Contract contingency in vertically related markets

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In this appendix we provide

1. The conditions under which the joint profits of the firms are concave in the input prices (Section 1).

2. The “proofs available upon request” concerning the derivations of the optimal contracts (Sections 2 to 5).

3. An explanations of the mechanics linking (non-)contingency and the determination of the outside options (Section 6).

For the sake of readability, we have not introduced further notation unless strictly necessary to avoid confusion. The ensuing pages are landscape-oriented to contain the cumbersome formulas.
1 Concavity of the joint profits

Here we show the conditions under which the concavity of the joint profits of the two production channels in the input prices is obtained under passive beliefs. The profits accruing to the firms (gross of the transfers $t_i$, $i = h, l$ which do not influence neither the input nor the retail prices) are

$$X_i(p_h, p_l) = D_i(p_h, p_l)(p_i - w_i), \quad i = h, l,$$

$$Y(p_h, p_l) = D_h(p_h, p_l)w_h + D_l(p_h, p_l)w_l.$$  \hspace{1cm} (1)

The optimal retail prices

$$\hat{p}_h(w_h, w_l) = \frac{u_h[2(u_h - u_l + w_h) + w_l]}{4u_h - u_l}, \quad \hat{p}_l(w_h, w_l) = \frac{u_l(u_h - u_l + w_h) + 2u_hw_l}{4u_h - u_l}.  \hspace{1cm} (3)$$

At these prices, the profits to the downstream firms under passive beliefs, gross of transfers, are

$$X_h(w_h, w_l^*) = \frac{(2u_h^2 + u_h(w_l^* - 2(u_l + w_h)) + u_l w_h)^2}{(u_h - u_l)(4u_h - u_l)^2},$$  \hspace{1cm} (4)

$$X_l(w_h^*, w_l) = \frac{u_h[(u_h - 2w_l) + u_l(w_h^* + w_l - w_h)]^2}{u_l(u_h - u_l)(4u_h - u_l)^2},$$  \hspace{1cm} (5)

where $(w_h^*, w_l^*)$ are the candidate equilibrium input prices. The profits of each of the two channels under passive beliefs are thus

$$C_h(w_h, w_l) = D_h(\hat{p}_h(w_h, w_l), \hat{p}_l(w_h, w_l))w_h + X_h(w_h, w_l^*) = \frac{u_h\{4u_h^2 + 4u_h^2(w_l^* - 2u_l) + u_h\{4u_l^2 + 4u_l^2(u_h + u_l) + 2u_h w_h - 4w_h - w_l + (w_l^*)^2\} - u_l w_h (2u_l - 2w_h + w_l - 2w_l^*)\}}{(u_h - u_l)(4u_h - u_l)^2}.  \hspace{1cm} (6)$$
and

\[ C_l(w_h, w_l) = D_l(\hat{p}_h(w_h, w_l), \hat{p}_l(w_h, w_l))w_l + X_l(w_h^*, w_l) = \]
\[ = \frac{w_h[u_l^2(w_h - w_l)^2 + u_h u_l w_l (u_l + 4w_h - u_l^2) - 2u_h u_l^2 (2u_h - u_l^2) w_l (u_l + w_h - 2w_l^2)]}{u_l(u_h - u_l)(4u_h - u_l^2)^2} \]

Under a contract equilibrium the solution to the system

\[
\begin{cases}
\frac{\partial C_h}{\partial w_h} = 0 \\
\frac{\partial C_l}{\partial w_l} = 0
\end{cases}
\]

returns the candidate optimal input prices, namely \( w^*_h = \frac{u_l}{4} \) and \( w^*_l = \frac{u_l^2}{4u_h} \). The joint concavity of the profits relative to the input prices requires the evaluation of the leading principal minors of the following matrix

\[
\mathcal{H} = \begin{bmatrix}
\frac{\partial^2(C_h(\cdot) + C_l(\cdot))}{\partial w_h^2} & \frac{\partial^2(C_h(\cdot) + C_l(\cdot))}{\partial w_l \partial w_h} \\
\frac{\partial^2(C_h(\cdot) + C_l(\cdot))}{\partial w_l \partial w_h} & \frac{\partial^2(C_h(\cdot) + C_l(\cdot))}{\partial w_l^2}
\end{bmatrix} = \begin{bmatrix}
-\frac{4u_h(2u_h - u_l)}{(u_h - u_l)(4u_h - u_l)^2} & \frac{2u_h}{4u_h^2 - 5u_h u_l + u_l^2} \\
\frac{2u_h}{4u_h^2 - 5u_h u_l + u_l^2} & -\frac{4u_h^2 - 5u_h u_l + u_l^2}{u_l(u_h - u_l)(4u_h - u_l)^2}
\end{bmatrix}.
\]

Direct inspection reveals that the first-order leading principal minors are negative, whereas the determinant of the matrix, namely

\[
\det[\mathcal{H}] = \frac{4u_h^2(16u_h^3 - 32u_h^2 u_l + 12u_h u_l^2 - u_l^4)}{u_l(u_h - u_l)^2(4u_h - u_l)^4}
\]

is positive, in the admissible parameter range \( 0 < u_l < u_h \) for \( u_h > 1.53908u_l \approx 1.54u_l \).
2 Derivation of \((T_H^N, T_I^N)\)

Consider the first order conditions on the logarithm of \(NP_h(\cdot)\) with respect to \(t_h\) and \(w_h\).

\[
\frac{\partial \log[NP_h(\cdot)]}{\partial t_h} = -\Delta(u_h - u_i)(w_i - u_i)\left(\frac{2t_h A - w_i B + 2w_h \Gamma + w_i (u_i - 3u_h)}{2t_h A - w_i B + 2w_h \Gamma + w_i (u_i - 3u_h)}\right) = 0, \quad (11)
\]

\[
\frac{\partial \log[NP_h(\cdot)]}{\partial w_h} = \frac{4t_h \Delta(u_h - u_i)[(2u_h - u_i)A + w_h (w_i - 4w_h + 2u_i - u_i)]}{2t_h A - w_i B + 2w_h \Gamma + w_i (u_i - 3u_h)} = 0 \quad (12)
\]

where \(A \equiv (4u_h^2 - 5u_h u_i + u_i^2), B \equiv (2u_h^2 - 3u_h u_i - 4u_h w_h + u_i^2)\) and \(\Gamma \equiv [2u_h^2 - 2u_h (u_i + w_h) + u_i w_h], \Delta \equiv u_h - u_i, E \equiv [2u_h^2 + u_h [w_i - 2(u_i + w_h)] + u_i w_h]\).

The solution of (11) with respect to \(t_h\) is

\[
t_h(w_h, w_i) = \frac{(4\Delta)(w_i B - 2w_h \Gamma - w_i^2 (u_i - 3u_h)) - \mu}{2\Delta(u_i - 4u_h)^2} = 0, \quad (13)
\]

This can be plugged back into (12) which, after simplification, reduces to

\[
\frac{4u_h}{w_i^2} \left[ - (10u_h^2 - 7u_h u_i + u_i^2) \right] + w_i [u_h^2 (6u_i + 8w_h) - 7u_h u_i^2 + u_i^2] + 4u_h (\Delta + w_h)^\Gamma = 0, \quad (14)
\]

its solution with respect to \(w_h\) is

\[
w_h(w_h) = \frac{u_h u_i + 2u_h w_i - u_i^2}{2(2w_h - u_i)}. \quad (15)
\]
Now consider the first order conditions of the logarithm of $NP_t^N(\cdot)$ relative to $t_l$ and $w_l$.

\[
\frac{\partial \log [NP_t^N(\cdot)]}{\partial t_l} = -\frac{\mu}{2} \left[ u_l \Delta (4u_h - u_l) - u_l^2 \left[ 2u_h \Delta (u_l - u_l) - u_l^2 \right] \right] = 0, \quad (16)
\]

\[
\frac{\partial \log [NP_t^N(\cdot)]}{\partial w_l} = 2u_h \left[ 4u_h^2 u_l \left[ u_l^2 - u_h (u_l + 3w_h) + u_l w_h \right] + 2u_h^2 u_l w_h (\Delta + 2w_h) - 2u_h^2 w_h^2 (2u_h - u_l) \right], \quad (17)
\]

where $Z \equiv \left\{ u_l^2 w_h \left[ u_l^2 - u_h (u_l + 3w_h) + u_l w_h \right] + 2u_h^2 u_l w_h (\Delta + 2w_h) - 2u_h^2 w_h^2 (2u_h - u_l) \right\}$,

$H \equiv \left\{ 4u_h^2 [u_l - 2(\mu + 1) w_l] + u_l u_h [(\mu - 6) u_l + 4(\mu + 1) w_h + 4(\mu + 2) w_l] - u_l^2 [(\mu - 2) u_l + 2(w_h + w_l)] \right\}$,

$I \equiv \left[ u_h (u_l - 2w_l) + u_l (-u_l + w_h + w_l) \right]$, $K \equiv -2 \mu u_l^2 + (5\mu - 2) u_l w_h + 3(\mu + 2) u_l w_l - 8w_l (w_h + w_l)$.

The solution of (16) with respect to $t_l$ is

\[
t_l (w_h, w_l) = \frac{2u_h^2 (u_l - 2w_l) [(\mu - 2) u_l - 2(\mu + 1) w_l] - 2u_h^2 u_l \Lambda + u_l^2 \left\{ 2 \mu u_l^2 - u_l [(\mu - 5) w_h + 2(\mu + 1) w_l] + 2(6 - 5\mu) w_h^2 + 2w_h w_l + w_l^2 \right\} - (1 - \mu) u_h u_l^2 w_h (u_l + 7w_h) + (1 - \mu) u_l^2 w_l^2}{2u_h \Delta (4u_h - u_l)^2}.
\]

(18)

where $\Lambda \equiv \left\{ 2 \mu u_l^2 - u_l [(4\mu - 2) w_h + (\mu + 5) w_l] + 2w_l (2(\mu - 2) w_h - (3 - \mu) w_l) \right\}$. This can be plugged back into (17), which, after simplification, writes

\[
\frac{2u_h^2 \left\{ -8u_h^2 w_h + u_h u_l [u_l + 4(w_h + w_l)] \right\} - u_l^2}{4u_h^2 w_h^2 (u_l - 2u_h) + u_l^2 \left\{ -8w_h^2 + u_l (u_l - 5w_h) + u_l w_h \right\} - 2u_h^2 + 2u_h (u_l + w_l) - u_l w_h + 2u_h^2 u_l w_l (u_l \Delta + 4u_h w_h) = 0,
\]

(19)

whose solution with respect to $w_l$ is

\[
w_l (w_h) = \frac{u_h u_l^2 + 4u_h u_l w_h - u_l^2}{4u_h (2u_h - u_l)}.
\]

(20)

Solving the system defined by (15) and (20) returns

\[
w_h^N = \frac{u_l}{4}, \quad w_l^N = \frac{u_l^2}{4u_h}.
\]

(21)
The last step to obtain \((T_h^N, T_l^N)\) is to substitute (21) back into (13) and (18) and simplify.

As far as the second-order conditions are concerned, the Hessian matrices relative to the two maximizations, evaluated at the optimal contracts \((T_h^N, T_l^N)\) are:

\[
\mathcal{H}_h^N = \begin{bmatrix}
\frac{\partial^2 \log[N P_h^N(\cdot)]}{\partial w_h^2} & \frac{\partial^2 \log[N P_h^N(\cdot)]}{\partial w_h \partial t_h} \\
\frac{\partial^2 \log[N P_h^N(\cdot)]}{\partial t_h \partial w_h} & \frac{\partial^2 \log[N P_h^N(\cdot)]}{\partial t_h^2}
\end{bmatrix}_{(T_h^N, T_l^N)} = \begin{bmatrix}
\frac{128u_h}{\Delta(4u_h - u_l)} & \frac{1024u_h^4}{(2u_h - u_l)^3(2u_h + u_l)^2(1 - \mu)} \\
\frac{1024u_h^4}{(2u_h - u_l)^3(2u_h + u_l)^2(1 - \mu)} & \frac{512u_h^2(2u_h - u_l)}{(4u_h - u_l)(2u_h + u_l)^2(1 - \mu)}
\end{bmatrix}
\]

and

\[
\mathcal{H}_l^N = \begin{bmatrix}
\frac{\partial^2 \log[N P_l^N(\cdot)]}{\partial w_l^2} & \frac{\partial^2 \log[N P_l^N(\cdot)]}{\partial w_l \partial t_l} \\
\frac{\partial^2 \log[N P_l^N(\cdot)]}{\partial t_l \partial w_l} & \frac{\partial^2 \log[N P_l^N(\cdot)]}{\partial t_l^2}
\end{bmatrix}_{(T_h^N, T_l^N)} = \begin{bmatrix}
\frac{128u_l^2}{\Delta(4u_h - u_l)} & \frac{512u_l^2(2u_h - u_l)}{(4u_h - u_l)(2u_h + u_l)^2(1 - \mu)} \\
\frac{512u_l^2(2u_h - u_l)}{(4u_h - u_l)(2u_h + u_l)^2(1 - \mu)} & \frac{131072u_l^7}{(1 - \mu)\Delta(2u_h - u_l)^3(4u_h - u_l)^2(2u_h + u_l)^3}
\end{bmatrix}
\]

It is a matter of simple calculations to ascertain that first-order principal principal minors at \((T_h^N, T_l^N)\) of \(\mathcal{H}_h\) and \(\mathcal{H}_l\) are negative for all \(u_h > u_l > 0\) and \(0 < \mu < 1\), and that the determinant at \((T_h^N, T_l^N)\) of the matrices

\[
det[\mathcal{H}_h]_{(T_h^N, T_l^N)} = \frac{131072u_h^{7}}{(1 - \mu)\Delta(2u_h - u_l)^3(4u_h - u_l)^2(2u_h + u_l)^3}
\]

and

\[
det[\mathcal{H}_l]_{(T_h^N, T_l^N)} = \frac{131072u_l^{7}(2u_h - u_l)}{(1 - \mu)\mu u_l^4\Delta(4u_h - u_l)(2u_h + u_l)^3}
\]
are positive instead, ensuring the local concavity of the Nash Products, which, together with the uniqueness of the solution, guarantees its optimality.

3 Derivation of \((T^C_h, T^C_l)\)

Consider the first order conditions on the logarithm of \(NP^N_h(\cdot)\) with respect to \(t_h\) and \(w_h\).

\[
\frac{\partial \log[NP^C_h(\cdot)]}{\partial t_h} = \frac{A\left\{4\Delta_{\mu\Delta} \left[4(\Delta_{4u_h - 4w_h}(u_h - 2w_h)) - (2u_h - u_l)M\right] + 2M\left[4(1-\mu)\Delta_{\mu\Delta} \left[-8u_h^2 + 14u_h^2 u_l - 7u_h w^2 + u_l^2\right] + 4(2u_h - u_l)\Xi + \mu\Delta_{\mu\Delta} - \mu^2 u_l^2 \Delta_{4u_h - 4w_h}(u_h - 4u_h)\right]\right\}}{[t_h \Delta_{4u_h - 4w_h}(u_h - 4u_h) - \mu^2 u_l^2 \Delta_{4u_h - 4w_h}(u_h - 4u_h)]} = 0, \tag{26}
\]

\[
\frac{\partial \log[NP^C_h(\cdot)]}{\partial w_h} = \frac{-8t_h u_l A\left\{\mu u_h(u_h^2 - 2w_h) - u_h(u_h - 4w_h) + u_h w^2(u_h - 2u_l)\right\} - \mu^2 u_l^2 (4u_h - u_l)\Delta}{[t_h \Delta_{4u_h - 4w_h}(u_h - 4u_h) - \mu^2 u_l^2 \Delta_{4u_h - 4w_h}(u_h - 4u_h)\Delta]} = 0 \tag{27}
\]

where \(M \equiv \{2u_h^2 + u_h(w_l - 2(u_l + w_h)) + u_l w_h\}\), \(N \equiv 4\left\{u_l \left(t_h + t_l\right) - w_h^2(2u_h - u_l) + 2u_h w_l\Delta\right\} + u_h^2 w_l(u_h - u_l + 2w_h) + u_l w_l^2(u_l - 2u_h)\} - \mu^2 u_l^2 (4u_h - u_l)\Delta\), \(\Xi \equiv \left[u_l w_l \left(-2u_h^2 + 2u_h(u_l + w_h) - u_l w_h\right) - u_h u_l w_h(u_h - u_l + 2w_h) + u_h w_l^2(2u_h - u_l)\right]\), \(O \equiv \left[u_l \left(-\Delta \left[4t_l(4u_h - u_l)^2 + 16u_h^3 - 8u_h^2 u_l^2 + u_l^3\right] + 8u_h u_l^2(2u_h - u_l) - 8u_h u_l w_l\Delta\right] - 4u_h u_l w_l \left(8u_h^2 - 9u_h u_l + 4u_h w_h + u_l^2\right) + 4u_h w_l^2 \left(8u_h^2 - 7u_h u_l + u_l^2\right)\}\), \(R \equiv \left[u_l \left(8u_h^3 - 8u_h^2 w_h + 2u_h u_l(2w_h - 3u_l) + u_l^3\right) - 4u_h w_l^2 \left(4u_h - u_l\right) + u_l w_l^2 (5u_h - u_l)\right]\).

The solution of (26) with respect to \(t_h\) is

\[
t_h(u_h, w_l, t_l) = \frac{-4(4u_h - u_l)\left\{t_l u_l A + u_l u_l w_l(u_h - u_l + 2w_h) - u_h w_l^2(2u_h - u_l)\right\} + \mu \left\{u_l \left(D\Sigma - 8u_h^2 w_l^2(2u_h - u_l) + 8u_h u_l w_l\Delta\right) + 4u_h u_l w_l \left(8u_h^2 - 9u_h u_l + 4u_h w_h + u_l^2\right) - 4u_h w_l^2 Y\right\} - \mu^2 u_l^2 \Delta_{4u_h - 4w_h}(u_h - 4u_h)^2}{4u_l \Delta_{4u_h - 4w_h}(u_h - 4u_h)^2 Y - \mu^2 u_l^2 \Delta_{4u_h - 4w_h}(u_h - 4u_h)^2} = 0, \tag{28}
\]

where \(\Sigma \equiv \left(4t_l(u_l - 4u_h)^2 + 16u_h^3 - 8u_h^2 u_l^2 + u_l^3\right), Y \equiv \left(8u_h^2 - 7u_h u_l + u_l^2\right)\). This can be plugged back into (27), which, after simplification, writes

\[
\frac{8u_h u_l \left(u_l - 4u_h\right) - u_h(u_l - 2w_h)}{4t_l u_l \Delta_{4u_h - 4w_h}(u_h - 4u_h)^2 + 4u_l \left\{u_l w_l \left(8u_h^2 - 9u_h u_l + 4u_h w_h + u_l^2\right) - w_l^2 Y + 2u_l \left(D + w_l\right) \left(2u_h - u_l\right) + u_l w_l\right\} - \mu^2 u_l^2 \Delta_{4u_h - 4w_h}(u_h - 4u_h)^2} = 0. \tag{29}
\]
Its solution with respect to $w_h$ is
\[
w_h(w_l) = \frac{u_h u_l + 2u_h w_l - u_l^2}{2(2u_h - u_l)}.
\] (30)

Move now on the first order conditions of the logarithm of $NP_{t_1}^C(\cdot)$ relative to $t_l$ and $w_l$.

\[
\frac{\partial \log [NP_{t_l}^C(\cdot)]}{\partial t_l} = -u_A \left\{ -4(1-\mu)t_h u_l \Delta (4u_h - u_l)^2 - 4u_h u_l - 4u_h w_l \Delta \Phi + u_l \left\{ -8u_h^2 \Delta (8u_h - u_l^2) + 2u_h w_l (u_h - u_l) - 6u_h^2 (u_h - u_l) \right\} - \Delta (2u_h - u_l) + 4u_h w_l (u_h - u_l) \right\} = 0, \tag{31}
\]

\[
\frac{\partial \log [NP_{t_l}^C(\cdot)]}{\partial w_l} = 2u_h \left\{ 4(1-\mu)t_h u_l \left\{ -8u_h^2 \Delta (8u_h - u_l^2) - 8u_h w_l (u_h - u_l) \right\} + 2u_h AX + I \right\} = 0, \tag{32}
\]

where $\Phi \equiv \{ t_l u_l A + u_h w_h \Gamma + u_h u_l w_l (\Delta + 2w_h) - u_l w_l^2 (2u_h - u_l) \}$, $X \equiv \{ \mu \left[ -8u_h^2 w_l + u_h u_l (u_l + (w_h + w_l)) \right] - w_l^3 \} + 2(2u_h - u_l) I$, $\Psi \equiv \{ u_l w_h \Gamma + u_h u_l w_l (\Delta + 2w_h) - u_h w_l^2 (2u_h - u_l) \}$, $\Omega \equiv \{ -8u_h^3 + 4u_h^2 (u_l - 4w_h + w_l) + u_h \left\{ u_l^2 + 2u_l w_h - 2u_l w_l + 16w_h^2 \right\} \}$.

The solution of (16) with respect to $t_l$ is
\[
t_l(w_h, w_l) = \frac{(1-\mu) \left\{ 16u_h^3 u_l (4u_h - u_l)^2 + 4u_h^2 \Delta u_h \Phi + u_h w_l \left\{ \frac{\mu (\mu + 2) u_l^2 - 4u_l (2w_h + \mu w_l + w_l) + 4 \left[ (\mu - 6) w_h^2 + 16w_h + w_l \right] \} + 4(1-\mu) w_l^3 \right\} \right\}}{4u_h \left\{ (u_h - u_l)^2 \right\}} \tag{33}
\]

where $F \equiv \{ \mu (6\mu - 5) w_l^2 - 8(1-\mu) u_l w_h - 4w_l w_h + 4(2-\mu) w_l^2 \}$ and $G \equiv \{ \mu (9-\mu) w_l^2 + 4w_l \left[ (10 - \mu) w_h + (\mu + 5) w_l \right] + 8 \left[ 4(1-\mu) w_h^2 - 2(2-\mu) w_h w_l + (3 - \mu) w_l^2 \right] \}$. As before, this can be plugged into (32) to obtain, after simplifying
\[
\frac{4u_h \left\{ 8u_h^2 w_l - 4u_h \left[ (4w_l + w_h) \right] + u_l^2 \right\} \right\}}{4u_l \left\{ -\Delta (4u_h - u_l)^2 + u_h \left[ 8u_h^2 - 7u_h u_l + u_l^2 \right] - 8u_h w_l \Delta \right\} + 2u_h u_l \left[ u_l^2 - u_h (u_l + 4w_h) \right] + \mu u_h u_l \Delta (4u_h - u_l)^2} = 0, \tag{34}
\]

its solution with respect to $w_l$ is
\[
w_l(w_h) = \frac{u_h u_l^2 + 4u_h u_l w_h - u_l^3}{4u_h (2u_h - u_l)} \tag{35}
\]
The solution of the system defined by (30) and (35) is

\[ w^C_h = \frac{u_t}{4}, \quad w^C_l = \frac{u_l^2}{4u_h}. \] (36)

The last step to obtain \((T^C_h, T^C_l)\) requires, as above, to substitute (36) back into (28) and (33) and simplify.

Let us now move to the second-order conditions as before, the Hessian matrices evaluated at the optimal contracts are the following.

\[
H^C_h = \begin{bmatrix}
\frac{\partial^2 \log[N P^C_h(\cdot)]}{\partial w_h^2} & \frac{\partial^2 \log[N P^C_h(\cdot)]}{\partial w_h \partial t_h} \\
\frac{\partial^2 \log[N P^C_h(\cdot)]}{\partial t_h \partial w_h} & \frac{\partial^2 \log[N P^C_h(\cdot)]}{\partial t_h^2}
\end{bmatrix}_{(T^C_h, T^C_h)} = \begin{bmatrix}
-64(2u_h - u_l)(2 - \mu)(4(2 - \mu)u_t^2 + u_h(\mu + 3)(5u_h - 8)u_t - 4u_l^2(2 - \mu)^2(1 + \mu)) \\
\Delta(u_h - 4u_l^2(5u_l + 4u_h(2 - \mu))^2(1 - \mu)\mu
\end{bmatrix}
\]

and

\[
H^C_l = \begin{bmatrix}
\frac{\partial^2 \log[N P^C_l(\cdot)]}{\partial w_l^2} & \frac{\partial^2 \log[N P^C_l(\cdot)]}{\partial w_l \partial t_l} \\
\frac{\partial^2 \log[N P^C_l(\cdot)]}{\partial t_l \partial w_l} & \frac{\partial^2 \log[N P^C_l(\cdot)]}{\partial t_l^2}
\end{bmatrix}_{(T^C_l, T^C_l)} = \begin{bmatrix}
-64(2u_h - u_l)(2 - \mu)[4u_t^3 - 7u_t^2 + u_t(3u_l - 2\mu)u_h + u_l^2(2 - \mu)] \\
-\Delta(4u_h - u_l)^2[3(3 - 4\mu)^2(1 - \mu)\mu
\end{bmatrix}
\]

Their determinants are, respectively

\[
\det[H^C_h_{(T^C_h, T^C_l)}] = -\frac{16384(2 - \mu)^3u_h(u_h - u_l)}{(1 - \mu)(u_h - u_l)(4u_h - u_l)^2[5u_l - 4(2 - \mu)u_h]^3}
\] (37)
and

$$\det[H_i|_{T^H_i,T^C_i}] = -\frac{16384(2-\mu)^3 u_0^2(2u_h - u_l)}{(1-\mu)(4\mu - 3)^3 u_1^4 (u_h - u_l)(4u_h - u_l)^2}$$

(40)

Inspection reveals that a necessary and sufficient condition for (40) to be positive is that $0 < \mu < \frac{3}{4}$. If this condition is met, (39) is positive as well and the first-order principal minors of (37) and (38) are negative, which guarantee concavity of the two Nash products. When $\frac{3}{4} < \mu < 1$ the bargaining power distribution is such that firm $D_l$ suffers losses at the contract described above. Non-exclusive contingent contract can be constructed by imposing that the fixed fee $t_l$ is set so as to satisfy firm $D_l$'s participation constraint with equality. This amounts to solving the following program (for the sake of readability, we are not going to introduce further notation).

$$\max_{u_h,t_h} NP^C_h(T_h, T^C_i), \quad \max_{u_i} [\tilde{\Pi}(T^C_i, T_l) + \tilde{\pi}_l(T_l, w^C_h)], \quad \text{and} \quad \tilde{\pi}_l(T_l, w^C_h) \leq 0.$$ 

(41)

It is clear that the first-order conditions $NP^C_h(\cdot)$ w.r.t $t_h$ and $u_h$ coincide with those above analyzed. The last two are, instead

$$w_l(u_h) = \frac{u_l^2(\Delta + w_h)}{4u_h(2u_h - u_l)}, \quad t_l(w_h, u_l) = \frac{u_h[u_h(u_l - 2w_l) + u_l(4u_h + w_l - u_l)]^2}{u_l \Delta(4u_h - u_l)^2}.$$ 

(42)

By solving the system defined by this set of equations one obtains the optimal contracts in this case, which write

$$T^C_h = (w^C_h, t^C_h) = \left(\frac{u_l}{4}, \frac{4u_h u_l + u_l(-3 + (3-4\mu)\mu)}{16}\right),$$

$$T^C_l = (w^C_l, t^C_l) = \left(\frac{u_l^2}{4u_h}, \frac{u_l(u_h - u_l)}{16u_h}\right).$$

(43)  (44)

In the appendix 1 of the paper we show that the profit to firm $U$ under (43) and (44) is lower than that with an exclusive contract with firm $D_h$. 
4 Derivation of \((T^M_h, T^M_I)\)

The first order conditions on the logarithm of \(NP^N_h(\cdot)\) with respect to \(t_h\) and \(w_h\) are as follows

\[
\frac{\partial \log [NP^N_h(\cdot)]}{\partial t_h} = A \left\{ 4t_hw_1 \Delta (4u_h - u_l)^2 + 4(4u_h - u_l) \left[ \{t_hu_1 + u_lw_1, \Gamma + u_hu_1w_1(4u_h - u_l) + 2w_h\}^2 (u_l - 2u_h)\right] + \mu O + \mu^2 u_h^2 \Delta (u_l - 4u_h)^2 \right\} = 0, \tag{45}
\]

\[
\frac{\partial \log [NP^N_h(\cdot)]}{\partial w_h} = -8t_hw_1A\left[ \mu u_h(u_l(u_l - 2w_h) - u_h(u_l - 4w_h + 2w_l) - 2(u_h - u_l) - M - 2M \left[ 4(1 - \mu) t_h u_1 \left( -8u_h^2 + 14u_h w_l - 7u_h^2 + w_l^2 \right) + 4(2u_h - u_l) \Xi + \mu \Delta P - \mu^2 u_h^2 \Delta (2u_h - u_l)(4u_h - u_l) \right] \right] = 0 \tag{46}
\]

The solution of (45) with respect to \(t_h\) is

\[
t_h(w_h, w_l, t_l) = \frac{4(4u_h - u_l) \left[ t_h \{ A + u_lw_h \Gamma + u_hu_1w_1(4u_h - u_l - 2w_h) - u_hw_1^2 (2u_h - u_l) \} + \mu \left\{ u_l \Delta \Sigma - 8u_h w_l^2 (2u_h - u_l) - 8u_hu_1w_h \Delta \right\} \right] + 4u_hu_1w_1 \left( 8u_h^2 - 9u_hu_l + 4u_h w_h + w_h^2 \right) - 4u_hw_h^2 Y }{ 4u_1 \Delta (4u_h - u_l)^2 }.
\]

This can be plugged back into (46), which, after simplification, writes

\[
\frac{8u_hu_1u_h(u_l - 4w_h + 2u_l) - u_h(u_l - 2w_h))}{4t_hu_1 \Delta (4u_h - u_l)^2 + 4u_h \left\{ u_l w_1 \left( 8u_h^2 - 9u_hu_l + 4u_hw_h + w_l^2 \right) - w_l^2 Y + 2u_l \{ \Delta + w_h \left[ 2u_h^2 - 2u_h(4u_h - u_l) \right] \} - \mu u_h^2 \Delta (4u_h - u_l)^2 \right\} \right] = 0. \tag{48}
\]

Its solution with respect to \(w_h\) is

\[
w_h(w_l) = \frac{u_hu_1 + 2u_hw_l - w_l^2}{2(2u_h - u_l)}. \tag{49}
\]

Notice here that (45) and (46) coincide with (26) and (27) respectively, therefore also (47) coincides with (28) and (49) to (30).

Consider now the set of first order conditions in the negotiation for \(T^M_i\).
\[ \frac{\partial \log [NP^M_h]}{\partial u} = -\frac{u_i \Delta (4u_h - u_i) \{ \mu \{ -4u_h^3w^7(2u_h - u_i) - w_i^2 + u_h(5w_h - w_i) - u_iw_h \} - 2u_h^2 + 2u_h(u_i + w_h) - u_iw_h \} + 2u_h^2w_h^2(u_i \Delta + 4u_h w_h) \} - (4u_h - u_i)(2u_iu_hw_i) + Z \} }{2(u_iu_hw_i + Z) \} = 0, \] (50)

\[ \frac{\partial \log [NP^M_h]}{\partial w} = 2u_h \{ t_{1i}u_hu_hA - I \{ u_h^4(u_i - u_i)(\mu u_i - 4w_i) + u_h^2K + u_h^2w_i^2 \} - (1 - \mu)u_h w_h \} \] (51)

The solution of (50) with respect to \( I_i \) is

\[ I_i(w_h, w_i) = \frac{2u_h^4(u_i - u_i)[\mu u_i - 2(2 - \mu)w_i] - 2u_h^4u_h + u_h^2w_i^2 \} \] (52)

This can be plugged back into (51), which, after simplification, writes

\[ \frac{2u_h^2 \{ (9\mu - 5)w_i + 2(\mu + 1)w_i + 2 \} \} - (1 - \mu)u_h^3w_i(17w_h) + (1 - \mu)u_i^2w_h^2 }{2u_h^2w_i(14u_iw_i - u_i^3) } = 0, \] (53)

whose solution with respect to \( w_i \) is

\[ w_i(w_h) = \frac{u_iu_i^2 + 4u_hu_iw_h - u_i^3 }{4u_h (2u_h - u_i) } \] (54)

In the case, (50) and (51) coincide with (16) and (17), whence (52) coincides with (18) and (54) with (20). Solving the system defined by (49) and (54) returns

\[ w_h^M = \frac{u_i}{4}, \quad w_i^M = \frac{u_i^2}{4u_h} \] (55)

As above, substitution back of (55) into (47) and (52) and simplification yields the optimal contract. Let us now consider the second-order conditions. The Hessian matrices evaluated at \( (T_h, T_i^M) \) are
Their determinants are
\[
\det[H_h|_{T_{MN}^M, T_{L}^L}] = \frac{131072u_2^4(2u_h - u_1)}{(1 - \mu)(u_h - u_1)(4u_h - u_1)^2(8u_2^3 - 2(3\mu + 2)u_2u_h - (1 - \mu)u_1)^3},
\]
and
\[
\det[H_l|_{T_{MN}^M, T_{L}^L}] = \frac{131072u_2^4(2u_h - u_1)}{(1 - \mu)u_1^4(4u_h - u_1)^2(2u_h + u_1)^3},
\]
It is easily checked that, while the first-order principal minors of $H_l|_{T_{MN}^M, T_{L}^L}$ are negative and its determinant is positive for all $u_h > u_l > 0$ and $0 < \mu < 1$, this is not always the case for $H_h|_{T_{MN}^M, T_{L}^L}$. However, it is a matter of calculations to ascertain that its determinant is positive for $u_l < u_h < \frac{5}{4}u_l$ and $\mu < \frac{8u_2^3 - 4u_2u_h - u_h^2}{u_1(6u_h - u_1)} < 1$ or $u_h > \frac{3}{4}u_l$ and $0 < \mu < 1$ and that, under either of these conditions, its first-order principal minors are indeed negative. In
the remaining parametric constellation $u_l < u_h < \frac{5}{4} u_l$ and $\frac{8u_h^2 - 4u_h u_l - u_l^2}{u_l(6u_h - u_l)} < \mu < 1$ firm $D_h$ earns negative profits, yet, this contract configuration can be constructed by imposing that $t_h$ satisfies with equality the participation constraint of this firm. The new maximization program is thus

$$\max_{w_l, t_l} NP^M_l(T_h^M, T_l^M), \quad \max_{w_h} [\hat{\Pi}(T_h, T_l^M) + \hat{\pi}_h(T_h, w_l^M)], \quad \text{and } \hat{\pi}_h(T_h, w_l^M) \leq 0. \tag{60}$$

The first-order conditions on $NP^M_l(\cdot)$ coincide with those above analyzed, the remaining two are

$$w_h(w_l) = \frac{u_h u_l + 2u_h w_l - u_l^2}{4u_h - 2u_l}, \quad t_h(w,h,w_l) = \frac{\{2u_h^2 + u_h[w_l - 2(u_l + w_h)] + u_l w_h\}^2}{\Delta(4u_h - u_l)^2}. \tag{61}$$

The solution to this set of FOCs is

$$T_h^M = (w_h^M, t_h^M) = \left(\frac{u_l}{4}, \frac{u_h - u_l}{4}\right), \tag{62}$$

$$T_l^M = (w_l^M, t_l^M) = \left(\frac{u_l^2}{4u_h}, \frac{u_l[2\mu u_h - (3 - \mu)u_l]}{32u_h}\right). \tag{63}$$

As pointed out in the paper, the profit to firm $U$ under these contracts is lesser than that obtained with an exclusive contract with firm $D_h$.

5 Derivation of $(T_h^Z, T_l^Z)$

This mixed case as well is a combination of cases $C$ and $N$. The first-order conditions of $NP^Z_h$ w.r.t. $t_h$ and $w_h$ are
\[
\frac{\partial \log[N P^2_{\Omega}(\cdot)]}{\partial w_h} = - \frac{\Delta(4u_h - u_l)(u_l - 4u_h)[2u_hA - u_lB + 2w_h \Gamma + w_l^2(u_l - 3u_h)] + \mu \{w_l^2[-(10u_h^2 - 7u_hu_l + u_l^2)] + w_l \{w_h^2(6u_h + 8w_h) - 7u_hu_l^2 + u_l^2\} + 4u_h(u_h - u_l)\Gamma\}}{2w_h(\Delta - 4u_h)^2 - E^2} = 0, \quad (64) \\
\frac{\partial \log[N P^2_{\Omega}(\cdot)]}{\partial w_l} = \frac{4u_h\Delta(4u_h - u_l)(u_l - 4u_h)[2u_hA - u_lB + 2w_h \Gamma + w_l^2(u_l - 3u_h)] - 2E\{\mu \Delta \{4u_h^4 + 2u_h^2[u_l - 2(u_l + w_l)] + 2u_h(u_lw_h + 2u_lw_l - 2w_l^2) + u_lw_l(u_l - u_l)\} - (2u_h - u_l)(-w_lB + 2w_h \Gamma + w_l^2(u_l - 3u_h))\}}{2w_h(\Delta - 4u_h)^2 - E^2} = 0, \quad (65)
\]

The solution of (64) w.r.t. \( t_h \) is
\[
t_h(w_h, w_l) = \frac{(4\Delta)(u_l - 2u_h\Gamma - w_l^2(u_l - 3u_h)) - \mu \{ -w_l^2(10u_h^2 - 7u_hu_l + u_l^2) + w_l \{w_h^2(6u_h + 8w_h) - 7u_hu_l^2 + u_l^2\} + 4u_h(\Delta - w_h)\Gamma\}}{2w_h(\Delta - 4u_h)^2}.
\quad (66)
\]

This can be plugged back into (65) which, after simplification, reduces to
\[
\frac{4u_h[u_h(u_l - 4u_h + 2w_l) - u_l(u_l - 2w_l)]}{w_l^2[-(10u_h^2 - 7u_hu_l + u_l^2)] + w_l \{u_h^2(6u_l + 8w_h) - 7u_hu_l^2 + u_l^2\} + 4u_h(\Delta + w_h)\Gamma} = 0, \quad (67)
\]
whose solution with respect to \( w_h \) is
\[
w_h(w_h) = \frac{u_hu_l + 2u_hw_l - u_l^2}{2(2u_h - u_l)}. \quad (68)
\]

Equations (64) and (65) coincide with (11) and (12) respectively, therefore (66) coincides with (13) and (68) with (15).
Let us now consider the first-order conditions on the logarithm of to the

\[
\frac{\partial \log[NP^2(\cdot)]}{\partial t} = - w_i A \left\{ -4(1-\mu)u_i \Delta (4u_{\text{h}} - u_i)^2 - 4(4u_{\text{h}} - u_i)\Phi + \mu \left\{ u_i \left\{ -4u^2_i Y - u_i \Delta \left[ 32u_{\text{h}} w_{\text{h}} - 16u^2_i + 4u_{\text{h}} u_i + 3u^2_{\text{h}} \right]\right\} - 8u^2_i w_{\text{h}} (2u_{\text{h}} - u_i) + 4u_{\text{h}} u_i w_{\text{h}} (u_i \Delta + u_{\text{h}} w_{\text{h}}) \right\} - \mu^2 u_i u_{\text{h}} \Delta (4u_{\text{h}} - u_i)^2 \right\} = 0, (69)
\]

\[
\frac{\partial \log[NP^2(\cdot)]}{\partial w_{\text{h}}} = 2u_i \left\{ [4(1-\mu)u_i \left\{ -8u^2_i + 14u^2_i u_i - 7u_{\text{h}} u_{\text{h}}^2 + w_{\text{h}} \right\}] (w_{\text{h}} (2u_{\text{h}} - u_i) - u_i (u_{\text{h}} - u_{\text{i}} + w_{\text{h}})) + 2u_{\text{h}} AX + I \left\{ [4(2u_{\text{h}} - u_i) \Phi + \mu u_{\text{h}} \Delta \Omega + \mu^2 u_i u_{\text{h}} \Delta (2u_{\text{h}} - u_i) (4u_{\text{h}} - u_i)] \right\} \right\} = 0, (70)
\]

The solution of (69) with respect to \(t_i\) is

\[
l_i(w_h, w_i) = \frac{(1-\mu) \left\{ 16 \mu u_i^4 (u_i - 4u_{\text{h}} u_i \Delta (4u_{\text{h}} - u_i)^2) + 4u_i F + u_i G + u_i u_i \{ \mu (\mu + 3) u^2_i - 4u_i (2\mu w_{\text{h}} + \mu w_{\text{h}} + w_{\text{h}}) + 4 [(7\mu - 6) w_{\text{h}}^2 + 2w_{\text{h}} w_{\text{h}} + w_{\text{h}}^2] \} + (1-\mu) u^2_i w_{\text{h}}^2 \right\}}{4u_i \Delta (4u_{\text{h}} - u_i)^2}
\]

(71)

As before, this can be plugged into (70) to obtain, after simplifying

\[
\frac{4u_i \left\{ 8u^2_i w_{\text{h}} - u_i u_i [u_i + 4(w_{\text{h}} + w_i)] + w_{\text{h}}^2 \right\}}{4u_i \left\{ -\Delta [4(4u_{\text{h}} - u_i)^2 + u_i u_i \Delta] + w_{\text{h}} (8u^2_i - 7u_{\text{h}} u_i + u_{\text{h}}^2) - 8u^2_i u_{\text{h}} \Delta + 8u^2_i (2u_{\text{h}} - u_i) + 4u_{\text{h}} u_i u_i [u_i^2 - u_i (u_i + 4w_{\text{h}})] + \mu u_{\text{h}} u_i \Delta (4u_{\text{h}} - u_i)^2 \right\} = 0,
\]

(72)

its solution with respect to \(w_i\) is

\[
w_i(w_h) = \frac{u_i u_{\text{h}}^2 + 4u_i u_i w_{\text{h}} - u_{\text{h}}^2}{4u_i (2u_{\text{h}} - u_i)}.
\]

(73)

The solution of the system defined by (68) and (73) is

\[
w_{\text{h}}^c = \frac{u_i}{4}, \quad w_i^c = \frac{u_{\text{h}}^2}{4u_{\text{h}}}.
\]

(74)

The last step to obtain \((T_{\text{h}}^Z, T_{\text{i}}^Z)\) requires, as above, to substitute (74) back into (66) and (71) and simplify.
Let us now move to the second-order conditions as before, the Hessian matrices evaluated at the optimal contracts are the following.

\[
\begin{align*}
\mathcal{H}_n^Z &= \begin{bmatrix}
\frac{\partial^2 \log[N P^Z_n(\cdot)]}{\partial w^2_n} & \frac{\partial^2 \log[N P^Z_n(\cdot)]}{\partial w_n \partial h_n} \\
\frac{\partial^2 \log[N P^Z_n(\cdot)]}{\partial h_n \partial w_n} & \frac{\partial^2 \log[N P^Z_n(\cdot)]}{\partial h^2_n}
\end{bmatrix}
\bigg|_{(T_n^Z, T_n^Z)}
\end{align*}
\]

and

\[
\begin{align*}
\mathcal{H}_l^Z &= \begin{bmatrix}
\frac{\partial^2 \log[N P^Z_l(\cdot)]}{\partial w^2_l} & \frac{\partial^2 \log[N P^Z_l(\cdot)]}{\partial w_l \partial t_l} \\
\frac{\partial^2 \log[N P^Z_l(\cdot)]}{\partial t_l \partial w_l} & \frac{\partial^2 \log[N P^Z_l(\cdot)]}{\partial t^2_l}
\end{bmatrix}
\bigg|_{(T_l^Z, T_l^Z)}
\end{align*}
\]

heir determinants are

\[
\det[\mathcal{H}_h|_{(T_h^Z, T_h^Z)}] = \frac{131072 u_h}{(1 - \mu) \mu \Delta(2u_h - u_l)^3(2u_h - u_l)^3}
\]

and

\[
\det[\mathcal{H}_l|_{(T_l^Z, T_l^Z)}] = -\frac{131072 u_l^5 (2u_h - u_l)}{(1 - \mu) \mu u_l^4 \Delta(4u_h - u_l)^2 [(4u - 2) u_h^2 - 2(1 - \mu) u_h u_l + (1 - \mu) u_l^2]^3}
\]

Direct inspection reveals that while the first-order principal minors of \(H_h|_{(T_h^Z, T_h^Z)}\) are negative and its determinant positive for all \(u_h > u_l > 0\) and \(0 < \mu < 1\), this is not the case for \(H_l|_{(T_l^Z, T_l^Z)}\). Yet, calculations show that, in this case, a necessary and sufficient condition to guarantee concavity is \(\mu < \frac{2u_h^2 + 2u_h u_l - u_l^2}{2u_h^2 + 2u_h u_l - u_l^2}\). When this condition is not met, the low-quality downstream firm reaps a negative profit at the above contracts. In this case, the
optimization problem is modified such that \( t_l \) satisfies the participation constraint of firm \( D_l \):

\[
\max_{w, t_l} NP^Z_h(T_h, T_l^Z), \quad \max_{w_l} \hat{\Pi}(T_h^Z, T_l) + \hat{\pi}_l(w^Z_h, T_l), \quad \text{and} \quad \hat{\pi}_l(w^Z_h, T_l) \triangleq 0. \tag{79}
\]

The FOCs relative to \( NP^Z_h(\cdot) \) coincide with those above, the remaining two are

\[
w_l(w_h) \frac{u_h u_l^2 + 4 u_h u_l w_h - w_l^3}{8 u_h^2 - 4 u_h u_l}, \quad t_h(w_h, w_l) = \frac{u_h [u_h (u_l - 2 w_l) + u_l (-u_l + w_h + w_l)^2]}{u_l \Delta (u_l - 4 u_h)^2} \tag{80}
\]

The solution to (79) is

\[
T^Z_h = (w^Z_h, t^Z_h) = \left( \frac{u_l}{4}, \frac{8 \mu u_h^3 - 4 (1 + \mu) u_h^2 u_l + 2 (1 - \mu) u_h u_l^2 - (1 - \mu) u_l^3}{32 u_h^2} \right), \tag{81}
\]

\[
T^Z_l = (w^Z_l, t^Z_l) = \left( \frac{u_l^2}{4 u_h}, \frac{u_l (u_h - u_l)}{16 u_h} \right). \tag{82}
\]

Calculations show that the profit reaped by firm \( U \) under this contract falls short of that earned with an exclusive relationship with firm \( D_h \).
6 Behavior of Outside Options

Let us consider the total profits of firm $U$ (res. firms $D_h$ and $D_l$) at the bargaining stage as the sum of the profits from the sales of the input to the downstream firms (res. the profits from the sales of the variants of the good to the consumers) and of the fixed fees as in the following.\(^1\)

$$\hat{\Pi}(T_h, T_l) = \hat{V}(w_h, w_l) + t_h + t_l, \quad \hat{\pi}_h(T_h, w_l) = \hat{v}_h(w_h, w_l) - t_h, \quad \hat{\pi}_l(T_l, w_h) = \hat{v}_l(w_h, w_l) - t_l. \quad (83)$$

As pointed out in the paper, neither $\hat{V}(\cdot)$ nor $\hat{v}_i(\cdot)$, $i = h, l$ depend on the fixed fees $t_h$ and $t_l$. Let $T^*_i \equiv (w^*_i, t^*_i), i \in \{N, C, M\}$ be the sub-game equilibrium pre-contractual arrangement executed between firm $U$ and $D_i$, and $d^*_i$ the induced outside option for firm $U$ in the negotiation with firm $D_i$, $i = h, l, i \neq j$. The Nash products write

$$NP_i(T_i, T^*_j) = \left[\hat{V}(w_i, w^*_j) + t_i + t^*_j - d^*_i\right]^\mu [\hat{v}_i(w_i, w^*_j) - t_i]^{1-\mu} \quad (84)$$

It is here worth noticing that, within each $NP_i(\cdot)$, no outside option depends on the ongoing negotiation between firm $U$ and $D_i$, indeed, in the case of contingent contracts $d^*_C = \frac{w_{ij}^*}{\mu}$, in the case of non-contingent contracts $d^*_N = \hat{\Pi}_m(T^*_N) = \hat{V}_m(w^*_j) + t^*_N$ and the case of mixed contracts being a combination of case $N$ and $C$.\(^2\)

As is well known (see e.g. ?), the maximization of (84) with respect to $T_i, i = h, l$ can be split in two steps: first identify the $w_i$ that maximizes the joint surplus, then apportion the maximized surplus it according to the bargaining weights. The first-order conditions with respect to $t_i$ yield

$$t_i(w_i, w^*_j) = \mu \hat{v}_i(w_i, w^*_j) - (1 - \mu) \left[\hat{V}(w_i, w^*_j) + t^*_j - d^*_i\right], \quad i = h, l, i \neq j. \quad (85)$$

\(^1\)See the paper, eqs. (13) - (15).
\(^2\)See the paper, eq. (7)
which can be plugged back into (84) that, in turn, reduces to

\[ \mu^\mu (1 - \mu)^{1-\mu} \left[ \hat{V}(w_i, w_j^*) + \hat{v}(w_i, w_j^*) + t_j^* - d_i^* \right] \]  

(86)

It is straightforward to observe that choosing \( w_i \) to maximize (86) amounts to choosing the input price to maximize the sum \( \hat{V}(w_i, w_j^*) + v_i(w_i, w_j^*), i = h, l, i \neq j \), which, as observed above, does not depend on the transfers and thus on the type of non-exclusive pre-contractual arrangement. At the optimal input prices \( (w_i^*, w_j^*) \), equations (85) define the subgame equilibrium transfers \( (t_i^*, t_j^*) \).

6.1 Non-contingent contracts

Under non-contingent contracts \( d_i^N = \hat{V}_m(w_j^N) + t_j^N \), so that, in \( NP_i^N(\cdot) \), \( t_j^* \) cancels out, which implies that

\[ t_i^N = \mu \hat{v}_i(w_i^N, w_j^N) - (1 - \mu) \left[ \hat{V}(w_i^N, w_j^N) - \hat{V}_m(w_j^N) \right], \quad i = h, l, i \neq j. \]  

(87)

This last equation shows that, under non-contingent contracts, the fixed fees are independent one from the other. Furthermore, it is easy to ascertain that, as \( \mu \) tends to zero, the subgame equilibrium value of each outside option tends to

\[ \lim_{\mu \to 0} d_i^N = \hat{V}_m(w_j^N) + \hat{V}_m(w_i^N) - \hat{V}(w_i^N, w_j^N) \]  

(88)

which is positive by Lemmata 1 and 2 (see the paper).

6.2 Contingent contracts

Under contingent contracts, we have \( \lim_{\mu \to 0} d_i^C = 0 \).
6.3 Mixed contracts

In this case we have that $d^M_h = \mu \frac{u_h}{4}$ and $d^M_l = \hat{V}_m(w^M_h) + t^M_h$, whence in $NP^M_l(\cdot) t^M_h$ cancels out, while in $NP^M_h(\cdot) t^M_l$ does not. The subgame equilibrium transfers are

$$
t^M_h = \mu \{ (1 - \mu) \left[ \frac{u_l}{4} - V(w^M_h, w^M_l) - \hat{v}_l(w^M_h, w^M_l) \right] + \hat{v}_h(w^M_h, w^M_l) \} - (1 - \mu)^2 \hat{V}_m(w^M_h),
$$

$$
t^M_l = \mu \hat{v}_l(w^M_h, w^M_l) - (1 - \mu) \left[ \hat{V}(w^M_h, w^M_l) - \hat{V}_m(w^M_h) \right].
$$

(89)

It is a matter of simple calculations to observe that, in this case as well the values of the outside options for firm $U$ tend to zero as $\mu$ tends to zero:

$$
\lim_{\mu \to 0} d^M_i = 0, i = h, l.
$$

(90)