p_B - C(r)] \equiv \pi_B(p_A, p_B; q, r). \quad (2)$$

In case $q = r$, and $p_A = p_B$, each firm sells to one half of the mass of consumers.

We study subgame-perfect equilibria of the standard multistage game of quality-price competition:

Stage 0: Consumers' valuations are drawn from respective distributions.

Stage 1: Firm A and Firm B simultaneously choose their product quality vectors q and r , respectively.

Stage 2: Firm A and Firm B simultaneously choose their product prices, p_A and p_B , respectively. Then each consumer picks a firm to buy from.

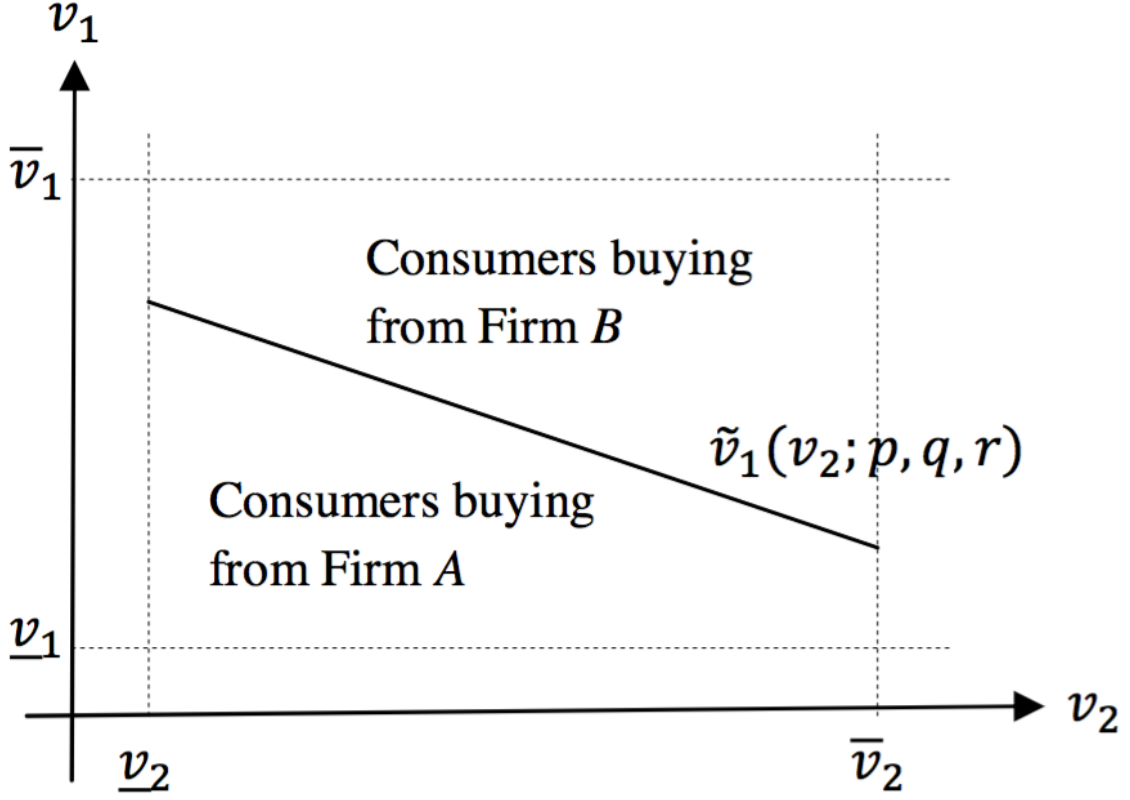


Figure 1: Consumers' choices given prices and qualities

support over the entire real line, but set $f_1(x) = 0$ whenever x lies outside the support $[\underline{v}_1, \bar{v}_1]$. For easier exposition, we will stick with the current notation.

We use the following shorthand to simplify the notation:

$$\int_{v_{-1}} \text{ stands for } \int_{\underline{v}_N}^{\bar{v}_N} \dots \int_{\underline{v}_2}^{\bar{v}_2} \text{ and } dF_{-1} \text{ stands for } \prod_{k=2}^N dF_k(v_k).$$

Profits of Firms A and B are, respectively:

$$\pi_A(p_A, p_B; q, r) = \int_{v_{-1}} F_1(\tilde{v}_1(v_{-1}; p, q, r)) dF_{-1} \times [p_A - C(q)] \quad (4)$$

$$\pi_B(p_A, p_B; q, r) = \int_{v_{-1}} [1 - F_1(\tilde{v}_1(v_{-1}; p, q, r))] dF_{-1} \times [p_B - C(r)]. \quad (5)$$

3.1 Subgame-perfect equilibrium prices

In subgame (q, r) , if $q = r$, the equilibrium in Stage 2 is the standard Bertrand equilibrium so each firm will charge its unit production cost: $p_A = p_B = C(q) = C(r)$.

We now turn to subgames in which $q \neq r$. By a permutation of quality indexes and interchanging the firms' indexes if necessary, we let $q_1 < r_1$. A price equilibrium in subgame (q, r) is a pair of prices (p_A^*, p_B^*) that are best responses: $p_A^* = \operatorname{argmax}_{p_A} \pi_A(p_A, p_B^*; q, r)$ and $p_B^* = \operatorname{argmax}_{p_B} \pi_B(p_A^*, p_B; q, r)$, where the profit functions are defined by (4) and (5). The existence of a price equilibrium follows from Caplin and Nalebuff (1991). Furthermore, because of the logconcavity assumption on the densities, a firm's profit function is quasi-concave in its own price.

As we will show, the characterization of equilibrium prices boils down to the properties of the solution of an integral equation. We begin with differentiating the profit functions with respect to prices:

$$\begin{aligned} \frac{\partial \pi_A}{\partial p_A} &= \int_{v_{-1}} F_1(\tilde{v}_1(v_{-1}; p, q, r)) dF_{-1} - \int_{v_{-1}} f_1(\tilde{v}_1(v_{-1}; p, q, r)) dF_{-1} \times \left[\frac{p_A - C(q)}{r_1 - q_1} \right] \\ \frac{\partial \pi_B}{\partial p_B} &= \int_{v_{-1}} [1 - F_1(\tilde{v}_1(v_{-1}; p, q, r))] dF_{-1} - \int_{v_{-1}} f_1(\tilde{v}_1(v_{-1}; p, q, r)) dF_{-1} \times \left[\frac{p_B - C(r)}{r_1 - q_1} \right], \end{aligned}$$

where we have used the derivatives of $\tilde{v}_1(v_{-1}; p, q, r)$ in (3) with respect to p_A and p_B .

The equilibrium prices $(p_A^*, p_B^*) \equiv p^*$ satisfy the first-order conditions:

$$p_A^* - C(q) = \frac{\int_{v_{-1}} F_1(\tilde{v}_1(v_{-1}; p^*, q, r)) dF_{-1}}{\int_{v_{-1}} f_1(\tilde{v}_1(v_{-1}; p^*, q, r)) dF_{-1}} \times (r_1 - q_1) \quad (6)$$

$$p_B^* - C(r) = \frac{\int_{v_{-1}} [1 - F_1(\tilde{v}_1(v_{-1}; p^*, q, r))] dF_{-1}}{\int_{v_{-1}} f_1(\tilde{v}_1(v_{-1}; p^*, q, r)) dF_{-1}} \times (r_1 - q_1) \quad (7)$$

where

$$\tilde{v}_1(v_{-1}; p^*, q, r) = \frac{p_B^* - p_A^*}{r_1 - q_1} - \sum_{k=2}^N v_k \frac{r_k - q_k}{r_1 - q_1}, \quad v_k \in [\underline{v}_k, \bar{v}_k], \quad k = 2, \dots, N. \quad (8)$$

Equations (6) and (7) say that the price-cost margins follow the usual inverse elasticity rule, a standard

result.⁵ The complication is that (8) sets up a function \tilde{v}_1 that depends on $N - 1$ continuous variables v_k , $k = 2, \dots, N$, and the quality vector (q, r) , and this is to be determined simultaneously with the prices in (6) and (7).

Let $p^* = (p_A^*, p_B^*)$ be the subgame-perfect equilibrium prices in Stage 2 in subgame (q, r) . The equilibrium prices are functions of the quality vector, so we write them as $p^*(q, r)$. Let $\tilde{v}_1(v_{-1}; p^*(q, r), q, r)$ be the solution to (8) at the subgame-perfect equilibrium. Now we define $\tilde{v}_1^*(v_{-1}; q, r) \equiv \tilde{v}_1(v_{-1}; p^*(q, r), q, r)$, which describes the set of consumers who are indifferent between buying from Firm A and Firm B in an equilibrium in subgame (q, r) .

By substituting the equilibrium prices (6) and (7) into $\tilde{v}_1(v_{-1}; p^*(q, r), q, r)$ in (8) above, we have:

$$\tilde{v}_1^*(v_{-1}; q, r) = \frac{\int_{x_{-1}} [1 - 2F_1(\tilde{v}_1^*(x_{-1}; q, r))] dF_{-1}}{\int_{x_{-1}} f_1(\tilde{v}_1^*(x_{-1}; q, r)) dF_{-1}} + \frac{C(r) - C(q)}{r_1 - q_1} - \sum_{k=2}^N v_k \frac{r_k - q_k}{r_1 - q_1}, \quad (9)$$

where for the variables of the integrals we have used the notation x_{-1} to denote (x_2, \dots, x_N) with $x_k \in [\underline{v}_k, \bar{v}_k]$ following distribution F_k , $k = 2, \dots, N$. This is an integral equation in \tilde{v}_1^* , a function that maps v_{-1} and quality vectors q and r to a real number, and the solution holds the key to the characterization of the price equilibrium. Indeed, we have decomposed the system in (6), (7) and (8) into two systems: a single integral equation (9), and those two equations in (6) and (7).

The integral equation (9) is independent of prices. Using the solution to (9), we can then proceed to solve for the equilibrium prices in (6) and (7). We are unaware that any paper in the extant literature of multiple qualities has decomposed the equilibrium prices and demand characterization in this fashion. Yet, using this decomposition, we can characterize the functional relationship between quality and equilibrium prices. Notice that because the price equilibrium in subgame (q, r) exists (from Caplin and Nalebuff (1991)), a solution to (9) must exist.

Lemma 1 *The solution of the integral equation (9) takes the form $\tilde{v}_1^*(v_{-1}; q, r) = \alpha(q, r) - \sum_{k=2}^N v_k \beta_k(q, r)$,*

⁵If we divide (6) by p_A^* , it can easily be seen that the right-hand side is the inverse elasticity of demand, which is obtained from the demand $\int_{v_{-1}} F_1(\tilde{v}_1(v_{-1}; p, q, r)) dF_{-1}$.

for $v_k \in [\underline{v}_k, \bar{v}_k]$, $k = 2, \dots, N$, where the functions $\alpha(q, r)$ and $\beta_k(q, r)$ are defined by

$$\alpha(q, r) = \frac{\int_{x_{-1}} \left[1 - 2F_1(\alpha(q, r) - \sum_{k=2}^N x_k \beta_k(q, r)) \right] dF_{-1}}{\int_{x_{-1}} f_1(\alpha(q, r) - \sum_{k=2}^N x_k \beta_k(q, r)) dF_{-1}} + \frac{C(r) - C(q)}{r_1 - q_1} \quad (10)$$

$$\beta_k(q, r) = \frac{r_k - q_k}{r_1 - q_1}, \quad k = 2, \dots, N. \quad (11)$$

From (6) and (7), a firm's qualities affect equilibrium prices of both firms. In turn, when equilibrium prices change, the set of indifferent consumers changes accordingly. The composition of the quality effect on equilibrium prices, and then the effect of equilibrium prices on the equilibrium set of indifferent consumers is the solution in Lemma 1. The equilibrium set of indifferent consumers takes the linear form, so the intercept α and all the slopes β_k , $k = 2, \dots, N$ are functions of the qualities.

Lemma 1 is a remarkable result. First, the solution to the integral equation (9) takes a manageable form: it consists of one implicit function $\alpha(q, r)$ in (10) and $N - 1$ explicit (and simple) functions $\beta_k(q, r)$, $k = 2, \dots, N$, in (11). Equation (10) is no longer an integral equation (for a solution $\tilde{v}_1^*(v_{-1}; q, r)$). Equation (10) defines implicitly one function $\alpha(q, r)$ whose arguments are qualities q and r but not v_{-1} .

Lemma 1 is a sort of aggregative result. The set of indifferent consumers determines firms' market shares, and qualities determine the intercept and slopes of the multi-dimensional line for the set of indifferent consumers. Although the integral equation (9) can be likened to a continuum of equations, Lemma 1 says that the solution can be aggregated into just N equations.

From Lemma 1, the equilibrium prices boil down to solving for the solutions of just three equations. By substituting the expressions for (10) and (11) to the right-hand side of (6) and (7), we can state the following proposition (proof omitted):

Proposition 1 *In subgame (q, r) , equilibrium prices are the solution of p_A^* in (12) and p_B^* in (13):*

$$p_A^* - C(q) = \frac{\int_{v_{-1}} F_1(\alpha(q, r) - v_{-1} \cdot \beta_{-1}(q, r)) dF_{-1}}{\int_{v_{-1}} f_1(\alpha(q, r) - v_{-1} \cdot \beta_{-1}(q, r)) dF_{-1}} (r_1 - q_1) \quad (12)$$

$$p_B^* - C(r) = \frac{\int_{v_{-1}} [1 - F_1(\alpha(q, r) - v_{-1} \cdot \beta_{-1}(q, r))] dF_{-1}}{\int_{v_{-1}} f_1(\alpha(q, r) - v_{-1} \cdot \beta_{-1}(q, r)) dF_{-1}} (r_1 - q_1), \quad (13)$$

with $\alpha(q, r)$ implicitly defined by (10), and $\beta_k(q, r) = \frac{r_k - q_k}{r_1 - q_1}$, $k = 2, \dots, N$.

The importance of Proposition 1 is this. The equilibrium price p_A^* is given by (12), a function of qualities. Thus, a direct differentiation of p_A^* with respect to qualities yields all the relevant information of how any of Firm B 's quality choices changes Firm A 's equilibrium price. The same applies to p_B^* and (13). The common link between p_A^* in (12) and p_B^* in (13) is the implicit function (10), the explicit functions (11), and the distributions of quality valuations.

3.2 Qualities and equilibrium prices

We now determine how qualities change equilibrium prices, and begin with writing equilibrium prices p_A^* in (12) and p_B^* in (13) as

$$\frac{p_A^* - C(q)}{r_1 - q_1} = G(\alpha, \beta_{-1}) \quad \text{and} \quad \frac{p_B^* - C(r)}{r_1 - q_1} = H(\alpha, \beta_{-1}),$$

where the functions: $G(\alpha, \beta_{-1}) : \mathfrak{R}^N \rightarrow \mathfrak{R}$, and $H(\alpha, \beta_{-1}) : \mathfrak{R}^N \rightarrow \mathfrak{R}$ are defined by

$$G \equiv \frac{\int_{v_{-1}} F_1(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1}}{\int_{v_{-1}} f_1(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1}} \quad \text{and} \quad H \equiv \frac{\int_{v_{-1}} [1 - F_1(\alpha - v_{-1} \cdot \beta_{-1})] dF_{-1}}{\int_{v_{-1}} f_1(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1}}. \quad (14)$$

The functions G and H (with their arguments omitted) are firms' equilibrium price-cost markups per unit of quality difference. The numerators of G and H are, respectively, Firm A 's and Firm B 's demands.

By Proposition 1, we directly differentiate p_A^* with respect to Firm B 's qualities r_i , $i = 1, \dots, N$, and differentiate p_B^* with respect to Firm A 's qualities q_i , and these derivatives, $\frac{\partial p_A^*}{\partial r_i}$ and $\frac{\partial p_B^*}{\partial q_i}$, are the *price-reaction effects*. In the Appendix, we show these derivatives right after the proof of Lemma 1. There, we

present two intermediate results (Lemmas 2 and 3) that help us to simplify the price-reaction effects. As it turns out, product differentiation is determined by differences between price-reaction effects, and the next proposition presents them.

Proposition 2 *In subgame (q, r) , for quality j , $j = 2, \dots, N$, the difference in the price-reaction effects can be written in two ways:*

$$\begin{aligned} & \frac{\partial p_B^*(q, r)}{\partial q_j} - \frac{\partial p_A^*(q, r)}{\partial r_j} \\ &= \frac{(r_1 - q_1) \frac{\partial}{\partial r_j} \left[\int_{v_{-1}} f_1(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1} \right]}{\left(\int_{v_{-1}} f_1(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1} \right)^2} + Z [C_j(q) - C_j(r)] \end{aligned} \quad (15)$$

$$\begin{aligned} &= - \frac{(r_1 - q_1) \frac{\partial}{\partial q_j} \left[\int_{v_{-1}} f_1(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1} \right]}{\left(\int_{v_{-1}} f_1(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1} \right)^2} + Z [C_j(q) - C_j(r)], \end{aligned} \quad (16)$$

where

$$Z = \frac{1 + H(\alpha, \beta_{-1}) \frac{\int_{v_{-1}} f_1'(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1}}{\int_{v_{-1}} f_1(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1}}}{3 - G(\alpha, \beta_{-1}) \frac{\int_{v_{-1}} f_1'(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1}}{\int_{v_{-1}} f_1(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1}}},$$

and C_j denotes the j^{th} partial derivative of C .

The proposition says that the difference in firms' price-reaction effects of a quality is determined by i) how the quality changes the total density of the equilibrium set of indifferent consumers, $\int_{v_{-1}} f_1(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1}$, and ii) by the difference in the marginal costs of quality $[C_j(q) - C_j(r)]$. Indeed, the sum of the markups per unit of quality difference is $G + H = \left\{ \int_{v_{-1}} f_1(\tilde{v}_1^*) dF_{-1} \right\}^{-1}$. The first term in each of two equivalent expressions in Proposition 2 is the derivative of the sum of markups with respect to a quality j , $j = 2, \dots, N$.

This is point i). Also, Firm A 's quality on the total density of the set of indifferent consumers is equal and opposite to that of Firm B 's, so this accounts for the equivalence of (15) and (16). For ii), we just note that

the second term in each of the two expressions is the difference in firms' marginal costs of a quality adjusted by Z .

Finally, when f_1 is a step function, $f_1'(\alpha - v_{-1} \cdot \beta_{-1}) = 0$, expressions (15) and (16) simplify to the term related to the difference in marginal costs:

$$\frac{\partial p_B^*(q, r)}{\partial q_j} - \frac{\partial p_A^*(q, r)}{\partial r_j} = \frac{1}{3} [C_j(q) - C_j(r)]. \quad (17)$$

That is, when the differentiated dimension has a uniform-distribution valuation, the leading case in the literature, each firm's quality raises the markup by the same amount, so the relative price-reaction effects fall entirely on the marginal-cost difference.

3.3 Equilibrium qualities

Now we characterize equilibrium qualities. When firms produce the same qualities, $q = r$, the continuation is a strict Bertrand game, so each firm makes a zero profit. Clearly, there is no equilibrium in which firms choose identical qualities. We use the convention that firms' qualities differ in the first dimension, $q_1 < r_1$. The profit functions in Stage 1 in terms of qualities are:

$$\pi_A(p_A^*(q, r), p_B^*(q, r); q, r) = \int_{v_{-1}} F_1(\tilde{v}_1(v_{-1}; p^*, q, r)) dF_{-1} \times [p_A^*(q, r) - C(q)] \quad (18)$$

$$\pi_B(p_A^*(q, r), p_B^*(q, r); q, r) = \int_{v_{-1}} [1 - F_1(\tilde{v}_1(v_{-1}; p^*, q, r))] dF_{-1} \times [p_B^*(q, r) - C(r)], \quad (19)$$

where $p_A^*(q, r)$ and $p_B^*(q, r)$ are equilibrium prices in Stage 2, and $\tilde{v}_1(v_{-1}; p^*, q, r) \equiv \frac{p_B^*(q, r) - p_A^*(q, r)}{r_1 - q_1} - \sum_{k=2}^N v_k \frac{r_k - q_k}{r_1 - q_1}$ is the set of equilibrium indifference consumers. Given subgame-perfect equilibrium prices, p^* , equilibrium qualities are q^* and r^* that are mutual best responses:

$$q^* \equiv (q_1^*, \dots, q_N^*) = \operatorname{argmax}_q \int_{v_{-1}} F_1(\tilde{v}_1(v_{-1}; p^*(q, r^*), q, r^*)) dF_{-1} \times [p_A^*(q, r^*) - C(q)]$$

$$r^* \equiv (r_1^*, \dots, r_N^*) = \operatorname{argmax}_r \int_{v_{-1}} [1 - F_1(\tilde{v}_1(v_{-1}; p^*(q^*, r), q^*, r))] dF_{-1} \times [p_B^*(q^*, r) - C(r)],$$

where $p^*(q, r) \equiv (p_A^*(q, r), p_B^*(q, r))$.

Qualities q_i , $i = 1, \dots, N$, affect Firm A 's profit (18) in three ways. First, they have a direct effect through costs and demand. Second, they affect the profit through Firm A 's own equilibrium price $p_A^*(q, r)$. Third, they affect the profit through Firm B 's equilibrium price $p_B^*(q, r)$, captured by $\partial p_B^*/\partial q_i$. Because the equilibrium prices $p_A^*(q, r)$ and $p_B^*(q, r)$ are mutual best responses in the price subgame in Stage 2, the envelope theorem applies. That is, Firm A 's qualities q_i , $i = 1, \dots, N$, have second-order effects on its own profit (18) through its own equilibrium price; the second effect can be ignored.

The first-order derivative of (18) with respect to q_i is

$$- \left[\int_{v_{-1}} F_1(\tilde{v}_1(v_{-1}; p^*(q, r^*), q, r^*)) dF_{-1} \right] C_i(q) + \underbrace{\frac{\partial}{\partial q_i} \left\{ \int_{v_{-1}} F_1(\tilde{v}_1(v_{-1}; p^*(q, r^*), q, r^*)) dF_{-1} \right\}}_{\text{effects of quality } q_i \text{ on cost and demand}} \times [p_A^*(q, r^*) - C(q)] \quad (20)$$

$$+ \underbrace{\frac{\partial}{\partial p_B^*} \left\{ \int_{v_{-1}} F_1(\tilde{v}_1(v_{-1}; p^*(q, r^*), q, r^*)) dF_{-1} \right\}}_{\text{effect of quality on Firm } B \text{'s price}} \frac{\partial p_B^*}{\partial q_i} \times [p_A^*(q, r^*) - C(q)], \quad i = 1, \dots, N, \quad (21)$$

where the partial derivative of profit with respect to p_A has been ignored. The terms in (20) describe how a quality affects cost and demand, whereas the term in (21) describes the strategic effect of a quality on the rival's price. We can also write out the derivative of profit (19) with respect to r_i to obtain a similar expression. For brevity, we have omitted the expressions.

We now state the main result on equilibrium qualities. We obtain the set of equations in the next proposition by first simplifying the first-order derivatives and then setting them to zero. For simplification, we use the basic demand function (3) and equilibrium prices (12) and (13) in Proposition 1, and finally drop common factors in the first-order derivatives. (Details are in the proof.)

Proposition 3 *For the quality-price, multistage game in Subsection 2.2, equilibrium qualities (q^*, r^*) (under*

the convention that $q_1^* < r_1^*$) must satisfy the following $2N$ equations:

$$\frac{\partial p_B^*}{\partial q_1} + \frac{\int_{v_{-1}} f_1(\alpha - v_{-1} \cdot \beta_{-1}) \tilde{v}_1^* dF_{-1}}{\int_{v_{-1}} f_1(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1}} - C_1(q^*) = 0 \quad (22)$$

$$\frac{\partial p_A^*}{\partial r_1} + \frac{\int_{v_{-1}} f_1(\alpha - v_{-1} \cdot \beta_{-1}) \tilde{v}_1^* dF_{-1}}{\int_{v_{-1}} f_1(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1}} - C_1(r^*) = 0, \quad (23)$$

and for $j = 2, \dots, N$,

$$\frac{\partial p_B^*}{\partial q_j} + \frac{\int_{v_{-1}} f_1(\alpha - v_{-1} \cdot \beta_{-1}) v_j dF_{-1}}{\int_{v_{-1}} f_1(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1}} - C_j(q^*) = 0 \quad (24)$$

$$\frac{\partial p_A^*}{\partial r_j} + \frac{\int_{v_{-1}} f_1(\alpha - v_{-1} \cdot \beta_{-1}) v_j dF_{-1}}{\int_{v_{-1}} f_1(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1}} - C_j(r^*) = 0, \quad (25)$$

where α and β_j are the functions in (10) and (11), respectively, and \tilde{v}_1^* is $\tilde{v}_1^*(v_{-1}; q^*, r^*)$, the solution of the integral equation in Lemma 1.

The properties of equilibrium qualities in (22) and (24) can be explained as follows. There are two effects. The first term in each expression is the *price-reaction effect*: it describes how Firm A 's qualities q_1 and q_j , $j = 2, \dots, N$ affect the rival's price in the continuation subgame (see (33) to (36) in the Appendix).

The second effect concerns the average valuation of the j^{th} quality among the equilibrium set of indifferent consumers—the integrals in (22) and (24)—and the j^{th} quality's marginal contribution to the per-unit cost, $C_j(q) \equiv \frac{\partial C(q)}{\partial q_j}$. These two terms together form the *Spence effect*.⁶ Indeed, Spence (1975) shows that a profit-maximizing firm chooses the efficient quality for the marginal consumer (and then raises the price to extract the marginal consumer's surplus).⁷ The same price-reaction and Spence effects apply to Firm B 's

⁶In the literature, various authors have used such terms as demand and market-share effects to describe the direct effect of a quality (or a location) on marginal consumers. See, for example, Tirole (1988, pp.281-2) and Vandenbosch and Weinberg (1995, p.226). These earlier works, however, have assumed either zero or linear quality cost, so must limit the quality to a bounded interval.

⁷Let $P(D, q)$ be the price a firm can charge when it sells D units of its good at quality $q = (q_1, \dots, q_N)$. Let $C(D, q)$ be the cost when the firm produces D units at quality q . Profit is $DP(D, q) - C(D, q)$. The profit-maximizing quality q_i is given by $D \frac{\partial P}{\partial q_i} = \frac{\partial C}{\partial q_i}$. Hence, the quality valuation of the marginal consumer $\frac{\partial P}{\partial q_i}$ is equal to the marginal contribution of quality i to per-unit cost $\frac{\partial C / \partial q_i}{D}$. See Spence (1975, p.419; equation (8)).

equilibrium quality choices described in (23) and (25).

Because the two firms face the same equilibrium set of indifferent consumers, the Spence effect pushes them to choose the same qualities. The price-reaction effects generally put the firms in a race situation. Prices are strategic complements, so each firm wants to use its qualities to raise the rival's price. The price-reaction effect dictates how much a firm's equilibrium quality deviates from the efficient quality for the equilibrium set of indifferent consumers. The firm that has a stronger price-reaction effect deviates more.

If the two firms were playing another game in which prices and qualities were chosen concurrently (one with merged Stages 1 and 2 in the extensive form in Subsection 2.2), the price-reaction effect would vanish. Then the Spence effect would dictate equilibrium strategies. Each firm would choose qualities optimal for the average valuations of the common set of marginal consumers, so would choose the same level for each quality attribute. Firms must then set their prices at marginal cost. (For an illustration of a game with firms choosing prices and qualities concurrently, see Ma and Burgess (1993).)

We have stated Proposition 3 in terms of a set of equations, so we implicitly assume that firms do not set qualities at zero. If "corner" equilibrium qualities are to be included, the equalities in the Proposition will be replaced by weak inequalities. If firms choose to have zero level of a certain quality, then (trivially) product differentiation does not happen in that quality. Obviously, when a quality valuation support has a high lower bound, firms will find it optimal to produce a strictly positive quality, so zero quality can be avoided simply by raising the support.

3.4 Quality differentiation

Proposition 3 draws a connection between the price-reaction effects and qualities' marginal contributions to unit production cost. Recalling that $C_j(q) \equiv \partial C(q)/\partial q_j$, we state this formally:

Corollary 1 *At the equilibrium (q^*, r^*) , a firm's j^{th} quality contributes more to its own unit production cost than a rival's j^{th} quality contributes to the rival's unit production cost if and only if the firm's price-reaction effect of that quality is stronger than the rival's. That is, for each $j = 2, \dots, N$, the following are equivalent:*

i) $C_j(q^) < C_j(r^*)$,*

$$ii) \frac{\partial p_B^*(q^*, r^*)}{\partial q_j} < \frac{\partial p_A^*(q^*, r^*)}{\partial r_j},$$

iii) the two equivalent expressions on the right-hand side of (15) and (16) in Proposition 2 are negative at equilibrium (q^*, r^*) .

Corollary 1 offers a general perspective. What matter are not quality levels. The key is how each quality contributes to the unit production cost. From Proposition 3 we have for any quality:

$$\frac{\partial p_B^*(q^*, r^*)}{\partial q_j} - C_j(q^*) = \frac{\partial p_A^*(q^*, r^*)}{\partial r_j} - C_j(r^*). \quad (26)$$

The statements in the Corollary simply reflect this property: how product quality raises the rival's price and its own unit production cost must be equalized among the two firms in an equilibrium.

However, the corollary does not directly address the equilibrium quality levels. We have used a general cost function, so it is quite possible that $C_j(q^*) = C_j(r^*)$ but the qualities q_j^* and r_j^* are different.⁸ Sharper results can be obtained from the following (with proof omitted):

Corollary 2 *Suppose that the cost function C is additively separable:*

$$C(q) = C(q_1, q_2, \dots, q_N) = \gamma_1(q_1) + \gamma_2(q_2) + \dots + \gamma_N(q_N),$$

where γ_i is an increasing, differentiable and strictly convex function, so $C_i(q) = \gamma'_i(q_i)$, $i = 1, 2, \dots, N$. In an equilibrium (q^*, r^*) , for $j = 2, \dots, N$,

$$q_j^* < r_j^* \iff \frac{\partial p_B^*(q^*, r^*)}{\partial q_j} < \frac{\partial p_A^*(q^*, r^*)}{\partial r_j}.$$

With separable cost, a quality's contribution to the unit production cost is independent of other qualities. A firm having a stronger price-reaction effect at a quality than its rival's must choose a higher quality than the rival's quality. Notice that the Corollary gives a sufficient condition. In particular, when a cost function is separable in some, but not all, qualities, differentiation in some qualities may still be manifested according to the relative strength of the price-reaction effects.

⁸For example, if the cost function is $C(q_1, q_2) = q_1^2 + \theta q_1 q_2 + \frac{1}{2} q_2^2$, for some parameter $\theta \neq 0$, then $C_1(q_1, q_2) = 2q_1 + \theta q_2$, and $C_1(r_1, r_2) = 2r_1 + \theta r_2$. Even if $2q_1 + \theta q_2 = 2r_1 + \theta r_2$, q_i may not be equal to r_i , $i = 1, 2$.

In the literature, the separable-cost assumption has been adopted. According to Corollaries 1 and 2 the fundamental issue is how a quality contributes to the production cost. For a model with many qualities, a quality's contribution to production cost depends on the entire quality profile, and a cost function that assumes away cost spillover is restrictive. We illustrate this point by some examples below, but first we consider specific quality-valuation density functions commonly used in the literature:

Corollary 3 *Suppose that f_1 is a step function, so $f_1' = 0$ almost everywhere. In an equilibrium (q^*, r^*) , $C_j(q^*) = C_j(r^*)$ and $\frac{\partial p_B^*(q^*, r^*)}{\partial q_j} = \frac{\partial p_A^*(q^*, r^*)}{\partial r_j}$, $j = 2, \dots, N$. Furthermore, if C is additively separable, then $q_j^* = r_j^*$, $j = 2, \dots, N$; in other words, qualities 2 through N are nondifferentiated.*

Corollary 3 presents a striking result. Recall that in an equilibrium, there must be at least one quality for which firms produce at different levels. This is our convention for labeling that equilibrium differentiated quality as quality 1. Now if consumers' valuations of quality 1 is a step function (of which the uniform distribution is a common example in the literature), Proposition 2 says that the difference in price-reaction effects, from (17), is: $\frac{\partial p_B^*(q, r)}{\partial q_j} - \frac{\partial p_A^*(q, r)}{\partial r_j} = \frac{1}{3} [C_j(q) - C_j(r)]$, $j = 2, \dots, N$. Corollary 1 also says that this price-reaction difference is equal to $[C_j(q) - C_j(r)]$, $j = 2, \dots, N$ (see (26)), so we have $C_j(q) = C_j(r)$, $j = 2, \dots, N$. Next, if the cost function is additively separable (another common assumption in the literature), Corollary 2 applies, so in equilibrium, firms produce identical qualities 2 through N !

Uniform valuation distributions *and* additively separable cost are the drivers for quality nondifferentiation. Here is a simple example to show that even when valuations are uniformly distributed, cost consideration will give rise to equilibrium product differentiation. Use the following cost function for two qualities: $C(q_1, q_2) = \frac{1}{2}q_1^2 + \theta q_1 q_2 + \frac{1}{2}q_2^2$. The marginal costs are $C_1(q_1, q_2) = q_1 + \theta q_2$, and $C_2(q_1, q_2) = \theta q_1 + q_2$. Suppose that f_1 is a step function, so firms' price-reaction effects satisfy: $\frac{\partial p_B^*}{\partial q_2} - \frac{\partial p_A^*}{\partial r_2} = \frac{1}{3} [C_2(q_1, q_2) - C_2(r_1, r_2)]$ (see (17)). According to Corollary 3, $C_2(q_1^*, q_2^*) = C_2(r_1^*, r_2^*)$. In other words, $\theta q_1^* + q_2^* = \theta r_1^* + r_2^*$, and $\theta(r_1^* - q_1^*) = q_2^* - r_2^*$. By assumption, we have $q_1^* < r_1^*$ in the equilibrium. We conclude that $q_2^* > r_2^*$ if and only if $\theta > 0$. When qualities have positive spillover on cost ($\theta > 0$), then Firm A 's product has one superior quality and one inferior quality compared to Firm B 's. By contrast, when qualities have negative spillover ($\theta < 0$), Firm A 's qualities are always lower than Firm B 's.

Whereas Corollary 3 presents a set of sufficient conditions for equilibria with minimum differentiation, we also present, as an addendum, a set of necessary conditions in Corollary 5 at the end of the Appendix.

Corollary 3 has used the convention that, in the equilibrium, $q_1^* < r_1^*$. The following explains the scope of the convention.

Corollary 4 *Consider the game defined by valuation densities f_i , $i = 1, \dots, N$, and the separable cost function $C = \sum_{i=1}^N \gamma_i$. Suppose that at least one of the densities is a step function.*

- i) In every subgame-perfect equilibrium, firms choose an identical level for at least one quality.*
- ii) If in an equilibrium (q^*, r^*) there is differentiation in the j^{th} quality, so that $q_j^* \neq r_j^*$, and f_j is a step function, $j = 1, \dots, N$, then there is no differentiation in any other quality, so $q_k^* = r_k^*$, for $k = 1, 2, \dots, N$, and $k \neq j$*
- iii) For the special case of $N = 2$, in every subgame-perfect equilibrium, one and only one quality will be differentiated.*

In this corollary, we have gotten rid of the convention that in equilibrium Firm A chooses a first quality different from Firm B 's. Consider all equilibria of the multistage game, given valuation densities and the cost function. The effect of *any* uniform quality-valuation distribution and the separable cost function is strong. Suppose that the j^{th} quality has a uniform valuation distribution. If it so happens that in equilibrium firms choose $q_j^* \neq r_j^*$, then Corollary 3 applies to the j^{th} quality, so all equilibrium qualities except the j^{th} must be identical. The only case in which equilibrium differentiation happens in more than one quality is when $q_j^* = r_j^*$. Then Corollary 3 does not apply to the j^{th} quality. But this means that there is (at least) one nondifferentiated quality.

Corollary 4 clarifies the “Max-Min-Min...-Min” results in Irmen and Thisse (1998). They consider an N -dimensional Hotelling model (which can be translated into our N -dimensional quality model). Consumers’ locations are uniformly distributed on the N -dimensional unit hypercube. Consumers’ mismatch disutility is the (weighted) N -dimensional Euclidean distance. Irmen and Thisse derive a subgame-perfect equilibrium in which the two firms choose the maximum distance between themselves in one dimension but zero distance in all other dimensions (p.90, Proposition 2). Although Corollary 4 does not address existence of equilibria,

it is consistent with the Irmen-Thisse result. To see this, we can rewrite Corollary 4 as follows: if M of the qualities have uniformly distributed valuations, $1 \leq M \leq N$, then at least $\min\{M, N - 1\}$ qualities will be nondifferentiated. This is a slightly more general result than in Irmen and Thisse (1998). In their model, all valuations are uniformly distributed, so $M = N$. Therefore exactly $N - 1$ qualities will be nondifferentiated.

Vandenbosch and Weinberg (1995) and Lauga and Ofek (2011) are two related papers in the vertical differentiation literature. They use a linear cost function, and restrict each of two qualities to be in its own bounded interval. In the notation here, in both papers, $N = 2$, valuation density f_i is uniform, quality q_i is to be chosen from interval $[q_i, \bar{q}_i]$, $i = 1, 2$, and unit production cost at quality q is $C(q) = c_1q_1 + c_2q_2$, for constants c_1 and c_2 . (In fact, the values of c_1 and c_2 are set at 0 in some cases.) The linear cost function does not satisfy our assumption of strict convexity. Equilibria with maximum or minimum differentiation arise due to corner solutions of firms' profit maximization.⁹

We can interpret these results in terms of price-reaction and Spence effects. First, according to Corollary 3, because quality valuations are uniformly distributed, price-reaction effects differ according to the marginal-cost difference, independent of the density of the set of indifferent consumers. Second, because the cost is linear, a quality's marginal contribution to unit production cost $C_i(q)$ is constant. The Spence effect generically does not specify an interior solution under linear cost. Hence, the maximum-minimum differentiation results are driven by the combination of linear costs and uniform valuation distributions.

Garella and Lambertini (2014) use a discontinuous cost function: a firm producing z units of the good at quality (q_1, q_2) has a total cost of $cz + T(q_1, q_2)$ if $q_1 > \underline{q}_1$, but only $T(q_1, q_2)$ if $q_1 = \underline{q}_1$, where $\underline{q}_1 > 0$ and $c > 0$ are fixed parameters, and T is increasing when $q_1 > \underline{q}_1$. Consumers have homogenous preferences on the second quality, but their valuations on the first quality follow a uniform distribution. They derive equilibria in which firms choose different levels in both qualities. We use a continuous cost function, so our results do not apply to their model.

⁹However, for some parameter configurations, Vandenbosch and Weinberg (1995) exhibit an interior choice of one quality.

4 Examples on two quality dimensions

In this section, we present two sets of examples of a model with two quality dimensions ($N = 2$). Besides verifying results, we also use these examples to address existence of equilibria. For these examples, we assume that cost is quadratic: $C(q_1, q_2) = \frac{1}{2}q_1^2 + \theta q_1 q_2 + \frac{1}{2}q_2^2$, which exhibits a positive cost spillover if and only if $\theta > 0$, and which is separable if $\theta = 0$. Next, we assume that consumers' valuations of both qualities belong to the interval $[1, 2]$, and that v_1 follows the uniform distribution. Consumers' valuation on the second quality, v_2 , follows a trapezoid distribution. For a parameter k , $-1 \leq k \leq 1$, the density function is $f_2(v_2) = 1 - k + 2k(v_2 - 1)$ (so f_2 is a straight line, and has densities $1 - k$ at $v_2 = 1$, and $1 + k$ at $v_2 = 2$), with the corresponding distribution $F_2(v_2) = (1 - k)(v_2 - 1) + k(v_2 - 1)^2$. Notice that at $k = 0$, the distribution is uniform, and at $k = 1$, the density is triangular on $[1, 2]$.

Solving for $\alpha(q, r)$ in (10) in Lemma 1, we obtain a unique solution (see the *Mathematica* program in the online supplements):

$$\alpha(q, r) = \left[\frac{3q_1(6 + q_1) + 2(9 + k)q_2 + 3q_2^2 - 3r_1(6 + r_1) - 2(9 + k)r_2 - 3r_2^2 - 6\theta(q_1 q_2 + r_1 r_2)}{18(q_1 - r_1)} \right], \quad (27)$$

which will be used in the following two subsections.

4.1 Quadratic cost function and two uniform distributions

The first set of examples uses the general quadratic cost function with spillover ($\theta \neq 0$) and two uniform distributions of consumers' valuations ($k = 0$). Setting $k = 0$ and rearranging terms in (27), we obtain

$$\alpha(q, r) = \frac{1}{6} \left[\frac{q_1^2 + q_2(6 + q_2) - r_2(6 + r_2) + 2q_1(3 + \theta q_2) - r_1(6 + r_1 + 2\theta r_2)}{q_1 - r_1} \right]. \quad (28)$$

The equilibrium set of marginal consumers is

$$\tilde{v}_1^*(v_2; q, r) = \left[\frac{q_1^2 + q_2(6 + q_2) - r_2(6 + r_2) + 2q_1(3 + \theta q_2) - r_1(6 + r_1 + 2\theta r_2)}{6(q_1 - r_1)} \right] - \frac{r_2 - q_2}{r_1 - q_1} v_2.$$

The expressions for $G(\alpha, \beta_2)$ and $H(\alpha, \beta_2)$ are

$$\begin{aligned} G(\alpha, \beta_2) &= \left[\frac{q_1^2 + q_2(q_2 - 3) - r_2(r_2 - 3) + 2\theta q_1 q_2 - r_1(r_1 + 2\theta r_2)}{6(q_1 - r_1)} \right] \\ H(\alpha, \beta_2) &= 1 - \left[\frac{q_1^2 + q_2(q_2 - 3) - r_2(r_2 - 3) + 2\theta q_1 q_2 - r_1(r_1 + 2\theta r_2)}{6(q_1 - r_1)} \right]. \end{aligned}$$

Price effects are:

$$\begin{aligned}\frac{\partial p_A^*}{\partial r_1} &= G + (r_1 - q_1) \frac{\partial G}{\partial r_1} = \frac{r_1 + \theta r_2}{3}, & \frac{\partial p_B^*}{\partial q_1} &= -H + (r_1 - q_1) \frac{\partial H}{\partial q_1} = \frac{-3 + q_1 + \theta q_2}{3} \\ \frac{\partial p_A^*}{\partial r_2} &= (r_1 - q_1) \frac{\partial G}{\partial r_2} = \frac{-3 + 2r_2 + 2\theta r_1}{6}, & \text{and } \frac{\partial p_B^*}{\partial q_2} &= (r_1 - q_1) \frac{\partial H}{\partial q_2} = \frac{-3 + 2q_2 + 2\theta q_1}{6}.\end{aligned}\quad (29)$$

A firm's influence on the rival's price is independent of the rival's qualities, a consequence of the uniform-distribution assumption. We verify (17):

$$\frac{\partial p_B^*(q, r)}{\partial q_2} - \frac{\partial p_A^*(q, r)}{\partial r_2} = \frac{C_2(q_1, q_2) - C_2(r_1, r_2)}{3} = \frac{q_2 - r_2 + \theta(q_1 - r_1)}{3}.$$

Solving the system of equations of the first-order conditions in Proposition 3 we find:

$$q_1^* = \left(\frac{3}{4}\right) \left(\frac{1-2\theta}{1-\theta^2}\right); \quad q_2^* = \left(\frac{3}{4}\right) \left(\frac{2-\theta}{1-\theta^2}\right); \quad r_1^* = \left(\frac{3}{4}\right) \left(\frac{3-2\theta}{1-\theta^2}\right); \quad r_2^* = \left(\frac{3}{4}\right) \left(\frac{2-3\theta}{1-\theta^2}\right). \quad (30)$$

Any equilibrium qualities must be in (30). Moreover, substituting the qualities in (30) into the expressions for the price effects, $\frac{\partial p_A^*(q, r)}{\partial r_2}$ and $\frac{\partial p_B^*(q, r)}{\partial q_2}$, in (29), respectively, we verify that price effects of the second qualities are identical at an equilibrium.

Finally, for some specific values of the parameter θ , we have

$$\begin{aligned}(1) \text{ If } \theta &= 0, & q_1^* &= \frac{3}{4} & q_2^* &= \frac{3}{2} & r_1^* &= \frac{9}{4} & r_2^* &= \frac{3}{2} \\ (2) \text{ If } \theta &= \frac{1}{4}, & q_1^* &= \frac{2}{5} & q_2^* &= \frac{7}{5} & r_1^* &= 2 & r_2^* &= 1 \\ (3) \text{ If } \theta &= -\frac{1}{2}, & q_1^* &= 2 & q_2^* &= \frac{5}{2} & r_1^* &= 4 & r_2^* &= \frac{7}{2}.\end{aligned}\quad (31)$$

Case (1) in (31) illustrates Corollary 3 and Corollary 4(iii): with a separable cost function and uniform distributions, only the first dimension of quality is differentiated. Moreover, in equilibrium, the two firms must choose $q_2^* = r_2^* = E[v_2]$. This is the same result in Irmen and Thisse (1998). In Case (2), for positive cost spillover, Firm A produces a superior second quality than Firm B. Conversely, in Case (3), for negative cost spillover, Firm A produces an inferior second quality than Firm B. Cases (2) and (3) confirm that the nondifferentiation result in Irmen and Thisse (1998) depends on the separable mismatch disutility.

4.2 Separable quadratic cost function, and one uniform distribution and one trapezoid distribution

This second set of examples uses a separable quadratic cost function ($\theta = 0$), and one uniform distribution and one trapezoid distribution ($k \neq 0$). Setting θ at 0 in (27), we obtain the equilibrium set of marginal consumers:

$$\tilde{v}_1^*(v_2; q, r) = \left[\frac{3q_1(6 + q_1) + 2(9 + k)q_2 + 3q_2^2 - 3r_1(6 + r_1) - 2(9 + k)r_2 - 3r_2^2}{18(q_1 - r_1)} \right] - \frac{r_2 - q_2}{r_1 - q_1} v_2.$$

Using equation $\tilde{v}_1^*(v_2; q, r)$ to compute the equilibrium prices p_A^* in (12) and p_B^* in (13), we derive the profits as functions of qualities and the parameter k . We then solve the system of equations of the first-order conditions with respect to the qualities to obtain:

$$\begin{aligned} -1 \leq k \leq 1 \quad q_1^* &= \frac{3}{4} \quad q_2^* = \frac{3}{2} + \frac{k}{6} \quad r_1^* = \frac{9}{4} \quad r_2^* = \frac{3}{2} + \frac{k}{6} \\ (4) \text{ If } k = 0 \quad q_1^* &= \frac{3}{4} \quad q_2^* = \frac{3}{2} \quad r_1^* = \frac{9}{4} \quad r_2^* = \frac{3}{2} \\ (5) \text{ If } k = 1 \quad q_1^* &= \frac{3}{4} \quad q_2^* = \frac{3}{2} + \frac{1}{6} \quad r_1^* = \frac{9}{4} \quad r_2^* = \frac{3}{2} + \frac{1}{6}. \end{aligned} \tag{32}$$

The results are consistent with Corollary 3 and Corollary 4(iii). Because $q_1^* < r_1^*$, and f_1 is uniform, the second quality dimension must not be differentiated for any members of the trapezoid distributions. Solution (4) matches case (1) ($\theta = 0$) of (31) in the previous subsection. In fact, the consequence of a valuation distribution that puts more density on higher valuations than the uniform is higher equilibrium quality $q_2^*(= r_2^*)$. Also, in Solution (5), firms have equal shares of the market ($\alpha = \frac{3}{2}$, and $\tilde{v}_1^*(v_2; q^*, r^*) = \alpha$); Firm A sells a product with a lower quality at a lower price, and the opposite is true for Firm B : $p_A^* = 2.42014 < p_B^* = 4.67014$. However, because unit costs are increasing in qualities, profits are the same for the two firms: $\pi_A^* = \pi_B^* = 0.375$.

4.2.1 Existence of equilibria

Our characterization results can be interpreted as necessary conditions for subgame-perfect equilibria. Our results do not provide a proof of the existence of equilibria. The general difficulty regarding existence has to do with multiple qualities that maximize profits. Consider for now just the first quality. For some given Firm B 's quality r_1 , Firm A 's profit-maximizing quality may not be unique; say, they are qualities q_1' and

q_1'' , and $q_1' < r_1 < q_1''$ (but any convex combination of q_1' and q_1'' does not maximize profit). For other values of r_1 , Firm A 's profit-maximizing quality may be unique, but it may be larger than r_1 , or it may be smaller. Thus, the requirement that a candidate equilibrium has q_1 smaller than r_1 may be difficult to verify. We are unaware of results for the existence of fixed points that serve as mutual quality best responses.

Nevertheless, our results allow us to construct candidate equilibria, as in the previous two subsections. Therefore, one may verify that they are mutual best responses. We have in fact done that for Solution (5) in the previous subsection: at $k = 1$ for the trapezoid distribution $(q_1, q_2) = (\frac{3}{4}, \frac{3}{2} + \frac{1}{6})$ and $(r_1, r_2) = (\frac{9}{4}, \frac{3}{2} + \frac{1}{6})$ are mutual best responses. To do so we have written a *Mathematica* program to compute profits $\pi_A(q_1, q_2; r_1^*, r_2^*)$ and $\pi_B(r_1, r_2; q_1^*, q_2^*)$ in (18) and (19) for all demand configurations.¹⁰ The program is in the online supplement.

It is important to note that some candidate equilibria may fail to be equilibria. For the case of $k = 1$ in the previous subsection, another candidate equilibrium could have firms producing identical qualities in the first dimension (valuations following a uniform distribution), but different qualities in the second (valuations following a trapezoid distribution). We have used *Mathematica* to compute such a candidate equilibrium. The numerical method yields $(q_1^*, q_2^*) = (1.50, 0.62)$, and $(r_1^*, r_2^*) = (1.50, 1.96)$. However, our computation indicates that Firm A has a profitable deviation. In other words, for $k = 1$ there is no equilibrium in which firms produce identical qualities in the dimensions where valuations are uniform.

5 Concluding remarks

We reexamine the principle of product differentiation relaxing price competition in the classical quality-price game. The environment for analysis in our model is more general than existing works. Yet, we are able to characterize equilibria without having to solve for equilibria explicitly. Product quality is used by each firm to raise a rival firm's equilibrium price, so firms engage in a race. A product quality's contribution to

¹⁰We make sure that for any combination of q_i and r_i , valuations of the indifferent consumer $\tilde{v}_1^*(v_2; q, r)$ must reside in $[1, 2]$ as v_2 varies over $[1, 2]$. The *Mathematica* program computes $\max_{q_1, q_2} \pi_A(q_1, q_2; r_1^*, r_2^*)$ at $r_1^* = \frac{9}{4}, r_2^* = \frac{5}{3}$. (We do not place any restriction on q_1 or q_2 .) We have found that indeed the maximum profit is achieved at $(q_1, q_2) = (\frac{3}{4}, \frac{5}{3})$. Then we perform the corresponding computation for Firm B 's profit and have found that the maximum profit is achieved at $(r_1, r_2) = (\frac{9}{4}, \frac{5}{3})$.

marginal costs is equal among firms when firms are equal rivals in this race. Generally, firms have different capabilities of raising the rival's equilibrium price, due to nonuniform consumer valuations or cost spillover, so product differentiation tends to be common. The outcome of minimum differentiation in all but one dimension in earlier works can be attributed to consumers' quality valuations (or location) being uniformly distributed and quality cost (or mismatch disutility) being separable.

Various open questions remain. First, we have used only necessary conditions of equilibria, not conditions of existence of equilibria. Existence of price equilibrium in any quality subgame is guaranteed by the logconcavity of the valuation functions (as in Caplin and Nalebuff (1991)). Further restrictions may need to be imposed for existence of the equilibrium qualities (as in the case of Anderson, Goeree and Ramer (1997) for the single-dimension Hotelling model with a general location distribution and quadratic transportation). Our necessary conditions of equilibria characterize all candidate equilibria. Hence, one may develop algorithms to check if a candidate equilibrium constitutes best responses. Uniqueness of equilibrium seems too much to expect in our general setting, but our characterization applies to each equilibrium.

Second, we assume linear preferences: each quality benefits a consumer at a constant rate. The linearity assumption is so ubiquitous in modern microeconomics that relaxing this is both challenging and consequential. Third, our approach does make use of the independence of consumer valuations across different qualities. However, there are actually two ways for qualities to become related. We have already incorporated one way—that the cost function allows for positive or negative spillover between qualities. Correlation in valuations is the other way. A full model that allows valuation correlation and cost spillover will be for future research.

Fourth, we have assumed that consumers must buy a product from a firm. Although the “fully-covered-market” assumption is convenient, it obviously imposes restrictions on specific applications. Also, in a single dimensional model without production costs, existence of duopoly equilibria is not easily established when consumers have the nonpurchase option (Benassi, Chirco, and Colombo (2015)). The consumer nonpurchase option can be formally likened to a model with three firms: the two original firms, and one (new but artificial) firm that produces a good at zero quality and sells it a zero price. This covered-market issue is related to

the duopoly assumption. If there are more than two firms, obviously strategic interactions become complex. Future research may shed light on these problems.

Appendix: Proofs of Lemmas, Propositions, and Corollaries

Proof of Lemma 1: Equilibrium prices p_A^* and p_B^* depend on qualities (q, r) , so the right-hand side of (8) is (affine) linear in v_2, \dots, v_N . The solution $\tilde{v}_1^*(v_{-1}; q, r)$ of (9) also satisfies (8), so it must also be linear in v_2, \dots, v_N . Therefore, we write $\tilde{v}_1^*(v_{-1}; q, r) = \alpha(q, r) - \sum_{k=2}^N v_k \beta_k(q, r)$, $v_k \in [v_k, \bar{v}_k]$, $k = 2, \dots, N$ for some functions α , and β_k , $k = 2, \dots, N$. Then we substitute $\tilde{v}_1^*(v_{-1}; q, r)$ by $\alpha(q, r) - \sum_{k=2}^N v_k \beta_k(q, r)$ in (9) to get

$$\begin{aligned} \alpha(q, r) - \sum_{k=2}^N v_k \beta_k(q, r) &= \frac{\int_{x_{-1}} \left[1 - 2F_1(\alpha(q, r) - \sum_{k=2}^N x_k \beta_k(q, r)) \right] dF_{-1}}{\int_{x_{-1}} (f_1(\alpha(q, r) - \sum_{k=2}^N x_k \beta_k(q, r))) dF_{-1}} \\ &+ \frac{C(r) - C(q)}{r_1 - q_1} - \sum_{k=2}^N v_k \frac{r_k - q_k}{r_1 - q_1}, \quad \text{for } v_k \in [v_k, \bar{v}_k], \quad k = 2, \dots, N. \end{aligned}$$

Because this is true for every v_2, \dots, v_N , the equations (10) and (11) in the lemma follow. ■

Steps and Lemmas for Proposition 2:

By partially differentiating p_A^* and p_B^* in (12) and (13) with respect to qualities, we obtain

$$\frac{\partial p_A^*}{\partial r_1} = \frac{\partial(r_1 - q_1)G(\alpha, \beta_{-1})}{\partial r_1} = G(\alpha, \beta_{-1}) + (r_1 - q_1) \left[\frac{\partial G}{\partial \alpha} \frac{\partial \alpha}{\partial r_1} + \sum_{k=2}^N \frac{\partial G}{\partial \beta_k} \frac{\partial \beta_k}{\partial r_1} \right] \quad (33)$$

$$\frac{\partial p_B^*}{\partial q_1} = \frac{\partial(r_1 - q_1)H(\alpha, \beta_{-1})}{\partial q_1} = -H(\alpha, \beta_{-1}) + (r_1 - q_1) \left[\frac{\partial H}{\partial \alpha} \frac{\partial \alpha}{\partial q_1} + \sum_{k=2}^N \frac{\partial H}{\partial \beta_k} \frac{\partial \beta_k}{\partial q_1} \right], \quad (34)$$

and

$$\frac{\partial p_A^*}{\partial r_j} = (r_1 - q_1) \frac{\partial G(\alpha, \beta_{-1})}{\partial r_j} = (r_1 - q_1) \left[\frac{\partial G}{\partial \alpha} \frac{\partial \alpha}{\partial r_j} + \frac{\partial G}{\partial \beta_j} \frac{\partial \beta_j}{\partial r_j} \right], \quad j = 2, \dots, N \quad (35)$$

$$\frac{\partial p_B^*}{\partial q_j} = (r_1 - q_1) \frac{\partial H(\alpha, \beta_{-1})}{\partial q_j} = (r_1 - q_1) \left[\frac{\partial H}{\partial \alpha} \frac{\partial \alpha}{\partial q_j} + \frac{\partial H}{\partial \beta_j} \frac{\partial \beta_j}{\partial q_j} \right], \quad j = 2, \dots, N. \quad (36)$$

Because we label a differentiated quality attribute as the first attribute ($q_1 < r_1$), there is a slight difference between the form of price-reaction effects of the first quality and the other qualities. We now present two lemmas that are used for Proposition 2.

Lemma 2 *In any subgame (q, r) , the sum of the proportional changes in the firms' equilibrium price-cost markups and the proportional change in the total density of the equilibrium set of indifferent consumers must vanish:*

$$d \ln[G(\alpha, \beta_{-1}) + H(\alpha, \beta_{-1})] + d \ln \int_{v_{-1}} f_1(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1} = 0. \quad (37)$$

It follows that the sum of the partial derivatives of $G(\alpha, \beta_{-1})$ and $H(\alpha, \beta_{-1})$ with respect to α and β_j , $j = 2, \dots, N$ are

$$\frac{\partial G}{\partial \alpha} + \frac{\partial H}{\partial \alpha} = - \frac{\int_{v_{-1}} f_1'(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1}}{\left(\int_{v_{-1}} f_1(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1} \right)^2} \quad (38)$$

$$\frac{\partial G}{\partial \beta_j} + \frac{\partial H}{\partial \beta_j} = \frac{\int_{v_{-1}} f_1'(\alpha - v_{-1} \cdot \beta_{-1}) v_j dF_{-1}}{\left(\int_{v_{-1}} f_1(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1} \right)^2}. \quad (39)$$

Proof of Lemma 2: From the definitions of G and H in (14), at each (q, r) , we have:

$$G(\alpha, \beta_{-1}) + H(\alpha, \beta_{-1}) = \frac{1}{\int_{v_{-1}} f_1(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1}}.$$

Hence

$$d \ln(G + H) + d \ln \int_{v_{-1}} f_1(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1} = 0,$$

so the first statement of the lemma follows.

Because (37) holds for each (q, r) , we can partially differentiate it with respect to α and β_j , $j = 2, \dots, N$, to obtain (38) and (39). ■

Lemma 2 allows us to present how qualities change the equilibrium price markups. This then allows us to find how the intercept and slopes of the equation for the equilibrium set of indifferent consumers are impacted by qualities.

Lemma 3 In any subgame (q, r) , in equilibrium, for $i = 1, \dots, N$,

$$\frac{\partial \beta_j(q, r)}{\partial q_i} + \frac{\partial \beta_j(q, r)}{\partial r_i} = 0 \quad \text{and} \quad \frac{\partial \alpha(q, r)}{\partial q_i} + \frac{\partial \alpha(q, r)}{\partial r_i} = \frac{C_i(r) - C_i(q)}{(r_1 - q_1) \left[1 + \frac{\partial G}{\partial \alpha} - \frac{\partial H}{\partial \alpha} \right]}. \quad (40)$$

Proof of Lemma 3: From (11), the functions $\beta_j(q, r)$ are $\beta_j = \frac{r_j - q_j}{r_1 - q_1}$, $j = 2, \dots, N$. Hence,

$$\frac{\partial \beta_j}{\partial q_1} = \frac{r_j - q_j}{(r_1 - q_1)^2} = -\frac{\partial \beta_j}{\partial r_1}, \quad \text{and} \quad \frac{\partial \beta_j}{\partial q_j} = -\frac{1}{r_1 - q_1} = -\frac{\partial \beta_j}{\partial r_j}, \quad j = 2, \dots, N, \quad (41)$$

and all partial derivatives of β_j with respect to q_k or r_k , $k = 2, \dots, N$, $k \neq j$, vanish. These prove the first equality in (40).

From definitions of G and H in (14), we write (10) as

$$\alpha + G(\alpha, \beta_{-1}) - H(\alpha, \beta_{-1}) = \frac{C(r) - C(q)}{r_1 - q_1}. \quad (42)$$

We totally differentiate (42) to obtain

$$\left[1 + \frac{\partial G}{\partial \alpha} - \frac{\partial H}{\partial \alpha} \right] d\alpha + \sum_{k=2}^N \left(\frac{\partial G}{\partial \beta_k} - \frac{\partial H}{\partial \beta_k} \right) d\beta_k = d \left[\frac{C(r) - C(q)}{r_1 - q_1} \right].$$

Then we have

$$\begin{aligned} \left[1 + \frac{\partial G}{\partial \alpha} - \frac{\partial H}{\partial \alpha} \right] \frac{\partial \alpha}{\partial q_1} + \sum_{k=2}^N \left(\frac{\partial G}{\partial \beta_k} - \frac{\partial H}{\partial \beta_k} \right) \frac{\partial \beta_k}{\partial q_1} &= \frac{\partial}{\partial q_1} \left[\frac{C(r) - C(q)}{r_1 - q_1} \right] = -\frac{C_1(q)}{(r_1 - q_1)} + \left[\frac{C(r) - C(q)}{(r_1 - q_1)^2} \right] \\ \left[1 + \frac{\partial G}{\partial \alpha} - \frac{\partial H}{\partial \alpha} \right] \frac{\partial \alpha}{\partial r_1} + \sum_{k=2}^N \left(\frac{\partial G}{\partial \beta_k} - \frac{\partial H}{\partial \beta_k} \right) \frac{\partial \beta_k}{\partial r_1} &= \frac{\partial}{\partial r_1} \left[\frac{C(r) - C(q)}{r_1 - q_1} \right] = \frac{C_1(r)}{(r_1 - q_1)} - \left[\frac{C(r) - C(q)}{(r_1 - q_1)^2} \right], \end{aligned}$$

where $C_i(q) \equiv \frac{\partial C(q)}{\partial q_i}$ denotes the i^{th} partial derivative of the cost function C . Using (41), we obtain

$$\frac{\partial \alpha}{\partial q_1} + \frac{\partial \alpha}{\partial r_1} = \frac{C_1(r) - C_1(q)}{(r_1 - q_1) \left[1 + \frac{\partial G}{\partial \alpha} - \frac{\partial H}{\partial \alpha} \right]}.$$

Next, we have, for $j = 2, \dots, N$,

$$\begin{aligned} \left[1 + \frac{\partial G}{\partial \alpha} - \frac{\partial H}{\partial \alpha} \right] \frac{\partial \alpha}{\partial q_j} + \sum_{k=2}^N \left(\frac{\partial G}{\partial \beta_k} - \frac{\partial H}{\partial \beta_k} \right) \frac{\partial \beta_k}{\partial q_j} &= \frac{\partial}{\partial q_j} \left[\frac{C(r) - C(q)}{r_1 - q_1} \right] = -\frac{C_j(q)}{(r_1 - q_1)} \\ \left[1 + \frac{\partial G}{\partial \alpha} - \frac{\partial H}{\partial \alpha} \right] \frac{\partial \alpha}{\partial r_j} + \sum_{k=2}^N \left(\frac{\partial G}{\partial \beta_k} - \frac{\partial H}{\partial \beta_k} \right) \frac{\partial \beta_k}{\partial r_j} &= \frac{\partial}{\partial r_j} \left[\frac{C(r) - C(q)}{r_1 - q_1} \right] = \frac{C_j(r)}{(r_1 - q_1)} \end{aligned}$$

Using (41), we obtain

$$\frac{\partial \alpha}{\partial q_j} + \frac{\partial \alpha}{\partial r_j} = \frac{C_j(r) - C_j(q)}{(r_1 - q_1) \left[1 + \frac{\partial G}{\partial \alpha} - \frac{\partial H}{\partial \alpha} \right]}.$$

We have proven the second equality in (40).■

Proof of Proposition 2: From (35) and (36), we have

$$\frac{\partial p_B^*(q, r)}{\partial q_j} - \frac{\partial p_A^*(q, r)}{\partial r_j} = (r_1 - q_1) \left\{ \left[\frac{\partial H}{\partial \alpha} \frac{\partial \alpha}{\partial q_j} + \frac{\partial H}{\partial \beta_j} \frac{\partial \beta_j}{\partial q_j} \right] - \left[\frac{\partial G}{\partial \alpha} \frac{\partial \alpha}{\partial r_j} + \frac{\partial G}{\partial \beta_j} \frac{\partial \beta_j}{\partial r_j} \right] \right\}, \quad j = 2, \dots, N.$$

Using Lemma 3, we have $\frac{\partial \alpha}{\partial q_j} = \frac{C_j(r) - C_j(q)}{(r_1 - q_1) \left[1 + \frac{\partial G}{\partial \alpha} - \frac{\partial H}{\partial \alpha} \right]} - \frac{\partial \alpha}{\partial r_j}$ and $\frac{\partial \beta_j}{\partial q_j} = -\frac{\partial \beta_j}{\partial r_j}$, and substitute them

into the above to obtain:

$$\begin{aligned} & \frac{\partial p_B^*(q, r)}{\partial q_j} - \frac{\partial p_A^*(q, r)}{\partial r_j} \\ &= (r_1 - q_1) \left\{ \left[\frac{\partial H}{\partial \alpha} \left(\frac{C_j(r) - C_j(q)}{(r_1 - q_1) \left[1 + \frac{\partial G}{\partial \alpha} - \frac{\partial H}{\partial \alpha} \right]} - \frac{\partial \alpha}{\partial r_j} \right) - \frac{\partial H}{\partial \beta_j} \frac{\partial \beta_j}{\partial r_j} \right] - \left[\frac{\partial G}{\partial \alpha} \frac{\partial \alpha}{\partial r_j} + \frac{\partial G}{\partial \beta_j} \frac{\partial \beta_j}{\partial r_j} \right] \right\} \\ &= (r_1 - q_1) \left\{ \left[-\frac{\partial H}{\partial \alpha} \frac{\partial \alpha}{\partial r_j} - \frac{\partial H}{\partial \beta_j} \frac{\partial \beta_j}{\partial r_j} \right] - \left[\frac{\partial G}{\partial \alpha} \frac{\partial \alpha}{\partial r_j} + \frac{\partial G}{\partial \beta_j} \frac{\partial \beta_j}{\partial r_j} \right] \right\} + \frac{\partial H}{\partial \alpha} \frac{[C_j(r) - C_j(q)]}{\left[1 + \frac{\partial G}{\partial \alpha} - \frac{\partial H}{\partial \alpha} \right]} \\ &= -(r_1 - q_1) \left\{ \left[\frac{\partial G}{\partial \alpha} + \frac{\partial H}{\partial \alpha} \right] \frac{\partial \alpha}{\partial r_j} + \left[\frac{\partial G}{\partial \beta_j} + \frac{\partial H}{\partial \beta_j} \right] \frac{\partial \beta_j}{\partial r_j} \right\} + \frac{\partial H}{\partial \alpha} \frac{[C_j(r) - C_j(q)]}{\left[1 + \frac{\partial G}{\partial \alpha} - \frac{\partial H}{\partial \alpha} \right]}. \end{aligned}$$

Next, we define

$$Z \equiv \frac{\partial H}{\partial \alpha} \frac{1}{\left[1 + \frac{\partial G}{\partial \alpha} - \frac{\partial H}{\partial \alpha} \right]}.$$

We use (38) and (39) in Lemma 2 to obtain

$$\begin{aligned} & \frac{\partial p_B^*(q, r)}{\partial q_j} - \frac{\partial p_A^*(q, r)}{\partial r_j} \\ &= (r_1 - q_1) \left[\frac{\int_{v_{-1}} f'_1(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1}}{\left(\int_{v_{-1}} f_1(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1} \right)^2} \right] \frac{\partial \alpha}{\partial r_j} \\ & \quad - (r_1 - q_1) \left[\frac{\int_{v_{-1}} f'_1(\alpha - v_{-1} \cdot \beta_{-1}) v_j dF_{-1}}{\left(\int_{v_{-1}} f_1(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1} \right)^2} \right] \frac{\partial \beta_j}{\partial r_j} + Z [C_j(q) - C_j(r)] \\ &= \frac{(r_1 - q_1) \left(\int_{v_{-1}} f'_1(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1} \frac{\partial \alpha}{\partial r_j} - \int_{v_{-1}} f'_1(\alpha - v_{-1} \cdot \beta_{-1}) v_j dF_{-1} \frac{\partial \beta_j}{\partial r_j} \right)}{\left(\int_{v_{-1}} f_1(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1} \right)^2} \\ & \quad + Z [C_j(q) - C_j(r)]. \end{aligned} \tag{43}$$

Then we further write the first term in (43) as

$$(r_1 - q_1) \frac{\frac{\partial}{\partial r_j} \left[\int_{v_{-1}} f_1(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1} \right]}{\left(\int_{v_{-1}} f_1(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1} \right)^2}. \quad (44)$$

Hence, we have shown (15). Next, from Lemma 3, we have $\frac{\partial \alpha}{\partial r_j} = \frac{C_j(r) - C_j(q)}{(r_1 - q_1) \left[1 + \frac{\partial G}{\partial \alpha} - \frac{\partial H}{\partial \alpha} \right]} - \frac{\partial \alpha}{\partial q_j}$ and

$\frac{\partial \beta_j}{\partial r_j} = -\frac{\partial \beta_j}{\partial q_j}$, so (44) also equals

$$-\frac{(r_1 - q_1) \frac{\partial}{\partial q_j} \left[\int_{v_{-1}} f_1(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1} \right]}{\left(\int_{v_{-1}} f_1(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1} \right)^2}.$$

Therefore, (15) equals (16).

Finally, from the definition of $G(\alpha, \beta_{-1})$ and $H(\alpha, \beta_{-1})$ in (14), we have:

$$\begin{aligned} \frac{\partial}{\partial \alpha} G(\alpha, \beta_{-1}) &= 1 - \frac{\int_{v_{-1}} f'_1(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1} \int_{v_{-1}} F_1(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1}}{\left(\int_{v_{-1}} f_1(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1} \right)^2} \\ \frac{\partial}{\partial \alpha} H(\alpha, \beta_{-1}) &= -1 - \frac{\int_{v_{-1}} f'_1(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1} \int_{v_{-1}} [1 - F_1(\alpha - v_{-1} \cdot \beta_{-1})] dF_{-1}}{\left(\int_{v_{-1}} f_1(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1} \right)^2}. \end{aligned}$$

After we substitute these into the definition of Z , we obtain the same expression for Z in the Proposition. ■

Proof of Proposition 3: We begin by simplifying Firm A 's first-order derivatives with respect to qualities. First, for (20) we use (3) to obtain

$$\begin{aligned} &\frac{\partial}{\partial q_j} \int_{v_{-1}} F_1(\tilde{v}_1(v_{-1}; p^*(q, r^*), q, r^*)) dF_{-1} \\ &= \frac{1}{r_1 - q_1} \int_{v_{-1}} f_1(\tilde{v}_1(v_{-1}; p^*(q, r^*), q, r^*)) v_j dF_{-1} \quad j = 2, \dots, N. \end{aligned}$$

Second, for (21), again we use (3) to obtain

$$\begin{aligned} &\frac{\partial}{\partial p_B^*} \int_{v_{-1}} F_1(\tilde{v}_1(v_{-1}; p^*(q, r^*), q, r^*)) dF_{-1} \\ &= \frac{1}{r_1 - q_1} \int_{v_{-1}} f_1(\tilde{v}_1(v_{-1}; p^*(q, r^*), q, r^*)) dF_{-1}. \end{aligned}$$

We then substitute these expressions into (20) and (21), and the first-order derivative of Firm A 's with respect to quality q_j , $j = 2, \dots, N$, becomes

$$\begin{aligned}
& - \left[\int_{v_{-1}} F_1(\tilde{v}_1(v_{-1}; p^*(q, r^*), q, r^*)) dF_{-1} \right] C_i(q) \\
& + \frac{1}{r_1 - q_1} \int_{v_{-1}} f_1(\tilde{v}_1(v_{-1}; p^*(q, r^*), q, r^*)) v_j dF_{-1} [p_A^*(q, r^*) - C(q)] \\
& + \frac{1}{r_1 - q_1} \int_{v_{-1}} f_1(\tilde{v}_1(v_{-1}; p^*(q, r^*), q, r^*)) dF_{-1} \frac{\partial p_B^*}{\partial q_j} [p_A^*(q, r^*) - C(q)].
\end{aligned} \tag{45}$$

We now evaluate (45) at the equilibrium qualities, so replace $\tilde{v}_1(v_{-1}; p^*(q^*, r^*), q^*, r^*)$ as $\tilde{v}_1^*(v_{-1}; q^*, r^*) = \alpha(q^*, r^*) - v_{-1} \cdot \beta_{-1}(q^*, r^*)$. Using the equilibrium price (12) in Proposition 1

$$\frac{p_A^*(q^*, r^*) - C(q^*)}{r_1^* - q_1^*} = \frac{\int_{v_{-1}} F_1(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1}}{\int_{v_{-1}} f_1(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1}},$$

we simplify the first-order derivative of Firm A 's profit with respect to q_j to

$$\left[\int_{v_{-1}} F_1(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1} \right] \left[\frac{\partial p_B^*}{\partial q_j} + \frac{\int_{v_{-1}} f_1(\alpha - v_{-1} \cdot \beta_{-1}) v_j dF_{-1}}{\int_{v_{-1}} f_1(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1}} - C_j(q) \right], \quad j = 2, \dots, N,$$

where we have omitted the arguments in α and β_{-1} . We set this to zero to obtain the first-order condition for q_j^* :

$$\frac{\partial p_B^*}{\partial q_j} + \frac{\int_{v_{-1}} f_1(\alpha - v_{-1} \cdot \beta_{-1}) v_j dF_{-1}}{\int_{v_{-1}} f_1(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1}} - C_j(q^*) = 0 \quad j = 2, \dots, N.$$

For brevity we do not lay out all the steps for obtaining the first-order condition of Firm A 's equilibrium quality q_1 , but the key difference is that (3) yields $\frac{\partial \tilde{v}_1}{\partial q_1} = \frac{\tilde{v}_1}{r_1 - q_1}$. The effect of quality q_1 on demand now becomes

$$\begin{aligned}
& \frac{\partial}{\partial q_1} \int_{v_{-1}} F_1(\tilde{v}_1(v_{-1}; p^*(q, r^*), q, r^*)) dF_{-1} \\
& = \frac{1}{r_1 - q_1} \int_{v_{-1}} f_1(\tilde{v}_1(v_{-1}; p^*(q, r^*), q, r^*)) \tilde{v}_1 dF_{-1}.
\end{aligned}$$

Following the same steps, we obtain the following first-order condition for q_1^*

$$\frac{\partial p_B^*}{\partial q_1} + \frac{\int_{v_{-1}} f_1(\alpha - v_{-1} \cdot \beta_{-1}) \tilde{v}_1^* dF_{-1}}{\int_{v_{-1}} f_1(\alpha - v_{-1} \cdot \beta_{-1}) dF_{-1}} - C_1(q^*) = 0.$$

The first-order conditions for Firm B 's equilibrium qualities are derived analogously.■

Proof of Corollary 1: For each $j = 2, \dots, N$, the terms with the integrals are the same in the two equations in (24) and (25). Taking their difference, we have

$$\frac{\partial p_B^*}{\partial q_j} - \frac{\partial p_A^*}{\partial r_j} = C_j(q^*) - C_j(r^*),$$

and the equivalence of i) and ii) in the Corollary follows. Then we simply apply Proposition 2 on the equilibrium (q^*, r^*) for the equivalence of ii) and iii).■

Proof of Corollary 3: Consider the subgame defined by equilibrium quality (q^*, r^*) . The difference in the firms' price-reaction effects is in Proposition 2. Obviously, $f'_1 = 0$ by assumption, so (15) becomes $\frac{\partial p_B^*(q^*, r^*)}{\partial q_j} - \frac{\partial p_A^*(q^*, r^*)}{\partial r_j} = \frac{1}{3} [C_j(q^*) - C_j(r^*)]$. Under the step-function assumption, Corollary 1 then says that $\frac{\partial p_B^*(q^*, r^*)}{\partial q_j} - \frac{\partial p_A^*(q^*, r^*)}{\partial r_j} = \frac{1}{3} [C_j(q^*) - C_j(r^*)] = C_j(q^*) - C_j(r^*)$. Hence it must be $C_j(q^*) = C_j(r^*)$, $j = 2, \dots, N$. We conclude that $\frac{\partial p_B^*(q^*, r^*)}{\partial q_j} - \frac{\partial p_A^*(q^*, r^*)}{\partial r_j} = 0$. Finally, we apply Corollary 2 to obtain the nondifferentiation result.■

Proof of Corollary 4: Let the valuation density of quality j be a step function. Consider an equilibrium (q^*, r^*) . If $q_j^* = r_j^*$, then the first part of the corollary is trivially true. Suppose that $q_j^* \neq r_j^*$. Without loss of generality we let $q_j^* < r_j^*$. Now we relabel the indexes so that $j = 1$. Then Corollary 3 applies, and the firms choose identical qualities for all quality attributes $k = 2, \dots, N$. Finally, the last part of the Corollary is a special case of i) and ii).■

Corollary 5 *Suppose that the cost function C is additively separable. If in equilibrium (q^*, r^*) , $q_j^* = r_j^*$ at some j , $2 < j < N$, then*

$$\int_{v_{-1}} f'_1(\alpha(q^*, r^*) - v_{-1} \cdot \beta_{-1}(q^*, r^*)) dF_{v_{-1}} \times \left[(r_1^* - q_1^*) \frac{\partial \alpha(q^*, r^*)}{\partial r_j} - E(v_j) \right] = 0, \quad (46)$$

where $E(v_j)$ is the expected value of v_j . Furthermore, if $q_j^* = r_j^*$ for each $j = 2, \dots, N$, then

$$f'_1(\alpha(q^*, r^*)) \times \left[(r_1^* - q_1^*) \frac{\partial \alpha(q^*, r^*)}{\partial r_j} - E(v_j) \right] = 0. \quad (47)$$

Corollary 5 can be understood as follows. Absent differentiation at quality j , firms have identical price-reaction effects. The slope of the line defining the set of indifferent consumers has a zero slope at quality

j ($\beta_j = 0$). If f_1 's derivative does not vanish, the derivative of the intercept α with respect to r_j (or q_j), evaluated at the equilibrium (q^*, r^*) , exactly equals the mean of the quality- j valuation distribution divided by $(r_1^* - q_1^*)$, independent of other qualities or distributions. Corollary 5 does not imply any global properties of the key α function. Neither does Corollary 5 imply any global properties that f_1 must satisfy.

Proof of Corollary 5: From Corollary 2, $q_j^* = r_j^*$ implies $\frac{\partial p_B^*(q^*, r^*)}{\partial q_j} = \frac{\partial p_A^*(q^*, r^*)}{\partial r_j}$. Next, from Proposition 2, we use (15) and $C_j(q^*) = C_j(r^*)$ to obtain

$$\begin{aligned}
& \frac{\partial p_B^*(q^*, r^*)}{\partial q_j} - \frac{\partial p_A^*(q^*, r^*)}{\partial r_j} \\
&= \frac{(r_1^* - q_1^*) \frac{\partial}{\partial r_j} \left[\int_{v_{-1}} f_1(\alpha(q^*, r^*) - v_{-1} \cdot \beta_{-1}(q^*, r^*)) dF_{-1} \right]}{\left(\int_{v_{-1}} f_1(\alpha(q^*, r^*) - v_{-1} \cdot \beta_{-1}(q^*, r^*)) dF_{-1} \right)^2} \\
&= \frac{(r_1^* - q_1^*) \int_{v_{-1}} f_1'(\alpha(q^*, r^*) - v_{-1} \cdot \beta_{-1}(q^*, r^*)) \left[\frac{\partial \alpha(q^*, r^*)}{\partial r_j} - \frac{v_j}{r_1^* - q_1^*} \right] dF_{v_{-1}}}{\left(\int_{v_{-1}} f_1(\alpha(q^*, r^*) - v_{-1} \cdot \beta_{-1}(q^*, r^*)) dF_{-1} \right)^2}. \tag{48}
\end{aligned}$$

In the numerator of (48), the term in the integrand involving $\frac{v_j}{r_1^* - q_1^*}$ is

$$(r_1^* - q_1^*) \int_{v_{-1}} f_1'(\alpha(q^*, r^*) - v_{-1} \cdot \beta_{-1}(q^*, r^*)) \left[\frac{v_j}{r_1^* - q_1^*} \right] dF_{v_{-1}}.$$

Observe that when $q_j^* = r_j^*$, $\alpha(q^*, r^*) - v_{-1} \cdot \beta_{-1}(q^*, r^*)$ is independent of v_j , so $f_1'(\alpha(q^*, r^*) - v_{-1} \cdot \beta_{-1}(q^*, r^*))$ is also independent of v_j . Therefore, we can simplify this to

$$\begin{aligned}
& \int_{v_{-1}} f_1'(\alpha(q^*, r^*) - v_{-1} \cdot \beta_{-1}(q^*, r^*)) dF_{v_{-1}} \times \int_{v_{-1}} v_j dF_{v_{-1}} \\
&= \int_{v_{-1}} \left[f_1'(\alpha(q^*, r^*) - v_{-1} \cdot \beta_{-1}(q^*, r^*)) \times \mathbb{E}(v_j) \right] dF_{v_{-1}}.
\end{aligned}$$

Using this and then setting (48) to 0, we obtain (46). Finally, if, for each $j = 2, \dots, N$, we have $q_j^* = r_j^*$, then

$$\beta_j(q^*, r^*) = \frac{r_j^* - q_j^*}{r_1^* - q_1^*} = 0, \text{ so } v_{-1} \cdot \beta_{-1}(q^*, r^*) = 0. \text{ Simplifying the argument inside } f_1', \text{ we obtain (47).} \blacksquare$$

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