Bulíček, Miroslav, Giovanni Cupini, Bianca Stroffolini, e Anna Verde. «Existence and regularity results for weak solutions to (p,q)-elliptic systems in divergence form». *Advances in Calculus of Variations* 11, n. 3 (2018). https://www.degruyter.com/view/journals/acv/11/3/article-p273.xml.

DE GRUYTER Adv. Calc. Var. 2018; 11(3): 273–288

Research Article

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Existence and regularity results for weak solutions to (p, q)-elliptic systems in divergence form

DOI: 10.1515/acv-2016-0054
Received November 10, 2016; revised January 26, 2017; accepted March 15, 2017

Abstract: We prove existence and regularity results for weak solutions of non-linear elliptic systems with non-variational structure satisfying (p, q)-growth conditions. In particular, we are able to prove higher differentiability results under a dimension-free gap between p and q.

Keywords: Elliptic system, existence of solutions, (p, q)-growth conditions, regularity

MSC 2010: Primary 35J47; secondary 35J25

Communicated by: Frank Duzaar

1 Introduction

In this paper we focus on the existence and the regularity results for solutions u to the Dirichlet problems associated with the following nonlinear system in divergence form (here $\alpha = 1, \dots, N$):

$$\begin{cases} \displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i} A_i^\alpha(Du) = 0 & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where the functions $A_i^\alpha(\xi)$ are locally $\mathcal{C}^1$ in $\mathbb{R}^{nN}$, Ω is an open bounded subset of $\mathbb{R}^n$ and $Du : \Omega \to \mathbb{R}^{nN}$ represents the gradient of a (vector-valued) function $u : \Omega \to \mathbb{R}^N$.

We equip the problem with the general (p, q)-growth conditions, i.e., we assume that there are p, q with $1 < p \le q < \infty$ and two positive constants m, M such that for all $\xi, \lambda \in \mathbb{R}^{nN}$ and for all $i, j = 1, \dots, n$, and $\alpha, \beta = 1, \dots, N$ there holds

$$m(1 + |\xi|^2)^{\frac{p-2}{2}} |\lambda|^2 \le \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} \frac{\partial A_i^\alpha}{\partial \xi_j^\beta}(\xi) \lambda_i^\alpha \lambda_j^\beta, \tag{1.2}$$

$$\left| \frac{\partial A_i^\alpha}{\partial \xi_j^\beta}(\xi) \right| \le M(1 + |\xi|^2)^{\frac{q-2}{2}}. \tag{1.3}$$

Notice that (1.2) is the usual ellipticity condition and (1.3) is the q-growth condition, from which the name of (p, q)-growth comes from. Under these assumptions, one can easily observe (see Lemma 2.1)

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that

$$|A_i^\alpha(\xi)| \le C(1+|\xi|)^{q-1}$$

with some generic constant C and therefore we can naturally define a notion of a weak solution to (1.1) in the following way:

Let $u_0 \in W^{1,p}(\Omega;\mathbb{R}^N) \cap W^{1,q}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$. We say that u is a weak solution to (1.1) if

$$u - u_0 \in W_0^{1,p}(\Omega;\mathbb{R}^N) \cap W^{1,q}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$$

and for all open Ω' fulfilling $\overline{\Omega'} \subset \Omega$ and for all $\varphi \in W_0^{1,q}(\Omega';\mathbb{R}^N)$ there holds

$$\int_\Omega \sum_{i=1}^n \sum_{\alpha=1}^N A_i^\alpha(Du)\varphi^\alpha_{x_i}(x)\,dx = 0. \tag{1.4}$$

Here, and also in what follows, we use the abbreviation $\varphi^\alpha_{x_i} := \frac{\partial \varphi^\alpha}{\partial x_i}$.

Our main task in the paper is to establish the existence of such a solution and further some regularity of arbitrary weak solutions. However, contrary to the classical result, we do not in general assume any symmetry condition on the derivative of A_i^α and so we do not assume that the system is in variational form. Nevertheless, as done in [23] in the scalar framework, we will need to compensate this lack of symmetry by the following assumption on the asymptotic behavior of the skew-symmetric part, namely, for all $\xi, \lambda \in \mathbb{R}^{nN}$ and for all $i,j = 1,\dots,n$, and $\alpha,\beta = 1,\dots,N$ there holds

$$\left|\frac{\partial A_i^\alpha}{\partial \xi_j^\beta}(\xi) - \frac{\partial A_j^\beta}{\partial \xi_i^\alpha}(\xi)\right| \le M(1+|\xi|^2)^{\frac{q+p-4}{4}}. \tag{1.5}$$

If $p = q$, the existence of weak solutions to (1.1) can be established using the theory of coercive, monotone operators, see Leray–Lions [19], Browder [3] and Hartman–Stampacchia [15]. Also the regularity issue has been studied extensively, see the monographs [12, 14] and the surveys [24, 25]. Notice also that, without any further additional structural assumptions, the best[1] known regularity information about the solution is that $V(Du) \in W^{1,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^{nN})$, where

$$V(\xi) := (1+|\xi|^2)^{\frac{p-2}{4}}\xi.$$

On the other hand, if $p < q$, the above classical existence results cannot be applied due to the lack of coercivity in $W^{1,q}$. Moreover, the request $u \in W^{1,q}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ in the definition of weak solution, needed to have a well-defined integral, is an additional difficulty. Notice that such a request is a priori assumed in some regularity results under the (p,q)-growth, see for example [2, 7, 17].

The first result of the paper is that any weak solution is in fact twice weakly differentiable.

Theorem 1.1. *Let $1 < p \le q < \infty$ be arbitrary and suppose that A satisfies* (1.2), (1.3) *and* (1.5). *Then any $u \in W^{1,\max\{q,2\}}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ fulfilling* (1.4) *satisfies for all $\eta \in C_c^\infty(\Omega)$ the estimate*

$$\int_\Omega (1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2\eta^2\,dx \le c\int_\Omega (1+|Du|^2)^{\frac{q}{2}}|D\eta|^2\,dx, \tag{1.6}$$

where the constant c depends only on m and M. In particular, we also have

$$\int_\Omega |DV(Du)|^2\eta^2\,dx \le c\int_\Omega (1+|Du|^2)^{\frac{q}{2}}|D\eta|^2\,dx. \tag{1.7}$$

The above theorem provides the existence of the second derivatives for arbitrary $1 < p \le q < \infty$ but the right-hand side of (1.6) or (1.7) still depends on the $W^{1,q}$ norm of u. We shall improve this estimate provided that p and q are sufficiently close to each other. Thus, the second main theorem of the paper is the following.

1 This information can be as usual slightly improved by the Gehring lemma.

Theorem 1.2. *Let $1 < p \le q < \infty$ be arbitrary, let A satisfy* (1.2), (1.3) *and* (1.5) *and let $u \in W^{1,\max\{q,2\}}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ satisfy* (1.4). *Then for all open $\Omega' \subset \overline{\Omega'} \subset \Omega$ the following hold:*

(i) *If*

$$q < p\frac{n+2}{n}, \tag{1.8}$$

then

$$\int_{\Omega'} \Big(|V(Du)|^{\frac{2q}{p}} + |DV(Du)|^2 + (1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2\Big)\,dx \le C(\Omega', n, N, p, q, m, M, \|Du\|_{L^p(\Omega)}).$$

(ii) *If $u \in L^\infty(\Omega;\mathbb{R}^N)$ and*

$$q < p+2 \quad \textit{and} \quad p < n, \tag{1.9}$$

then

$$\int_{\Omega'} \Big(|V(Du)|^{\frac{2q}{p}} + |DV(Du)|^2 + (1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2\Big)\,dx \le C(\Omega', n, N, p, q, m, M, \|Du\|_{L^p(\Omega)}, \|u\|_{L^\infty(\Omega)}).$$

In particular, in both cases we have $V(Du) \in W^{1,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^{nN})$, which, due to the embedding theorem, leads to $Du \in L^{p2^*/2}_{\mathrm{loc}}(\Omega;\mathbb{R}^{nN})$.

Finally, we state our last main result of the paper. It is an existence result for the Dirichlet problem (1.1). For this purpose, we need to consider a regularity assumption on the boundary datum. We shall require in what follows that

$$u_0 \in W^{1,r}(\Omega;\mathbb{R}^N), \quad \text{with } r := \max\Big\{2, \frac{p(q-1)}{p-1}\Big\}. \tag{1.10}$$

Theorem 1.3. *Let $1 < p \le q < \infty$ be arbitrary and let A satisfy* (1.2), (1.3) *and* (1.5). *Moreover, let u_0 fulfill* (1.10). *Then there exists a weak solution to the problem* (1.1) *provided that at least one of the following conditions holds:*

(i) *p and q satisfy* (1.8).

(ii) *p and q satisfy* (1.9), *$u_0 \in L^\infty(\partial\Omega;\mathbb{R}^N)$ and*

$$\sum_{i=1}^n A_i^\alpha(\xi)\xi_i^\alpha \ge 0 \quad \textit{for all } \xi \in \mathbb{R}^{nN} \textit{ and all } \alpha \in \{1,\dots,N\}. \tag{1.11}$$

As far as the regularity of solutions is concerned, the obstructions are essentially two: we are dealing with systems and under non-standard growth ($p < q$). Indeed, in the vectorial case, even under the standard growth, the everywhere regularity of solutions for systems, or of minimizers of integrals, cannot be expected unless some structure conditions are assigned, and this holds also for the local boundedness, see e.g. the counterexamples by De Giorgi [8] and Šverák–Yan [26]. Assumption (1.11) is the structural conditions that leads to locally bounded solution and was used e.g. in [18] in context of the regularity theory for nonlinear elliptic systems or in [28] for existence theory in problems with the right-hand side being a Radon measure. Since the pioneering paper by Marcellini [22], the theory of regularity in the framework of non-standard growth has been deeply investigated. The results and the contributions to regularity are so many, that it is a hard task to provide a comprehensive overview of the issue. For this, we refer to the survey of Mingione [24] for an accurate and interesting account on this subject. A common feature is that to get regularity results p and q must be not too far apart, as examples of irregular solutions by Giaquinta [13], Marcellini [21] and Hong [16] show. On the other hand, many regularity results are available if the ratio $\frac{q}{p}$ is bounded above by a suitable constant that in general depends on the dimension n, and converges to 1 when n tends to infinity [1, 5, 9–11]. Moreover, the condition on the distance between the exponents p and q can usually be relaxed if the solutions/minimizers are assumed locally bounded.

Let us observe that the local higher differentiability results for bounded minimizers of integral functionals satisfying (p, q)-growth conditions is more studied than the analogous issue for systems of PDEs. In particular, recently, the authors, in [4], considered integral functionals with convex integrand satisfying (p, q)-growth conditions. They proved local higher differentiability results for bounded minimizers under dimension-free conditions on the gap between the growth and the coercivity exponents; i.e., (1.9) restricted to the case $p \geq 2$, using an improved Gagliardo–Nirenberg's inequality. We also observe that an existence result in the (p, q)-framework was proved in [6] for a Dirichlet problem (1.1) with monotone operators possibly depending on the x-variable, but for $p \geq 2$ only. As a novel feature, the main results are achieved through uniform higher differentiability estimates for solutions to a class of auxiliary problems, constructed adding higher order perturbations to the integrand. Here we achieve the same result for systems with non-variational structure with control on the skew-symmetric part (see (1.5)).

The plan of the paper is the following. In Section 2 we prove some preliminary algebraic inequalities. In Sections 3 and 4 we prove the higher differentiability results Theorem 1.1 and Theorem 1.2, respectively. In the last section, we prove the existence result (Theorem 1.2) for problem (1.1).

2 Auxiliary algebraic inequalities

In this part, we recall several algebraic inequalities related to the mapping A. Although their proof can be in some simplified setting found in many works, see e.g. [23, Lemmas 4.4, Lemma 2.4], [27, Lemma 1], [7, Lemma 5.1] or [20, Chapter 5], we provide for the sake of clarity a detailed proof here. We start with the first auxiliary result based on assumptions (1.2)–(1.3).

Lemma 2.1. *Let $A : \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ be a continuous mapping fulfilling* (1.2) *and* (1.3). *Then there exists a positive constant K such that for all $\xi, \eta \in \mathbb{R}^{nN}$ there hold*

$$|\xi|^p \leq K\left\{(1+|\eta|^2)^{\frac{p(q-1)}{2(p-1)}} + \sum_{i=1}^{n}\sum_{\alpha=1}^{N} A_i^\alpha(\xi)(\xi_i^\alpha - \eta_i^\alpha)\right\}, \tag{2.1}$$

$$|A_i^\alpha(\xi)| \leq K(1+|\xi|^2)^{\frac{q-1}{2}} \quad \textit{for all } \alpha = 1,\dots,N \textit{ and } i = 1,\dots,n, \tag{2.2}$$

$$|\xi-\eta|^p \leq K\sum_{i=1}^{n}\sum_{\alpha=1}^{N}(A_i^\alpha(\xi) - A_i^\alpha(\eta))(\xi_i^\alpha - \eta_i^\alpha) \quad \textit{for } p \geq 2, \tag{2.3}$$

$$(1+|\xi|^2+|\eta|^2)^{\frac{p-2}{2}}|\xi-\eta|^2 \leq K\sum_{i=1}^{n}\sum_{\alpha=1}^{N}(A_i^\alpha(\xi) - A_i^\alpha(\eta))(\xi_i^\alpha - \eta_i^\beta) \quad \textit{for } p \in (1,2). \tag{2.4}$$

Proof. We start the proof with (2.2). Since

$$A_i^\alpha(\xi) - A_i^\alpha(0) = \int_0^1 \sum_{j=1}^{n}\sum_{\beta=1}^{N} \frac{\partial A_i^\alpha(t\xi)}{\partial \xi_j^\beta}\xi_j^\beta\,dt,$$

we can use assumption (1.3), to get

$$|A_i^\alpha(\xi)| \leq |A_i^\alpha(0)| + M\int_0^1 \sum_{j=1}^{n}\sum_{\beta=1}^{N}(1+t^2|\xi|^2)^{\frac{q-2}{2}}|\xi_j^\beta|\,dt \leq |A_i^\alpha(0)| + MnN\int_0^1 (1+t^2|\xi|^2)^{\frac{q-2}{2}}|\xi|\,dt.$$

Thus, in case $q \geq 2$, inequality (2.2) immediately follows.

If $q \in (1,2)$, we can continue with estimating the last integral in the following way:

$$\int_0^1 (1+t^2|\xi|^2)^{\frac{q-2}{2}}|\xi|\,dt = \int_0^{|\xi|}(1+t^2)^{\frac{q-2}{2}}\,dt \leq 2^{\frac{2-q}{2}}\int_0^{|\xi|}(1+t)^{q-2}\,dt \leq \frac{2^{\frac{2-q}{2}}}{q-1}(1+|\xi|)^{q-1}$$

and we again see that (2.2) follows directly.

To show (2.3)–(2.4), we write

$$\sum_{i=1}^{n}\sum_{\alpha=1}^{N}(A_i^\alpha(\xi) - A_i^\alpha(\eta))(\xi_i^\alpha - \eta_i^\alpha) = \int_0^1 \sum_{i,j=1}^{n}\sum_{\alpha=1}^{N}\frac{\partial A_i^\alpha(t\xi+(1-t)\eta)}{\partial \xi_j^\beta}(\xi_j^\beta - \eta_j^\beta)(\xi_i^\alpha - \eta_i^\alpha)\,dt$$

$$\geq m|\xi-\eta|^2\int_0^1 (1+|t\xi+(1-t)\eta|^2)^{\frac{p-2}{2}}\,dt,$$

using (1.2) for the estimate. Then following step by step proof of [20, Chapter 5, Lemma 1.19], we deduce inequalities (2.3)–(2.4).

To show (2.1), we first consider the case $p \geq 2$. Then by using (2.3) and (2.2) and also Young's inequality, we can observe that for all $\epsilon > 0$ and all $\xi, \eta \in \mathbb{R}^{nN}$, we have

$$\begin{aligned}
|\xi|^p \leq c(|\xi-\eta|^p + |\eta|^p) &\leq c\Big\{\sum_{i=1}^{n}\sum_{\alpha=1}^{N}(A_i^\alpha(\xi)-A_i^\alpha(\eta))(\xi_i^\alpha-\eta_i^\alpha) + |\eta|^p\Big\}\\
&\leq c\Big\{|\eta|^p + \sum_{i=1}^{n}\sum_{\alpha=1}^{N}A_i^\alpha(\xi)(\xi_i^\alpha-\eta_i^\alpha) + \bar{C}(1+|\eta|^2)^{\frac{q-1}{2}}(|\xi|+|\eta|)\Big\}\\
&\leq c\Big\{(1+|\eta|^2)^{\frac{p}{2}} + \sum_{i=1}^{n}\sum_{\alpha=1}^{N}A_i^\alpha(\xi)(\xi_i^\alpha-\eta_i^\alpha) + c_\epsilon(1+|\eta|^2)^{\frac{p(q-1)}{2(p-1)}} + \epsilon(|\xi|+|\eta|)^p\Big\};
\end{aligned}$$

thus if ϵ is small enough, we get (2.1).

In the case $1 < p < 2$, we proceed slightly differently. By using Young's inequality with complementary exponents $\frac{2}{p}$ and $\frac{2}{2-p}$ we get for $\epsilon > 0$,

$$\begin{aligned}
|\xi|^p \leq c(|\xi-\eta|^p + |\eta|^p) &\leq c\Big(|\eta|^p + (|\xi-\eta|^2)^{\frac{p}{2}}(1+|\xi|^2+|\eta|^2)^{\frac{p(p-2)}{4}+\frac{p(2-p)}{4}}\Big)\\
&\leq c\Big\{(1+|\eta|^2)^{\frac{p}{2}} + c_\epsilon(1+|\xi|^2+|\eta|^2)^{\frac{p-2}{2}}|\xi-\eta|^2 + \epsilon(1+|\xi|^2+|\eta|^2)^{\frac{p}{2}}\Big\}.
\end{aligned}$$

Therefore, by (2.4), with a proper choice of (small) $\epsilon > 0$, we get

$$|\xi|^p \leq c\Big\{(1+|\eta|^2)^{\frac{p}{2}} + \sum_{i=1}^{n}\sum_{\alpha=1}^{N}(A_i^\alpha(\xi)-A_i^\alpha(\eta))(\xi_i^\alpha-\eta_i^\alpha)\Big\}$$

and we conclude by proceeding as above. □

The following estimate will play a crucial role for getting the information about the second derivatives of the weak solutions to (1.4).

Lemma 2.2. *Let A be a continuous mapping fulfilling* (1.2), (1.3) *and* (1.5). *Then there exists a positive constant K such that for all $\xi, \eta, \zeta \in \mathbb{R}^{nN}$ we have*

$$\frac{m}{2}(1+|\zeta|^2)^{\frac{p-2}{2}}|\xi|^2 \leq \sum_{i,j=1}^{n}\sum_{\alpha,\beta=1}^{N}\frac{\partial A_i^\alpha(\zeta)}{\partial\zeta_j^\beta}(\xi_i^\alpha-\eta_i^\alpha)\xi_j^\beta + K(1+|\zeta|^2)^{\frac{q-2}{2}}|\eta|^2. \tag{2.5}$$

Proof. For arbitrary $\zeta, \xi, \eta \in \mathbb{N}$, we define a bilinear form (for fixed ζ)

$$(\xi,\eta)_\zeta := \frac{1}{2}\sum_{i,j=1}^{n}\sum_{\alpha,\beta=1}^{N}\Big(\frac{\partial A_i^\alpha(\zeta)}{\partial\zeta_j^\beta} + \frac{\partial A_j^\beta(\zeta)}{\partial\zeta_i^\alpha}\Big)\eta_i^\alpha\xi_j^\beta.$$

Trivially, $(\xi,\eta)_\zeta = (\eta,\xi)_\zeta$. Moreover, using assumption (1.2), we get

$$(\xi,\xi)_\zeta = \sum_{i,j=1}^{n}\sum_{\alpha,\beta=1}^{N}\frac{\partial A_i^\alpha(\zeta)}{\partial\zeta_j^\beta}\xi_i^\alpha\xi_j^\beta \geq m(1+|\zeta|^2)^{\frac{p-2}{2}}|\xi|^2,$$

and consequently, we see that for any fixed ζ, the relation $(\xi,\eta)_\zeta$ is a scalar product on $\mathbb{R}^{nN}$ and therefore the Cauchy–Schwarz inequality holds, i.e.,

$$|(\xi,\eta)_\zeta| \leq (\xi,\xi)_\zeta^{\frac{1}{2}}(\eta,\eta)_\zeta^{\frac{1}{2}}.$$

Thus, by assumption (1.2) and taking into account that

$$\sum_{i,j=1}^{n}\sum_{\alpha,\beta=1}^{N}\Big(\frac{\partial A_i^\alpha(\zeta)}{\partial\zeta_j^\beta} - \frac{\partial A_j^\beta(\zeta)}{\partial\zeta_i^\alpha}\Big)\xi_i^\alpha\xi_j^\beta = 0,$$

we have

$$
\begin{aligned}
m(1+|\zeta|^2)^{\frac{p-2}{2}}|\xi|^2 &\le (\xi,\xi)_\zeta \\
&= -(\xi-\eta,\xi-\eta)_\zeta + 2(\xi,\xi-\eta)_\zeta + (\eta,\eta)_\zeta \\
&\le 2(\xi,\xi-\eta)_\zeta + (\eta,\eta)_\zeta \\
&= \sum_{i,j=1}^n \sum_{\alpha,\beta=1}^N \Big(\frac{\partial A_i^\alpha(\zeta)}{\partial \zeta_j^\beta} + \frac{\partial A_j^\beta(\zeta)}{\partial \zeta_i^\alpha}\Big)(\xi_i^\alpha - \eta_i^\alpha)\xi_j^\beta + \sum_{i,j=1}^n \sum_{\alpha,\beta=1}^N \frac{\partial A_i^\alpha(\zeta)}{\partial \zeta_j^\beta}\eta_i^\alpha \eta_j^\beta \\
&= 2\sum_{i,j=1}^n \sum_{\alpha,\beta=1}^N \frac{\partial A_i^\alpha(\zeta)}{\partial \zeta_j^\beta}(\xi_i^\alpha - \eta_i^\alpha)\xi_j^\beta - \sum_{i,j=1}^n \sum_{\alpha,\beta=1}^N \Big(\frac{\partial A_i^\alpha(\zeta)}{\partial \zeta_j^\beta} - \frac{\partial A_j^\beta(\zeta)}{\partial \zeta_i^\alpha}\Big)(\xi_i^\alpha - \eta_i^\alpha)\xi_j^\beta \\
&\qquad + \sum_{i,j=1}^n \sum_{\alpha,\beta=1}^N \frac{\partial A_i^\alpha(\zeta)}{\partial \zeta_j^\beta}\eta_i^\alpha \eta_j^\beta \\
&= 2\sum_{i,j=1}^n \sum_{\alpha,\beta=1}^N \frac{\partial A_i^\alpha(\zeta)}{\partial \zeta_j^\beta}(\xi_i^\alpha - \eta_i^\alpha)\xi_j^\beta + \sum_{i,j=1}^n \sum_{\alpha,\beta=1}^N \Big(\frac{\partial A_i^\alpha(\zeta)}{\partial \zeta_j^\beta} - \frac{\partial A_j^\beta(\zeta)}{\partial \zeta_i^\alpha}\Big)\eta_i^\alpha \xi_j^\beta \\
&\qquad + \sum_{i,j=1}^n \sum_{\alpha,\beta=1}^N \frac{\partial A_i^\alpha(\zeta)}{\partial \zeta_j^\beta}\eta_i^\alpha \eta_j^\beta \\
&\le 2\sum_{i,j=1}^n \sum_{\alpha,\beta=1}^N \frac{\partial A_i^\alpha(\zeta)}{\partial \zeta_j^\beta}(\xi_i^\alpha - \eta_i^\alpha)\xi_j^\beta + MnN(1+|\zeta|^2)^{\frac{q+p-4}{4}}|\eta||\xi| + MnN(1+|\zeta|^2)^{\frac{q-2}{2}}|\eta|^2,
\end{aligned}
$$

using (1.3) and (1.5) in the last inequality. Taking into account that

$$(1+|\zeta|^2)^{\frac{q+p-4}{4}}|\eta||\xi| = \big((1+|\zeta|^2)^{\frac{p-2}{4}}|\xi|\big)\big((1+|\zeta|^2)^{\frac{q-2}{4}}|\eta|\big)$$

and using Young's inequality, we get

$$m(1+|\zeta|^2)^{\frac{p-2}{2}}|\xi|^2 \le 2\sum_{i,j=1}^n \sum_{\alpha,\beta=1}^N \frac{\partial A_i^\alpha(\zeta)}{\partial \zeta_j^\beta}(\xi_i^\alpha - \eta_i^\alpha)\xi_j^\beta + \frac{m}{2}(1+|\zeta|^2)^{\frac{p-2}{2}}|\xi|^2 + C(1+|\zeta|^2)^{\frac{q-2}{2}}|\eta|^2$$

for a suitable constant C. Then (2.5) easily follows. □

3 Proof of Theorem 1.1

We proceed via difference quotients technique. Due to the assumed regularity of the solution u and thanks to (2.2), it follows from (1.4) that

$$\int_\Omega \sum_{i=1}^n \sum_{\alpha=1}^N (A_i^\alpha(Du(x+he_k)) - A_i^\alpha(Du(x)))\varphi_{x_i}^\alpha \, dx = 0$$

for all $\varphi \in W_0^{1,q}(\Omega_h;\mathbb{R}^N)$, all $h \in (0,1)$ and all $k = 1,\dots,n$, where $\Omega_h := \{x \in \Omega : B_{2h}(x) \subset \Omega\}$ and e_k is a unit vector in the kth direction. Hence, setting

$$\varphi(x) := (u(x+he_k) - u(x))\tau^2(x)$$

with $\tau \in \mathcal{C}_c^\infty(\Omega_{2h})$ (which is an admissible choice), we obtain the starting identity

$$0 = \int_\Omega \sum_{i=1}^n \sum_{\alpha=1}^N (A_i^\alpha(Du(x+he_k)) - A_i^\alpha(Du(x)))\tau(x)\big((u_{x_i}^\alpha(x+he_k) - u_{x_i}^\alpha(x))\tau(x) + 2(u^\alpha(x+he_k) - u^\alpha(x))\tau_{x_i}\big)\, dx. \quad (3.1)$$

Since

$$A_i^\alpha(Du(x+he_k)) - A_i^\alpha(Du(x)) = \int_0^t \sum_{j=1}^n \sum_{\beta=1}^N \int_0^1 \frac{\partial A_i^\alpha(tDu(x+he_k) + (1-t)Du(x))}{\partial \zeta_j^\beta}(u_{x_j}^\beta(x+he_k) - u_{x_j}^\beta(x))\, dt,$$

identity (3.1) can be equivalently rewritten as

$$0=\int_\Omega \sum_{i,j=1}^{n}\sum_{\alpha,\beta=1}^{N}\int_0^1 \frac{\partial A_i^\alpha(tDu(x+he_k)+(1-t)Du(x))}{\partial \zeta_j^\beta}(u^\beta_{x_j}(x+he_k)-u^\beta_{x_j}(x))\tau(x)$$
$$\cdot\left((u^\alpha_{x_i}(x+he_k)-u^\alpha_{x_i}(x))\tau(x)+2(u^\alpha(x+he_k)-u^\alpha(x))\tau_{x_i}\right)dt\,dx. \tag{3.2}$$

Abbreviating for the moment

$$\xi_i^\alpha:=\tau(x)(u^\alpha_{x_i}(x+he_k)-u^\alpha_{x_i}(x)),\quad \eta_i^\alpha:=-2(u^\alpha(x+he_k)-u^\alpha(x))\tau_{x_i}(x),\quad \zeta:=tDu(x+he_k)+(1-t)Du(x),$$

we can formally rewrite (3.2) as

$$0=\int_\Omega\int_0^1\sum_{i,j=1}^{n}\sum_{\alpha,\beta=1}^{N}\frac{\partial A_i^\alpha(\zeta)}{\partial\zeta_j^\beta}\xi_j^\beta(\xi_i^\alpha-\eta_i^\alpha)\,dt\,dx.$$

Thus, using (2.5), we obtain (here C is some constant depending only on m, M, n, N, p, q)

$$\int_\Omega\int_0^1(1+|\zeta|^2)^{\frac{p-2}{2}}|\xi|^2\,dt\,dx\le C\int_\Omega\int_0^1(1+|\zeta|^2)^{\frac{q-2}{2}}|\eta|^2\,dt\,dx,$$

which in terms of original variables after division by h^2 means that

$$\int_\Omega\int_0^1(1+|tDu(x+he_k)+(1-t)Du(x)|^2)^{\frac{p-2}{2}}\frac{|Du(x+he_k)-Du(x)|^2}{h^2}\tau^2(x)\,dt\,dx$$
$$\le 4C\int_\Omega\int_0^1(1+|tDu(x+he_k)+(1-t)Du(x)|^2)^{\frac{q-2}{2}}\frac{|u(x+he_k)-u(x)|^2}{h^2}|D\tau(x)|^2\,dt\,dx. \tag{3.3}$$

Finally, we let $h\to 0_+$. First, we focus on the limit in the term on the right-hand side of (3.3). In case that $q\le 2$, we use the assumption that $u\in W^{1,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ and therefore, we can use the Lebesgue dominated convergence theorem to conclude that

$$\limsup_{h\to 0}\int_\Omega\int_0^1(1+|tDu(x+he_k)+(1-t)Du(x)|^2)^{\frac{q-2}{2}}\frac{|u(x+he_k)-u(x)|^2}{h^2}|D\tau(x)|^2\,dt\,dx$$
$$=\int_\Omega(1+|Du|^2)^{\frac{q-2}{2}}|u_{x_k}|^2|D\tau|^2\,dx\le\int_\Omega(1+|Du|^2)^{\frac{q}{2}}|D\tau|^2\,dx.$$

Next, if $q>2$, we use the Hölder inequality, the assumption $u\in W^{1,q}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ and the Lebesgue dominated convergence theorem to conclude

$$\limsup_{h\to 0}\int_\Omega\int_0^1(1+|tDu(x+he_k)+(1-t)Du(x)|^2)^{\frac{q-2}{2}}\frac{|u(x+he_k)-u(x)|^2}{h^2}|D\tau(x)|^2\,dt\,dx$$
$$=\limsup_{h\to 0}\int_\Omega\int_0^1\left(((1+|tDu(x+he_k)+(1-t)Du(x)|^2)^{\frac{q-2}{2}}|D\tau(x)|^{2\frac{q-2}{q}}\right)$$
$$\cdot\frac{|u(x+he_k)-u(x)|^2}{h^2}|D\tau(x)|^{\frac{4}{q}}\,dt\,dx$$
$$\le\limsup_{h\to 0}\int_0^1\left(\int_\Omega((1+|tDu(x+he_k)+(1-t)Du(x)|^2)^{\frac{q}{2}}|D\tau(x)|^2\,dx\right)^{\frac{q-2}{q}}$$
$$\cdot\left(\int_\Omega\frac{|u(x+he_k)-u(x)|^q}{h^q}|D\tau(x)|^2\,dx\right)^{\frac{2}{q}}dt$$
$$\le\int_\Omega(1+|Du|^2)^{\frac{q}{2}}|D\tau|^2\,dx.$$

Consequently, substituting these limits into (3.3), we have

$$\begin{aligned}\limsup_{h\to 0}\int_\Omega\int_0^1 &(1+|tDu(x+he_k)+(1-t)Du(x)|^2)^{\frac{p-2}{2}}\frac{|Du(x+he_k)-Du(x)|^2}{h^2}\tau^2(x)\,dt\,dx\\ &\le 4C\int_\Omega (1+|Du|^2)^{\frac{q}{2}}|D\tau|^2\,dx. \end{aligned}\tag{3.4}$$

From this estimate it immediately follows that $u\in W^{2,\min\{2,p\}}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$; in particular, we know that D^2u exists and that for almost all x,

$$\frac{Du(x+he_k)-Du(x)}{h}\to D^2u_{x_k}(x),$$

where $D^2u_{x_k}$ stands for $\frac{\partial Du}{\partial x_k}$. Therefore, we can use the Fatou lemma in (3.4) to conclude

$$\int_\Omega (1+|Du|^2)^{\frac{p-2}{2}}|D^2u_{x_k}(x)|^2\tau^2\,dx\le 4C\int_\Omega (1+|Du|^2)^{\frac{q}{2}}|D\tau|^2\,dx.$$

Since k is arbitrary, relation (1.6) obviously follows. In addition, using the following algebraic inequality

$$|DV(Du)|^2\le K(1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2,$$

we see that (1.7) holds as well. Hence the proof is complete.

4 Proof of Theorem 1.2

We shall start by recalling the definition of the Sobolev embedding exponent:

$$2^*=\begin{cases}\frac{2n}{n-2} & \text{if } n\ge 3,\\ \text{arbitrary}>2 & \text{if } n=2.\end{cases}$$

The value 2^* in dimension $n=2$ will be finally chosen sufficiently large. Since u is assumed to be a weak solution belonging to $W^{1,\max\{q,2\}}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$, we can use Theorem 1.1 and after summing (1.7) and (1.6), we obtain the starting inequality valid for all $\tau\in\mathcal{C}^\infty_c(\Omega)$:

$$\int_\Omega ((1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2\tau^2+|DV(Du)|^2\tau^2)\,dx\le K\int_\Omega (1+|Du|^2)^{\frac{q}{2}}|D\tau|^2\,dx.\tag{4.1}$$

Moreover, we remark that

$$\int_\Omega (1+|Du|^2)^{\frac{q}{2}}|D\tau|^2\,dx\le c\int_\Omega ((1+|Du|^2)^{\frac{p}{2}}+|V(Du)|^{\frac{2q}{p}})|D\tau|^2\,dx.\tag{4.2}$$

Indeed, in $\{|Du|\le 1\}$ we have

$$(1+|Du|^2)^{\frac{q}{2}}\le 2^{\frac{q-p}{2}}(1+|Du|^2)^{\frac{p}{2}}$$

and, in $\{|Du|>1\}$,

$$(1+|Du|^2)^{\frac{q}{2}}=\{(1+|Du|^2)^{\frac{p-2}{2}}(1+|Du|^2)\}^{\frac{q}{p}}\le 2^{\frac{q}{p}}|V(Du)|^{\frac{2q}{p}}.$$

Next, we split the proof for cases (i) and (ii).

4.1 The case $q<p\frac{n+2}{n}$

In this case, we first use the Sobolev embedding to conclude that (with some C depending on 2^*)

$$\begin{aligned}\|V(Du)\tau\|^2_{2^*}\le C\|D(V(Du)\tau)\|^2_2&\le 2C\int_\Omega (|DV(Du)|^2\tau^2+|V(Du)|^2|D\tau|^2)\,dx\\ &\le 2C\int_\Omega (|DV(Du)|^2\tau^2+(1+|Du|^2)^{\frac{p}{2}}|D\tau|^2)\,dx.\end{aligned}$$

Using this inequality in (4.1), and taking into account (4.2), we get

$$\begin{aligned}\|V(Du)\tau\|_{2^*}^2 &+ \int_\Omega \big((1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2\tau^2 + |DV(Du)|^2\tau^2\big)\,dx\\ &\le K_1\int_\Omega \big((1+|Du|^2)^{\frac{p}{2}}|D\tau|^2 + (1+|Du|^2)^{\frac{q}{2}}|D\tau|^2\big)\,dx\\ &\le K_2\int_\Omega \big((1+|Du|^2)^{\frac{p}{2}}|D\tau|^2 + |V(Du)|^{\frac{2q}{p}}|D\tau|^2\big)\,dx. \qquad (4.3)\end{aligned}$$

In particular, we have $V(Du) \in L^{2^*}_{\mathrm{loc}}$.

Let us now estimate the last integral on the right-hand side. Since $q \in (p, p\frac{2^*}{2})$, which follows from the assumption that $q < p\frac{n+2}{n}$ (note here that the value of 2^* in dimension $n = 2$ has to be chosen greater than $\frac{2q}{p}$), there exists a unique $\theta \in (0, 1)$ such that

$$\frac{q}{2} = \frac{p}{2}(1-\theta) + \frac{p2^*}{4}\theta, \qquad \theta := \frac{q-p}{p(\frac{2^*}{2}-1)}.$$

As we will prove below, under our assumptions on the exponents p and q and, if $n = 2$, with a suitable choice of 2^*, we have

$$2 > 2^*\theta. \qquad (4.4)$$

Consider $\eta \in \mathcal{C}_c^\infty(\Omega)$ an arbitrary nonnegative cut-off function and set

$$\tau := \eta^\gamma \quad \text{with } \gamma := \frac{2}{2-2^*\theta}.$$

We have

$$\frac{|D\tau|^2}{\tau^{2^*\theta}} = \gamma^2\eta^{\gamma(2-2^*\theta)-2}|D\eta|^2 = \gamma^2|D\eta|^2. \qquad (4.5)$$

Then by the Hölder inequality, we have

$$\begin{aligned}\int_\Omega |V(Du)|^{\frac{2q}{p}}|D\tau|^2\,dx &= \int_\Omega |V(Du)|^{2(1-\theta)}(V(Du)\tau)^{2^*\theta}\frac{|D\tau|^2}{\tau^{2^*\theta}}\,dx\\ &\le \|V(Du)\|_2^{2(1-\theta)}\|V(Du)\tau\|_{2^*}^{2^*\theta}\Big\|\frac{|D\tau|^2}{\tau^{2^*\theta}}\Big\|_\infty\end{aligned}$$

and we can apply Young's inequality to deduce that for arbitrary $\varepsilon > 0$ we have

$$\int_\Omega |V(Du)|^{\frac{2q}{p}}|D\tau|^2\,dx \le \varepsilon\|V(Du)\tau\|_{2^*}^2 + C(\varepsilon,\gamma)\|V(Du)\|_2^{2(1-\theta)\gamma}\Big\|\frac{|D\tau|^2}{\tau^{2^*\theta}}\Big\|_\infty^\gamma. \qquad (4.6)$$

Therefore, combining (4.3), (4.6), (4.5) and taking into account that $|V(Du)| \le (1+|Du|^2)^{\frac{p}{4}}$, with a proper choice of $\varepsilon > 0$, we obtain

$$\|V(Du)\tau\|_{2^*}^2 + \int_\Omega \big((1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2\eta^{2\gamma} + |DV(Du)|^2\eta^{2\gamma}\big)\,dx \le K(\gamma, \|D\eta\|_\infty)\Big(\int_\Omega (1+|Du|^2)^{\frac{p}{2}}\,dx\Big)^{\tilde q},$$

with some power $\tilde q$ whose value depends on p, q and γ. From this inequality statement (i) of Theorem 1.2 follows directly.

Now, we check the validity of (4.4), which, by using of definition of θ, it can be written as

$$q < 2p\Big(1 - \frac{1}{2^*}\Big).$$

If $n = 2$, we can choose 2^* arbitrarily large, therefore in this case condition (4.4) reduces to $q < 2p$, which is exactly assumption (1.8) for $n = 2$. If $n \ge 3$, we have $2^* = \frac{2n}{n-2}$ and the above condition is then equivalent to

$$q < p\frac{n+2}{n},$$

which is nothing else than assumption (1.8). Hence the proof of statement (i) is finished.

4.2 The case $q < p + 2$ and $p < n$

We again start to estimate the integral on the right-hand side of (4.1). Using a simple inequality and the integration by parts, we find that (here K is again a generic constant depending only on q, n, $\|D\tau\|_2$ and N, see also [1, the bottom of p. 147])

$$
\begin{aligned}
\int_\Omega (1+|Du|^2)^{\frac{q}{2}}|D\tau|^2\,dx &\le K + 2^q \sum_{k=1}^{n}\sum_{\alpha=1}^{N}\int_\Omega (1+|Du|^2)^{\frac{q-2}{2}} u^\alpha_{x_k}u^\alpha_{x_k}|D\tau|^2\,dx\\
&= K - 2^q \sum_{k=1}^{n}\sum_{\alpha=1}^{N}\int_\Omega ((1+|Du|^2)^{\frac{q-2}{2}} u^\alpha_{x_k}|D\tau|^2)_{x_k} u^\alpha\,dx\\
&\le K + K\|u\|_\infty \int_\Omega ((1+|Du|^2)^{\frac{q-2}{2}}|D^2u||D\tau|^2 + (1+|Du|^2)^{\frac{q-1}{2}}|D\tau||D^2\tau|)\,dx.
\end{aligned}
$$

Let us now set $\tau := \eta^\gamma$, $\gamma \ge 2$ to be chosen later, where $\eta \in \mathcal{C}^\infty_c(\Omega)$ is an arbitrary nonnegative cut-off function. By Young's inequality,

$$
\begin{aligned}
&K\|u\|_\infty \int_\Omega (1+|Du|^2)^{\frac{q-2}{2}}|D^2u||D\tau|^2\,dx\\
&\quad= \int_\Omega \{(1+|Du|^2)^{\frac{p-2}{4}}|D^2u|\tau\}\Big\{\|u\|_\infty(1+|Du|^2)^{\frac{2q-p-2}{4}}\frac{|D\tau|^2}{\tau}\Big\}\,dx\\
&\quad\le \varepsilon\int_\Omega (1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2\tau^2\,dx + c_{\varepsilon,K}\|u\|^2_\infty \int_\Omega (1+|Du|^2)^{\frac{2q-p-2}{2}}\frac{|D\tau|^4}{\tau^2}\,dx.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\int_\Omega (1+|Du|^2)^{\frac{q}{2}}|D\tau|^2\,dx &\le \varepsilon\int_\Omega (1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2\tau^2\,dx + K + K\|u\|^2_\infty\int_\Omega (1+|Du|^2)^{\frac{2q-p-2}{2}}\frac{|D\tau|^4}{\tau^2}\,dx\\
&\quad + K\|u\|_\infty\int_\Omega (1+|Du|^2)^{\frac{q-1}{2}}|D\tau||D^2\tau|\,dx,
\end{aligned}
\tag{4.7}
$$

with a possibly different positive constant $K(\varepsilon)$ than before depending also on $\varepsilon > 0$.

Let us now discuss first the case $q \in [p, p+1]$. If q belongs to this range, the above inequality immediately reduces to

$$
\begin{aligned}
\int_\Omega (1+|Du|^2)^{\frac{q}{2}}|D\tau|^2\,dx &\le \varepsilon\int_\Omega (1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2\tau^2\,dx + K + K\|u\|^2_\infty\int_\Omega (1+|Du|^2)^{\frac{p}{2}}\frac{|D\tau|^4}{\tau^2}\,dx\\
&\quad + K\|u\|_\infty\int_\Omega (1+|Du|^2)^{\frac{p}{2}}|D\tau||D^2\tau|\,dx.
\end{aligned}
$$

Let us now choose $\gamma = 2$, that is $\tau := \eta^2$. Thus we get

$$
\int_\Omega (1+|Du|^2)^{\frac{q}{2}}|D\tau|^2\,dx \le \varepsilon\int_\Omega (1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2\tau^2\,dx + K(\varepsilon) + C(\varepsilon,\|u\|_\infty,\|\eta\|_{2,\infty})\int_\Omega (1+|Du|^2)^{\frac{p}{2}}\,dx. \tag{4.8}
$$

Hence by (4.1) and taking a proper $\varepsilon > 0$, so that we can absorb the first term on the right-hand side in (4.8) by the left-hand side in (4.1), it is not difficult to arrive to statement (ii) of Theorem 1.2 for $q \in [p, p+1]$. Notice here that once fixing ε, the constants $K(\varepsilon)$, $C(\varepsilon, \|u\|_\infty, \|\eta\|_{2,\infty})$ depend again only on data, in particular through the numbers m and M.

Next, we focus on the case when $q \in (p+1, p+2)$. There exist $\theta_1, \theta_2 \in (0,1)$ such that

$$
\begin{aligned}
q-1 &= p(1-\theta_1) + q\theta_1, \quad \theta_1 := \frac{q-1-p}{q-p},\\
2q-p-2 &= p(1-\theta_2) + q\theta_2, \quad \theta_2 := \frac{2(q-1-p)}{q-p}.
\end{aligned}
$$

In addition, considering $\tau = \eta^\gamma$ with

$$\gamma = \frac{2-\theta_2}{1-\theta_2}, \tag{4.9}$$

we have $\frac{|D\tau|^{2-\theta_2}}{\tau} = \gamma^{2-\theta_2}|D\eta|^{2-\theta_2}$.

With this setting, we can now estimate the remaining integrals on the right-hand side of (4.7) by means of the Hölder inequality as follows:

$$\begin{aligned}\int_\Omega (1+|Du|^2)^{\frac{2q-p-2}{2}} \frac{|D\tau|^4}{\tau^2}\,dx &= \int_\Omega ((1+|Du|^2)^{\frac{q}{2}}|D\tau|^2)^{\theta_2}(1+|Du|^2)^{\frac{p(1-\theta_2)}{2}} \frac{|D\tau|^{4-2\theta_2}}{\tau^2}\,dx\\ &\le C\bigg\|\frac{|D\tau|^{4-2\theta_2}}{\tau^2}\bigg\|_\infty \|(1+|Du|)\|_p^{p(1-\theta_2)}\bigg(\int_\Omega (1+|Du|^2)^{\frac{q}{2}}|D\tau|^2\,dx\bigg)^{\theta_2}.\end{aligned}$$

Then the above estimate reduces to

$$\int_\Omega (1+|Du|^2)^{\frac{2q-p-2}{2}} \frac{|D\tau|^4}{\tau^2}\,dx \le C(\theta_2, \|\eta\|_{1,\infty})\|(1+|Du|)\|_p^{p(1-\theta_2)}\bigg(\int_\Omega (1+|Du|^2)^{\frac{q}{2}}|D\tau|^2\,dx\bigg)^{\theta_2}. \tag{4.10}$$

We proceed similarly also with the remaining integral in (4.7), i.e., using the Hölder inequality, we have

$$\begin{aligned}\int_\Omega (1+|Du|^2)^{\frac{q-1}{2}}|D\tau||D^2\tau|\,dx &= \int_\Omega (1+|Du|^2)^{\frac{p(1-\theta_1)}{2}}((1+|Du|^2)^{\frac{q}{2}}|D\tau|^2)^{\theta_1}|D\tau|^{1-2\theta_1}|D^2\tau|\,dx\\ &\le K\|(1+|Du|)\|_p^{p(1-\theta_1)}\||D\tau|^{1-2\theta_1}|D^2\tau|\|_\infty\bigg(\int_\Omega (1+|Du|^2)^{\frac{q}{2}}|D\tau|^2\,dx\bigg)^{\theta_1}\\ &\le C(\|\tau\|_{2,\infty}, \theta_2)\|(1+|Du|)\|_p^{p(1-\theta_1)}\bigg(\int_\Omega (1+|Du|^2)^{\frac{q}{2}}|D\tau|^2\,dx\bigg)^{\theta_1},\end{aligned} \tag{4.11}$$

where the last inequality follows from the fact that $1-2\theta_1 = 1-\theta_2 > 0$.

Finally, using (4.10) and (4.11) in (4.7), keeping in mind the special choice of τ in (4.9) and applying Young's inequality (notice that $\theta_1, \theta_2 < 1$), we observe that

$$\int_\Omega (1+|Du|^2)^{\frac{q}{2}}|D\tau|^2\,dx \le \varepsilon\int_\Omega (1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2\tau^2\,dx + C(\varepsilon, \theta_2, \|\eta\|_{2,\infty}, \|u\|_\infty, \|Du\|_p).$$

Thus, going back to (4.1), choosing $\varepsilon > 0$ sufficiently small to absorb the term involving the second derivatives by the left-hand side, we finally get statement (ii) of Theorem 1.2.

5 Proof of Theorem 1.3

In this final section we establish the existence of a weak solution to the Dirichlet problem (1.1), under assumption (1.10) on the boundary datum u_0; i.e.,

$$u_0 \in W^{1,r}(\Omega;\mathbb{R}^N), \quad r := \max\Big\{2, p\frac{q-1}{p-1}\Big\}.$$

We use an approximation procedure. For arbitrary $\epsilon \in (0,1)$ we introduce the approximate problem $(\alpha = 1,\dots,N)$

$$\begin{cases}\displaystyle\sum_{i=1}^n \frac{\partial}{\partial x_i}(A_{\epsilon,i}^\alpha(Du_\epsilon)) = 0 & \text{in } \Omega,\\ u_\epsilon = u_0 & \text{on } \partial\Omega,\end{cases} \tag{5.1}$$

where $A_{\epsilon,i}^\alpha : \mathbb{R}^{nN} \to \mathbb{R}$ is defined as

$$A_{\epsilon,i}^\alpha(\xi) := A_i^\alpha(\xi) + \epsilon(1+|\xi|^2)^{\frac{\max\{q,2\}-2}{2}}\xi_i^\alpha. \tag{5.2}$$

In addition, in case we deal with statement (ii) of the theorem, we shall require that $u_0 \in L^\infty(\partial\Omega)$. Due to (2.1) used with $\eta \equiv 0$ and (2.2), we have $A^\alpha_{\epsilon,i}(\xi)$ satisfies the following properties:

$$\sum_{i=1}^{n}\sum_{\alpha=1}^{N} A^\alpha_{\epsilon,i}(\xi)\xi^\alpha_i \geq \epsilon|\xi|^{\max\{q,2\}} - \lambda$$

and

$$|A^\alpha_{\epsilon,i}(\xi)| \leq M'(1+|\xi|)^{\max\{q,2\}-1}$$

for some positive λ and M' independent on ϵ. Furthermore, thanks to (2.3) and (2.4), we can apply the theory of monotone operators (see e.g. [3, 15, 19]) to prove the existence of a unique solution to (5.1), i.e., the existence of $u_\epsilon \in u_0 + W^{1,\max\{q,2\}}_0(\Omega;\mathbb{R}^N)$ fulfilling

$$\int_\Omega \sum_{\alpha=1}^{N}\sum_{i=1}^{n} A^\alpha_{\epsilon,i}(Du_\epsilon)\varphi^\alpha_{x_i}\,dx = 0 \quad \text{for all } \varphi \in W^{1,\max\{q,2\}}_0(\Omega;\mathbb{R}^N). \tag{5.3}$$

5.1 First a priori estimates

We now derive estimates for u_ϵ independent of ϵ. Using $\varphi := u_\epsilon - u_0$ as a test function in (5.3), we get

$$\begin{aligned} 0 &= \int_\Omega \sum_{\alpha=1}^{N}\sum_{i=1}^{n} A^\alpha_{\epsilon,i}(Du_\epsilon)((u_\epsilon)^\alpha_{x_i} - (u_0)^\alpha_{x_i})\,dx \\ &= \int_\Omega \sum_{\alpha=1}^{N}\sum_{i=1}^{n} \Big\{A^\alpha_i(Du_\epsilon)((u_\epsilon)^\alpha_{x_i} - (u_0)^\alpha_{x_i}) + \epsilon(1+|Du_\epsilon|^2)^{\frac{\max\{q,2\}-2}{2}}(u_\epsilon)^\alpha_{x_i}((u_\epsilon)^\alpha_{x_i} - (u_0)^\alpha_{x_i})\Big\}\,dx \\ &\geq \int_\Omega \Big(K^{-1}|Du_\epsilon|^p - (1+|Du_0|^2)^{\frac{p(q-1)}{2(p-1)}} + \epsilon(1+|Du_\epsilon|^2)^{\frac{\max\{q,2\}-2}{2}}|Du_\epsilon|(|Du_\epsilon| - |Du_0|)\Big)\,dx, \end{aligned} \tag{5.4}$$

where we have used (2.1) for the last inequality. Since

$$(1+|Du_\epsilon|^2)^{\frac{\max\{q,2\}-2}{2}}|Du_\epsilon|(|Du_\epsilon| - |Du_0|) = (1+|Du_\epsilon|^2)^{\frac{\max\{q,2\}-2}{2}}|Du_\epsilon|^2 - (1+|Du_\epsilon|^2)^{\frac{\max\{q,2\}-2}{2}}|Du_\epsilon||Du_0|,$$

inequality (5.4) implies

$$\begin{aligned} &\int_\Omega \big(|Du_\epsilon|^p + \epsilon(1+|Du_\epsilon|^2)^{\frac{\max\{q,2\}-2}{2}}|Du_\epsilon|^2\big)\,dx \\ &\qquad \leq c\int_\Omega \big((1+|Du_0|^2)^{\frac{p(q-1)}{2(p-1)}} + \epsilon(1+|Du_\epsilon|^2)^{\frac{\max\{q,2\}-2}{2}}|Du_\epsilon||Du_0|\big)\,dx. \end{aligned} \tag{5.5}$$

We claim that (5.5) implies

$$\int_\Omega \big(|Du_\epsilon|^p + \frac{\epsilon}{2}(1+|Du_\epsilon|^2)^{\frac{\max\{2,q\}-2}{2}}|Du_\epsilon|^2\big)\,dx \leq c\int_\Omega (1+|Du_0|^2)^{\frac{r}{2}}\,dx. \tag{5.6}$$

If $q \leq 2$, we can conclude using Young's inequality with exponent 2 on the last term in (5.5):

$$|Du_\epsilon||Du_0| \leq \frac{1}{2c}|Du_\epsilon|^2 + c'|Du_0|^2.$$

Therefore, recalling that $r = \max\{2, \frac{p(q-1)}{p-1}\}$, inequality (5.6) follows. Otherwise, if $q > 2$, the last term in (5.5) can be estimate as follows:

$$\epsilon(1+|Du_\epsilon|^2)^{\frac{\max\{q,2\}-2}{2}}|Du_\epsilon||Du_0| \leq \epsilon\{c(1+|Du_0|^2)^{\frac{r}{2}} + c(1+|Du_\epsilon|^2)^{\frac{q-2}{4}}|Du_\epsilon|^{\frac{q}{2}}|Du_0|\}. \tag{5.7}$$

Indeed, in $\{|Du_\epsilon| \leq 1\}$ we have

$$(1+|Du_\epsilon|^2)^{\frac{\max\{q,2\}-2}{2}}|Du_\epsilon||Du_0| \leq 2^{\frac{q-2}{2}}|Du_0| \leq c(1+|Du_0|^2)^{\frac{r}{2}}$$

and, in $\{|Du_\epsilon| > 1\}$,

$$(1+|Du_\epsilon|^2)^{\frac{\max\{q,2\}-2}{2}}|Du_\epsilon||Du_0| \leq 2^{\frac{q-2}{4}}(1+|Du_\epsilon|^2)^{\frac{q-2}{4}}|Du_\epsilon|^{\frac{q}{2}}|Du_0|$$

and (5.7) follows.

To estimate the last term in (5.7), we use Young's inequality with exponents $2, q, \frac{2q}{q-2}$. Recalling that $\epsilon < 1$, we have

$$\begin{aligned}\epsilon c(1+|Du_\epsilon|^2)^{\frac{q-2}{4}}|Du_\epsilon|^{\frac{q}{2}}|Du_0| &= \epsilon c\{(1+|Du_\epsilon|^2)^{\frac{q-2}{4}}|Du_\epsilon|\}|Du_\epsilon|^{\frac{q-2}{2}}|Du_0| \\ &\le \frac{\epsilon}{8}(1+|Du_\epsilon|^2)^{\frac{q-2}{2}}|Du_\epsilon|^2 + \frac{\epsilon}{8}|Du_\epsilon|^q + \epsilon c|Du_0|^q \\ &\le \frac{\epsilon}{4}(1+|Du_\epsilon|^2)^{\frac{\max\{q,2\}-2}{2}}|Du_\epsilon|^2 + c(1+|Du_0|^2)^{\frac{r}{2}} \end{aligned} \tag{5.8}$$

with c independent of ϵ. Therefore, collecting (5.5), (5.7) and (5.8), inequality (5.6) follows also in the case $q > 2$. Thus, we can find a universal constant $C > 0$ such that (using also the Poincaré inequality)

$$\|u_\epsilon\|_{1,p} + \epsilon\|u_\epsilon\|_{1,\max\{q,2\}}^{\max\{q,2\}} \le C. \tag{5.9}$$

If assumption (1.11) holds, then for every $\alpha \in \{1,\dots,N\}$ we have

$$\sum_{i=1}^{n} A^\alpha_{\epsilon,i}(\xi)\xi^\alpha_i \ge \epsilon(1+|\xi|^2)^{\frac{\max\{2,q\}-2}{2}}|\xi^\alpha|^2 \ge \epsilon|\xi^\alpha|^{\max\{q,2\}} \tag{5.10}$$

and

$$|A^\alpha_{\epsilon,i}(\xi)| \le (K+1)(1+|\xi|^2)^{\frac{\max\{2,q\}-1}{2}},$$

where K is as in (2.2). Next we denote $\tilde{M} := \|u_0\|_{L^\infty(\partial\Omega)}$ and define

$$\varphi^\alpha := \max\{u^\alpha_\epsilon - \tilde{M}, 0\}, \quad \alpha \in \{1,\dots,N\}.$$

Evidently, $\varphi = (\varphi^1,\dots,\varphi^N) \in W^{1,\max\{2,q\}}_0(\Omega;\mathbb{R}^N)$ and can be used as a test function in (5.3). Doing so, and using the definition of φ we obtain (here $\chi_{u^\alpha_\epsilon \ge \tilde{M}}$ denotes the characteristic function of the set, where $u^\alpha_\epsilon \ge \tilde{M}$)

$$0 = \int_\Omega \sum_{\alpha=1}^{N}\sum_{i=1}^{n} A^\alpha_{\epsilon,i}(Du_\epsilon)\varphi^\alpha_{x_i}\,dx = \int_\Omega \sum_{\alpha=1}^{N}\sum_{i=1}^{n} A^\alpha_{\epsilon,i}(Du_\epsilon)Du^\alpha_\epsilon\chi_{u^\alpha_\epsilon \ge \tilde{M}}\,dx.$$

Using finally (5.10), we see that

$$\begin{aligned} 0 &= \int_\Omega \sum_{\alpha=1}^{N}\sum_{i=1}^{n} A^\alpha_{\epsilon,i}(Du_\epsilon)Du^\alpha_\epsilon\chi_{u^\alpha_\epsilon \ge \tilde{M}}\,dx \\ &\ge \epsilon\int_\Omega \sum_{\alpha=1}^{N} |Du^\alpha_\epsilon|^{\max\{q,2\}}\chi_{u^\alpha_\epsilon \ge \tilde{M}}\,dx \\ &= \epsilon\int_\Omega \sum_{\alpha=1}^{N} |D\varphi^\alpha|^{\max\{q,2\}}\,dx. \end{aligned}$$

Consequently, $D\varphi \equiv 0$ and since it has zero trace, due to the Poincaré inequality, it must be identically zero and it directly follows from its definition that $u^\alpha_\epsilon \le \tilde{M} = \|u_0\|_{L^\infty(\partial\Omega)}$ for all $\alpha \in \{1,\dots,N\}$. The minimum principle can be obtained by repeating step by step the above procedure for a test function defined as

$$\varphi^\alpha := \min\{u^\alpha_\epsilon + \tilde{M}, 0\}, \quad \alpha \in \{1,\dots,N\}.$$

Therefore, we conclude that, for every $\epsilon \in (0,1)$,

$$\|u_\epsilon\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\partial\Omega)}. \tag{5.11}$$

5.2 Uniform higher order estimates

Due to the proof of a priori estimates, we can use Theorem 1.1 to get the existence of the second order derivatives of u_ϵ, but with their estimates depending on ϵ. Nevertheless, we can repeat step by step the estimates in Theorem 1.1 to get the inequality

$$\begin{aligned}\int_\Omega (1+|Du_\epsilon|^2)^{\frac{p-2}{2}}|D^2u_\epsilon|^2\tau^2\,dx\\ \le c\int_\Omega (1+|Du_\epsilon|^2)^{\frac{q}{2}}|D\tau|^2\,dx - c\epsilon\int_\Omega \sum_{i,k=1}^n\sum_{\alpha=1}^N\big((1+|Du_\epsilon|^2)^{\frac{\max\{q,2\}-2}{2}}(u_\epsilon)^\alpha_{x_i}\big)_{x_k}\big((u_\epsilon)^\alpha_{x_k}\tau^2\big)_{x_i}\,dx\end{aligned}\tag{5.12}$$

for every $\tau\in\mathcal{C}_c^\infty(\Omega)$. To provide more details of the proof of the above inequality, we shall first re-denote the parameters p and q appearing in Theorem 1.1 by $\tilde q$ and $\tilde p$. Then using the definition (5.2), we see that all assumptions of Theorem 1.1 are satisfied with $\tilde p=\tilde q=\max\{2,q\}$. Therefore, $DV(Du_\epsilon)\in L^2_{\text{loc}}$ and also u_ϵ has square integrable second derivatives. In particular, all integrals appearing in (5.12) exist and are finite. Next, we follow the proof of Theorem 1.1. It follows from (5.3) that for any compactly supported $\varphi\in W_0^{1,\max\{q,2\}}(\Omega;\mathbb{R}^N)$, any $k=1,\dots,n$ and sufficiently small $h>0$, we have

$$\int_\Omega\sum_{\alpha=1}^N\sum_{i=1}^n(A^\alpha_{\epsilon,i}(Du_\epsilon(x+he_k))-A^\alpha_{\epsilon,i}(Du_\epsilon(x)))\varphi^\alpha_{x_i}\,dx=0.$$

Setting $\varphi(x):=(u_\epsilon(x+he_k)-u_\epsilon(x))\tau^2(x)$ and using the definition of A_ϵ, we obtain

$$\begin{aligned}&h^{-2}\int_\Omega\sum_{\alpha=1}^N\sum_{i=1}^n(A^\alpha_i(Du_\epsilon(x+he_k))-A^\alpha_i(Du_\epsilon(x)))((u^\alpha_\epsilon(x+he_k)-u^\alpha_\epsilon(x))\tau^2(x))_{x_i}\,dx\\ &\quad=-\epsilon h^{-2}\int_\Omega\sum_{\alpha=1}^N\sum_{i=1}^n\Big((1+|Du_\epsilon(x+he_k)|^2)^{\frac{\max\{2,q\}-2}{2}}(u_\epsilon)^\alpha_{x_i}(x+he_k)\\ &\qquad\qquad-(1+|Du_\epsilon(x)|^2)^{\frac{\max\{2,q\}-2}{2}}(u_\epsilon)^\alpha_{x_i}(x)\Big)((u^\alpha_\epsilon(x+he_k)-u^\alpha_\epsilon(x))\tau^2(x))_{x_i}\,dx.\end{aligned}$$

Finally, we let $h\to0_+$. For the term on the left-hand side, we can directly use the proof of Theorem 1.1 (see the resulting estimate after (3.4)) to get

$$\begin{aligned}&\limsup_{h\to0_+}h^{-2}\int_\Omega\sum_{\alpha=1}^N\sum_{i=1}^n(A^\alpha_i(Du_\epsilon(x+he_k))-A^\alpha_i(Du_\epsilon(x)))((u^\alpha_\epsilon(x+he_k)-u^\alpha_\epsilon(x))\tau^2(x))_{x_i}\,dx\\ &\quad\ge c^{-1}\int_\Omega(1+|Du|^2)^{\frac{p-2}{2}}|D^2u_{x_k}(x)|^2\tau^2\,dx-4Cc^{-1}\int_\Omega(1+|Du|^2)^{\frac{q}{2}}|D\tau|^2\,dx.\end{aligned}$$

Similarly, for the term on the right-hand side, we get

$$\begin{aligned}&\lim_{h\to0_+}\epsilon h^{-2}\int_\Omega\sum_{\alpha=1}^N\sum_{i=1}^n\Big((1+|Du_\epsilon(x+he_k)|^2)^{\frac{\max\{2,q\}-2}{2}}(u_\epsilon)^\alpha_{x_i}(x+he_k)\\ &\qquad\qquad-(1+|Du_\epsilon(x)|^2)^{\frac{\max\{2,q\}-2}{2}}(u_\epsilon)^\alpha_{x_i}(x)\Big)((u^\alpha_\epsilon(x+he_k)-u^\alpha_\epsilon(x))\tau^2(x))_{x_i}\,dx\\ &\quad=\epsilon\int_\Omega\sum_{\alpha=1}^N\sum_{i=1}^n((1+|Du_\epsilon|^2)^{\frac{\max\{2,q\}-2}{2}}(u_\epsilon)^\alpha_{x_i})_{x_k}((u_\epsilon)^\alpha_{x_k}\tau^2(x))_{x_i}\,dx.\end{aligned}$$

Combining the resulting inequalities and summing with respect to $k=1,\dots,n$, we obtain (5.12).

Thus, we need to bound uniformly the last integral in (5.12). By a rather standard manipulation and using the Young inequality, it is not difficult to check that

$$\begin{aligned}&\sum_{i,k=1}^n\sum_{\alpha=1}^N\big((1+|Du_\epsilon|^2)^{\frac{\max\{q,2\}-2}{2}}(u_\epsilon)^\alpha_{x_i}\big)_{x_k}\big((u_\epsilon)^\alpha_{x_k}\tau^2\big)_{x_i}\\ &\quad\ge(1+|Du_\epsilon|^2)^{\frac{\max\{q,2\}-2}{2}}|D^2u_\epsilon|^2\tau^2-2\max\{q,2\}(1+|Du_\epsilon|^2)^{\frac{\max\{q,2\}-2}{2}}|D^2u_\epsilon|\tau|Du_\epsilon||D\tau|\\ &\quad\ge-C(1+|Du_\epsilon|^2)^{\frac{\max\{q,2\}}{2}}|D\tau|^2\end{aligned}$$

with C independent of ϵ. Substituting this into (5.12), we derive

$$\begin{aligned}\int_\Omega (1+|Du_\epsilon|^2)^{\frac{p-2}{2}}|D^2u_\epsilon|^2\tau^2\,dx &\le c\int_\Omega (1+|Du_\epsilon|^2)^{\frac{q}{2}}|D\tau|^2\,dx + c\epsilon\int_\Omega (1+|Du_\epsilon|^2)^{\frac{\max\{q,2\}}{2}}|D\tau|^2\,dx\\ &\le c + c\int_\Omega (1+|Du_\epsilon|^2)^{\frac{q}{2}}|D\tau|^2\,dx,\end{aligned}$$

where we have used (5.9) in the last inequality. Hence, we are in the same starting position as in the proof of Theorem 1.2 and due to uniform (ϵ-independent) uniform bounds (5.9) and (5.11), we deduce that for arbitrary open $\Omega' \subset \overline{\Omega'} \subset \Omega$,

$$\int_{\Omega'} \big(|Du_\epsilon|^q + |DV(Du_\epsilon)|^2 + (1+|Du_\epsilon|^2)^{\frac{p-2}{2}}|D^2u_\epsilon|^2\big)\,dx \le C(\Omega', u_0). \tag{5.13}$$

Further, it is then not difficult to observe with the help of the Hölder inequality that

$$\int_{\Omega'} |D^2u_\epsilon|^{\min\{2,p\}}\,dx \le C(\Omega', u_0). \tag{5.14}$$

5.3 Limit $\epsilon \to 0$

Using the uniform bounds (5.9), (5.13) and (5.14), the compact Sobolev embedding and the diagonal procedure, we can find a subsequence, that we do not relabel, and it exists $u \in (u_0 + W_0^{1,p}(\Omega;\mathbb{R}^N)) \cap W^{1,q}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ such that for arbitrary open $\Omega' \subset \overline{\Omega'} \subset \Omega$, we have

$$u^\epsilon \rightharpoonup u \quad \text{weakly in } W^{1,p}(\Omega;\mathbb{R}^N), \tag{5.15}$$
$$u^\epsilon \rightharpoonup u \quad \text{weakly in } W^{1,q}(\Omega';\mathbb{R}^N), \tag{5.16}$$
$$Du^\epsilon \to Du \quad \text{strongly in } L^p(\Omega';\mathbb{R}^N), \tag{5.17}$$
$$Du^\epsilon \to Du \quad \text{almost everywhere in } \Omega, \tag{5.18}$$
$$\epsilon(1+|Du^\epsilon|^2)^{\frac{\max(2,q)-2}{2}}Du^\epsilon \to 0 \quad \text{strongly in } L^1(\Omega';\mathbb{R}^{nN}). \tag{5.19}$$

Having (5.15)–(5.19), it is easy to let $\epsilon \to 0$ in (5.3) with arbitrary $\varphi \in \mathcal{C}_c^\infty(\Omega;\mathbb{R}^N)$ to deduce (1.4) for the same class of functions φ. The density result then leads to the validity of (1.4) in the full generality. This finishes the proof.

Acknowledgment: We thank the anonymous referee for his/her careful reading of the paper and valuable comments.

Funding: Miroslav Bulíček's work was supported by the ERC-CZ project LL1202 financed by the Ministry of Education, Youth and Sports, Czech Republic. Miroslav Bulíček is a member of the Nečas center for Mathematical Modeling. The other authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

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