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Quantum Mechanics and Hidden Superconformal Symmetry

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Solvability of the ubiquitous quantum harmonic oscillator relies on a spectrum generating $\mathfrak{osp}(1|2)$ superconformal symmetry. We study the problem of constructing all quantum mechanical models with a hidden $\mathfrak{osp}(1|2)$ symmetry on a given space of states. This problem stems from interacting higher spin models coupled to gravity. In one dimension, we show that the solution to this problem is the Plyushchay family of quantum mechanical models with hidden superconformal symmetry obtained by viewing the harmonic oscillator as a one dimensional Dirac system, so that Grassmann parity equals wavefunction parity. These models—both oscillator and particle-like—realize all possible unitary irreducible representations of $\mathfrak{osp}(1|2)$.

I. INTRODUCTION

The quantum harmonic oscillator

$$H = \frac{1}{2}(p^2 + q^2)\,, \qquad [p, q] = -i\,,$$

is solvable because the ladder operators

$$a = \frac{q + ip}{\sqrt{2}}\,, \qquad a^\dagger = \frac{q - ip}{\sqrt{2}}\,, \tag{1}$$

generate the spectrum. This is perhaps the simplest example of the Lie superalgebra $\mathfrak{osp}(1|2)$: to see this, one treats the ladder operators as supercharges [28]

$$S^+ := a^\dagger\,, \qquad S^- := a\,.$$

Then defining the $\mathfrak{sp}(2)$ generators [29]

$$Q^{++} = (a^\dagger)^2\,, \quad Q^{+-} = H\,, \quad Q^{--} = a^2\,,$$

the five generators $\{S^\pm, Q^{\pm\pm}, Q^{+-}\}$ generate the algebra [30] $\mathfrak{osp}(1|2)$:

$$\begin{aligned} &\{S^\pm, S^\pm\} = 2Q^{\pm\pm}\,, \quad \{S^+, S^-\} = 2Q^{+-}\,, \\ &[S^\mp, Q^{\pm\pm}] = \pm 2S^\pm\,, \quad [Q^{+-}, S^\pm] = \pm S^\pm\,. \end{aligned} \tag{2}$$

Strangely enough, here one assigns the ladder operators a Grassmann odd grading, even though these are the standard complex combinations of position and momentum given in Eq. (1). Thus, the fermion number operator F that grades the $\mathfrak{osp}(1|2)$ algebra counts one for odd powers of ladder operators (and zero for even powers) and therefore labels wavefunction parity [31].

The basic question we address is the existence of operator quintuples acting on the harmonic oscillator Fock space obeying the $\mathfrak{osp}(1|2)$ Lie superalgebra. The solution to this problem is a class of quantum mechanical models that have been studied in detail by Plyushchay [1]. We also answer this question for generalized particle models with plane wave-normalizable spectra for which the $\mathfrak{osp}(1|2)$ algebra acts as a generalized one dimensional superconformal symmetry.

Our study is motivated by a proposal of Bars *et al* [2], who suggested that the space of operators obeying an $\mathfrak{sp}(2)$ algebra and acting on functions of a $d+2$ dimensional spacetime with two times, could describe gravitating, interacting higher spin theories. We have shown [3, 4] that this proposal is intimately linked to the study of d dimensional conformal geometries in terms of a $d+2$ dimensional ambient space initiated by Fefferman and Graham [5]. The inclusion of fermions in such models leads to an $\mathfrak{osp}(1|2)$ generalization of Bars' theory [6] (see also [7]). The study of the detailed spectra, interactions, ultraviolet and unitarity properties of such models is a complicated problem commensurate with that of string field theories, as one is dealing with field equations for operator-valued fields. Although the solution we

find in one dimension is largely controlled by orthosymplectic representation theory, the existence of a mathematically well-defined answer in this setting is an important first step towards analyzing models in $d+2$ dimensions, for which the solution space already includes all d-dimensional conformal geometries. Moreover D'Hoker and Vinet [9] have analyzed a hidden $\mathfrak{osp}(1|2)$ symmetry of the Dirac equation in monopole backgrounds, which indicates tractability for models in higher dimensions.

Our analysis begins in Section II with the "master" equations of motion and gauge symmetries for the supercharges $S^{\pm}$. Sections III-VII are devoted to solving these equations on a harmonic oscillator Fock space while Sections VIII-X focus on particle models with hidden superconformal symmetry. Appendix B reviews $\mathfrak{osp}(1|2)$ representations.

II. EQUATIONS OF MOTION

To answer the question posed in the introduction, we view the supercharges $S^{\pm}$ as the fundamental "fields" and study "equations of motion" for these that guarantee that the algebra $\mathfrak{osp}(1|2)$ of Eq. (2) holds. These have been formulated in [6]; the result is [32]:

$$\begin{aligned}[S^-, S^+S^+] &= 2S^+\,,\\ [S^-S^-, S^+] &= 2S^-\,.\end{aligned} \tag{3}$$

The statement here is that if the pair of operators $S^{\pm}$ obey these equations, then the operator quintuple $\{S^{\pm},\, Q^{\pm\pm}=S^{\pm}S^{\pm},\, Q^{+-}=\frac{1}{2}[S^+S^- + S^+S^-]\}$ satisfies the $\mathfrak{osp}(1|2)$ Lie superalgebra (2).

Clearly, if $S^{\pm}$ solve Eqs. (3), then so too do $U^{-1}S^{\pm}U$ for any invertible operator U. Linearizing U around the identity $U\approx \mathrm{Id}+\varepsilon$ gives the gauge invariance

$$S^{\pm}\sim S^{\pm}+[S^{\pm},\varepsilon]\,, \tag{4}$$

of the equations of motion (3). Here the gauge parameter ε is itself also an operator.

The problem of solving Eqs. (3) for operators $S^{\pm}$ is not defined without specifying the state space $\mathcal{H}$ on which these operators act [33]. The set of possible choices for an underlying Hilbert space $\mathcal{H}$ is clearly enormous. We commence with perhaps the simplest case in which $\mathcal{H}$ is the harmonic oscillator Fock space.

III. THE SPACE OF OPERATORS

We now let $\mathcal{H}$ equal the quantum harmonic oscillator Hilbert space with Fock basis $\{|n\rangle : n\in\mathbb{Z}_{\geq 0}\}$. We employ the slightly non-standard normalization $\langle m|n\rangle = n!\,\delta_{mn}$ for states $|n\rangle$, since this allows us to identify $|n\rangle$ with the monomial z^n and in turn study wavefunctions given by polynomials, or more generally suitable analytic functions, in z [34]. Thus we study operators

$$S^{\pm}=s_0^{\pm}(z)+s_1^{\pm}(z)\frac{\partial}{\partial z}+s_2^{\pm}(z)\frac{\partial^2}{\partial z^2}+\cdots\,, \tag{5}$$

where $s_i^{\pm}(z)$ are analytic functions of z in a neighborhood of the origin. In terms of ladder operators, this amounts to studying operators given by sums of normal ordered products of a's and $a^{\dagger}$'s. More precisely, we are looking for the most general set of formal power series in ladder operators obeying the $\mathfrak{osp}(1|2)$ superalgebra.

IV. GAUGE CHOICES

To simplify our problem we fix a gauge using the freedom in Equation (4). A propitious choice is

$$S^+=z\,. \tag{6}$$

To verify gauge reachability, we consider

$$\varepsilon=\epsilon_0(z)+\epsilon_1(z)\frac{\partial}{\partial z}+\epsilon_2(z)\frac{\partial^2}{\partial z^2}+\cdots\,.$$

Then a short computation gives

$$[\varepsilon,z]=\epsilon_1(z)+2\epsilon_2(z)\frac{\partial}{\partial z}+3\epsilon_3(z)\frac{\partial^2}{\partial z^2}+\cdots\,.$$

Thus by solving for $\epsilon_1(z)$, $\epsilon_2(z)$,... we can bring $S^+=z$ to an operator of the general form (5) by a gauge transformation (4). The function $\epsilon_0(z)$ remains undetermined because there are still residual gauge transformations, respecting our choice $S^+=z$, of the form $S^{\pm}\mapsto \big(1/U(z)\big)\,S^{\pm}U(z)$. The beauty of the gauge choice (6) is that the first equation of motion in (3) is now linear.

V. THE LINEAR EQUATION

The linear equation for S^- reads

$$[S^-,z^2]=2z\,. \tag{7}$$

Using the identity

$$\Big[\frac{\partial^k}{\partial z^k},z^2\Big]=k\Big[2z\frac{\partial}{\partial z}+(k-1)\Big]\frac{\partial^{k-2}}{\partial z^{k-2}}\,,$$

we can solve this order by order for S^- and find

$$S^-=\frac{\partial}{\partial z}+A(z)+B(z)\Big[1-z\frac{\partial}{\partial z}+\frac{2}{3}z^2\frac{\partial^2}{\partial z^2}+\cdots\Big]\frac{\partial}{\partial z}\,.$$

In the above $A(z)$ and $B(z)$ are arbitrary functions. Defining the number operator $N:=z\frac{\partial}{\partial z}$, and denoting normal ordering by $:\bullet:$ (*e.g.*, $:N^2:\;=z^2\frac{\partial^2}{\partial z^2}=N(N-1)$), the above display becomes [35]

$$S^-=\frac{\partial}{\partial z}+A(z)+B(z)\,:\Big[\frac{1-e^{-2N}}{2N}\Big]:\,\frac{\partial}{\partial z}\,.$$

Using the identity

$$z:f(N):\frac{\partial}{\partial z}=:Nf(N):\ ,$$

we have

$$S^-=\frac{\partial}{\partial z}+A(z)+\frac{B(z)}{z}:\Big[\frac{1-e^{-2N}}{2}\Big]:\ . \tag{8}$$

The normal ordered operator in the above expression is related to the Klein operator of [12]. It has an interesting action on number operator eigenstates

$$:\Big[\frac{1-e^{-2N}}{2}\Big]:|n\rangle=\frac{1}{2}(1-(-1)^n)|n\rangle\,,$$

i.e., it vanishes on the space of even number operator eigenstates $\mathcal{B}$ and is unity on the space of odd number operator eigenstates $\mathcal{F}$. This means that the operator $1/z$ appearing in Eq. (8) is well-defined. Also, the operator in the above display is the fermion number operator

$$F:=\frac{1}{2}(1-(-1)^N)=F^2\,.$$

This obeys $\{F,z\}=z$ and $[F,z^2]=0$. In addition to providing a $\mathbb{Z}_2$ grading of the Hilbert space

$$\mathcal{H}=\mathcal{B}\oplus\mathcal{F}\,,$$

we may demand that F also coincides with the $\mathbb{Z}_2$ grading of the Lie superalgebra $\mathfrak{osp}(1|2)=\mathfrak{sp}(2)\oplus\mathbb{R}^2$. In the following we focus on the case where the two gradings coincide, since it leads quickly to the solution space; we prove that this yields the most general solution in Appendix A.

VI. HARMONIC OSCILLATOR SOLUTION

Requiring coincidence of $\mathbb{Z}_2$ gradings in conjunction with the solution to the linear equation (8) forces us to consider an ansatz of the form

$$\begin{aligned}S^+&=\ z\,,\\ S^-&=\frac{\partial}{\partial z}+\alpha(z)+(-1)^F\beta(z)\,,\end{aligned}\tag{9}$$

where $\alpha(z)$ and $\beta(z)$ are both odd with respect to the $\mathbb{Z}_2$ grading (*i.e.*, even and odd functions of z). Here we also used that $(-1)^F=1-2F$.

It remains to solve the second, non-linear equation in (3) which we rewrite as

$$[H,S^-]+S^-=0\,,$$

where the Hamiltonian is easily computed from Eq. (9):

$$H=\frac{1}{2}\{S^+,S^-\}=N+\frac{1}{2}+z\alpha(z)\,.$$

The above leads to the relation

$$z\beta'(z)+\beta(z)=0\ \Rightarrow\beta(z)=\frac{c}{2z}\,,$$

for some constant c. Requiring that S^- acting on the Fock space $\mathcal{H}$ (and in particular on the vacuum $|0\rangle$) is well-defined we set

$$\alpha(z)=\frac{c}{2z}+A(z)\,,$$

where $A(z)$ is analytic and odd. Thus

$$S^-=\frac{\partial}{\partial z}+A(z)+\frac{c}{z}\,F\,. \tag{10}$$

First observe that since $F|0\rangle=0$, the operator $\frac{1}{z}F$ is, as promised, well-defined. Moreover, since $A(z)$ is odd, the function $U(z)=\exp\big(-\int^z A(z)\big)$ is even and thus commutes with F. Hence $(1/U(z))\,S^-U(z)=\frac{\partial}{\partial z}+\frac{c}{z}F$.

The constant $(c+1)/2$ measures the zero point energy E_0 of the vacuum $|0\rangle$, so we now call $c=2E_0-1$. Altogether then, we find a one parameter family of solutions

$$S^+=z,\quad S^-=\frac{\partial}{\partial z}+\frac{2E_0-1}{z}\,F, \tag{11}$$

$$Q^{+-}=N+E_0\,,$$

$$Q^{++}=z^2,\quad Q^{--}=\frac{\partial^2}{\partial z^2}+\frac{2E_0-1}{z}\frac{\partial}{\partial z}-\frac{2E_0-1}{z^2}\,F\,.$$

Although the Hamiltonian $H=Q^{+-}$ only receives a shift in its zero point energy. The commutator of the deformed oscillators $S^\pm$ is easily calculated to be

$$[S^-,S^+]=1-(2E_0-1)(2F-1)\,. \tag{12}$$

This is exactly the model proposed by Plyushchay [1] (although basic quantum commutators were already studied in [10]). The operator S^- is a Yang–Dunkl type operator [11]. The $\mathfrak{osp}(1|2)$ representation obeyed by this model was analyzed in [12] (and also recently discussed in [13]), this is summarized in the next section [36].

VII. OSCILLATOR ORTHOSYMPLECTIC REPRESENTATION

Our solution (11) obeys the $\mathfrak{osp}(1|2)$ Lie superalgebra and therefore provides a representation thereof. To analyze this we start by searching for states annihilated by S^- so consider $\psi(z)$ subject to

$$S^-\psi(z)=0\,,$$

which we decompose as

$$\psi(z)=\psi_+(z)+\psi_-(z)\,,$$

where the two terms on the right hand side are analytic and even/odd respectively. Since S^- is odd we must separately have

$$\begin{cases} \psi'_+(z) = 0\,, \\ \psi'_-(z) + \dfrac{2E_0-1}{z}\,\psi_-(z) = 0\,. \end{cases}$$

Thus $\psi_+(z) = 1 = |0\rangle$, the standard Fock vacuum. There is in addition the possibility of a second solution $\psi_-(z) = z^{1-2E_0}$. Because $\psi_-(z)$ is analytic and odd this occurs only when $E_0 = -n$ with $n \in \mathbb{Z}_{\geq 0}$, whence $\psi_-(z) = |2n+1\rangle = S_+^{2n+1}|0\rangle$. Thus

$$\ker S^- = \begin{cases} \mathrm{span}\big\{|0\rangle\,,\ |2n+1\rangle\big\}\,, & E_0 \in \mathbb{Z}_{\leq 0}\,, \\ \mathrm{span}\big\{|0\rangle\big\}\,, & E_0 \notin \mathbb{Z}_{\leq 0}\,. \end{cases}$$

Thus $|0\rangle$ is always a highest weight state subject to

$$H|0\rangle = E_0|0\rangle\,,$$

while $|2n+1\rangle$ is a singular vector when $E_0 = -n$ and then obeys

$$H|2n+1\rangle = (n+1)|2n+1\rangle\,.$$

At the harmonic oscillator value $E_0 = \frac{1}{2}$, we have $S^- = \partial/\partial z = a = (S^+)^\dagger$ and $Q^{--} = (Q^{++})^\dagger$. The Hilbert space is then the unitary irreducible representation $\mathcal{S}(1/2) = \mathcal{D}(1/2) \oplus \mathcal{D}(3/2)$ given by a direct sum of two discrete series unitary irreducible $\mathfrak{sp}(2)$ representations. Indeed, unlike $\mathfrak{sp}(2)$, which also has supplementary and principal series representations, the Lie superalgebra $\mathfrak{osp}(1|2)$ only has discrete series unitary irreducible representations [16] (see Appendix B for further details).

When $E_0 \notin \mathbb{Z}_{\leq 0}$, the even and odd states $\mathcal{B} = \{|0\rangle, |2\rangle, |4\rangle, \ldots\}$ and $\mathcal{F} = \{|1\rangle, |3\rangle, |5\rangle \ldots\}$, respectively, separately diagonalize the $\mathfrak{sp}(2)$ Casimir

$$c_{\mathfrak{sp}(2)} = \frac{1}{4}(Q^{+-})^2 - \frac{1}{8}\{Q^{++}, Q^{--}\}$$

which takes values

$$c_{\mathfrak{sp}(2)}(\mathcal{B}) = \tfrac{E_0(E_0-2)}{4} \ \text{ and } \ c_{\mathfrak{sp}(2)}(\mathcal{F}) = \tfrac{(E_0-1)(E_0+1)}{4}\,.$$

When $E_0 > 0$, these precisely match the Casimirs of the discrete series representations $\mathcal{D}(E_0)$ and $\mathcal{D}(E_0+1)$. Moreover, the direct sum of these representations yields the $\mathfrak{osp}(1|2)$ discrete series representation $\mathcal{S}(E_0)$. Indeed, the orthosymplectic Casimir

$$c_{\mathfrak{osp}(1|2)} = \frac{1}{4}(Q^{+-})^2 - \frac{1}{8}\{Q^{++}, Q^{--}\} - \frac{1}{8}[S^+, S^-]\,,$$

obeys

$$c_{\mathfrak{osp}(1|2)}(\mathcal{H}) = \frac{E_0(E_0-1)}{4} = c_{\mathfrak{osp}(1|2)}\big(\mathcal{S}(E_0)\big)$$

on the harmonic oscillator state space $\mathcal{H} = \mathcal{B} \oplus \mathcal{F}$. However, when $E_0 \neq 1/2$, the operators Q^{--} and S^- are no longer the hermitean conjugates of Q^{++} and S^+ with respect to the standard Fock space inner product. But, since the $\mathfrak{osp}(1|2)$ action on the harmonic oscillator Fock space is isomorphic to that of the orthosymplectic discrete series, there exists a corresponding inner product with respect to which this is a unitary representation. This inner product can be computed as follows:

First observe that with respect to the Fock norm the state $|E_0, n\rangle = (S^+)^n|0\rangle = |n\rangle$ obeys

$$\big\||E_0, n\rangle\big\|^2_{\text{Fock}} = \langle 0|a^n(a^\dagger)^n|0\rangle = n!\langle 0|0\rangle = n!\,.$$

However, with respect to the unitary discrete series norm,

$$\begin{aligned} \big\||E_0, n\rangle\big\|^2_{\mathfrak{osp}} &= \big\langle |E_0, n\rangle, |E_0, n\rangle \big\rangle_{\mathfrak{osp}} \\ &= \langle E_0, 0|(S^-)^n(S^+)^n|E_0, 0\rangle \\ &= \langle E_0, 0|(S^-)^{n-1}S^-|E_0, n\rangle \\ &= \big((E_0 - \tfrac{1}{2})(1-(-1)^n) + n\big)\big\||E_0, n-1\rangle\big\|^2_{\mathfrak{osp}} \\ &= 2^n (E_0)_{[\frac{n+1}{2}]}\big([\tfrac{n}{2}]\big)!\,. \end{aligned}$$

Here we have employed the standard Pochhammer notation and used the identity (valid for $n \in \mathbb{Z}_{\geq 1}$)

$$S^-|E_0, n\rangle = \big((2E_0-1)(1-F) + n\big)|E_0, n-1\rangle\,. \quad (13)$$

The operator version of this identity is given in (12).

Importantly, the above derivation uses only the $\mathfrak{osp}(1|2)$ algebra. Hence we have the relation between Fock and discrete series inner products [37] for the complete set of states $\{|E_0, n\rangle \,|\, n \in \mathbb{Z}_{\geq 0}\}$

$$\begin{aligned} \big\langle |E_0, n\rangle, |E_0, m\rangle \big\rangle_{\mathfrak{osp}} &= \frac{2^{[\frac{n+1}{2}]}(E_0)_{[\frac{n+1}{2}]}}{(2[\frac{n+1}{2}]-1)!!}\langle E_0, n|E_0, m\rangle \\ &= 2E_0 n!\,\delta_{n,m}\big(1+\tfrac{2E_0-1}{3}\big)\big(1+\tfrac{2E_0-1}{5}\big)\cdots\big(1+\tfrac{2E_0-1}{2[\frac{n+1}{2}]-1}\big)\,. \end{aligned}$$

We would like to encode this using an operator built from the Casimir and number operators, and therefore note that

$$\sqrt{4c_{\mathfrak{osp}(1|2)} + \tfrac{1}{4}}\,\big|E_0, n\big\rangle = |E_0 - \tfrac{1}{2}|\,\big|E_0, n\big\rangle\,. \quad (14)$$

Thus, by virtue of the identity (13), we introduce the operator

$$\mathcal{I} = \frac{2^{\frac{N+F}{2}}\,(\hat{E}_0)_{\frac{N+F}{2}}}{(N+F-1)!!}\,,$$

where the operator-valued Pochhammer and double factorial are defined by expanding in eigenstates of N, while the operator $\hat{E}_0$ returns E_0 on all states and can be expressed in terms of the Casimir via (14). By construction $a\mathcal{I}|E_0, n\rangle = \mathcal{I}S^-|E_0, n\rangle$ whence

$$a\mathcal{I} = \mathcal{I}S^-\,.$$

Thus, the discrete series unitary inner product $\langle\cdot,\cdot\rangle_{\mathfrak{osp}}$ between states $\Psi = |\psi\rangle$ and $\Phi = |\phi\rangle$ then reads

$$\langle\Psi,\Phi\rangle_{\mathfrak{osp}} = \langle\psi|\mathcal{I}|\phi\rangle\,.$$

Hence, when $E_0 > 0$ we have found a realization of the unitary orthosymplectic discrete series representations $\mathcal{S}(E_0)$ in terms of the harmonic oscillator state space.

Finally, note that when $E_0 = -n \in Z_{\leq 0}$ the harmonic oscillator no longer gives an irreducible orthosymplectic representation. However, the space of descendants $\mathcal{H}_-$ of the singular vector

$$|n{+}1,0\rangle := |2n{+}1\rangle\,,\ \text{where } H|n{+}1,0\rangle = (n{+}1)|n{+}1,0\rangle\,,$$

form a unitary discrete series representation $\mathcal{S}(n+1)$ (with respect to the $E_0 = n{+}1$ inner product). The quotient $\mathcal{H}/\mathcal{H}_-$ then gives a finite dimensional (non-unitary) orthosymplectic representation.

VIII. SUPERCONFORMAL QUANTUM MECHANICS

We now want to repurpose our harmonic oscillator analysis for a study of novel superconformal theories. For that we will modify our Hilbert space such that the operator $-\frac{1}{2}Q^{--}$ is self-adjoint and plays the *rôle* of the Hamiltonian H. We may then view $\mathfrak{osp}(1|2)$ as a conformal superalgebra:

$$H = -\frac{1}{2}(S^-)^2\,,\quad D = \frac{1}{2}\{S^+,S^-\}\,,\quad K = \frac{1}{2}(S^+)^2\,,$$
$$iQ = S^-\,,\quad S = S^+\,.$$

Here, because $\mathfrak{osp}(1|2)$ imposes

$$Q^2 = 2H\,,$$

the operator Q is the SUSY generator. Also D and K correspond to dilations and conformal boosts while S is the conformal SUSY charge.

We now need to build the Hilbert space on which H and Q act. For that we begin by studying the space of wavefunctions $\psi(x)$ on the line $\mathbb{R}$. Since the de Rham cohomology of this space is trivial, we will *assume* that the abelian gauge field A appearing in Eq. (10) can be gauged away in the following analysis. Thus the SUSY charge is

$$iQ = \frac{\partial}{\partial x} + \frac{2E_0-1}{x}\,F\,.$$

while half its square gives the Hamiltonian

$$H = -\frac{1}{2}\frac{\partial^2}{\partial x^2} - \big(E_0 - \tfrac{1}{2}\big)\Big(\frac{1}{x}\frac{\partial}{\partial x} - \frac{1}{x^2}\,F\Big)\,.$$

In the above displays, the fermion occupation number F equals unity on odd wavefunctions $\psi_-(x) = -\psi_-(-x)$ and vanishes on even wavefunctions $\psi_+(x) = \psi_+(-x)$. The remaining $\mathfrak{osp}(1|2)$ generators are obtained by the replacement $z \mapsto x$ in the solution given in Eq. (11). Observe, that the $\mathbb{Z}_2$ grading $\mathfrak{osp}(1|2) = \mathcal{B}\oplus\mathcal{F}$ with $\mathcal{B} = \text{span}\{Q^{\pm\pm}, Q^{+-}\}$ and $\mathcal{F} = \text{span}\{S^\pm\}$ still holds when F is defined by wavefunction parity.

The inverse square potential in the above Hamiltonian is typical of conformal quantum mechanical models [8]. Supersymmetry charges and Hamiltonians of this type were also studied by Plyushchay [1, 12, 17]. Our next task is to develop an inner product with respect to which they are self-adjoint. This will require a careful analysis of the space of self-adjoint extensions for these operators [38].

IX. THE INNER PRODUCT

Our first task is to ensure definite hermiticity for the supercharge Q (thereafter we will examine its self-adjointness). For that, first observe that acting on odd functions iQ simply acts as $\frac{\partial}{\partial x} + \frac{2E_0-1}{x}$. Therefore it is convenient to define

$$\psi_-(x) =: x^{1-2E_0}\tilde{\psi}(x) \tag{15}$$

so that we have the identity

$$iQ\psi_-(x) = x^{1-2E_0}\,\frac{\partial}{\partial x}\tilde{\psi}(x)\ .$$

Note that E_0 is an, *a priori* arbitrary, complex number. Firstly let decompose wavefunctions into even and odd parts according to

$$\psi = \psi_+ + \psi_-\,,$$

and then use that the information of ψ is stored by $\psi_\pm$ on the positive half line $x > 0$. On the whole line we thus define [39]

$$\psi_\pm(x) := \begin{cases} \psi_\pm(x)\,, & x > 0\,,\\ \pm\psi_\pm(-x)\,, & x < 0\,.\end{cases}$$

Using the parametrization (15) for the odd part we may thus define the inner product

$$\begin{aligned}\langle\varphi,\psi\rangle :&= 2\int_0^\infty dx\, x^{2E_0-1}\Big[\varphi_+^*\psi_+ + \varphi_-^*\psi_-\Big]\\ &= 2\int_0^\infty dx\Big[x^{2E_0-1}\,\varphi_+^*\psi_+ + x^{1-2E_0}\,\tilde{\varphi}^*\tilde{\psi}\Big]\,.\end{aligned} \tag{16}$$

For $E_0 \in \mathbb{R}$, this inner product is positive definite and sesquilinear, but restricts the allowed behavior of wavefunctions at $x = 0, \infty$. In particular $\psi_\pm$ must be square integrable with respect to the measure x^{2E_0-1} on $\mathbb{R}$. In particular this requires that for small x, the fastest decay behavior of $\psi_\pm$ is $\psi_\pm \sim x^{a_\pm}$, with

$$a_\pm > -E_0\,. \tag{17}$$

We denote the space of functions with square integrable behavior at large x and decay rate at the origin satisfying the above bound by $\mathcal{H}_{a_+,a_-}$. We next examine the SUSY charge on these spaces.

Now since $iQ\psi = \psi'_+ + x^{1-2E_0}\tilde{\psi}'$ (primes denote x derivatives), it follows that

$$(iQ\psi)_+ = x^{1-2E_0}\tilde{\psi}'\,,\ \ (iQ\psi)_- = \psi'_+\,.$$

A wavefunction ψ sits inside the domain $\mathrm{dom}(Q)$ of Q provided it has the following small-x behavior

$$\psi_\pm \sim x^{a_\pm}\,, \quad a_\pm > 1 - E_0\,. \tag{18}$$

The operator Q is hermitian, since

$$\begin{aligned}\langle\theta, Q\psi\rangle^* &= 2i\int_0^\infty dx\big[\theta_+^*\tilde{\psi}' + \tilde{\theta}^*\psi'_+\big]^* \\ &= -2i\int_0^\infty dx\big[\psi_+^*\tilde{\theta}' + \tilde{\psi}^*\theta'_+\big] - 2i\big[\psi_+^*\tilde{\theta} + \tilde{\psi}^*\theta_+\big]|_{x=0} \\ &= \langle\psi, Q\theta\rangle\,, \quad \forall\theta,\,\psi\in\mathrm{dom}(Q)\,.\end{aligned} \tag{19}$$

In the above, the condition (18) guarantees cancellation of the boundary term, which only requires the (weaker) condition $a_+ + a_- > 1 - 2E_0$. Thus the SUSY charge is hermitean (indeed we chose the inner product (16) precisely for this reason). It remains to examine whether Q is (essentially) self-adjoint, or more precisely whether it admits self-adjoint extensions. The following analysis is standard and follows classical work by Von Neumann [19].

The space $\mathrm{dom}(Q)$ is dense in $\mathcal{H}$ so Q possibly has self-adjoint extensions. The dimension of the space of extensions equals the dimensions of $[\mathrm{ran}(Q\pm i\lambda)]^\perp$ for λ real and positive—if these dimensions differ for $\pm\lambda$ the operator Q has no self-adjoint extensions—these dimensions are known as *deficiency indices*. It is, of course, equivalent to compute the dimensions of $\ker(Q\pm i\lambda)$ and the condition $Q\psi = -i\lambda\psi$ amounts to

$$\psi'_+ = \lambda\psi_-\,, \quad x^{1-2E_0}\big(x^{2E_0-1}\psi_-\big)' = \lambda\psi_+\,. \tag{20}$$

These can be reduced to a pair of modified Bessel equations: We call $y = \lambda x$ and $\psi_\pm(x) = x^{1-E_0}u_\pm(y)$ and feed the two equations into one another which gives

$$u''_\pm(y) + \frac{1}{y}u'_\pm(y) - \left[1 + \frac{\alpha_\pm^2}{y^2}\right]u_\pm(y) = 0\,, \tag{21}$$

where $\alpha_+ = E_0 - 1$ and $\alpha_- = E_0$.

Equations (21) are identical for both $\pm\lambda$, so that the deficiency indices are equal. Solutions to (21) are modified Bessel functions (I_α, K_α) with indices $\alpha_\pm$. Of these solutions only $K_\alpha(\lambda x)$ has a good behavior at $x\to\infty$. On the other hand, for small, positive, x it behaves (up to a non-zero coefficient) as [40]

$$K_\alpha(\lambda x) \sim x^{-|\alpha|}\,,$$

so that

$$\psi_+(x) \sim x^{1-E_0-|E_0-1|}\,, \quad \psi_-(x) \sim x^{1-E_0-|E_0|}\,.$$

Hence, solutions to the kernel condition (20) are in $\mathcal{H}$ if the above exponents satisfy the condition (17), which amounts to

$$0 < E_0 < 1\,. \tag{22}$$

In other words, when the parameter E_0 satisfies the above condition both deficiency indices are unity and there is a one-parameter set of self-adjoint extensions [41]. On the other hand if E_0 does not satisfy (22) there is a unique extension. Since $2H = Q^2$, it follows that the Hamiltonian also has a unique self-adjoint extension in the latter case. Moreover, we immediately learn that the spectrum of H is bounded below by zero. This can also be seen by explicitly computing the expectation value of the Hamiltonian for some state $\psi = \psi_+ + \psi_- := \psi_+ + x^{\frac{1}{2}-E_0}\chi$:

$$\begin{aligned}\langle\psi, H\psi\rangle &= -\int_0^\infty dx\, x^{2E_0-1}\left[\psi_+^*\psi''_+ + \frac{2E_0-1}{x}\psi_+^*\psi'_+\right] - \int_0^\infty dx\, x^{2E_0-1}\left[\psi_-^*\psi''_- + \frac{2E_0-1}{x}\left(\psi_-^*\psi'_- - \frac{|\psi_-|^2}{x}\right)\right] \\ &= \int_0^\infty dx\, x^{2E_0-1}|\psi'_+|^2 + \int_0^\infty dx\left[|\chi'|^2 + \frac{E_0^2-\frac{1}{4}}{x^2}|\chi|^2\right] = \int_0^\infty dx\, x^{2E_0-1}|\psi'_+|^2 + \int_0^\infty dx\left|\chi' + \frac{E_0-\frac{1}{2}}{x}\chi\right|^2.\end{aligned}$$

Here we have used that ψ is in the domain of H to kill boundary terms at the origin generated by integrations by parts in the above computation. The final result is manifestly positive for all E_0 (even though the Hamiltonian has non-positive potential term for $E_0 < \frac{1}{2}$ acting on odd wavefunctions).

X. THE SPECTRUM

To compute the spectrum of the model we diagonalize the SUSY charge Q in order to solve the Schrödinger equation $H\psi = E\psi$. The "BPS" states obeying $Q\psi = 0$ are constants which are not finite norm. This indicates that we expect to find a plane wave normalizable spectrum, just as for the free particle on a line.

Indeed, we may recycle our deficiency index computation to solve $H\psi = E\psi$ by replacing $\lambda \to i\sqrt{2E}$. We find $\psi_\pm = x^{1-E_0} v_\pm(\sqrt{2E}x)$ where $v_\pm(y)$ obeys the Bessel equation

$$v''_\pm(y) + \frac{1}{y} v'_\pm(y) + \left[1 - \frac{\nu_\pm^2}{y^2}\right] v_\pm(y) = 0\,,$$

with indexes

$$\nu_+ = |E_0 - 1|\,, \quad \nu_- = |E_0|\,. \tag{23}$$

Here we have chosen $\nu_\pm \geq 0$ in order that we get plane wave normalizable solutions. Thus we have

$$\psi_E(x) = \frac{\beta_+ J_{|E_0-1|}(\sqrt{2E}x) + \beta_- J_{|E_0|}(\sqrt{2E}x)}{x^{E_0-1}}\,,$$

where the complex constants $\beta_\pm$ multiply the even/odd solutions. It follows from our previous deficiency index computations that these solutions are not normalizable, nonetheless, they obey an analog of plane wave normalizability by virtue of the closure relation for Bessel functions (valid for $\nu > -1/2$ and hence for any values of the positive indexes $\nu_\pm$ in Eq. (23))

$$\int_0^\infty x dx\, J_\nu(\sqrt{2E}x) J_\nu(\sqrt{2E'}x) = \frac{\delta(\sqrt{2E} - \sqrt{2E'})}{\sqrt{2E}} = \delta(E - E')\,.$$

Indeed, if we define Bose and Fermi scattering states by

$$|E, +\rangle = \frac{J_{|E_0-1|}(\sqrt{2E}x)}{\sqrt{2}\, x^{E_0-1}}\,, \quad |E, -\rangle = \frac{J_{|E_0|}(\sqrt{2E}x)}{\sqrt{2}\, x^{E_0-1}}\,,$$

then $\langle E, - \,|\, E, +\rangle = 0$ and

$$\langle E, + \,|\, E', +\rangle = \delta(E - E') = \langle E, - \,|\, E', -\rangle\,.$$

In addition to particle scattering states, it is interesting to look for the $\mathfrak{osp}(1|2)$ analog of the $\mathfrak{sp}(2)$ spherical vector. Indeed recall that the spherical vector for the metaplectic representation of $Sp(2,\mathbb{R})$ is the state with minimal eigenvalue of the generator $H + K$ of the maximal compact subgroup $SO(2)$. Indeed, this is none other than the harmonic oscillator ground state $\psi_0 = \exp(-\frac{1}{2}x^2)$. When $E_0 = \frac{1}{2}$, this state is annihilated by $S + iQ$. For the $\mathfrak{osp}(1|2)$ algebra, we therefore search for states in the kernel of $S + iQ$. For bosonic (even) states, the only solution is again

$$\psi_0^B = \exp(-\frac{1}{2}x^2)\,,$$

which is in the Hilbert space $\mathcal{H}$ so long as $E_0 > 0$. For fermionic (odd) states, we must solve

$$\psi' + \frac{2E_0 - 1}{x}\psi + x\psi = 0$$

and find

$$\psi_0^F = \begin{cases} \dfrac{e^{-\frac{1}{2}x^2}}{x^{2E_0-1}}\,, & x > 0\,, \\ -\dfrac{e^{-\frac{1}{2}x^2}}{|x|^{2E_0-1}}\,, & x < 0\,. \end{cases}$$

The above state is in $\mathcal{H}$ whenever $E_0 < 1$. Note that strictly speaking, for values of the parameter E_0 with $0 < E_0 < 1$ a detailed analysis of the self-adjoint extensions of Q is required to decide which combination(s) of the above two states is actually in the kernel of $S + iQ$. The above states will play the *rôle* of highest weights in the next section.

XI. PARTICLE ORTHOSYMPLECTIC REPRESENTATION

It remains to identify the orthosymplectic representations realized by the particle solutions to the deformation equations.

First we compute the Casimir operator for the $\mathfrak{sp}(2)$ subalgebra (H, K, D), which reads

$$c_{\mathfrak{sp}(2)} = \frac{1}{4}D^2 + \frac{1}{2}\{H, K\} = \frac{1}{16}[iQ, S]\big([iQ, S] - 4\big)\,. \tag{24}$$

Using $[F, x] = x(1 - 2F)$, we here have

$$\big[iQ, S\big] = \Big[\frac{\partial}{\partial x} + \frac{2E_0 - 1}{x}\, F, x\Big] = 1 - (2E_0 - 1)(2F - 1), \tag{25}$$

so once again find

$$c_{\mathfrak{sp}(2)}(\mathcal{B}) = \tfrac{E_0(E_0-2)}{4} \text{ and } c_{\mathfrak{sp}(2)}(\mathcal{F}) = \tfrac{(E_0-1)(E_0+1)}{4}\,,$$

and in turn $\mathcal{H} = \mathcal{B} \oplus \mathcal{F}$ obeys

$$c_{\mathfrak{osp}(1|2)}(\mathcal{H}) = \frac{E_0(E_0 - 1)}{4} = c_{\mathfrak{osp}(1|2)}\big(\mathcal{S}(E_0)\big)\,.$$

Unitarity requires that the generators $\{iQ, S, H, iD, K\}$ are self-adjoint. Our deficiency index analysis shows that this holds for all E_0, modulo the choice of self-adjoint extension for $0 < E_0 < 1$.

To analyze the $\mathfrak{osp}(1|2)$ content of the model, we can consider an oscillator-like basis for the generators with

the reality condition (B6) by employing the map (B7). Indeed, calling

$$A := \frac{S + iQ}{\sqrt{2}}\,, \qquad A^\dagger = \frac{S - iQ}{\sqrt{2}}\,,$$

we have (using (25))

$$[A, A^\dagger] = 1 - (2E_0 - 1)(2F - 1)\,,$$

and $S^+ = A^\dagger$, $S^- = A$ obey the $\mathfrak{osp}(1|2)$ algebra (2). (Note that this is a different solution to that given in Eq. (11).) At this point the operators A and $A^\dagger$ obey the same algebra as analyzed for the oscillator models in Section VII, so we can inherit that analysis, however some care is required when $0 < E_0 < 1$.

Firstly when $|E_0 - \frac{1}{2}| \geq 1/2$ the self-adjoint extension problem gives a unique answer and indeed there is a unique highest weight state

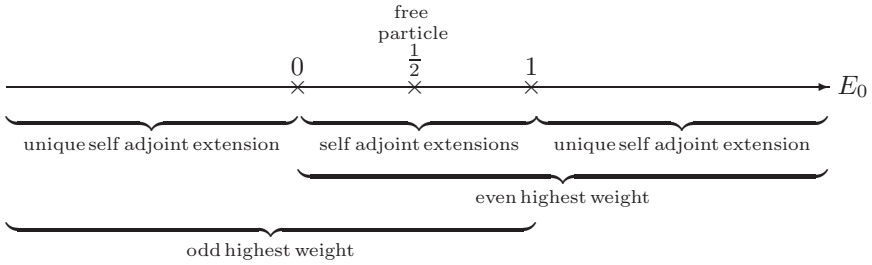
$$|E_0, 0\rangle = \begin{cases} \psi_0^B\,, & E_0 \geq 1\,, \\ \psi_0^F\,, & E_0 \leq 0\,. \end{cases}$$

The descendants of $|E_0, 0\rangle$ (generated by acting with $A^\dagger$) then span the irreducible representation $\mathcal{S}(E_0)$.

When $0 < E_0 < 1$ there are potentially two highest weight states ψ_0^B and ψ_0^F, however, we *conjecture* that only one combination of these is a zero mode of A for a given choice of self-adjoint extension of Q.

As an example consider the undeformed models with $E_0 = \frac{1}{2}$ and $Q = \frac{d}{dx}$. Here, the Hilbert space is $\mathcal{H} = L^2(\mathbb{R}^+) \oplus L^2(\mathbb{R}^+)$. There are of course no self-adjoint extensions of $\frac{d}{dx}$ on the half-line, but it is easy to find one for $\frac{d}{dx}$ defined on two copies of $\mathbb{R}^+$, namely by viewing pairs of wavefunctions there as the even and odd parts of wavefunctions in $L^2(\mathbb{R})$, on which $\frac{d}{dx}$ is essentially self-adjoint. In that case $A\psi_0^B = 0$ because ψ_0^B is the usual harmonic oscillator ground state, while $A\psi_0^F(x) = 2\delta(x) \neq 0$. The descendants of ψ_0^B then give the unitary irreducible orthosymplectic representation $\mathcal{S}(\frac{1}{2})$.

We have summarized the orthosymplectic representations realized by deformations of superconformal quantum mechanics in the diagram below:



XII. SUMMARY AND CONCLUSIONS

Although supersymmetric quantum mechanics has a long history [42], its presence in even the simplest of quantum mechanical models is often underappreciated—both the free particle and harmonic oscillator enjoy a hidden $\mathfrak{osp}(1|2)$ superconformal symmetry realized by employing wave-function parity for the Bose–Fermi $\mathbb{Z}_2$-grading. Given a particle/oscillator Hilbert space, we studied the natural question whether other sets of operators realize this algebra. In higher dimensions the moduli space of such operators has a particularly interesting geometric structure: For example, on *any* (pseudo)-Riemannian manifold whose metric $g_{\mu\nu}$ is the gradient of a covector $g_{\mu\nu} = \nabla_\mu \xi_\nu$, the triplet of operators $\{\xi_\mu \xi^\mu,\, \xi^\mu \nabla_\mu,\, \nabla^\mu \nabla_\mu\}$ generate the algebra $\mathfrak{sp}(2)$. Including spinors and the Dirac operator, this algebra can be extended to the $\mathfrak{osp}(1|2)$ superalgebra studied here and indeed our study is the special case when the underlying manifold is one-dimensional. The fact that we were able to give a detailed classification of this space of operators in a one-dimensional setting suggests that similar general results ought be obtainable in higher dimensions. This is exciting because of its relevance to interacting higher spin and quantum gravity models [2–4, 6, 7].

The one dimensional solutions to the $\mathfrak{osp}(1|2)$ operator question are parameterized by a one (complex) parameter moduli space. It would be interesting to try and mimic these results for higher hidden quantum mechanical SUSY algebras, the results of [24] indicates that this ought be possible [43]. Here, once one studies Hilbert spaces for mechanics in higher dimensions, one expects a moduli space of solutions with more constraining geometric structures than conformal geometries.

One might wonder whether our results contravene the Stone–Von Neumann theorem on unitary equivalence of

Heisenberg representations. This is not the case because the Plyushchay-type models generate $\mathfrak{osp}(1|2)$ representations with differing values of E_0 and inner product by modifying the commutation relation $[a, a^\dagger] = 1$ to $[S, S^\dagger] = 1 - (2E_0 - 1)(2F - 1)$, where $E_0 = 1/2$ gives the standard harmonic oscillator model. It interesting to note that that this deformation is important for deformations higher spin algebras leading to interactions [26, 27].

The $E_0 = 1/2$ orthosymplectic representation is a sum of two discrete series $\mathfrak{sl}(2, \mathbb{R})$ representations analogous to the double cover half integer spin representations in the theory of angular momentum. Indeed, this is the so-called metaplectic representation of $Sl(2, \mathbb{R})$. It would be interesting to exponentiate these realizations of $\mathfrak{osp}(1|2)$ representations to give analogs of the metaplectic representation.

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Appendix A: General Parity Solutions

To show that the solution (11) is general, we must relax the requirement that the $\mathbb{Z}_2$ gradings of the $\mathfrak{osp}(1|2)$ Lie superalgebra and the Hilbert space $\mathcal{H}$ are coincident. Thus we study a general version of the ansatz Eq. (9), namely

$$S^- = \frac{\partial}{\partial z} + \alpha_+(z) + \alpha_-(z) + (-1)^F \big[\beta_+(z) + \beta_-(z)\big] .$$

Here and in what follows, we denote even/odd functions of z by a subscript $\pm$. The second, nonlinear equation in (3) now yields a Dirac-like equation

$$\big(z\frac{\partial}{\partial z} + 1\big) \begin{pmatrix} \beta_+(z) \\ \beta_-(z) \end{pmatrix} - 2z\alpha_+(z) \begin{pmatrix} \beta_-(z) \\ \beta_+(z) \end{pmatrix} = 0 .$$

Notice that $\alpha_-(z)$ is completely free while we can solve for $\beta(z) = \beta_+(z) + \beta_-(z)$ in terms of $\alpha_+(z)$ as

$$\beta(z) = \frac{E_0 - \frac{1}{2}}{z} \exp\Big(2\int^z \alpha_+\Big) .$$

Hence we find

$$S^- = \frac{\partial}{\partial z} + \alpha(z) + (-)^F \frac{E_0 - \frac{1}{2}}{z} \exp\Big(2\int^z \alpha_+\Big) .$$

Here $\alpha(z) = \alpha_+(z) + \alpha_-(z)$ and we must set $\alpha_-(z) = \frac{2E_0-1}{2z} + a_-(z)$ (with $a_-(z)$ odd and analytic) to cancel the $1/z$ pole in S^-. Again, evenness of $U(z) = \exp\big(-\int^z a_-(z)\big)$ allows use to gauge away $a_-(z)$. This yields

$$S^- = \frac{\partial}{\partial z} + \frac{2E_0 - 1}{z} F + \alpha_+(z) + (-1)^F \frac{E_0 - \frac{1}{2}}{z} \Big[\exp\Big(2\int^z \alpha_+\Big) - 1\Big] ,$$

which is the sum of our previous $\mathfrak{osp}(1|2)$ odd solution and a mixed $\mathfrak{osp}(1|2)$ parity solution parameterized by the even, analytic function $\alpha_+(z)$.

The Hamiltonian for this class of models is given by

$$H = N + E_0 + z\alpha_+(z) .$$

The Casimir is again $c_{\mathfrak{osp}(1|2)} = E_0(E_0 - 1)/4$ which suggests that this solution is gauge equivalent to our previous one. Indeed the additional gauge transformation $U(z) = \exp\big(-\int^z a_+(z)\big)$ can be used to remove the $a_+(z)$ dependence of the Hamiltonian and the ladder operator S^-, whence $H = N + E_0$ and $S^- = \frac{\partial}{\partial z} + \frac{2E_0-1}{z}F$. Remembering that $S^+ = z$, we recognize our previous solution in Eq. (11).

Appendix B: Orthosymplectic representation theory

The following material reviews basic results from the representation theory of $\mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{osp}(1|2)$. We also provide a translation between common notations found in the literature and those used here.

The Lie algebra $\mathfrak{sl}(2, \mathbb{C}) = \{e, h, f\}$ where [44]

$$[h, e] = 2e , \quad [e, f] = h , \quad [f, h] = 2f , \tag{B1}$$

has two inequivalent real forms; since we are interested in quantum mechanical models with infinite dimensional Hilbert spaces, our focus is on the non-compact $\mathfrak{sl}(2, \mathbb{R}) \cong \mathfrak{sp}(2, \mathbb{R})$ form [45]

$$e^\dagger = -f , \quad h^\dagger = h , \quad f^\dagger = -e . \tag{B2}$$

For example, the harmonic oscillator obeys the above by setting $h = H = a^\dagger a + \frac{1}{2}$, $e = \frac{1}{2}(a^\dagger)^2$ and $f = -\frac{1}{2}a^2$. The real linear map

$$e \mapsto \tfrac{1}{2}(h + e - f) \quad h \mapsto -e - f , \quad f \mapsto \tfrac{1}{2}(h - e + f) , \tag{B3}$$

preserves the $\mathfrak{sl}(2)$ Lie algebra but gives reality conditions

$$e^\dagger = e , \quad h^\dagger = -h , \quad f^\dagger = f . \tag{B4}$$

This choice of $\mathfrak{sl}(2, \mathbb{R})$ generators corresponds to the free particle on a line with $e = \frac{1}{2}x^2$, $h = x\frac{\partial}{\partial x} + \frac{1}{2}$ and $f = H = -\frac{1}{2}\frac{\partial^2}{\partial x^2}$.

The Lie algebra (B1) is extended to the $\mathbb{Z}_2$ graded algebra $\mathfrak{osp}(1|2) \cong \mathfrak{sp}(2) \oplus \mathbb{R}^2$ by adding odd generators s and q that obey

$$\{s,s\} = e\,, \quad \{s,q\} = \frac{1}{2}h\,, \quad \{q,q\} = -f\,. \tag{B5}$$

In the notation of the introduction, $s = \frac{1}{2}S^+$, $q = \frac{1}{2}S^-$ so the remaining commutation relations may be read off the second line of (2) and read

$$[s,f] = q\,, \quad [h,s] = s\,, \quad [q,h] = q\,, \quad [q,e] = s\,.$$

Given the reality conditions (B2), there are two inequivalent reality conditions for the odd generators [14]

$$s^\dagger = \pm q\,, \quad q^\dagger = \pm s\,. \tag{B6}$$

The first choice above is realized by the harmonic oscillator with $s = \frac{1}{2}a^\dagger$ and $q = \frac{1}{2}a$. The real linear map

$$s \mapsto \tfrac{1}{\sqrt{2}}(s+q)\,, \quad q \mapsto \tfrac{1}{\sqrt{2}}(-s+q)\,, \tag{B7}$$

induces the map (B3) through the relations (B5) and preserves the $\mathfrak{osp}(1|2)$ algebra. It gives again the free particle-type reality conditions (B4) and reality conditions

$$s^\dagger = \pm s\,, \quad q^\dagger = \mp q\,.$$

The first case corresponds to a free particle on the line with $s = \frac{1}{2}x$ and $q = \frac{i}{2}Q = \frac{1}{2}\frac{\partial}{\partial x}$.

Unitary irreducible representations of $\mathfrak{sl}(2,\mathbb{R})$ are infinite dimensional [15] and fall into three series: principal, supplementary and discrete. Unitary irreducible representations of $\mathfrak{osp}(1|2)$ are also infinite dimensional and are built from a direct sum of discrete series representations [16]: Call

$$\mathcal{D}(E_0) := \mathrm{span}\big\{|E_0, 2k\rangle = e^k|E_0,0\rangle \,\big|\, k \in \mathbb{Z}_{\geq 0}\,,\, h|E_0,0\rangle = E_0|E_0,0\rangle\,, f|E_0,0\rangle = 0\big\}\,.$$

The reality conditions (B6) imply that

$$\big|\big||E_0,2k\rangle\big|\big|^2 = k!E_0(E_0+1)\cdots(E_0+k-1)\big|\big||E_0,0\rangle\big|\big|^2\,.$$

The right hand side above is certainly positive whenever the "ground state energy" $E_0 \in \mathbb{R}_{>0}$. Indeed the Hilbert space $\mathcal{D}(E_0)$ for real positive E_0 is the unitary irreducible (positive) discrete series representation of $\mathfrak{sl}(2,\mathbb{R})$. It has quadratic Casimir

$$c_{\mathfrak{sp}(2)} = \frac{1}{4}h^2 + \frac{1}{2}(ef+fe)$$

given by

$$c_{\mathfrak{sp}(2)}\big(\mathcal{D}(E_0)\big) = \frac{E_0(E_0-2)}{4} = \frac{1}{4}\big[(E_0-1)^2-1\big]\,.$$

Hence the representations $\mathcal{D}(E_0)$ and $\mathcal{D}(2-E_0)$ have the same Casimir. In particular, the harmonic oscillator Hilbert space is

$$\mathcal{D}(1/2) \oplus \mathcal{D}(3/2)\,,$$

where both discrete series representations have $c_{\mathfrak{sp}(2)} = -\frac{3}{16}$. Indeed $\mathcal{D}(1/2)$ is spanned by even number operator eigenstates $\{|0\rangle, |2\rangle, \ldots\}$ with $|1/2,0\rangle = |0\rangle$ while $\mathcal{D}(3/2)$ is spanned by odd eigenstates $\{|1\rangle, |3\rangle, \ldots\}$ with $|3/2,0\rangle = |1\rangle$. The above Hilbert space also forms the metaplectic representation of the group $Sl(2,\mathbb{R})$; this can be viewed as the non-compact analog of the double cover spin representations of $SU(2)$.

The unitary irreducible representations of $\mathfrak{osp}(1|2)$ generalize the harmonic oscillator example and are given by the $\mathbb{Z}_2$-graded vector space [16]

$$\begin{aligned}\mathcal{S}(E_0) &= \mathrm{span}\Big\{|E_0,2k\rangle = e^k|E_0,0\rangle, |E_0+1,2k\rangle = e^k s|E_0,0\rangle \,\Big|\, k \in \mathbb{Z}_{\geq 0}\,,\, h|E_0,0\rangle = E_0|E_0,0\rangle\,, f|E_0,0\rangle = 0 = q|E_0,0\rangle\Big\} \\ &= \mathcal{D}(E_0) \oplus \mathcal{D}(E_0+1) = \mathcal{D}(E_0) \oplus s\mathcal{D}(E_0)\,.\end{aligned}$$

where $E_0 > 0$. The respective $\mathfrak{sp}(2)$ Casimirs differ by $\frac{1}{2}\big(E_0 - \frac{1}{2}\big)$. The $\mathfrak{osp}(1|2)$ Casimir is

$$c_{\mathfrak{osp}(1|2)} = c_{\mathfrak{sp}(2)} + \frac{1}{2}\big(qs - sq\big)\,. \tag{B8}$$

This can be reexpressed in the enveloping algebra as

$$c_{\mathfrak{osp}(1|2)} = [q,s]\big([q,s] - \tfrac{1}{2}\big)\,.$$

On the orthosymplectic discrete series it takes the value

$$c_{\mathfrak{osp}(1|2)}\big(\mathcal{S}(E_0)\big) = \frac{1}{4}\,E_0(E_0-1) = \frac{1}{4}\left[(E_0-\tfrac{1}{2})^2 - \tfrac{1}{4}\right].$$

Observe that this is minimized by $E_0 = \frac{1}{2}$ which corresponds to the harmonic oscillator.

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[28] Readers familiar with the the Dirac operator $\gamma^\mu\nabla_\mu$ playing the *rôle* of a SUSY charge, may wish to call $a := \partial/\partial z$ and $a^\dagger := z$ and then introduce a single Grassmann coordinate γ obeying the Clifford algebra $\{\gamma,\gamma\} = 2$. Then the SUSY charge $S^- = \gamma\,\partial/\partial z$. In the above, γ has been represented by the 1×1 matrix 1.

[29] Indeed the space $\mathcal{B}$ of even number operator eigenstates of the harmonic oscillator form the metaplectic representation of $\mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{sp}(2)$. The full Fock space has the $\mathbb{Z}_2$ graded decomposition $\mathcal{H} = \mathcal{B}\oplus\mathcal{F}$.

[30] Here we have suppressed vanishing relations and the $\mathfrak{sp}(2)$ algebra obeyed by $\{Q^{++}, Q^{+-}, Q^{--}\}$.

[31] Recall that harmonic oscillator eigenstates $|n\rangle$ are given by a Gaussian multiplied by Hermite polynomials, which are parity even (odd) when n is even (odd). In a coherent state picture, we could alternatively view the operator a as a one dimensional Dirac operator $\gamma\frac{\partial}{\partial z}$ where γ obeys the Clifford algebra $\{\gamma,\gamma\} = 2$.

[32] Note that these derive from a simple action principle [6] $S = \mathrm{tr}\big[S^+S^- + \frac{1}{2}S^+S^+S^-S^-\big]$, where tr denotes an operator trace.

[33] Alternatively, one may first solve for a set of operators on some space and only thereafter search for an appropriate inner product.

[34] Put simply, as a convenient bookkeeping device, we identify $a^\dagger \leftrightarrow z$, $a \leftrightarrow \frac{\partial}{\partial z}$ and $|0\rangle \leftrightarrow 1$.

[35] Observe that $[1-\exp(-2N)]/N$ is analytic in N.

[36] The material presented in Section VII was actually developed independently of [12] and reproduces the results found there. The results on the singular vector are new.

[37] This relation was first uncovered in [17].

[38] A similar analysis for the case of the Dirac operator in monopole backgrounds has been performed in [18].
[39] *A posteriori* we will check that the behavior at $x = 0$ is smooth, *i.e.*, no powers of $|x|$ are involved.
[40] For $\alpha = 0$, K_0 behaves logarithmically and the corresponding wave functions are not normalizable.
[41] The problem of finding the boundary conditions for operators of the form $-\frac{\partial^2}{\partial x^2} + (\alpha - \frac{1}{4})\frac{1}{x^2}$ on the half-line is of topical interest in the analysis literature, see [20]
[42] See, for example [21–23].
[43] Another avenue for generating such model is to study the Dirac equation in more general backgrounds, such as dyonic ones [25].
[44] The conventions $X_+ = e$, $H = h/2$ and $X_- = -f$ for which $[H, X_+] = X_+$, $[X_+, X_-] = -2H$, $[X_-, H] = X_-$, and $X_0^\dagger = D_0$, $X_\pm^\dagger = X_\mp$ are also common.
[45] The congruence between $\mathfrak{sl}(2, \mathbb{R})$ and the worldline conformal algebra $\mathfrak{so}(2, 1) = \{J_0, J_1, J_2\}$ is often useful. This is given by $J_0 = h/2 = J_0^\dagger$, $J_1 = \frac{i}{2}(e + f) = J_1^\dagger$ and $J_2 = \frac{1}{2}(e - f) = J_2^\dagger$.