Contract contingency in vertically related markets

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Proofs not intended for publication

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In this appendix we provide

1. The conditions under which the joint profits of the firms are concave in the input prices (Section 1).

2. The "proofs available upon request" concerning the derivations of the optimal contracts (Sections 2 to 5).

3. An explanations of the mechanics linking (non-)contingency and the determination of the outside options (Section 6).

For the sake of readability, we have not introduced further notation unless strictly necessary to avoid confusion. The ensuing pages are landscape-oriented to contain the cumbersome formulas.

1 Concavity of the joint profits

Here we show the conditions under which the concavity of the joint profits of the two production channels in the input prices is obtained under passive beliefs. The profits accruing to the firms (gross of the the transfers $t_i, i = h, l$ which do not influence neither the input nor the retail prices) are

$$
\begin{aligned}
X_i(p_h, p_l) &= D_i(p_h, p_l)(p_i - w_i), \quad i = h, l, && (1) \\
Y(p_h, p_l) &= D_h(p_h, p_l)w_h + D_l(p_h, p_l)w_l. && (2)
\end{aligned}
$$

The optimal retail are prices

$$
\hat{p}_h(w_h, w_l) = \frac{u_h[2(u_h - u_l + w_h) + w_l]}{4u_h - u_l}, \quad \hat{p}_l(w_h, w_l) = \frac{u_l(u_h - u_l + w_h) + 2u_h w_l}{4u_h - u_l}. \tag{3}
$$

At these prices, the profits to the downstream firms under passive beliefs, gross of transfers, are

$$
\begin{aligned}
X_h(w_h, w_l^*) &= \frac{\left[2u_h^2 + u_h(w_l^* - 2(u_l + w_h)) + u_l w_h\right]^2}{(u_h - u_l)(4u_h - u_l)^2}, && (4) \\
X_l(w_h^*, w_l) &= \frac{u_h\left[u_h(u_l - 2w_l) + u_l(w_h^* + w_l - u_l)\right]^2}{u_l(u_h - u_l)(4u_h - u_l)^2}, && (5)
\end{aligned}
$$

where (w_h^*, w_l^*) are the candidate equilibrium input prices. The profits of each of the two channels under passive beliefs are thus

$$
\begin{aligned}
C_h(w_h, w_l) = D_h(\hat{p}_h(w_h, w_l), \hat{p}_l(w_h, w_l))w_h + X_h(w_h, w_l^*) = \\
= \frac{u_h\left\{4u_h^3 + 4u_h^2(w_l^* - 2u_l) + u_h\left[4u_l^2 - 4w_l^*(u_l + w_h) + 2u_l w_h - 4w_h(w_h - w_l) + (w_l^*)^2\right] - u_l w_h(2u_l - 2w_h + w_l - 2w_l^*)\right\}}{(u_h - u_l)(4u_h - u_l)^2}
\end{aligned} \tag{6}
$$

and

$$
\begin{aligned}
C_l(w_h, w_l) &= D_l(\hat{p}_h(w_h, w_l), \hat{p}_l(w_h, w_l))w_l + X_l(w_h^*, w_l) = \\
&= \frac{u_h\left[u_l^2(u_h-u_l+w_h^*)^2+u_h u_l w_l(u_l+4w_h-4w_h^*)-2u_h w_l^2(2u_h)-u_l^2 w_l(u_l+w_h-2w_h^*)\right]}{u_l(u_h-u_l)(4u_h-u_l)^2}
\end{aligned}
\tag{7}
$$

Under a contract equilibrium the solution to the system

$$
\begin{cases}
\dfrac{\partial C_h(\cdot)}{\partial w_h} = 0 \\
\dfrac{\partial C_l(\cdot)}{\partial w_l} = 0
\end{cases}
\tag{8}
$$

returns the candidate optimal input prices, namely $w_h^* = \frac{u_l}{4}$ and $w_l^* = \frac{u_l^2}{4u_h}$. The joint concavity of the profits relative to the input prices requires the evaluation of the leading principal minors of the following matrix

$$
\mathcal{H} = \begin{bmatrix} \dfrac{\partial^2(C_h(\cdot)+C_l(\cdot))}{\partial w_h^2} & \dfrac{\partial^2(C_h(\cdot)+C_l(\cdot))}{\partial w_h \partial w_l} \\ \dfrac{\partial^2(C_h(\cdot)+C_l(\cdot))}{\partial w_l \partial w_h} & \dfrac{\partial^2(C_h(\cdot)+C_l(\cdot))}{\partial w_l^2} \end{bmatrix} = \begin{bmatrix} -\frac{4u_h(2u_h-u_l)}{(u_h-u_l)(4u_h-u_l)^2} & \frac{2u_h}{4u_h^2-5u_h u_l+u_l^2} \\ \frac{2u_h}{4u_h^2-5u_h u_l+u_l^2} & -\frac{4u_h^2(2u_h-u_l)}{u_l(u_h-u_l)(4u_h-u_l)^2} \end{bmatrix}.
\tag{9}
$$

Direct inspection reveals that the first-order leading principal minors are negative, whereas the determinant of the matrix, namely

$$
\det[\mathcal{H}] = \frac{4u_h^2(16u_h^3 - 32u_h^2 u_l + 12u_h u_l^2 - u_l^3)}{u_l(u_h-u_l)^2(4u_h-u_l)^4}
\tag{10}
$$

is positive, in the admissible parameter range $0 < u_l < u_h$ for $u_h > 1.53908u_l \approx 1.54u_l$.

2 Derivation of (T_h^N, T_l^N)

Consider the first order conditions on the logarithm of $NP_h^N(\cdot)$ with respect to t_h and w_h.

$$\frac{\partial \log[NP_h^N(\cdot)]}{\partial t_h} = -\frac{\Delta(4u_h-u_l)\Big((u_l-4u_h)\big[2t_hA-w_lB+2w_h\Gamma+w_l^2(u_l-3u_h)\big]+\mu\big\{w_l^2\big[-\big(10u_h^2-7u_hu_l+u_l^2\big)\big]+w_l\big(u_h^2(6u_l+8w_h)-7u_hu_l^2+u_l^3\big)+4u_h(u_h-u_l+w_h)\Gamma\big\}\Big)}{\big[2t_hA-w_lB+2w_h\Gamma+w_l^2(u_l-3u_h)\big][t_h\Delta(u_l-4u_h)^2-E^2]} = 0, \quad (11)$$

$$\frac{\partial \log[NP_h^N(\cdot)]}{\partial w_h} = \frac{4t_h\Delta(4u_h-u_l)\{(2u_h-u_l)E+\mu u_h[u_h(u_l-4w_h+2w_l)-u_l(u_l-2w_h)]\}-2E\big\{\mu\Delta\big\{4u_h^3+2u_h^2[w_l-2(u_l+w_h)]+2u_h\big(u_lw_h+2u_lw_l-2w_l^2\big)+u_lw_l(w_l-u_l)\big\}-(2u_h-u_l)\big(-w_lB+2w_h\Gamma+w_l^2(u_l-3u_h)\big)\big\}}{\big[2t_hA-w_lB+2w_h\Gamma+w_l^2(u_l-3u_h)\big][t_h\Delta(u_l-4u_h)^2-E^2]} = 0 \quad (12)$$

where $A \equiv \left(4u_h^2 - 5u_hu_l + u_l^2\right)$, $B \equiv \left(2u_h^2 - 3u_hu_l - 4u_hw_h + u_l^2\right)$ and $\Gamma \equiv \left[2u_h^2 - 2u_h(u_l + w_h) + u_lw_h\right]$, $\Delta \equiv u_h - u_l$, $E \equiv \left\{2u_h^2 + u_h[w_l - 2(u_l + w_h)] + u_lw_h\right\}$.
The solution of (11) with respect to t_h is

$$t_h(w_h, w_l) = \frac{(4\Delta)\big(w_lB-2w_h\Gamma-w_l^2(u_l-3u_h)\big)-\mu\big\{-w_l^2\big(10u_h^2-7u_hu_l+u_l^2\big)+w_l\big[u_h^2(6u_l+8w_h)-7u_hu_l^2+u_l^3\big]+4u_h(\Delta+w_h)\Gamma\big\}}{2\Delta(u_l-4u_h)^2}. \quad (13)$$

This can be plugged back into (12) which, after simplification, reduces to

$$\frac{4u_h[u_h(u_l - 4w_h + 2w_l) - u_l(u_l - 2w_h)]}{w_l^2\left[-(10u_h^2 - 7u_hu_l + u_l^2)\right] + w_l\left[u_h^2(6u_l + 8w_h) - 7u_hu_l^2 + u_l^3\right] + 4u_h(\Delta + w_h)\Gamma} = 0, \quad (14)$$

its solution with respect to w_h is

$$w_h(w_h) = \frac{u_hu_l + 2u_hw_l - u_l^2}{2(2u_h - u_l)}. \quad (15)$$

Now consider the first order conditions of the logarithm of $NP_l^N(\cdot)$ relative to t_l and w_l.

$$\frac{\partial \log[NP_l^N(\cdot)]}{\partial t_l} = -\frac{u_l\Delta(4u_h-u_l)\Big\{\mu\Big\{-4u_h^3w_l^2(2u_h-u_l)-u_l^2\Big[u_h^2+u_h(5w_h-u_l)-u_lw_h\Big]\Big[-2u_h^2+2u_h(u_l+w_h)-u_lw_h\Big]+2u_h^2u_lw_l(u_l\Delta+4u_hw_h)\Big\}-(4u_h-u_l)(2t_lu_hu_lA+Z)\Big\}}{(2t_lu_hu_lA+Z)\Big\{t_lu_l\Delta(u_l-4u_h)^2-u_h[u_h(u_l-2w_l)+u_l(-u_l+w_h+w_l)]^2\Big\}} = 0, \quad (16)$$

$$\frac{\partial \log[NP_l^N(\cdot)]}{\partial w_l} = \frac{2u_h\Big\{t_lu_hu_lAH-I\Big\{u_h^4(u_l-2w_l)(\mu u_l-4w_l)+u_h^3u_lK+u_h^2u_l^2\Big\{\mu u_l^2-u_l[(6\mu-3)w_h+(\mu+2)w_l]+(6-4\mu)w_h^2+4w_hw_l+2w_l^2\Big\}-(1-\mu)u_hu_l^3w_h(u_l+5w_h)+(1-\mu)u_l^4w_h^2\Big\}\Big\}}{(2t_lu_hu_lA+Z)\Big\{t_lu_l\Delta(u_l-4u_h)^2-u_h[u_h(u_l-2w_l)+u_l(-u_l+w_h+w_l)]^2\Big\}} = 0, \quad (17)$$

where $Z \equiv \left\{u_l^2w_h\left[u_h^2-u_h(u_l+3w_h)+u_lw_h\right]+2u_h^2u_lw_l(\Delta+2w_h)-2u_h^2w_l^2(2u_h-u_l)\right\}$,
$H \equiv \left\{4u_h^2[u_l-2(\mu+1)w_l]+u_hu_l[(\mu-6)u_l+4(\mu+1)w_h+4(\mu+2)w_l]-u_l^2[(\mu-2)u_l+2(w_h+w_l)]\right\}$,
$I \equiv [u_h(u_l-2w_l)+u_l(-u_l+w_h+w_l)]$, $K \equiv [-2\mu u_l^2+(5\mu-2)u_lw_h+3(\mu+2)u_lw_l-8w_l(w_h+w_l)]$.

The solution of (16) with respect to t_l is

$$t_l(w_h,w_l) = \frac{2u_h^4(u_l-2w_l)[\mu u_l-2(2-\mu)w_l]-2u_h^3u_l\Lambda+u_h^2u_l^2\Big\{2\mu u_l^2-u_l[(9\mu-5)w_h+2(\mu+1)w_l]+2\Big[(6-5\mu)w_h^2+2w_hw_l+w_l^2\Big]\Big\}-(1-\mu)u_hu_l^3w_h(u_l+7w_h)+(1-\mu)u_l^4w_h^2}{2u_hu_l\Delta(4u_h-u_l)^2} \quad (18)$$

where $\Lambda \equiv \left\{2\mu u_l^2-u_l\left[(4\mu-2)w_h+(\mu+5)w_l\right]+2w_l(2(2-\mu)w_h-(3-\mu)w_l)\right\}$. This can be plugged back into (17), which, after simplification, writes

$$\frac{2u_h^2\Big\{\Big\{-8u_h^2w_l+u_hu_l[u_l+4(w_h+w_l)]\Big\}-u_l^3\Big\}}{4u_h^3w_l^2(u_l-2u_h)+u_l^2\Big[-u_h^2+u_h(u_l-5w_h)+u_lw_h\Big]\Big[-2u_h^2+2u_h(u_l+w_h)-u_lw_h\Big]+2u_h^2u_lw_l(u_l\Delta+4u_hw_h)} = 0, \quad (19)$$

whose solution with respect to w_l is

$$w_l(w_h) = \frac{u_hu_l^2+4u_hu_lw_h-u_l^3}{4u_h(2u_h-u_l)}. \quad (20)$$

Solving the system defined by (15) and (20) returns

$$w_h^N = \frac{u_l}{4}, \quad w_l^N = \frac{u_l^2}{4u_h}. \quad (21)$$

The last step to obtain (T_h^N, T_l^N) is to substitute (21) back into (13) and (18) and simplify.

As far as the second-order conditions are concerned, the Hessian matrices relative to the two maximizations, evaluated at the optimal contracts (T_h^N, T_l^N) are:

$$\mathcal{H}_h^N = \left[\begin{matrix} \dfrac{\partial^2 \log[NP_h^N(\cdot)]}{\partial w_h^2} & \dfrac{\partial^2 \log[NP_h^N(\cdot)]}{\partial w_h \partial t_h} \\ \dfrac{\partial^2 \log[NP_h^N(\cdot)]}{\partial t_h \partial w_h} & \dfrac{\partial^2 \log[NP_h^N(\cdot)]}{\partial t_h^2} \end{matrix}\right]\Bigg|_{(T_h^N, T_l^N)} = \left[\begin{matrix} -\dfrac{128u_h^3\left[4(2-\mu)(1+\mu)u_h^2 - 8u_l u_h - u_l^2\mu(1-\mu)\right]}{\Delta(4u_h-u_l)^2(2u_h-u_l)^2(2u_h+u_l)^2(1-\mu)\mu} & -\dfrac{1024u_h^4}{(2u_h-u_l)^3(4u_h-u_l)(2u_h+u_l)^2(1-\mu)\mu} \\ -\dfrac{1024u_h^4}{(2u_h-u_l)^3(4u_h-u_l)(2u_h+u_l)^2\mu(1-\mu)} & -\dfrac{1024u_h^4}{(2u_h-u_l)^4(2u_h+u_l)^2\mu(1-\mu)} \end{matrix}\right] \tag{22}$$

and

$$\mathcal{H}_l^N = \left[\begin{matrix} \dfrac{\partial^2 \log[NP_l^N(\cdot)]}{\partial w_l^2} & \dfrac{\partial^2 \log[NP_l^N(\cdot)]}{\partial w_l \partial t_l} \\ \dfrac{\partial^2 \log[NP_l^N(\cdot)]}{\partial t_l \partial w_l} & \dfrac{\partial^2 \log[NP_l^N(\cdot)]}{\partial t_l^2} \end{matrix}\right]\Bigg|_{(T_h^N, T_l^N)} = \left[\begin{matrix} -\dfrac{128u_h^2(2u_h-u_l)\left[2(2-\mu)(1+\mu)u_h^2 + u_l(\mu(1-\mu)-6)u_h + 2u_l^2\right]}{\Delta u_l^2(4u_h-u_l)^2(2u_h+u_l)^2(1-\mu)\mu} & -\dfrac{512u_h^2(2u_h-u_l)}{(4u_h-u_l)u_l^2(2u_h+u_l)^2(1-\mu)\mu} \\ -\dfrac{512u_h^2(2u_h-u_l)}{(4u_h-u_l)u_l^2(2u_h+u_l)^2(1-\mu)\mu} & -\dfrac{1024u_h^2}{u_l^2(2u_h+u_l)^2(1-\mu)\mu} \end{matrix}\right] \tag{23}$$

It is a matter of simple calculations to ascertain that first-order principal principal minors at (T_h^N, T_l^N) of $\mathcal{H}_h$ and $\mathcal{H}_l$ are negative for all $u_h > u_l > 0$ and $0 < \mu < 1$, and that the determinant at (T_h^N, T_l^N) of the matrices

$$\det[\mathcal{H}_h|_{(T_h^N, T_l^N)}] = \frac{131072u_h^7}{(1-\mu)\mu\Delta(2u_h-u_l)^5(4u_h-u_l)^2(2u_h+u_l)^3} \tag{24}$$

and

$$\det[\mathcal{H}_l|_{(T_h^N, T_l^N)}] = \frac{131072u_h^5(2u_h-u_l)}{(1-\mu)\mu u_l^4\Delta(4u_h-u_l)^2(2u_h+u_l)^3} \tag{25}$$

are positive instead, ensuring the local concavity of the Nash Products, which, together with the uniqueness of the solution, guarantees its optimality.

3 Derivation of (T_h^C, T_l^C)

Consider the first order conditions on the logarithm of $NP_h^N(\cdot)$ with respect to t_h and w_h.

$$\frac{\partial \log[NP_h^C(\cdot)]}{\partial t_h} = \frac{A\left\{4t_h u_l \Delta(4u_h-u_l)^2+4(4u_h-u_l)\left[t_l u_l A+u_l w_h \Gamma+u_h u_l w_l(u_h-u_l+2w_h)+u_h w_l^2(u_l-2u_h)\right]+\mu O+\mu^2 u_l^2 \Delta(u_l-4u_h)^2\right\}}{[t_h \Delta(u_l-4u_h)^2-M^2]\left[N+\mu u_l^2(u_l-4u_h)\Delta\right]} = 0, \tag{26}$$

$$\frac{\partial \log[NP_h^C(\cdot)]}{\partial w_h} = \frac{-8t_h u_l A\{\mu u_h[u_l(u_l-2w_h)-u_h(u_l-4w_h+2w_l)]-(2u_h-u_l)M\}-2M\left[4(1-\mu)t_l u_l\left(-8u_h^3+14u_h^2 u_l-7u_h u_l^2+u_l^3\right)+4(2u_h-u_l)\Xi+\mu\Delta P-\mu^2 u_l^2\Delta(2u_h-u_l)(4u_h-u_l)\right]}{[t_h \Delta(u_l-4u_h)^2-M^2]\left[N+\mu u_l^2(u_l-4u_h)\Delta\right]} = 0 \tag{27}$$

where $M \equiv \left\{2u_h^2+u_h(w_l-2(u_l+w_h))+u_l w_h\right\}$,
$N \equiv 4\left\{u_l\left[(t_h+t_l)A-w_h^2(2u_h-u_l)+2u_h w_h\Delta\right]+u_h u_l w_l(u_h-u_l+2w_h)+u_h w_l^2(u_l-2u_h)\right\}-\mu u_l^2(4u_h-u_l)\Delta$,
$\Xi \equiv \left[u_l w_h\left(-2u_h^2+2u_h(u_l+w_h)-u_l w_h\right)-u_h u_l w_l(u_h-u_l+2w_h)+u_h w_l^2(2u_h-u_l)\right]$,
$O \equiv \left\{u_l\left\{-\Delta\left[4t_l(4u_h-u_l)^2+16u_h^3-8u_h u_l^2+u_l^3\right]+8u_h w_h^2(2u_h-u_l)-8u_h u_l w_h\Delta\right\}-4u_h u_l w_l\left(8u_h^2-9u_h u_l+4u_h w_h+u_l^2\right)+4u_h w_l^2\left(8u_h^2-7u_h u_l+u_l^2\right)\right\}$,
$R \equiv \left[u_l\left(8u_h^3-8u_h^2 w_h+2u_h u_l(2w_h-3u_l)+u_l^3\right)-4u_h w_l^2(4u_h-u_l)+4u_h u_l w_l(5u_h-u_l)\right]$.

The solution of (26) with respect to t_h is

$$t_h(w_h,w_l,t_l) = \frac{-4(4u_h-u_l)\left[t_l u_l A+u_l w_h\Gamma+u_h u_l w_l(u_h-u_l+2w_h)-u_h w_l^2(2u_h-u_l)\right]+\mu\left\{u_l\left[\Delta\Sigma-8u_h w_h^2(2u_h-u_l)+8u_h u_l w_h\Delta\right]+4u_h u_l w_l\left(8u_h^2-9u_h u_l+4u_h w_h+u_l^2\right)-4u_h w_l^2 Y\right\}-\mu^2 u_l^2\Delta(u_l-4u_h)^2}{4u_l\Delta(4u_h-u_l)^2}, \tag{28}$$

where $\Sigma \equiv \left(4t_l(u_l-4u_h)^2+16u_h^3-8u_h u_l^2+u_l^3\right)$, $Y \equiv \left(8u_h^2-7u_h u_l+u_l^2\right)$. This can be plugged back into (27), which, after simplification, writes

$$\frac{8u_h u_l[u_h(u_l-4w_h+2w_l)-u_l(u_l-2w_h)]}{4t_l u_l\Delta(4u_h-u_l)^2+4u_h\left\{u_l w_l\left(8u_h^2-9u_h u_l+4u_h w_h+u_l^2\right)-w_l^2 Y+2u_l(\Delta+w_h)\left[2u_h^2-2u_h(u_l+w_h)+u_l w_h\right]\right\}-\mu u_l^2\Delta(4u_h-u_l)^2} = 0. \tag{29}$$

Its solution with respect to w_h is

$$w_h(w_l) = \frac{u_h u_l + 2u_h w_l - u_l^2}{2(2u_h - u_l)}. \tag{30}$$

Move now on the first order conditions of the logarithm of $NP_l^C(\cdot)$ relative to t_l and w_l.

$$\frac{\partial \log[NP_l^C(\cdot)]}{\partial t_l} = -\frac{u_l A\Big\{-4(1-\mu)t_h u_l \Delta(4u_h-u_l)^2 - 4(4u_h-u_l)\Phi + \mu\Big\{u_l\Big\{-4w_h^2 Y - u_h\Delta\Big[32u_h w_h - 16u_h^2 + 4u_h u_l + 3u_l^2\Big]\Big\} - 8u_h^2 w_l^2(2u_h-u_l) + 4u_h u_l w_l(u_l\Delta + 4u_h w_h)\Big\} - \mu^2 u_h u_l \Delta(4u_h-u_l)^2\Big\}}{[t_l u_l \Delta(4u_h-u_l)^2 - u_h I^2](N - \mu u_h u_l A)} = 0, \tag{31}$$

$$\frac{\partial \log[NP_l^C(\cdot)]}{\partial w_l} = \frac{2u_h\Big\{4(1-\mu)t_h u_l\Big(-8u_h^3 + 14u_h^2 u_l - 7u_h u_l^2 + u_l^3\Big)[w_l(2u_h-u_l) - u_l(u_h-u_l+w_h)] + 2t_l u_l A X + I\Big[4(2u_h-u_l)\Psi + \mu u_l \Delta\Omega + \mu^2 u_h u_l \Delta(2u_h-u_l)(4u_h-u_l)\Big]\Big\}}{[t_l u_l \Delta(4u_h-u_l)^2 - u_h I^2](N - \mu u_h u_l A)} = 0, \tag{32}$$

where $\Phi \equiv \left[t_l u_l A + u_l w_h \Gamma + u_h u_l w_l(\Delta + 2w_h) - u_h w_l^2(2u_h - u_l)\right]$, $X \equiv \left\{\left\{\mu\left[-8u_h^2 w_l + u_h u_l(u_l + 4(w_h + w_l)\right] - u_l^3\right\} + 2(2u_h - u_l)I\right\}$, $\Psi \equiv \left[u_l w_h \Gamma + u_h u_l w_l(\Delta + 2w_h) - u_h w_l^2(2u_h - u_l)\right]$, $\Omega \equiv \left[-8u_h^3 + 4u_h^2(u_l - 4w_h + w_l) + u_h\left(u_l^2 + 2u_l w_h - 2u_l w_l + 16w_h^2\right) - 4u_l w_h^2\right]$.

The solution of (16) with respect to t_l is

$$t_l(w_h, w_l) = \frac{(1-\mu)\Big[16\mu u_h^4 u_l - 4t_h u_l \Delta(4u_h-u_l)^2\Big] + 4u_h^3 F + u_h^2 u_l G + u_h u_l^2\Big\{\mu(\mu+3)u_l^2 - 4u_l(2w_h + \mu w_l + w_l) + 4\Big[(7\mu-6)w_h^2 + 2w_h w_l + w_l^2\Big]\Big\} + 4(1-\mu)u_l^3 w_h^2}{4u_l\Delta(4u_h-u_l)^2} \tag{33}$$

where $F \equiv \left[\mu(6\mu - 5)u_l^2 - 8(1-\mu)u_l w_h - 4u_l w_l + 4(2-\mu)w_l^2\right]$ and $G \equiv \left\{\left(\mu - 9\mu^2\right)u_l^2 + 4u_l\left[(10 - 8\mu)w_h + (\mu + 5)w_l\right] + 8\left[4(1-\mu)w_h^2 - 2(2-\mu)w_h w_l - (3-\mu)w_l^2\right]\right\}$. As before, this can be plugged into (32) to obtain, after simplifying

$$\frac{4u_h\Big\{8u_h^2 w_l - u_h u_l[u_l + 4(w_h + w_l)] + u_l^3\Big\}}{4u_l\Big\{-\Delta[t_h(4u_h-u_l)^2 + u_h u_l \Delta] + w_h^2\Big(8u_h^2 - 7u_h u_l + u_l^2\Big) - 8u_h^2 w_h \Delta\Big\} + 8u_h^2 w_l^2(2u_h-u_l) + 4u_h u_l w_l\Big[u_l^2 - u_h(u_l + 4w_h)\Big] + \mu u_h u_l \Delta(4u_h-u_l)^2} = 0, \tag{34}$$

its solution with respect to w_l is

$$w_l(w_h) = \frac{u_h u_l^2 + 4u_h u_l w_h - u_l^3}{4u_h(2u_h - u_l)}. \tag{35}$$

The solution of the system defined by (30) and (35) is

$$w_h^C = \frac{u_l}{4}, \quad w_l^C = \frac{u_l^2}{4u_h}. \tag{36}$$

The last step to obtain (T_h^C, T_l^C) requires, as above, to substitute (36) back into (28) and (33) and simplify.

Let us now move to the second-order conditions as before, the Hessian matrices evaluated at the optimal contracts are the following.

$$\mathcal{H}_h^C = \left.\begin{bmatrix} \dfrac{\partial^2 \log[NP_h^C(\cdot)]}{\partial w_h^2} & \dfrac{\partial^2 \log[NP_h^C(\cdot)]}{\partial w_h \partial t_h} \\ \dfrac{\partial^2 \log[NP_h^C(\cdot)]}{\partial t_h \partial w_h} & \dfrac{\partial^2 \log[NP_h^C(\cdot)]}{\partial t_h^2} \end{bmatrix}\right|_{(T_h^C, T_l^C)} = \begin{bmatrix} -\frac{-64(2u_h-u_l)(2-\mu)\left(4(2-\mu)u_l^2+u_h(\mu+3)(5\mu-8)u_l-4u_h^2(2-\mu)^2(1+\mu)\right)}{\Delta(u_l-4u_h)^2(5u_l+4u_h(2-\mu))^2(1-\mu)\mu} & -\frac{256(2u_h-u_l)(2-\mu)^2}{(4u_h-u_l)(5u_l+4u_h(2-\mu))^2(1-\mu)\mu} \\ \frac{256(2u_h-u_l)(2-\mu)^2}{(4u_h-u_l)(5u_l-4u_h(2-\mu))^2(1-\mu)\mu} & -\frac{256(2-\mu)^2}{(5u_l-4u_h(2-\mu))^2(1-\mu)\mu} \end{bmatrix} \tag{37}$$

and

$$\mathcal{H}_l^C = \left.\begin{bmatrix} \dfrac{\partial^2 \log[NP_l^C(\cdot)]}{\partial w_l^2} & \dfrac{\partial^2 \log[NP_l^C(\cdot)]}{\partial w_l \partial t_l} \\ \dfrac{\partial^2 \log[NP_l^C(\cdot)]}{\partial t_l \partial w_l} & \dfrac{\partial^2 \log[NP_l^C(\cdot)]}{\partial t_l^2}, \end{bmatrix}\right|_{(T_h^C, T_l^C)} = \begin{bmatrix} -\frac{-64(2u_h-u_l)(2-\mu)\left[\left(4\mu^3-7\mu^2+\mu+4\right)u_h^2-3u_l(2-\mu)u_h+u_l^2(2-\mu)\right]}{-\Delta u_l^2(4u_h-u_l)^2(3-4\mu)^2(1-\mu)\mu} & \frac{128(2u_h-u_l)(2-\mu)^2}{(4u_h-u_l)u_l^2(3-4\mu)^2(1-\mu)\mu} \\ \frac{128(2u_h-u_l)(2-\mu)^2}{-(4u_h-u_l)u_l^2(3-4\mu)^2(1-\mu)\mu} & \frac{256(2-\mu)^2}{-u_l^2(3-4\mu)^2(1-\mu)\mu}. \end{bmatrix} \tag{38}$$

Their determinants are, respectively

$$\det[\mathcal{H}_h|_{(T_h^C, T_l^C)}] = -\frac{16384(2-\mu)^3 u_h(2u_h-u_l)}{(1-\mu)\mu(u_h-u_l)(4u_h-u_l)^2[5u_l-4(2-\mu)u_h]^3} \tag{39}$$

and

$$\det[\mathcal{H}_l|_{(T_h^C,T_l^C)}] = -\frac{16384(2-\mu)^3 u_h^2(2u_h-u_l)}{(1-\mu)\mu(4\mu-3)^3 u_l^4(u_h-u_l)(4u_h-u_l)^2} \tag{40}$$

Inspection reveals that a necessary and sufficient condition for (40) to be positive is that $0<\mu<\frac{3}{4}$. If this condition is met, (39) is positive as well and the first-order principal minors of (37) and (38) are negative, which guarantee concavity of the two Nash products. When $\frac{3}{4}<\mu<1$ the bargaining power distribution is such that firm $\mathcal{D}_l$ suffers losses at the contract described above. Non-exclusive contingent contract can be constructed by imposing that the fixed fee t_l is set so as to satisfy firm $\mathcal{D}_l$'s participation constraint with equality. This amounts to solving the following program (for the sake of readability, we are not going to introduce further notation).

$$\max_{w_h,t_h} NP_h^C(T_h,T_l^C), \quad \max_{w_l}[\hat{\Pi}(T_h^C,T_l)+\hat{\pi}_l(T_l,w_h^C)], \text{ and } \hat{\pi}_l(T_l,w_h^C) \overset{t_l}{=} 0. \tag{41}$$

It is clear that the first-order conditions $NP_h^C(\cdot)$ w.r.t t_h and w_h coincide with those above analyzed. The last two are, instead

$$w_l(w_h) = \frac{u_l^2(\Delta+w_h)}{4u_h(2u_h-u_l)}, \qquad t_l(w_h,w_l) = \frac{u_h[u_h(u_l-2w_l)+u_l(+w_h+w_l-u_l)]^2}{u_l\Delta(4u_h-u_l)^2}). \tag{42}$$

By solving the system defined by this set of equations one obtains the optimal contracts in this case, which write

$$T_h^C = (w_h^C,t_h^C) = \left(\frac{u_l}{4}, \frac{4u_h\mu+u_l(-3+(3-4\mu)\mu)}{16}\right), \tag{43}$$

$$T_l^C = (w_l^C,t_l^C) = \left(\frac{u_l^2}{4u_h}, \frac{u_l(u_h-u_l)}{16u_h}\right). \tag{44}$$

In the appendix 1 of the paper we show that the profit to firm $\mathcal{U}$ under (43) and (44) is lower than that with an exclusive contract with firm $\mathcal{D}_h$.

4 Derivation of (T_h^M, T_l^M)

The first order conditions on the logarithm of $NP_h^N(\cdot)$ with respect to t_h and w_h are as follows

$$\frac{\partial \log[NP_h^M(\cdot)]}{\partial t_h} = \frac{A\Big\{4t_h u_l \Delta(4u_h-u_l)^2+4(4u_h-u_l)\big[t_l u_l A+u_l w_h \Gamma+u_h u_l w_l(u_h-u_l+2w_h)+u_h w_l^2(u_l-2u_h)\big]+\mu O+\mu^2 u_l^2 \Delta(u_l-4u_h)^2\Big\}}{[t_h\Delta(u_l-4u_h)^2-M^2]\big[N+\mu u_l^2(u_l-4u_h)\Delta\big]} = 0, \quad (45)$$

$$\frac{\partial \log[NP_h^M(\cdot)]}{\partial w_h} = \frac{-8t_h u_l A\{\mu u_h[u_l(u_l-2w_h)-u_h(u_l-4w_h+2w_l)]-(2u_h-u_l)M\}-2M\big[4(1-\mu)t_l u_l\big(-8u_h^3+14u_h^2 u_l-7u_h u_l^2+u_l^3\big)+4(2u_h-u_l)\Xi+\mu\Delta P-\mu^2 u_l^2\Delta(2u_h-u_l)(4u_h-u_l)\big]}{[t_h\Delta(u_l-4u_h)^2-M^2]\big[N+\mu u_l^2(u_l-4u_h)\Delta\big]} = 0 \quad (46)$$

The solution of (45) with respect to t_h is

$$t_h(w_h, w_l, t_l) = \frac{-4(4u_h-u_l)\big[t_l u_l A+u_l w_h \Gamma+u_h u_l w_l(u_h-u_l+2w_h)-u_h w_l^2(2u_h-u_l)\big]+\mu\Big\{u_l\big[\Delta\Sigma-8u_h w_h^2(2u_h-u_l)+8u_h u_l w_h \Delta\big]+4u_h u_l w_l\big(8u_h^2-9u_h u_l+4u_h w_h+u_l^2\big)-4u_h w_l^2 Y\Big\}-\mu^2 u_l^2\Delta(u_l-4u_h)^2}{4u_l\Delta(4u_h-u_l)^2}. \quad (47)$$

This can be plugged back into (46), which, after simplification, writes

$$\frac{8u_h u_l[u_h(u_l-4w_h+2w_l)-u_l(u_l-2w_h)]}{4t_l u_l\Delta(4u_h-u_l)^2+4u_h\Big\{u_l w_l\big(8u_h^2-9u_h u_l+4u_h w_h+u_l^2\big)-w_l^2 Y+2u_l(\Delta+w_h)\big[2u_h^2-2u_h(u_l+w_h)+u_l w_h\big]\Big\}-\mu u_l^2\Delta(4u_h-u_l)^2} = 0. \quad (48)$$

Its solution with respect to w_h is

$$w_h(w_l) = \frac{u_h u_l + 2u_h w_l - u_l^2}{2(2u_h - u_l)}. \quad (49)$$

Notice here that (45) and (46) coincide with (26) and (27) respectively, therefore also (47) coincides with (28) and (49) to (30).

Consider now the set of first order conditions in the negotiation for T_l^M.

$$\frac{\partial \log[NP_l^M(\cdot)]}{\partial t_l} = -\frac{u_l\Delta(4u_h-u_l)\Big\{\mu\Big\{-4u_h^3w_l^2(2u_h-u_l)-u_l^2\Big[u_h^2+u_h(5w_h-u_l)-u_lw_h\Big]\Big[-2u_h^2+2u_h(u_l+w_h)-u_lw_h\Big]+2u_h^2u_lw_l(u_l\Delta+4u_hw_h)\Big\}-(4u_h-u_l)(2t_lu_hu_lA+Z)\Big\}}{(2t_lu_hu_lA+Z)\Big\{t_lu_l\Delta(u_l-4u_h)^2-u_h[u_h(u_l-2w_l)+u_l(-u_l+w_h+w_l)]^2\Big\}} = 0, \quad (50)$$

$$\frac{\partial \log[NP_l^M(\cdot)]}{\partial w_l} = \frac{2u_h\Big\{t_lu_hu_lAH-I\Big\{u_h^4(u_l-2w_l)(\mu u_l-4w_l)+u_h^3u_lK+u_h^2u_l^2\Big\{\mu u_l^2-u_l[(6\mu-3)w_h+(\mu+2)w_l]+(6-4\mu)w_h^2+4w_hw_l+2w_l^2\Big\}-(1-\mu)u_hu_l^3w_h(u_l+5w_h)+(1-\mu)u_l^4w_h^2\Big\}\Big\}}{(2t_lu_hu_lA+Z)\Big\{t_lu_l\Delta(u_l-4u_h)^2-u_h[u_h(u_l-2w_l)+u_l(-u_l+w_h+w_l)]^2\Big\}} = 0, \quad (51)$$

The solution of (50) with respect to t_l is

$$t_l(w_h, w_l) = \frac{2u_h^4(u_l-2w_l)[\mu u_l-2(2-\mu)w_l]-2u_h^3u_l\Lambda+u_h^2u_l^2\Big\{2\mu u_l^2-u_l[(9\mu-5)w_h+2(\mu+1)w_l]+2\Big[(6-5\mu)w_h^2+2w_hw_l+w_l^2\Big]\Big\}-(1-\mu)u_hu_l^3w_h(u_l+7w_h)+(1-\mu)u_l^4w_h^2}{2u_hu_l\Delta(4u_h-u_l)^2} \quad (52)$$

This can be plugged back into (51), which, after simplification, writes

$$\frac{2u_h^2\Big\{\Big\{-8u_h^2w_l+u_hu_l[u_l+4(w_h+w_l)]\Big\}-u_l^3\Big\}}{4u_h^3w_l^2(u_l-2u_h)+u_l^2\Big[-u_h^2+u_h(u_l-5w_h)+u_lw_h\Big]\Big[-2u_h^2+2u_h(u_l+w_h)-u_lw_h\Big]+2u_h^2u_lw_l(u_l\Delta+4u_hw_h)} = 0, \quad (53)$$

whose solution with respect to w_l is

$$w_l(w_h) = \frac{u_hu_l^2 + 4u_hu_lw_h - u_l^3}{4u_h(2u_h - u_l)}. \quad (54)$$

In the case, (50) and (51) coincide with (16) and (17), whence (52) coincides with (18) and (54) with (20). Solving the system defined by (49) and (54) returns

$$w_h^M = \frac{u_l}{4}, \quad w_l^M = \frac{u_l^2}{4u_h}. \quad (55)$$

As above, substitution back of (55) into (47) and (52) and simplification yields the optimal contract. Let us now consider the second-order conditions. The Hessian matrices evaluated at (T_h^M, T_l^M) are

$$\mathcal{H}_h^M = \left[\begin{array}{cc} \dfrac{\partial^2 \log[NP_h^M(\cdot)]}{\partial w_h^2} & \dfrac{\partial^2 \log[NP_h^M(\cdot)]}{\partial w_h \partial t_h} \\ \dfrac{\partial^2 \log[NP_h^M(\cdot)]}{\partial t_h \partial w_h} & \dfrac{\partial^2 \log[NP_h^M(\cdot)]}{\partial t_h^2} \end{array}\right]\Bigg|_{(T_h^M, T_l^M)} = \left[\begin{array}{cc} \frac{128u_h^2(2u_h-u_l)\left\{-8(2-\mu)(\mu+1)u_h^2+2u_l\left[\mu\left(-3\mu^2+\mu+2\right)+12\right]u_h+u_l^2\left[(1-\mu)^2\mu-8\right]\right\}}{\Delta(4u_h-u_l)^2(1-\mu)\mu\left(8u_h^2-2u_l(3\mu+2)u_h-u_l^2(1-\mu)\right)^2} & \frac{1024u_h^2(2u_h-u_l)}{(4u_h-u_l)(1-\mu)\mu\left[8u_h^2-2u_l(3\mu+2)u_h-u_l^2(1-\mu)\right]^2} \\ -\frac{1024u_h^2(2u_h-u_l)}{(4u_h-u_l)(1-\mu)\mu\left(8u_h^2-2u_l(3\mu+2)u_h-u_l^2(1-\mu)\right)^2} & -\frac{1024u_h^2}{(1-\mu)\mu\left[8u_h^2-2u_l(3\mu+2)u_h-u_l^2(1-\mu)\right]^2} \end{array}\right] \tag{56}$$

and

$$\mathcal{H}_l^M = \left[\begin{array}{cc} \dfrac{\partial^2 \log[NP_l^M(\cdot)]}{\partial w_l^2} & \dfrac{\partial^2 \log[NP_l^M(\cdot)]}{\partial w_l \partial t_l} \\ \dfrac{\partial^2 \log[NP_l^M(\cdot)]}{\partial t_l \partial w_l} & \dfrac{\partial^2 \log[NP_l^M(\cdot)]}{\partial t_l^2}, \end{array}\right]\Bigg|_{(T_h^M, T_l^M)} = \left[\begin{array}{cc} -\frac{128u_h^2(2u_h-u_l)\left\{-2(2-\mu)(\mu+1)u_h^2+u_l[(1-\mu)\mu+6]u_h-2u_l^2\right\}}{\Delta u_l^2(u_l-4u_h)^2(2u_h+u_l)^2(\mu-1)\mu} & -\frac{512u_h^2(2u_h-u_l)}{(4u_h-u_l)u_l^2(2u_h+u_l)^2(1-\mu)\mu} \\ -\frac{512u_h^2(2u_h-u_l)}{(4u_h-u_l)u_l^2(2u_h+u_l)^2(1-\mu)\mu} & -\frac{1024u_h^2}{u_l^2(2u_h+u_l)^2(1-\mu)\mu}. \end{array}\right] \tag{57}$$

Their determinants are

$$\det[\mathcal{H}_h|_{(T_h^M, T_l^M)}] = \frac{131072u_h^4(2u_h-u_l)}{(1-\mu)\mu(u_h-u_l)(4u_h-u_l)^2\left(8u_h^2-2(3\mu+2)u_h u_l-(1-\mu)u_l^2\right)^3} \tag{58}$$

and

$$\det[\mathcal{H}_l|_{(T_h^M, T_l^M)}] = \frac{131072u_h^5(2u_h-u_l)}{(1-\mu)\mu u_l^4(u_h-u_l)(4u_h-u_l)^2(2u_h+u_l)^3}. \tag{59}$$

It is easily checked that, while the first-order principal minors of $\mathcal{H}_l|_{(T_h^M, T_l^M)}$ are negative and its determinant is positive for all $u_h > u_l > 0$ and $0 < \mu < 1$, this is not always the case for $\mathcal{H}_h|_{(T_h^M, T_l^M)}$. However, it is a matter of calculations to ascertain that its determinant is positive for $u_l < u_h < \frac{5}{4}u_l$ and $\mu < \frac{8u_h^2-4u_h u_l-u_l^2}{u_l(6u_h-u_l)} < 1$ or $u_h > \frac{5}{4}u_l$ and $0 < \mu < 1$ and that, under either of these conditions, its first-order principal minors are indeed negative. In the remaining parametric constellation $u_l < u_h < \frac{5}{4}u_l$ and $\frac{8u_h^2-4u_h u_l-u_l^2}{u_l(6u_h-u_l)} < \mu < 1$ firm $\mathcal{D}_h$ earns negative profits, yet, this contract configuration can be constructed by imposing that t_h satisfies with equality the participation constraint of this firm. The new maximization program is thus

$$\max_{w_l,t_l} NP_l^M(T_h^M, T_l), \quad \max_{w_h}[\hat{\Pi}(T_h, T_l^M) + \hat{\pi}_h(T_h, w_l^M)], \text{ and } \hat{\pi}_h(T_h, w_l^M) \overset{t_h}{=} 0. \tag{60}$$

The firs-order conditions on $NP_l^M(\cdot)$ coincide with those above analyzed, the remaining two are

$$w_h(w_l) = \frac{u_h u_l + 2u_h w_l - u_l^2}{4u_h - 2u_l}, \quad t_h(w_h, w_l) = \frac{\left\{2u_h^2 + u_h[w_l - 2(u_l + w_h)] + u_l w_h\right\}^2}{\Delta(4u_h - u_l)^2}. \tag{61}$$

The solution to this set of FOCs is

$$T_h^M = (w_h^M, t_h^M) = \left(\frac{u_l}{4}, \frac{u_h - u_l}{4}\right),, \tag{62}$$

$$T_l^M = (w_l^M, t_l^M) = \left(\frac{u_l^2}{4u_h}, \frac{u_l[2\mu u_h - (3-\mu)u_l]}{32u_h}\right). \tag{63}$$

As pointed out in the paper, the profit to firm $\mathcal{U}$ under these contracts is lesser than that obtained with an exclusive contract with firm $\mathcal{D}_h$.

5 Derivation of (T_h^Z, T_l^Z)

This mixed case as well is a combination of cases C and N. The first-order conditions of NP_h^Z w.r.t. t_h and w_h are

$$\frac{\partial \log[NP_h^Z(\cdot)]}{\partial t_h} = -\frac{\Delta(4u_h-u_l)\Big((u_l-4u_h)\big[2t_hA-w_lB+2w_h\Gamma+w_l^2(u_l-3u_h)\big]+\mu\Big\{w_l^2\big[-\big(10u_h^2-7u_hu_l+u_l^2\big)\big]+w_l\big(u_h^2(6u_l+8w_h)-7u_hu_l^2+u_l^3\big)+4u_h(u_h-u_l+w_h)\Gamma\Big\}\Big)}{\big[2t_hA-w_lB+2w_h\Gamma+w_l^2(u_l-3u_h)\big][t_h\Delta(u_l-4u_h)^2-E^2]} = 0, \quad (64)$$

$$\frac{\partial \log[NP_h^Z(\cdot)]}{\partial w_h} = \frac{4t_h\Delta(4u_h-u_l)\{(2u_h-u_l)E+\mu u_h[u_h(u_l-4w_h+2w_l)-u_l(u_l-2w_h)]\}-2E\Big\{\mu\Delta\Big\{4u_h^3+2u_h^2[w_l-2(u_l+w_h)]+2u_h\big(u_lw_h+2u_lw_l-2w_l^2\big)+u_lw_l(w_l-u_l)\Big\}-(2u_h-u_l)\big(-w_lB+2w_h\Gamma+w_l^2(u_l-3u_h)\big)\Big\}}{\big[2t_hA-w_lB+2w_h\Gamma+w_l^2(u_l-3u_h)\big][t_h\Delta(u_l-4u_h)^2-E^2]} \quad (65)$$

The solution of (64) w.r.t. t_h is

$$t_h(w_h, w_l) = \frac{(4\Delta)\big(w_lB-2w_h\Gamma-w_l^2(u_l-3u_h)\big)-\mu\big\{-w_l^2\big(10u_h^2-7u_hu_l+u_l^2\big)+w_l\big[u_h^2(6u_l+8w_h)-7u_hu_l^2+u_l^3\big]+4u_h(\Delta+w_h)\Gamma\big\}}{2\Delta(u_l-4u_h)^2}. \quad (66)$$

This can be plugged back into (65) which, after simplification, reduces to

$$\frac{4u_h[u_h(u_l-4w_h+2w_l)-u_l(u_l-2w_h)]}{w_l^2\left[-\left(10u_h^2-7u_hu_l+u_l^2\right)\right]+w_l\left[u_h^2(6u_l+8w_h)-7u_hu_l^2+u_l^3\right]+4u_h(\Delta+w_h)\Gamma} = 0, \quad (67)$$

whose solution with respect to w_h is

$$w_h(w_h) = \frac{u_hu_l+2u_hw_l-u_l^2}{2(2u_h-u_l)}. \quad (68)$$

Equations (64) and (65) coincide with (11) and (12) respectively, therefore (66) coincides with (13) and (68) with (15).

Let us now consider the first-order conditions on the logarithm of to the

$$\frac{\partial \log[NP_l^Z(\cdot)]}{\partial t_l} = -\frac{u_l A\Big\{-4(1-\mu)t_h u_l \Delta(4u_h-u_l)^2-4(4u_h-u_l)\Phi+\mu\Big\{u_l\Big\{-4w_h^2 Y-u_h\Delta\Big[32u_h w_h-16u_h^2+4u_h u_l+3u_l^2\Big]\Big\}-8u_h^2 w_l^2(2u_h-u_l)+4u_h u_l w_l(u_l\Delta+4u_h w_h)\Big\}-\mu^2 u_h u_l \Delta(4u_h-u_l)^2\Big\}}{[t_l u_l \Delta(4u_h-u_l)^2-u_h I^2](N-\mu u_h u_l A)} = 0, \quad (69)$$

$$\frac{\partial \log[NP_l^Z(\cdot)]}{\partial w_l} = \frac{2u_h\Big\{4(1-\mu)t_h u_l\Big(-8u_h^3+14u_h^2 u_l-7u_h u_l^2+u_l^3\Big)[w_l(2u_h-u_l)-u_l(u_h-u_l+w_h)]+2t_l u_l AX+I\Big[4(2u_h-u_l)\Psi+\mu u_l\Delta\Omega+\mu^2 u_h u_l \Delta(2u_h-u_l)(4u_h-u_l)\Big]\Big\}}{[t_l u_l \Delta(4u_h-u_l)^2-u_h I^2](N-\mu u_h u_l A)} = 0, \quad (70)$$

The solution of (69) with respect to t_l is

$$t_l(w_h, w_l) = \frac{(1-\mu)\Big[16\mu u_h^4 u_l-4t_h u_l\Delta(4u_h-u_l)^2\Big]+4u_h^3 F+u_h^2 u_l G+u_h u_l^2\Big\{\mu(\mu+3)u_l^2-4u_l(2w_h+\mu w_l+w_l)+4\Big[(7\mu-6)w_h^2+2w_h w_l+w_l^2\Big]\Big\}+4(1-\mu)u_l^3 w_h^2}{4u_l\Delta(4u_h-u_l)^2} \quad (71)$$

As before, this can be plugged into (70) to obtain, after simplifying

$$\frac{4u_h\Big\{8u_h^2 w_l-u_h u_l[u_l+4(w_h+w_l)]+u_l^3\Big\}}{4u_l\Big\{-\Delta[t_h(4u_h-u_l)^2+u_h u_l\Delta]+w_h^2\Big(8u_h^2-7u_h u_l+u_l^2\Big)-8u_h^2 w_h\Delta\Big\}+8u_h^2 w_l^2(2u_h-u_l)+4u_h u_l w_l\Big[u_l^2-u_h(u_l+4w_h)\Big]+\mu u_h u_l\Delta(4u_h-u_l)^2} = 0, \quad (72)$$

its solution with respect to w_l is

$$w_l(w_h) = \frac{u_h u_l^2 + 4u_h u_l w_h - u_l^3}{4u_h(2u_h - u_l)}. \quad (73)$$

The solution of the system defined by (68) and (73) is

$$w_h^C = \frac{u_l}{4}, \quad w_l^C = \frac{u_l^2}{4u_h}. \quad (74)$$

The last step to obtain (T_h^Z, T_l^Z) requires, as above, to substitute (74) back into (66) and (71) and simplify.

Let us now move to the second-order conditions as before, the Hessian matrices evaluated at the optimal contracts are the following.

$$\mathcal{H}_h^Z = \begin{bmatrix} \frac{\partial^2 \log[NP_h^Z(\cdot)]}{\partial w_h^2} & \frac{\partial^2 \log[NP_h^Z(\cdot)]}{\partial w_h \partial t_h} \\ \frac{\partial^2 \log[NP_h^Z(\cdot)]}{\partial t_h \partial w_h} & \frac{\partial^2 \log[NP_h^Z(\cdot)]}{\partial t_h^2} \end{bmatrix}\Bigg|_{(T_h^Z, T_l^Z)} = \begin{bmatrix} -\frac{128u_h^3\left(4(\mu-2)(\mu+1)u_h^2+8u_l u_h-u_l^2(\mu-1)\mu\right)}{(u_h-u_l)(u_l-4u_h)^2(u_l-2u_h)^2(2u_h+u_l)^2(\mu-1)\mu} & \frac{1024u_h^4}{(2u_h-u_l)^3(4u_h-u_l)(2u_h+u_l)^2(\mu-1)\mu} \\ \frac{1024u_h^4}{(2u_h-u_l)^3(4u_h-u_l)(2u_h+u_l)^2(\mu-1)\mu} & \frac{1024u_h^4}{(u_l-2u_h)^4(2u_h+u_l)^2(\mu-1)\mu} \end{bmatrix} \tag{75}$$

and

$$\mathcal{H}_l^Z = \begin{bmatrix} \frac{\partial^2 \log[NP_l^Z(\cdot)]}{\partial w_l^2} & \frac{\partial^2 \log[NP_l^Z(\cdot)]}{\partial w_l \partial t_l} \\ \frac{\partial^2 \log[NP_l^Z(\cdot)]}{\partial t_l \partial w_l} & \frac{\partial^2 \log[NP_l^Z(\cdot)]}{\partial t_l^2}, \end{bmatrix}\Bigg|_{(T_h^Z, T_l^Z)} = \begin{bmatrix} \frac{128u_h^4(2u_h-u_l)\left(2\left(2\mu^3-3\mu^2+\mu+2\right)u_h^2+2u_l\left((\mu-1)^2\mu-3\right)u_h-u_l^2(\mu-2)\left(\mu^2+1\right)\right)}{(u_h-u_l)u_l^2(u_l-4u_h)^2(\mu-1)\mu\left((4\mu-2)u_h^2+2u_l(\mu-1)u_h-u_l^2(\mu-1)\right)^2} & \frac{512u_h^4(2u_h-u_l)}{(4u_h-u_l)u_l^2(\mu-1)\mu\left((4\mu-2)u_h^2+2u_l(\mu-1)u_h-u_l^2(\mu-1)\right)^2} \\ \frac{512u_h^4(2u_h-u_l)}{(4u_h-u_l)u_l^2(\mu-1)\mu\left((4\mu-2)u_h^2+2u_l(\mu-1)u_h-u_l^2(\mu-1)\right)^2} & \frac{1024u_h^4}{u_l^2\left((2-4\mu)u_h^2-2u_l(\mu-1)u_h+u_l^2(\mu-1)\right)^2(\mu-1)\mu} \end{bmatrix} \tag{76}$$

heir determinants are

$$\det[\mathcal{H}_h|_{(T_h^Z, T_l^Z)}] = \frac{131072u_h^7}{(1-\mu)\mu\Delta(2u_h-u_l)^5(4u_h-u_l)^2(2u_h+u_l)^3} \tag{77}$$

and

$$\det[\mathcal{H}_l|_{(T_h^Z, T_l^Z)}] = -\frac{131072u_h^8(2u_h-u_l)}{(1-\mu)\mu u_l^4\Delta(4u_h-u_l)^2\left[(4\mu-2)u_h^2-2(1-\mu)u_h u_l+(1-\mu)u_l^2\right]^3} \tag{78}$$

Direct inspection reveals that while the first-order principal minors of $H_h|_{(T_h^Z, T_l^Z)}$ are negative and its determinant positive for all $u_h > u_l > 0$ and $0 < \mu < 1$, this is not the case for $H_l|_{(T_h^Z, T_l^Z)}$. Yet, calculations show that, in this case, a necessary and sufficient condition to guarantee concavity is $\mu < \frac{2u_h^2+2u_h u_l-u_l^2}{4u_h^2+2u_l u_l-u_l^2}$. When this condition is not met, the low-quality downstream firm reaps a negative profit at the above contracts. In this case, the optimization problem is modified such that t_l satisfies the participation constraint of firm $\mathcal{D}_l$:

$$\max_{w_h,t_h} NP_h^Z(T_h,T_l^Z), \quad \max_{w_l}[\hat{\Pi}(T_h^Z,T_l)+\hat{\pi}_l(w_h^Z,T_l)], \text{ and } \hat{\pi}_l(w_h^Z,T_l) \overset{t_l}{=} 0. \tag{79}$$

The FOCs relative to $NP_h^Z(\cdot)$ coincide with those above, the remaining two are

$$w_l(w_h)\frac{u_h u_l^2+4u_h u_l w_h-u_l^3}{8u_h^2-4u_h u_l}, \quad t_h(w_h,w_l)=\frac{u_h[u_h(u_l-2w_l)+u_l(-u_l+w_h+w_l]^2}{u_l\Delta(u_l-4u_h)^2} \tag{80}$$

The solution to (79) is

$$T_h^Z = (w_h^Z,t_h^Z)=\left(\frac{u_l}{4},\frac{8\mu u_h^3-4(1+\mu)u_h^2u_l+2(1-\mu)u_h u_l^2-(1-\mu)u_l^3}{32u_h^2}\right), \tag{81}$$

$$T_l^Z = (w_l^Z,t_l^Z)=\left(\frac{u_l^2}{4u_h},\frac{u_l(u_h-u_l)}{16u_h}\right). \tag{82}$$

Calculations show that the profit reaped by firm $\mathcal{U}$ under this contract falls short of that earned with an exclusive relationship with firm $\mathcal{D}_h$.

6 Behavior of Outside Options

Let us consider the total profits of firm $\mathcal{U}$ (res. firms $\mathcal{D}_h$ and $\mathcal{D}_l$) at the bargaining stage as the sum of the profits from the sales of the input to the downstream firms (res. the profits from the sales of the variants of the good to the consumers) and of the fixed fees as in the following.[1]

$$\hat{\Pi}(T_h, T_l) = \hat{V}(w_h, w_l) + t_h + t_l, \quad \hat{\pi}_h(T_h, w_l) = \hat{v}_h(w_h, w_l) - t_h, \quad \hat{\pi}_l(T_l, w_h) = \hat{v}_l(w_h, w_l) - t_l. \tag{83}$$

As pointed out in the paper, neither $\hat{V}(\cdot)$ nor $\hat{v}_i(\cdot), i = h, l$ depend on the fixed feeds t_h and t_l.

Let $T_i^* \equiv (w_i^*, t_i^*), * \in \{N, C, M\}$ be the sub-game equilibrium pre-contractual arrangement executed between firm $\mathcal{U}$ and $\mathcal{D}_i$, and d_i^* the induced outside option for firm $\mathcal{U}$ in the negotiation with firm $\mathcal{D}_i, i = h, l, i \neq j$. The Nash products write

$$NP_i(T_i, T_j^*) = \left[\hat{V}(w_i, w_j^*) + t_i + t_j^* - d_i^*\right]^{\mu} \left[\hat{v}_i(w_i, w_j^*) - t_i\right]^{1-\mu} \tag{84}$$

It is here worth noticing that, within each $NP_i(\cdot)$, no outside option depends on the ongoing negotiation between firm $\mathcal{U}$ and $\mathcal{D}_i$, indeed, in the case of contingent contracts $d_i^C = \frac{\mu u_j}{4}$, in the case of non-contingent contracts $d_i^N = \hat{\Pi}_m(T_j^N) = \hat{V}_m(w_j^N) + t_j^N$ and the case of mixed contracts being a combination of case N and C.[2]

As is well known (see e.g. **?**), the maximization of (84) with respect to $T_i, i = h, l$ can be split in two steps: first identify the w_i that maximizes the joint surplus, then apportion the maximized surplus it according to the bargaining weights. The first-order conditions with respect to t_i yield

$$t_i(w_i, w_j^*) = \mu \hat{v}_i(w_i, w_j^*) - (1 - \mu) \left[\hat{V}(w_i, w_j^*) + t_j^* - d_i^*\right], \quad i = h, l, i \neq j, \tag{85}$$

[1] See the paper, eqs. (13) - (15).
[2] See the paper, eq. (7)

which can be plugged back into (84) that, in turn, reduces to

$$\mu^{\mu}(1-\mu)^{1-\mu}\left[\hat{V}(w_i, w_j^*) + \hat{v}(w_i, w_j^*) + t_j^* - d_i^*\right] \tag{86}$$

It is straightforward to observe that choosing w_i to maximize (86) amounts to choosing the input price to maximize the sum $\hat{V}(w_i, w_j^*) + v_i(w_i, w_j^*), i = h, l, i \neq j$, which, as observed above, does not depend on the transfers and thus on the type of non-exclusive pre-contractual arrangement. At the optimal input prices (w_i^*, w_j^*), equations (85) define the subgame equilibrium transfers (t_i^*, t_j^*).

6.1 Non-contingent contracts

Under non-contingent contracts $d_i^N = \hat{V}_m(w_j^N) + t_j^N$, so that, in $NP_i^N(\cdot)$, t_j^* cancels out, which implies that

$$t_i^N = \mu\hat{v}_i(w_i^N, w_j^N) - (1-\mu)\left[\hat{V}(w_i^N, w_j^N) - \hat{V}_m(w_j^N)\right], \quad i = h, l, i \neq j. \tag{87}$$

This last equation shows that, under non-contingent contracts, the fixed fees are independent one from the other. Furthermore, it is easy to ascertain that, as μ tends to zero, the subgame equilibrium value of each outside option tends to

$$\lim_{\mu \to 0} d_i^N = \hat{V}_m(w_j^N) + \hat{V}_m(w_i^N) - \hat{V}(w_i^N, w_j^N) \tag{88}$$

which is positive by Lemmata 1 and 2 (see the paper).

6.2 Contingent contracts

Under contingent contracts, we have $\lim_{\mu \to 0} d_i^C = 0$.

6.3 Mixed contracts

In this case we have that $d_h^M = \mu\frac{u_l}{4}$ and $d_l^M = \hat{V}_m(w_h^M) + t_h^M$, whence in $NP_l^M(\cdot)$ t_h^M cancels out, while in $NP_h^M(\cdot)$ t_l^M does not. The subgame equilibrium transfers are

$$\begin{aligned} t_h^M =& \mu\{(1-\mu)[\frac{u_l}{4} - \hat{V}(w_h^M, w_l^M) - \hat{v}_l(w_h^M, w_l^M)] + \hat{v}_h(w_h^M, w_l^M)\} - (1-\mu)^2\hat{V}_m(w_h^M), \\ t_l^M =& \mu\hat{v}_l(w_h^M, w_l^M) - (1-\mu)[\hat{V}(w_h^M, w_l^M) - \hat{V}_m(w_h^M)]. \end{aligned} \tag{89}$$

It is a matter of simple calculations to observe that, in this case as well the values of the outside options for firm $\mathcal{U}$ tend to zero as μ tends to zero:

$$\lim_{\mu\to 0} d_i^M = 0, i = h, l. \tag{90}$$