Regularity results for the minimum time function with Hörmander vector fields

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REGULARITY RESULTS FOR THE MINIMUM TIME FUNCTION WITH HÖRMANDER VECTOR FIELDS

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ABSTRACT. In a bounded domain of $\mathbb{R}^n$ with smooth boundary, we study the regularity of the viscosity solution, T, of the Dirichlet problem for the eikonal equation associated with a family of smooth vector fields $\{X_1, \ldots, X_N\}$, subject to Hörmander's bracket generating condition. Due to the presence of characteristic boundary points, singular trajectories may occur in this case. We characterize such trajectories as the closed set of all points at which the solution loses point-wise Lipschitz continuity. We then prove that the local Lipschitz continuity of T, the local semiconcavity of T, and the absence of singular trajectories are equivalent properties. Finally, we show that the last condition is satisfied when the characteristic set of $\{X_1, \ldots, X_N\}$ is a symplectic manifold. We apply our results to Heisenberg's and Martinet's vector fields.

1. INTRODUCTION

In a bounded open set $\Omega \subset \mathbb{R}^n$ with smooth boundary, Γ, we study the regularity of the viscosity solution, T, to the Dirichlet problem

$$(1.1) \qquad \begin{cases} \sum_{j=1}^N (X_j T)^2(x) = 1 & \text{in} \quad \Omega \\ T = 0 & \text{on} \quad \Gamma, \end{cases}$$

where $\{X_1, \ldots, X_N\}$ is a system of smooth vector fields which satisfies Hörmander's bracket generating condition ([14]). It is well known that such a solution may be interpreted as the sub-Riemannian distance

from the boundary of Ω when $\{X_1, \ldots, X_N\}$ span a distribution or, in control theory, as a certain minimum time function. While structural properties of the singular set and the regularity of the Riemannian distance function have been widely investigated (see, e.g., [2], [5], [6], [8], [9]), similar issues have not been much addressed for the solution of (2.7) in the sub-Riemannian case or, more generally, when the quadratic form associated with the eikonal equation in (2.7) fails to be positive definite (see, e.g., [3]). The analysis of such questions is the main purpose of this paper.

In our approach, we will often use the characterization of the unique solution, T, of (2.7) as the value function of the time optimal control problem with target Γ and state equation

$$(1.2) \qquad \begin{cases} y'(t) = \sum_{j=1}^{N} u_j(t) X_j(y(t)) & (t \geq 0) \\ y(0) = x, \end{cases}$$

where $u : [0, +\infty[\to \overline{B}_1(0)$. In particular, a crucial role in our investigation will be played by the so-called *singular time-optimal trajectories* of (1.2), a well-known object in geometric control theory (see, e.g., [10]). Singular trajectories are time-optimal trajectories that can be completed to a solution of the characteristic system associated with (1.1), which belongs to the *characteristic set* of $\{X_1, \ldots, X_N\}$ and satisfies a suitable transversality condition (see Definition 3.1 below). Equivalently, singular time-optimal trajectories can be identified as those time-optimal trajectories that hit Γ at a *characteristic point*, that is, a point at which $X_1, \ldots, X_N$ are tangent to Γ (see Theorem 3.1 below).

The fact that the existence of singular trajectories may destroy the smoothness (subanalyticity) of a solution of a first order Hamilton-Jacobi equation was already observed in [19], [1], and [21] (see also Remark 4.3 below). In this paper, we interested in a more basic smoothness threshold—namely Lipschitz continuity—which, as we explain below, opens the way to higher regularity properties for the solution of (2.7). Therefore, we begin our analysis by giving a *necessary and sufficient condition* for point-wise Lipschitz continuity, showing that T fails to be Lipschitz continuous at a point $x \in \Omega$ if and only if x is the starting point of a singular time-optimal trajectory (Theorem 3.2). By dynamic programming, this result implies that the set of all singular time-optimal trajectories consists—as a point set—of all the points of $\overline{\Omega}$ at which T fails to be Lipschitz (Corollary 3.1).

We note that point-wise Lipschitz continuity is a very weak property. Nevertheless, Theorem 3.2 can be used to derive a classical Lipschitz regularity result whenever one can exclude the presence of singular time-optimal trajectories. This is the viewpoint leading to Theorem 4.1, which ensures the equivalence of the following three properties:

(a) system (1.2) admits no singular time-optimal trajectory;
(b) T locally is semiconcave in Ω;
(c) T is locally Lipschitz in Ω.

We observe that, in [7], the local semiconcavity of the sub-Riemannian distance to a point x_0, $d_{SR}(x_0, \cdot)$, is proved assuming the absence of singular time-optimal trajectories, but without giving any estimate of the dependence on x_0 of the semiconcavity constant of $d_{SR}(x_0, \cdot)$. Thus, the local semiconcavity of the minimum time function for a *general* target (i.e., (a) $\Rightarrow$ (b)) is not a direct consequence of the result of [7], even when $\{X_1, \ldots, X_N\}$ generates a distribution.

We conclude the introduction by quickly mentioning further results that complete our exposition, while outlining the structure of this paper. In Section 2, we recall all required notions concerning Hörmander vector fields, characteristic sets, time optimal control problems, and viscosity solutions of related eikonal equations. In Section 3, we introduce singular time-optimal trajectories and develop our analysis of the behaviour of T along such trajectories. Based on these results, in Section 4, we obtain the aforementioned characterization of the local Lipschitz regularity of T, we give sufficient conditions on the characteristic set of $\{X_1, \ldots, X_N\}$ in order to exclude the presence of singular trajectories, and we discuss the examples of Heisenberg's and Martinet's vector fields as applications of our results. We conclude the paper with Appendix A, where we prove a regularity result for T at the boundary of Ω (Theorem 4.2), which might be known to the reader in other forms than the one which fits the analysis of this paper.

2. HÖRMANDER VECTOR FIELDS AND MINIMUM TIME FUNCTION

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and assume that the boundary of Ω, Γ, is a smooth manifold of dimension $n - 1$.

We denote by $VF(\Omega)$ the space of all C^∞ vector fields on Ω. Given any two such vector fields,

$$X(x) = \sum_{i=1}^n f_i(x)\partial_{x_i} \quad \text{and} \quad Y(x) = \sum_{i=1}^n g_i(x)\partial_{x_i} \qquad (x \in \Omega),$$

with $f_i, g_i \in C^\infty(\Omega)$ $(i = 1, \ldots, n)$, we denote by $[X, Y]$ the Lie bracket

$$[X, Y](x) = \sum_{i=1}^{n} h_i(x)\partial_{x_i} \quad \text{where} \quad h_i = \sum_{j=1}^{n} \left(f_j \partial_{x_j} g_i - g_j \partial_{x_j} f_i \right).$$

Let $N \geq 2$ be an integer and let $X_1, \ldots, X_N \in VF(\Omega)$. The Lie algebra generated by $\{X_i\}_{i=1}^{N}$, $\text{Lie}(\{X_i\}_{i=1}^{N})$, is the smallest subspace of $VF(\Omega)$, containing $\{X_i\}_{i=1}^{N}$, which is invariant under the action of Lie brackets. So, we have that

$$\text{Lie}(\{X_i\}_{i=1}^{N}) = \bigcup_{k=1}^{\infty} \text{Lie}^k(\{X_i\}_{i=1}^{N}),$$

where $\text{Lie}^k(\{X_i\}_{i=1}^{N})$ is defined recursively by taking

$$\text{Lie}^1(\{X_i\}_{i=1}^{N}) = \text{span}\,\{X_i\}_{i=1}^{N}$$

and, for $k \geq 1$,

$$\text{Lie}^{k+1}(\{X_i\}_{i=1}^{N})$$
$$= \text{span}\left(\text{Lie}^k(\{X_i\}_{i=1}^{N}) \cup \left\{ [X, X_j] \ : \ X \in \text{Lie}^k(\{X_i\}_{i=1}^{N}), \ j = 1, \ldots, N \right\} \right).$$

For all $x \in X$ we also set

$$\begin{cases} \text{Lie}^k(\{X_i\}_{i=1}^{N})[x] = \left\{ X(x) \ : \ X \in \text{Lie}^k(\{X_i\}_{i=1}^{N}) \right\} & \forall k \geq 1 \\ \text{Lie}(\{X_i\}_{i=1}^{N})[x] = \left\{ X(x) \ : \ X \in \text{Lie}(\{X_i\}_{i=1}^{N}) \right\}. \end{cases}$$

We say that $\{X_i\}_{i=1}^{N} \subset VF(\Omega)$ is *a system of Hörmander vector fields* on Ω if the following bracket generating condition holds:

$$(2.3) \qquad\qquad \text{Lie}(\{X_i\}_{i=1}^{N})[x] = \mathbb{R}^n \qquad \forall x \in \Omega.$$

Moreover, we say that $\{X_i\}_{i=1}^{N}$ is a system of Hörmander vector fields on $\overline{\Omega}$ if the bracket generating condition holds true on some open set $\Omega' \supset \overline{\Omega}$. Finally, we say that $\{X_i\}_{i=1}^{N}$ is *strongly bracket generating* on Ω if for every $v = (v_1, \ldots, v_N) \in \mathbb{R}^n \setminus \{0\}$

$$(2.4) \quad \text{span}\{X_i\}_{i=1}^{N}[x] + \text{span}\left\{ \sum_{j=1}^{N} [X_j, X_i] \right\}_{i=1}^{N} [x] = \mathbb{R}^n \qquad \forall x \in \Omega.$$

The following hypotheses (H) will be assumed throughout:

$(H1)$ $\Omega \subset \mathbb{R}^n$ *is a bounded open set with boundary* Γ *of class* C^∞,
$(H2)$ $\{X_1, \ldots, X_N\}$ *is a system of Hörmander vector fields on* $\overline{\Omega}$.

Let us point out that, here, $X_1, \ldots, X_N$ need not be linearly independent, nor we suppose $N < n$.

We define the *Hamiltonian* associated with $\{X_1, \ldots, X_N\}$ by

$$(2.5) \qquad h(x,p) = \sum_{j=1}^{N} X_j(x,p)^2, \quad (x,p) \in T^*\Omega',$$

where we have set, with a slight abuse of notation,

$$X_j(x,p) := \langle X_j(x), p \rangle \qquad \forall\, (x,p) \in T^*\Omega', \forall j = 1, \ldots, N$$

(note that $T^*\Omega'$ can be identified with $\Omega' \times \mathbb{R}^n$).

The *characteristic set* of $\{X_1, \ldots, X_N\}$ is given by

$$(2.6) \quad \mathrm{Char}(X_1, \ldots X_N) = \big\{ (x,p) \in \Omega' \times (\mathbb{R}^n \setminus \{0\}) \ : \ h(x,p) = 0 \big\}.$$

Finally, a point $x \in \Gamma$ is called *characteristic* if the linear space generated by $X_1(x), \ldots, X_N(x)$ is contained in the tangent space to Γ at x. We denote by $E \subset \Gamma$ the set of all characteristic points. The following result, essential for this paper, is due to Derridj [11].

Theorem 2.1. *Under assumption* (H), *E is a closed subset of Γ of $(n-1)$-dimensional Hausdorff measure zero.*

Under assumption (H), consider the Dirichlet problem

$$(2.7) \qquad \begin{cases} \sum_{j=1}^{N} (X_j T)^2(x) = 1 & \text{in} \quad \Omega \\ T = 0 & \text{on} \quad \Gamma \end{cases}$$

and observe that, since the Hamiltonian $h(x,p)$ is not strictly convex in p, characteristic points may appear. It is well known that (2.7) admits a unique continuous viscosity solution $T : \overline{\Omega} \to \mathbb{R}$. Indeed, taking Γ as the target set, the minimum time function associated with $\{X_1, \ldots, X_N\}$ is a solution of the above equation. Such a function is defined as follows. Given $x \in \overline{\Omega}$ and a measurable control

$$u = (u_1, \ldots, u_N) : [0, +\infty[\to \mathbb{R}^N,$$

taking values in the unit ball of $\mathbb{R}^N$, let us denote by $y^{x,u}(\cdot)$ the unique solution of the Cauchy problem

$$(2.8) \qquad \begin{cases} y'(t) = \sum_{j=1}^{N} u_j(t) X_j(y(t)) & (t \geq 0) \\ y(0) = x. \end{cases}$$

Define the *transfer time* to Γ as

$$\tau_\Gamma(x,u) = \inf \big\{ t \geq 0 \ : \ y^{x,u}(t) \in \Gamma \big\}.$$

Clearly, $\tau_\Gamma(x,u) \in [0, +\infty]$. The *Minimum Time Problem* with target Γ is the following:

(MTP) To minimize $\tau_\Gamma(x,u)$ over all controls $u : [0, +\infty[\to \overline{B}_1(0)$.

Then, the *minimum time function*, defined as

$$T(x) = \inf_{u(\cdot)} \tau_\Gamma(x, u) \qquad (x \in \overline{\Omega}),$$

turns out to be the unique viscosity solution of the Dirichlet problem (2.7) (see, e.g., [4]). It is well-known that Hörmander's bracket generating condition implies that (2.8) is small time locally controllable. Hence, T is finite and continuous (see, e.g., [4]).

Remark 2.1. We recall that a $u(\cdot)$ is called an *optimal control* at a point $x \in \Omega$ if $T(x) = \tau_\Gamma(x, u)$. The corresponding solution of (2.8), $y^{x,u}$, is called the *time-optimal trajectory* at x associated with u.

A more precise description of the continuity properties of T can be given by looking at the maximal length of commutators needed to generate $\mathbb{R}^n$, that is, the function $k(\cdot)$ defined as follows:

$$(2.9) \quad k(x) = \min \left\{ k \geq 1 \; : \; \mathrm{Lie}^k j(\{X_i\}_{i=1}^N)[x] = \mathbb{R}^n \right\} \qquad (x \in \Omega).$$

We also define

$$(2.10) \qquad\qquad r_\Omega = \max\{k(x) \; : \; x \in \overline{\Omega}\}.$$

Then, thanks to the Baker-Campbell-Hausdorff formula (see, e.g., [18]), T turns out to be Hölder continuous of exponent $1/r_\Omega$ on $\overline{\Omega}$.

Next, observe that the equation in (2.7) can be recast in the form

$$(2.11) \qquad\qquad \langle A(x)DT(x), DT(x) \rangle = 1 \quad \text{in} \quad \Omega,$$

where $A(\cdot)$ is a suitable positive semidefinite $n \times n$ matrix with smooth entries and $DT = (\partial_{x_1} T, \ldots, \partial_{x_n} T)$. If $A(\cdot)$ is nondegenerate, then a classical viscosity argument ensures that any continuous viscosity solution of (2.11) is locally Lipschitz continuous in Ω. Furthermore, under very weak regularity conditions[1] on the entries of $A(\cdot)$, one can show that any continuous viscosity solution of (2.11) is locally semiconcave in Ω (see [2]). We recall that a function $T : \Omega \to \mathbb{R}$ is *locally semiconcave* in Ω if for every $V \Subset \Omega$ there exists a constant C such that $D^2 T \leq CI$ in $\mathcal{D}'(V)$ (in the sense of quadratic forms). We point out that, for the aforementioned regularity results, the strict convexity of the map $p \mapsto \langle A(x)p, p \rangle$ is a crucial assumption.

On the contrary, when $A(\cdot)$ in (2.11) is associated with a system of Hörmander's vector fields with $N < n$, known results mostly concern the continuity of solutions. For instance, it is known that any continuous viscosity solution of (2.11) is Hölder continuous of exponent $1/r_\Omega$

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[1]It suffices to assume that $x \mapsto \langle A(x)p, p \rangle$ is semiconvex (i.e. it can be locally written as the sum of a convex with a smooth function) uniformly w.r.t. p in the unit sphere of $\mathbb{R}^n$.

given by (2.10) (see, e.g., [12]). On the other hand, stronger regularity results are known under more restrictive assumption. For instance, as we recalled in the introduction, the following semiconcavity result holds for the sub-Riemannian setting ([7]): if $X_1(x), \ldots, X_N(x)$ are linearly independent for every $x \in \mathbb{R}^n$ and strongly bracket generating on $\mathbb{R}^n$, then for every $x_0 \in \mathbb{R}^n$ the minimum time function associated with the control system (2.8) and target $\{x_0\}$ is locally semiconcave—hence, locally Lipschitz—in $\mathbb{R}^n \setminus \{x_0\}$ (see also Remark 4.3 below).

3. Singular time-optimal trajectories

The notion of singular trajectory that we give below plays a key role in our analysis. For any boundary point $z \in \Gamma$ we denote by $\nu(z)$ the outward unit normal to Γ at z and we set

$$N_\Gamma(z) := \{\lambda \nu(z) \ : \ \lambda \geq 0\}.$$

Definition 3.1. *Let* $x \in \Omega$ *and let* $y(\cdot) = y^{x,u}(\cdot)$ *be a time-optimal trajectory, with* $u : [0, T(x)] \to \overline{B}_1(0)$. *We say that* $y(\cdot)$ *is* singular *if there exists an absolutely continuous arc* $p : [0, T(x)] \to \mathbb{R}^n \setminus \{0\}$ *such that, for a.e.* $t \in [0, T(x)]$,

$$(3.1) \quad p'_k(t) = -\sum_{j=1}^N u_j(t)\langle \partial_{x_k} X_j(y(t)), p(t)\rangle, \quad \langle X_k(y(t)), p(t)\rangle = 0,$$

for every $k = 1, \ldots, N$, *and*

$$(3.2) \qquad\qquad p(T(x)) \in N_\Gamma(y(T(x))).$$

Remark 3.2. (i) Our use of the adjective "singular" in Definition 3.1 is classical but it is also motivated by Proposition 3.1 below.

(ii) We recall that the strong bracket generating condition (2.4) ensures the absence of singular time-optimal trajectories (see, e.g., [7]).

(iii) We point out that, introducing the Control Theory Hamiltonian

$$(3.3) \qquad\qquad H(x, p, u) = \sum_{j=1}^N u_j X_j(x, p),$$

the optimal triple (y, u, p) arising from Definition 3.1 satisfies, for a.e. $t \in [0, T(x)]$, the Hamiltonian system

$$(3.4) \qquad \begin{cases} y'(t) = D_p H(y(t), p(t), u(t)) \\ p'(t) = -D_x H(y(t), p(t), u(t)) \end{cases}$$

In other words, a time-optimal trajectory is singular if it can be lifted in the phase space in such a way that the lifted trajectory:

- satisfies the Hamiltonian system (3.4) (with Hamiltonian given by (3.3)) together with the transversality condition (3.2), and
- lies in the characteristic set $\mathrm{Char}(X_1, \dots, X_N)$.

The following characterization of singular time-optimal trajectories will be used throughout the paper.

Theorem 3.1. *Assume (H). Let $x \in \Omega$ and let $y^{x,u}$ be a time-optimal trajectory. Then, $y^{x,u}$ is singular if and only if $y^{x,u}(T(x)) \in E$.*

Proof. Let $x \in \Omega$ and let $u(\cdot)$ a time-optimal control. Then, by the Pontryagin maximum principle there exists an absolutely continuous function $p : [0, T(x)] \to \mathbb{R}^n \setminus \{0\}$ (adjoint state) such that the pair $(y^{x,u}, p)$ satisfies the Hamiltonian system (3.4) and the transversality condition (3.2). Furthermore, for a.e. $t \in [0, T(x)]$,

$$(3.5) \quad H(y^{x,u}(t), p(t), u(t)) = \max_{u \in \overline{B}_1(0)} \sum_{j=1}^{N} u_j \langle X_j(y^{x,u}(t)), p(t) \rangle$$

$$= \Big(\sum_{j=1}^{N} \big\langle X_j(y^{x,u}(t)), p(t) \big\rangle^2 \Big)^{1/2} = \sqrt{h(y^{x,u}(t), p(t))}.$$

Also, the function $[0, T(x)] \ni t \mapsto h(y^{x,u}(t), p(t))$ is constant.

Now, suppose $y^{x,u}(T(x)) \notin E$. Then

$$h(y^{x,u}(T(x)), p(T(x))) \neq 0$$

So, by the constancy of the Hamiltonian, $y^{x,u}$ cannot be singular.

Vice versa, assume $y^{x,u}(T(x)) \in E$. Then, again by Pontryagin's principle, we deduce that $(y^{x,u}(t), p(t)) \in \mathrm{Char}(X_1, \dots, X_N)$, for every $t \in [0, T(x)]$, that is, $y^{x,u}$ is a singular time-optimal trajectory. $\square$

The following is a point-wise notion of Lipschitz continuity (see also Federer [13]).

Definition 3.2. *We say that a function $f : \overline{\Omega} \to \mathbb{R}$ is Lipschitz at a point $x_0 \in \overline{\Omega}$ if there exists a neighbourhood U of x_0 and a constant $L \geq 0$ such that*

$$|f(x) - f(x_0)| \leq L|x - x_0| \qquad \forall x \in U \cap \overline{\Omega}.$$

Equivalently, f is Lipschitz at x_0 if and only if

$$\limsup_{\overline{\Omega} \ni x \to x_0} \frac{|f(x) - f(x_0)|}{|x - x_0|} < \infty.$$

Observe that a function may well-be Lipschitz continuous at x_0 without being Lipschitz on any neighbourhood of x_0. However, the interest of such kind of continuity is made clear by our next result.

Theorem 3.2. *Assume (H) and let $x_0 \in \Omega$. Then T fails to be Lipschitz at x_0 if and only if there exists a singular time-optimal $y^{x_0, u}$*
.

For the proof Theorem 3.2 we borrow the following notion from non-smooth analysis.

Definition 3.3 (Proximal normals). *Let $S \subset \mathbb{R}^n$ be a closed set. A vector $v \in \mathbb{R}^n$ is called a proximal normal to S at x if there exist $\delta > 0$ and $C > 0$ (possibly depending on x, v) such that*

$$(3.6) \qquad \langle v, y - x \rangle \leq C|y - x|^2 \qquad \forall\, y \in B_\delta(x) \cap S.$$

The set of all proximal normals to S at x will be denoted by $N_S^P(x)$.

We recall that the the hypograph of a function $f : \overline{\Omega} \to \mathbb{R}$ is the set

$$\mathrm{hypo}(f) = \big\{ (x, \alpha) \in \overline{\Omega} \times \mathbb{R} \; : \; \alpha \leq f(x) \big\}.$$

Definition 3.4 (Horizontal proximal supergradients). *The set of horizontal proximal subgradients of f at a point $x \in \Omega$ is given by*

$$\partial^{P, \infty} f(x) = \big\{ p \in \mathbb{R}^n \; : \; (-p, 0) \in N_{\mathrm{hypo}(f)}(x, f(x)) \big\}.$$

We note that, in order for f to be Lipschitz at a point $x_0 \in \Omega$, it is necessary (but not sufficient) that $\partial^{P, \infty} f(x_0) = \{0\}$.

Proof of Theorem 3.2. Assuming that $y^{x_0, u}$ is a singular time-optimal trajectory, let us show that $\partial^{P, \infty} T(x_0)$ contains a nonzero vector, which implies that T cannot be Lipschitz at x_0. Here we adapt, to systems of Hörmanders vector fields, an argument used in [5, Theorem 4.1] to study time optimal control problems for differential inclusions. We observe that the present context is different from the one in [5] in that the nondegeneracy assumption of [5] is not satisfied by our control system. We will use the dual arc $p(\cdot)$ for $y^{x_0, u}$ provided by Pontryagin's principle. Observe that, in view of (3.2), $\exists\, \delta > 0$ such that

$$\overline{B}_\delta \big(y^{x_0, u}(T(x_0)) + \delta \nu(x_0) \big) \cap \overline{\Omega} = \{ y^{x_0, u}(T(x_0)) \},$$

or

$$(3.7) \quad \big\langle p(T(x_0)), y - y^{x_0, u}(T(x_0)) \big\rangle \leq \frac{1}{\delta} |y - y^{x_0, u}(T(x_0))|^2 \quad \forall\, y \in \overline{\Omega},$$

that is, $(p(T(x_0)), 0) \in N_{\mathrm{hypo}(T)}(y^{x_0, u}(T(x_0)), 0)$.

We now proceed to show that

$$(3.8) \qquad\qquad -p(0) \in \partial^{P, \infty} T(x_0),$$

or, equivalently,

$$(p(0), 0) \in N_{\mathrm{hypo}(T)}(x_0, T(x_0)).$$

We have to prove that there exists a constant $C_1 > 0$ such that, for every $x \in \overline{\Omega}$ with $|T(x) - T(x_0)| < 1$, we have that

$$(3.9) \quad \langle p(0), x - x_0 \rangle \leq C_1 \big(|x - x_0|^2 + (\alpha - T(x_0))^2 \big) \quad \forall \, \alpha \leq T(x).$$

Set $T_0 = T(x_0)$ and $T_1 = T(x)$. We shall analyse the two cases $T_1 \leq T_0$ and $T_1 > T_0$ separately.

Case 1: $\boxed{T_1 \leq T_0}$

It suffices to verify (3.9) with $\alpha = T_1$. Set

$$\overline{x}_0 = y^{x_0,u}(T_1) \quad \text{and} \quad \overline{x} = y^{x,u}(T_1).$$

Notice that $T(\overline{x}_0) = T_0 - T_1 \in [0, 1[$ and $\overline{x} \in \overline{\Omega}$. Furthermore

$$(3.10) \qquad |y^{x,u}(t) - y^{x_0,u}(t)| \leq C_1 |x - x_0| \quad \forall \, t \in [0, T_1]$$

for a suitable constant $C_1 > 0$ depending only on Ω, the C^1 norms of the vector fields $X_1, \ldots, X_N$, and $\max_{\overline{\Omega}} T$. Then, we find that

$$(3.11) \quad \langle p(0), x - x_0 \rangle = \langle p(T_1), \overline{x} - \overline{x}_0 \rangle$$

$$- \int_0^{T_1} \frac{d}{dt} \langle p(t), y^{x,u}(t) - y^{x_0,u}(t) \rangle \, dt.$$

We now claim that, for every $t \in [0, T_1]$,

$$(3.12) \qquad \left| \frac{d}{dt} \langle p(t), y^{x,u}(t) - y^{x_0,u}(t) \rangle \right| \leq C_2 |p(t)| \, |y(t) - y^{x,u}(t)|^2,$$

for a suitable constant C_2 depending on Ω, the C^2 norms of the vector fields $X_1, \ldots, X_N$, and $\max_{\overline{\Omega}} T$. Indeed, recalling (3.1) we deduce that

$$\frac{d}{dt} \langle p(t), y^{x,u}(t) - y^{x_0,u}(t) \rangle$$

$$= \sum_{j=1}^N u_j(t) \Big\langle p(t), X_j(y^{x,u}(t)) - X_j(y^{x_0,u}(t)) \Big\rangle$$

$$- \sum_{j=1}^N u_j(t) \Big\langle p(t), dX_j(y^{x_0,u}(t)))(y^{x,u}(t) - y^{x_0,u}(t)) \Big\rangle.$$

Hence, (3.12) follows.

Then, recalling (3.12) , (3.11), and (3.10), we find

$$(3.13) \qquad \langle p(0), x - x_0 \rangle \leq C_3 |x - x_0|^2 + \langle p(T_1), \overline{x} - \overline{x}_0 \rangle.$$

In order to bound the last term in the above inequality we observe that

$$\langle p(T_1), \overline{x} - \overline{x}_0 \rangle = \langle p(T_0), \overline{x} - \overline{x}_0 \rangle + \langle p(T_1) - p(T_0), \overline{x} - \overline{x}_0 \rangle$$

$$\leq \langle p(T_0), \overline{x} - \overline{x}_0 \rangle + C_4 \big(|T_1 - T_0|^2 + |\overline{x} - \overline{x}_0|^2 \big)$$

for a suitable constant C_4. Now, recall that

$$\langle p(T_0), \overline{x} - \overline{x}_0 \rangle = \langle p(T_0), y^{x,u}(T_1) - y^{x_0,u}(T_1) \rangle$$

Then, we have

$$\langle p(T_0), \overline{x} - \overline{x}_0 \rangle$$
$$= \langle p(T_0), y^{x_0,u}(T_0) - y^{x_0,u}(T_1) \rangle + \langle p(T_0), y^{x,u}(T_1) - y^{x_0,u}(T_0) \rangle$$

The second term can be estimated by (3.7):

$$\langle p(T_0), y^{x,u}(T_1) - y^{x_0,u}(T_0) \rangle$$
$$\leq \frac{1}{\delta} |y^{x,u}(T_1) - y^{x_0,u}(T_0)|^2 \leq C_5 \big(|x - x_0|^2 + (T_1 - T_0)^2 \big)$$

for some constant $C_5 > 0$. Moreover, we have that

$$\langle p(T_0), y^{x_0,u}(T_0) - y^{x_0,u}(T_1) \rangle = \sum_{j=1}^{N} \int_{T_1}^{T_0} u_j(t) \langle p(T_0), X_j(y^{x_0,u}(t)) \rangle dt.$$

Recalling that $y^{x_0,u}(T_0) \in E$, i.e.

$$\langle p(T_0), X_j(y^{x_0,u}(T_0)) \rangle = 0 \quad \text{for} \quad j = 1, \dots, N,$$

we deduce that

$$\langle p(T_0), y^{x_0,u}(T_0) - y^{x_0,u}(T_1) \rangle$$
$$= \sum_{j=1}^{N} \int_{T_1}^{T_0} u_j(t) \langle p(T_0), X_j(y^{x_0,u}(t)) - X_j(y^{x_0,u}(T_0)) \rangle \, dt \leq C_6 (T_1 - T)^2$$

for some constant $C_6 > 0$. Thus, (3.9) follows.

Case 2: $\boxed{T_1 > T_0}$
We have that

$$\langle p(0), x - x_0 \rangle$$
$$= \langle p(T_0), y^{x,u}(T_0) - y^{x_0,u}(T_0) \rangle - \int_0^{T_0} \frac{d}{dt} \langle p(t), y^{x,u}(t) - y^{x_0,u}(t) \rangle dt.$$

The second term on the right-hand side of the last identity can be bounded by a constant times $|x-x_0|^2$ like in (3.12). In order to estimate the remaining term it suffices to observe that $y^{x,u}(T_0) \in \overline{\Omega}$. So, (3.7) and (3.10) yield

$$\langle p(T_0), y^{x,u}(T_0) - y^{x_0,u}(T_0) \rangle \leq C_7 |x - x_0|^2.$$

This shows that, if $y^{x_0,u}$ is a singular time-optimal trajectory, then T fails to be Lipschitz at x_0.

In order to prove the converse, we argue by contradiction by assuming that T fails to be Lipschitz continuous at x_0 and that all optimal

trajectories from x_0 are not singular. Let $y^{x_0,u}$ be any time-optimal trajectory. So, $y^{x_0,u}(T(x_0)) \in \Gamma \setminus E$ by of Theorem 3.1. Then, appealing to Theorem 2.1 we can find $\delta > 0$ such that $B_\delta(y^{x_0,u}(T(x_0))) \cap E = \emptyset$. Furtheremore, taking δ small enough, we may assume that there exists a positive constant C such that

$$(3.14) \quad T(y) \le C \operatorname{dist}(y, \Gamma) \quad \text{for every} \quad y \in B_\delta(y^{x_0,u}(T(x_0))) \cap \overline{\Omega}.$$

Since T is not Lipschitz at x_0, there exists $\{x_j\} \subset \Omega$ such that

$$(3.15) \qquad \frac{|T(x_j) - T(x_0)|}{|x_j - x_0|} \ge j \text{ and } x_j \to x_0, \text{ as } j \to \infty.$$

Set

$$T_0 = T(x_0) \quad \text{and} \quad T_j = T(x_j).$$

Two possibilities my occur: either $T_0 \le T_j$ or $T_0 > T_j$. Observe that, in both cases,

$$(3.16) \qquad |y^{x_0,u}(t) - y^{x_j,u}(t)| \le C_1 |x_0 - x_j| \quad \forall t \in [0, \min\{T, T_j\}],$$

where C_1 is a positive constant depending on Ω, the C^1 norms of the vector fields $X_1, \ldots, X_N$, and $\max_\Omega T$.

Case 1: $\boxed{T_0 \le T_j}$
We have that

$$(3.17) \qquad j \le \frac{T_j - T_0}{|x_j - x_0|} \le \frac{T(y^{x_j,u}(T_0))}{|x_j - x_0|}.$$

Since $y^{x_0,u}(T_0) \in \Gamma$, by (3.16) we deduce that, for j large enough,

$$y^{x_j,u}(T_0) \in \overline{\Omega} \cap B_\delta(y^{x_0,u}(T_0)).$$

Then, by (3.17) and (3.14), we find

$$j \le C \frac{\operatorname{dist}(y^{x_j,u}(T_0), \Gamma)}{|x_j - x_0|} \le C \frac{|y^{x_j,u}(T_0) - y^{x_0,u}(T_0)|}{|x_j - x_0|}.$$

So, using (3.16) once again, we find the contradiction $j \le CC_1$.

Case 2: $\boxed{T_j < T_0}$
Denote by u_j a time-optimal control at x_j. After extending u_j to $[0, T_0]$ by setting $u_j(t) \equiv 0$ for $t \in [T_j, T_0]$, we choose a subsequence, still labeled $\{x_j\}$, such that y^{x_j,u_j} converges uniformly to y^{x_0,u_0} on all $[0, T_0]$, for a suitable $u_0(\cdot)$. Furthermore, because of $T_j \uparrow T_0$, we may assume that $y^{x_j,u_j}(T_j) \in \Gamma \cap B_\delta(y^{x_0,u_0}(T_0))$ for j large enough. Then, by dynamic programming, (3.15) yields

$$j \le \frac{T(y^{x_0,u_j}(T_j))}{|x_0 - x_j|}.$$

Using (3.16) (with u replaced by u_j) and (3.14) we find once more the contradiction

$$j \leq C \frac{|y^{x_0, u_j}(T_j) - y^{x_j, u_j}(T_j)|}{|x_0 - x_j|} \leq CC_1.$$

This implies that T cannot be Lipschitz continuous along y^{x_0, u_0}.

In order to show that y^{x_0, u_0} is singular observe that otherwise T would be Lipschitz continuous on a neighborhood of $y^{x_0, u_0}(T(x_0))$. Then Lipschitz regularity would propagate backwards along y^{x_0, u_0}, thus contradicting what we have just shown. □

Our next result is a straightforward consequence of Theorem 3.2 and dynamic programming.

Corollary 3.1. *Assume* (H). *Let* $x_0 \in \Omega$ *and let* $y^{x_0, u}$ *be a singular time-optimal trajectory. Then, for any* $t \in [0, T(x_0)[$, T *fails to be Lipschitz continuous at* $y^{x_0, u}(t)$.

4. REGULARITY

In this section we study the interior and boundary regularity of the solution of equation (2.7). Here is the main theorem of this section.

Theorem 4.1 (Interior regularity). *Under assumption* (H), *the following properties are equivalent:*

 (1) (MTP) *admits no singular time-optimal trajectory;*
 (2) T *is locally semiconcave in* Ω;
 (3) T *is locally Lipschitz continuous in* Ω.

Remark 4.3.   (i) The fact that the existence of singular optimal trajectories may destroy the regularity of a solution of a first order Hamilton-Jacobi equation was already observed by Sussmann (in an implicit form) in [19] and (explicitly) by Agrachev in [1]. The regularity these authors consider is subanalyticity of the point-to-point distance function associated with real analytic distributions. The aforementioned subanalyticity results were extended to solutions of the Dirichlet problem in [21].
  (ii) We observe that, as shown in [10], there exists an open dense set $\mathcal{O}$ in $(VF(\Omega))^N$ (equipped with the C^∞ topology) such that condition (1) in Theorem 4.1 holds whenever $(X_1, \ldots, X_N) \in \mathcal{O}$.
 (iii) In [7], the local semiconcavity of the sub-riemannian distance to a point x_0, $d_{SR}(x_0, x)$, is proved without giving any estimate of the dependance on x_0 of the semiconcavity constant of $d_{SR}(x_0, \cdot)$. Therefore, the semiconcavity of

$$T(x) = \min_{y \in \Gamma} d_{SR}(y, x)$$

is not a direct consequence of the result of [7].

The proof of Theorem 4.1 is based on methods from optimal control.

Proof. It is well-known that the local semiconcavity in Ω of T yields the local Lipschitz continuity of T in the same set. Thus, in order to prove the theorem, it suffices to show that the following implications hold true:

- (i) if (MTP) admits no singular time-optimal trajectories then T is locally semiconcave in Ω;
- (ii) if T is locally Lipschitz continuous in Ω then (MTP) admits no singular time-optimal trajectories.

Now, (ii) follows from Corollary 3.1. We therefore proceed to prove (i). First of all, let us recall that for a smooth controlled system the minimum time function is locally semiconcave whenever the target is a noncharacteristic smooth compact manifold. This result can be proved arguing as in [8]. For the reader convenience we give a sketch of the proof. It is well-known that since the target Γ is a smooth manifold, the Euclidean distance function is smooth on a neighborhood of Γ. Furthermore, since Γ is a noncharacteristic hypersurface, the minimum time function can be estimated from above by the Euclidean distance function in some neighborhood of Γ (see [8, Prop. 2.2]). Owing to these two facts, the minimum time function inherits the semiconcavity of the distance function in a neighborhood of Γ. Finally, by standard techniques which use time-optimal trajectories to "propagate" regularity, one can show that the semiconcavity of T holds true even away from the target.

Now, let $x_0 \in \Omega$ and consider all the time-optimal trajectories starting at x_0. We denote by $\{u_\alpha\}$, for α running in a suitable set of indices $\mathcal{A}(x_0)$, the set of all the optimal controls at x_0. By Theorem 3.1, we have that $y^{x_0,u_\alpha}(T(x_0)) \notin E$, for every $\alpha \in \mathcal{A}$. We claim that there exist a smooth compact submanifold $\Gamma_0 \subset \Gamma$, with $\Gamma_0 \cap E = \emptyset$, and a neighborhood of x_0, $V \Subset \Omega$, such that

$$(4.1) \qquad y^{x,u_\alpha}(T(x)) \in \Gamma_0, \quad \forall x \in V , \ \forall \alpha \in \mathcal{A}(x).$$

Indeed, thanks to Theorem 2.1, for any $x_0 \in \Omega$ there exists a smooth compact submanifold $\Gamma_0 \subset \Gamma$ such that

$$(4.2) \qquad \begin{cases} \Gamma_0 \cap E = \emptyset \\ y^{x_0,u_\alpha}(T(x_0)) \in \Gamma_0, \quad \forall \alpha \in \mathcal{A}(x_0) \\ \inf\left\{ |y^{x_0,u_\alpha}(T(x_0)) - y| \ : \ y \in \Gamma \setminus \Gamma_0, \ \alpha \in \mathcal{A}(x_0) \right\} > 0. \end{cases}$$

Arguing by contradiction, let us assume that there exist a sequence $x_j \in \Omega$, converging to x_0, and a sequence of optimal controls $u_j \in \mathcal{A}(x_j)$

such that $y^{x_j, u_j}(T(x_j)) \notin \Gamma_0$. Then, possibly taking a subsequence, we deduce that there exists $\bar{u} \in \mathcal{A}(x_0)$ such that

$$\lim_{j \to \infty} y^{x_j, u_j}(T(x_j)) = y^{x_0, \bar{u}}(T(x_0))$$

in contradiction with (4.2). Thus, (4.1) holds true.

This shows that the restriction of T to the set V coincides with the restriction to V of the minimum time function for the control system (2.8) with target Γ_0. Since Γ_0 is a smooth compact manifold, by the first part of this proof we deduce that T is semiconcave in V. A compactness argument then shows that T is semiconcave on every set $C \Subset \Omega$, thus completing the proof of (i). $\qquad\square$

Next, we give an example showing that singular time-optimal trajectories may well occur for the problem of interest to this paper.

Example 1. *In $\mathbb{R}^3$ consider vector fields*

$$X_1 = \partial_{x_1}, \qquad X_2 = (1 - x_1)\partial_{x_2} + x_1^2 \partial_{x_3}.$$

Then, there exists a bounded open set $\Omega \subset \mathbb{R}^3$, with C^∞ boundary, such that the viscosity solution of the Dirichlet problem

(4.3)
$$\begin{cases} (X_1 T)^2 + (X_2 T)^2 = 1 & in \quad \Omega, \\[2mm] T|_\Gamma = 0, \end{cases}$$

fails to be locally Lipschitz in Ω.

Proof. For fixed $a > 0$, consider the subset of $\mathbb{R}^3$

$$\Gamma_a = \left\{ (x_1, x_2, -(x_2 - a)^2) \ : \ x_1 \in \mathbb{R}, x_2 > 0 \right\}$$

and the controlled system

(4.4)
$$\begin{cases} \dot{y}_1 = u_1 \\ \dot{y}_2 = u_2 (1 - y_1) \\ \dot{y}_3 = u_2 y_1^2 \end{cases}$$

with initial condition $y(0) = (0, 0, 0)$. We denote by T_{Γ_a} the minimum time function associated with system (4.4) and target Γ_a.

We claim that, for $a < 1$, $T_{\Gamma_a}(0) = a$ and $(0, t, 0)$, $t \in [0, a]$, is a singular time-optimal trajectory. Indeed:

- taking $p(t) = (0, 0, -1)$, $y(t)$ can be lifted to a solution of (3.4) (in the characteristic set of the vector fields) satisfying the transversality condition (3.2);
- $y(t)$ reaches Γ_a in time a;

- in order to reach the target, starting at the origin, $\dot{y}_2$ should be positive whilst $\dot{y}_3$ should be negative (this can be done provided that $1-y_1 < 0$ but the minimal time, in order to have $1-y_1 < 0$, is larger than 1.)

Hence, for any $a \in]0, 1[$ we have that $T_{\Gamma_a}(0) = a$.

In order to complete the proof it remains to define the set Ω. In $\mathbb{R}^3$, consider the Euclidean open ball with center at $(0, a, 0)$ and radius $a/2$ and denote such a ball by B. Then, for every $b \in]a/2, a[$,

$$T_{\Gamma_a}(0, b, 0) = T_{\Gamma_a \cap B}(0, b, 0) = a - b.$$

It is clear that the hypersurface $\Gamma_a \cap B$ can be smoothly extended outside B and we can construct a C^∞ hypersurface Γ such that $\Gamma = \partial\Omega$ (for a suitable open bounded set Ω). For $b \in]a/2, a[$ small enough we have that $(0, b, 0) \in \Omega$ and $T_\Gamma(0, b, 0) = a - b$. As noted above, this implies that the (MTP) associated with the controlled system (4.4) and target Γ admits a singular time-optimal trajectory. Then, owing to Theorem 4.1, T_Γ is not a locally Lipschitz function. $\qquad\square$

We complete this section with the study of the regularity of T at the boundary of Ω. As we did for Lipschitz continuity, we begin by introducing a point-wise notion of Hölder regularity.

Definition 4.1. *We say that a function $f : \overline{\Omega} \to \mathbb{R}$ is Hölder continuous of exponent $\alpha \in]0, 1]$ at a point $x_0 \in \Omega$ if there exist a neighborhood $U \subset \Omega$ of x_0 and a constant $C \geq 0$ such that*

$$|f(x) - f(x_0)| \leq C|x - x_0|^\alpha \qquad \forall x \in U \cap \overline{\Omega}.$$

The following result shows that, for the Dirichlet problem, the maximal length of commutators $k(\cdot)$ gives the "optimal" Hölder exponent at characteristic boundary points.

Theorem 4.2 (Boundary regularity). *Assume (H). Then:*

(1) *for any $x \in \Gamma \setminus E$, T is C^∞ in a neighborhood of x;*
(2) *for any $x \in E$, T is Hölder (not Lipschitz) continuous at x of exponent $1/k(x)$, with $k(x)$ given by (2.9).*

Even though part of the above conclusions may be known to the reader, for completeness we provide a proof of Theorem 4.2 in Appendix A.

In order to state a more general regularity result, we introduce the set

$$\text{sing}_L T = \left\{ x \in \overline{\Omega} \ : \ \limsup_{\overline{\Omega} \ni y \to x} \frac{|T(y) - T(x)|}{|y - x|} = \infty \right\}$$

which consists of all points at which T fails to be Lipschitz.

Proposition 4.1. *Assume (H) and let T be the viscosity solution of (2.7). Then, $\mathrm{sing}_L T$ is a closed set.*

Proof. Let $x_j \in \mathrm{sing}_L T$, $j \in \mathbb{N}$, be a sequence of points converging to $x \in \Omega$. We claim that $x \in \mathrm{sing}_L T$. Indeed, since T fails to be Lipschitz at x_j, we have that there exist control functions u_j such that:

(1) y^{x_j, u_j} is a singular time-optimal trajectory (this is a consequence of Theorem 3.2);

(2) possibly taking a subsequence, y^{x_j, u_j} converges uniformly to an optimal trajectory $y^{x, u}$.

We claim that $y^{x, u}$ is singular. Indeed, Theorem 2.1 implies that E is a closed set and $y^{x_j, u_j}(T(x_j)) \in E$, by Theorem 3.1. Hence, we find that

$$E \ni \lim_{j \to \infty} y^{x_j, u_j}(T(x_j)) = y^{x, u}(T(x)).$$

So, by using once again Theorem 3.1, we deduce that $y^{x, u}$ is a singular time-optimal trajectory.

In order to complete the proof we observe that, if $x \in \Gamma$, then

$$x = \lim_{j \to \infty} y^{x_j, u_j}(T(x_j)) \in E$$

and the conclusion follows by Theorem 4.2(2). $\qquad\square$

Our next result fully describes the regularity of the solution to (2.7).

Theorem 4.3. *Assume (H) and let T be the viscosity solution of (2.7). Then, T fails to be Lipschitz at all points of the closed set $\mathrm{sing}_L T$ and is semiconcave on the relatively open set $\overline{\Omega} \setminus \mathrm{sing}_L T$.*

The proof of Theorem 4.3 is similar to the one of Theorem 4.1 and is omitted.

Remark 4.4. An immediate consequence of the above result is that $\mathrm{sing}_L T \setminus E$ coincides with the Lipschitz singular support of T. Indeed, $x \in \Omega \setminus \mathrm{sing}_L T$ if and only if there exists a neighborhood of x, V, such that T is Lipschitz continuous on V.

4.1. **Sufficient conditions for regularity.** In this section, we give conditions to guarantee the absence of singular time-optimal trajectories. Let us recall that, in canonical coordinates[2], the symplectic form in $T^*\Omega$ is the 2-form

$$(4.5) \qquad\qquad \sigma = \sum_{k=1}^{n} dp_k \wedge dx_k.$$

---

[2]More in general a symplectic form in a C^∞ manifold is a non-degenerate, closed C^∞ two form.

Furthermore, for any $\rho \in T^*\Omega$, given a vector space $W \subset T_\rho(T^*\Omega)$ we denote by W^σ the symplectic orthogonal to W, i.e.

$$W^\sigma = \{v \in T_\rho(T^*\Omega) \ : \ \sigma(v, w) = 0 \,, \ \forall w \in W\}.$$

Finally, we say that a manifold $M \subset T^*\Omega$ is *symplectic* if the restriction of σ to M is nondegenerate, i.e.

$$T_\rho M \cap (T_\rho M)^\sigma = \{0\} \qquad \forall \rho \in M.$$

We have the following result.

Theorem 4.4. *Assume* (H). *Then* (MTP) *admits no singular time-optimal trajectory if any of the following conditions is satisfied:*
  (i) Γ *is noncharacteristic (i.e.* $E = \emptyset$);
  (ii) $\mathrm{Char}(X_1, \ldots, X_N)$ *is a symplectic manifold.*

Proof. For noncharacteristic Γ the conclusion follows from Theorem 3.1.
  Let us assume that $\mathrm{Char}(X_1, \ldots, X_N)$ is a symplectic manifold. We have that

$$\mathrm{span}\{dX_1(x, p), \ldots, dX_N(x, p)\} \perp T_{(x,p)} \mathrm{Char}(X_1, \ldots, X_N)$$

(here "$\perp$" stands for orthogonality w.r.t. the Euclidean scalar product). Hence, we find

$$(\mathrm{span}\{dX_1(x, p), \ldots, dX_N(x, p)\})^\sigma \subset (T_{(x,p)} \mathrm{Char}(X_1, \ldots, X_N))^\sigma.$$

On the other hand, since $\mathrm{Char}(X_1, \ldots, X_N)$ is a symplectic manifold, using the identity

$$T_{(x,p)} \mathrm{Char}(X_1, \ldots, X_N) \cap (T_{(x,p)} \mathrm{Char}(X_1, \ldots, X_N))^\sigma = \{0\},$$

we conclude that either the broken Hamiltonian flow (3.4) is transveral to $\mathrm{Char}(X_1, \ldots, X_N)$, or it is a stationary flow, i.e., there are no singular time-optimal trajectories. This completes our proof. $\qquad \square$

Remark 4.5. As a consequence of our results and [21, Theorem 3.4], assuming that $\{X_1, \ldots, X_N\}$ are real analytic and $\partial\Omega$ is a real analytic submanifold, we have that, if $\mathrm{Char}(X_1, \ldots, X_N)$ is a symplectic manifold, then T is a subanalytic function in Ω (see, e.g. [20], for the definition and basic properties of subanalytic sets and functions).

4.2. **Examples.** In this section, we discuss problems to which Theorem 4.4 can be directly applied as well as examples where the assumptions of Theorem 4.4 are not fulfilled, but one can still use Theorem 4.1 to study the regularity of T.

  In the example below we consider Heisenberg vector fields and use Theorem 4.4 to show that this problem admits no singular time-optimal

trajectory. It is worth noting that the same conclusion follows from the fact that such vector fields are strongly bracket generating.

Example 2 (Heisenberg vector fields). *In $\mathbb{R}^3$ consider vector fields*

$$X_1 = \partial_{x_1}, \quad X_2 = \partial_{x_2} + x_1 \partial_{x_3}$$

and let Ω be a bounded open set with C^∞ boundary. We have that

$$\mathrm{Char}(X_1, X_2) = \big\{(x_1, x_2, x_3, 0, -x_1 p_3, p_3) : (x_1, x_2, x_3) \in \Omega,\ p_3 \neq 0\big\}$$

is a smooth submanifold of $\mathbb{R}^6$ of codimension 2. Furthermore, the restriction of σ to $\mathrm{Char}(X_1, X_2)$ is nondegenerate, i.e. $\mathrm{Char}(X_1, X_2)$ is a symplectic manifold. Then, by Theorem 4.4 (ii), we conclude that (MTP) has no singular time-optimal trajectory.

Since we consider a boundary value problem for the eikonal equation, due to the interaction of the boundary of Ω with $\mathrm{Char}(X_1, \ldots, X_N)$, even systems of vector fields admitting, in general, singular time-optimal trajectories may have a better behaviour when Ω enjoys specific properties. In this regard, we consider once more the system of vector fields in Example 1. We point out that such a system, introduced in [17], admits, in general, strictly abnormal geodesics—in the language of the geometric control theory. We will show that, whenever Ω is convex, we can exclude the occurrence of singular time-optimal trajectories.

Example 3. *In $\mathbb{R}^3$ consider vector fields*

$$X_1 = \partial_{x_1}, \quad X_2 = (1 - x_1)\partial_{x_2} + x_1^2 \partial_{x_3}$$

and let Ω be a bounded convex open set with C^∞ boundary. Then, (MTP) admits no singular time-optimal trajectory.

Proof. We argue by contradiction assuming the existence of a singular time-optimal trajectory. Let us write the two systems appearing in (3.4) in this specific case. We have

$$(4.6) \qquad \begin{cases} \dot{y}_1 = u_1 \\ \dot{y}_2 = u_2(1 - y_1) \\ \dot{y}_3 = u_2 y_1^2 \end{cases} \qquad \begin{cases} \dot{p}_1 = u_2(p_2 - 2y_1 p_3) \\ \dot{p}_2 = 0 \\ \dot{p}_3 = 0. \end{cases}$$

The only possible initial point $\bar{x} \in \Omega$ of a singular time-optimal trajectory must be of the form $(0, \bar{x}_2, \bar{x}_3)$, $\bar{x}_2, \bar{x}_3 \in \mathbb{R}$. Then the corresponding solution of (4.6) is given by

$$(y(t), p(t)) := \left(0, \bar{x}_2 + \int_0^t u_2(s), \bar{x}_3, 0, 0, \bar{p}_3\right)$$

where $(0, \bar{x}_2, \bar{x}_3, 0, 0, \bar{p}_3)$ are the "initial" conditions for the Hamiltonian system in (4.6). Let x_2^* be such that $y(T) = (0, x_2^*, \bar{x}_3)$ and note that

$(0, x_2^*, \bar{x}_3) \in \Gamma$ is a characteristic point. Since Ω is a smooth convex set, there exists a smooth function $\Phi = \Phi(x_1, x_2, x_3)$ such that

$$\overline{\Omega} \cap B_\delta(0, x_2^*, \bar{x}_3) = \big\{(x_1, x_2, x_3) \in B_\delta(0, x_2^*, \bar{x}_3) : \Phi(x_1, x_2, x_3) \leq 0\big\}$$

and

$$\partial\Omega \cap B_\delta(0, x_2^*, \bar{x}_3)$$
$$= \big\{(x_1, x_2, x_3) \in B_\delta(0, x_2^*, \bar{x}_3) : \Phi(x_1, x_2, x_3) = 0 , \ D\Phi(x_1, x_2, x_3) \neq 0\big\}$$

for a suitable $\delta > 0$. Now, the transversality condition (3.2) in Pontryagin's maximum principle implies that $D\Phi(y(T)) = \lambda(0, 0, 1)$, for some $\lambda \neq 0$. In particular, we deduce that $\partial_{x_3}\Phi(y(T)) \neq 0$. By appealing to the implicit function theorem and convexity assumption, we can suppose that, near $(0, x_2^*, \bar{x}_3)$, Ω is the hypograph of a smooth concave function $(x_1, x_2) \mapsto \varphi(x_1, x_2)$, which attains its maximum at $(0, x_2^*)$ and such that $\varphi(0, x_2^*) = \bar{x}_3$. This implies that $(0, x_2^*, \bar{x}_3)$ cannot be the final point of a trajectory starting from Ω because the line segment $(0, x_2^* + s, \bar{x}_3)$, $s \in \mathbb{R}$, can at most belong to the boundary of Ω. We have thus reached a contradiction. □

As shown in the next example, in some cases, even if the characteristic set is not a symplectic manifold but it can be splitted into a disjoint union of symplectic submanifolds, our approach can be applied.

Example 4. *Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with smooth boundary and let k be a positive integer. In $\mathbb{R}^3$, consider vector fields*

$$X_1 = \partial_{x_1} - x_2^{2k+1}\partial_{x_3} \quad and \quad X_2 = \partial_{x_2} + x_1^{2k+1}\partial_{x_3}.$$

Then no singular time-optimal trajectories exists.

Proof. The characteristic set is given by

$$\mathrm{Char}(X_1, X_2)$$
$$= \big\{(x_1, x_2, x_3, x_2^{2k+1}p_3, -x_1^{2k+1}p_3, p_3) \ : \ x_1, x_2, x_3 \in \mathbb{R}, \ p_3 \neq 0\big\}.$$

Therefore, $\mathrm{Char}(X_1, X_2)$ can be split into the connected submanifolds

$$\Sigma_{1,\pm} = \big\{(x_1, x_2, x_3, x_2^{2k+1}p_3, -x_1^{2k+1}p_3, p_3) \ :$$
$$x_1, x_2, x_3 \in \mathbb{R}, \ (x_1, x_2) \neq (0, 0), \ \pm p_3 > 0\big\}$$

and

$$\Sigma_{2,\pm} = \big\{(0, 0, x_3, 0, 0, p_3) \ : \ x_3 \in \mathbb{R}, \ \pm p_3 > 0\big\}.$$

Moreover, all these submanifolds are symplectic (the rank of the symplectic form is constant and the symplectic form is nondegenerate on these sets). So, there is no singular time-optimal trajectory. □

Observe that, in the above example, the characteristic set is a manifold of condimension 2 in $\mathbb{R}^3$, but the rank of the symplectic form is not constant, i.e., $\mathrm{Char}(X_1, X_2)$ is not a symplectic manifold. Then, since we are not assuming that $E = \emptyset$, none of the conditions in Theorem 4.4 is satisfied.

We complete this section with an application of Theorem 4.3.

Example 5. *Let $\Omega \subset$ be a bounded open set with smooth boundary and consider vector fields*

$$X_1 = \partial_{x_1}, \quad X_2 = (1 - x_1)\partial_{x_2} + x_1^2 \partial_{x_3}.$$

Then, $\mathrm{sing}_L T$ is a set of measure zero and T has a second order Taylor expansion at a.e. point of Ω.

Proof. We observe that, if we show that $\mathrm{sing}_L T$ is a set of measure zero, then T turns out to be semiconcave in $\Omega \setminus \mathrm{sing}_L T$ by Theorem 4.3. So, the conclusion follows by the Alexandrov theorem on twice differentiability of a convex function. Hence, let us show that $\mathrm{sing}_L T$ is a set of measure zero. We observe that all singular time-optimal trajectories can be lifted to the characteristic set

$$\mathrm{Char}(X_1, X_2) = \big\{(x_1, x_2, x_3, 0, p_2, p_3) \ : \ (x_1, x_2, x_3) \in \Omega,$$
$$(1 - x_1)p_2 + x_1^2 p_3 = 0, \quad p_3 \neq 0\big\}.$$

On the other hand, the characteristic set can be decomposed as follows

$$\mathrm{Char}(X_1, X_2) = V_1 \cup V_2 \cup V_3,$$

with

$$V_1 = \big\{(x_1, x_2, x_3, 0, p_2, p_3) \ : \ (x_1, x_2, x_3) \in \Omega,$$
$$(1 - x_1)p_2 + x_1^2 p_3 = 0 \quad - p_2 + 2x_1 p_3 \neq 0\big\},$$

$$V_2 = \big\{(2, x_2, x_3, 0, 4p_3, p_3) \ : \ (2, x_2, x_3) \in \Omega, \quad p_3 \neq 0\big\},$$

and

$$V_3 = \big\{(0, x_2, x_3, 0, 0, p_3) \ : \ (0, x_2, x_3) \in \Omega, \quad p_3 \neq 0\big\}.$$

We observe that the only solutions of (3.4) in the set V_1 are stationary points (it is a symplectic manifold). Hence, a nontrivial lifting of a singular time-optimal trajectory can only belong to the set $V_2 \cup V_3$. By Theorem 3.2, we deduce that

$$\mathrm{sing}_L T \subset \big\{(x_1, x_2, x_3) \in \Omega \ : \ x_1 = 0 \quad \text{or} \quad x_1 = 2\big\}$$

and the conclusion follows. $\qquad\square$

Appendix A. Proof of Theorem 4.2

The fact that T is of class C^∞ in a neighborhood of every non-characteristic point can be proved arguing as follows. Let $x_0 \in \Gamma \setminus E$. Then, by Theorem 2.1, there exists a neighborhood of x_0 in Ω', W, such that $W \cap E = \emptyset$. Consider the Dirichlet problem

$$h(x, Dv) = 1 \quad \text{in} \quad W \cap \Omega$$

with initial condition $v = 0$ in $\Gamma \cap W$. This is a noncharacteristic Dirichlet problem with smooth data. Then, possibly replacing W with a smaller neighborhood of x_0, we have that there exists a unique solution $v \in C^\infty(W \cap \Omega)$. Since the characteristics of the above problem are time-optimal trajectories, we conclude that $T \in C^\infty(W \cap \Omega)$.

Let us now prove Hölder regularity at a characteristic boundary point. Take $x_0 \in E$ and denote by $T_{\{x_0\}}$ the minimum time function for the controlled system (2.8) with target $\{x_0\}$. Let W be a neighborhood of x_0 in Ω' such that

$$\max_{x \in W \cap \overline{\Omega}} k(x) = k(x_0).$$

We want to show that T is Hölder continuous of exponent $1/k(x_0)$ at x_0. It is well known ([4, Theorem 1.15, Chap. IV]) that there exists a constant $C > 0$ such that

$$T_{\{x_0\}}(x) \le C|x - x_0|^{1/k(x_0)}, \quad \forall x \in W.$$

By possibly taking a smaller set W, we may assume that

$$T_{\{x_0\}}(x) \ge T(x), \quad \forall x \in \partial(W \cap \Omega)$$

and

$$h(x, DT_{\{x_0\}}(x)) = 1 = h(x, DT(x)) \quad \text{in} \quad W \cap \Omega$$

(in the viscosity sense). So, the comparison principle implies that

$$T(x) \le T_{\{x_0\}}(x) \quad \forall x \in W \cap \Omega.$$

Then, we deduce that

$$|T(x) - T(x_0)| = T(x) \le C|x - x_0|^{1/k(x_0)}, \quad \forall x \in W \cap \Omega.$$

Let us show that, if $x_0 \in E$, then T cannot be Lipschitz continuous at x_0. We argue by contradiction and assume $\exists\, C > 0$ such that

(A.1) $$T(x) \le C|x - x_0| \quad \forall x \in \overline{\Omega} \cap B_\delta(x_0),$$

for some $\delta > 0$. Then, by Theorem 2.1, there exists a sequence $x_j \in \Gamma \cap B_\delta(x_0) \setminus E$ such that $x_j \to x_0$ as $j \to \infty$. Then

$$DT(x_j) = \langle DT(x_j), \nu(x_j) \rangle \nu(x_j).$$

By (A.1), $\{\langle DT(x_j), \nu(x_j)\rangle\}_j$ is a bounded sequence. Hence, possibly taking a subsequence, we find that, as $j \to \infty$,

$$\langle DT(x_j), \nu(x_j)\rangle \to c \quad \text{and} \quad \nu(x_j) \to \nu(x_0),$$

for some $c \in \mathbb{R}$. Thus, we find the contradiction

$$1 = h(x_j, DT(x_j)) \quad \text{and} \quad \lim_{j\to\infty} h(x_j, DT(x_j)) = c^2 h(x_0, \nu(x_0)) = 0.$$

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