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NON-ISOTHERMAL AXIAL FLOW OF A RAREFIED GAS BETWEEN TWO COAXIAL CYLINDERS

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Abstract. A stationary problem of heat transfer in a rarefied gas confined between two coaxial cylinders is presented. The two cylinders are maintained at two different constant temperatures, so that a radial temperature gradient is created. The external cylinder is at rest, while the internal one moves in the axial direction with a constant velocity. A flow of the gas in the z-direction, orthogonal to the temperature gradient, is created by the motion of the internal boundary. Instead of the classical Navier-Stokes and Fourier theory, the field equations of extended thermodynamics with 13 moments are used in order to describe this physical problem. It turns out that, although only a radial temperature difference is imposed, the heat flux presents also an axial component. Moreover, some components of the stress tensor do not vanish, even though the axial velocity of the gas depends only on the radial coordinate r. The solution here obtained is compared with the classical one. Furthermore, the dependence of the solution on the boundary velocity is investigated.

1. Introduction

The description of physical phenomena in rarefied gases far from equilibrium is a relevant topic both from the theoretical and form the experimental point of view. It has motivated many studies and experiments in order to catch the main features of the problem. In particular, stationary heat transfer in rarefied gases has been the subject of many works, since it represents the simplest non-equilibrium example. So, starting from it, different approaches and methods have been tested and compared.

Historically, one referred to the classical thermodynamics in order to describe mathematically such a phenomenon. Following this classical theory, the field equations are based on the conservation laws of mass, momentum and energy, while the closure of the system is obtained thanks to the Navier-Stokes and the Fourier laws. However, it was observed that, while the Navier-Stokes-Fourier (NSF) approximations are in complete agreement with the experimental results for rather dense gases, they are not appropriate for rarefied cases or when strong deviations from equilibrium occur.

For stationary heat conduction in a rarefied gas several analyses have been carried out following different approaches. We recall for example the kinetic theory results (Bassanini *et al.* 1967; Ohwada *et al.* 1989; Ohwada 1996).

In this article we will refer to extended thermodynamics (Müller and Ruggeri 1998). This theory considers as field variables not only the classical ones (mass densities, momentum and energy) but also the stress tensor, the heat flux and, for a monatomic gas, the moments of the distribution function up to a given number. The corresponding field equations are balance laws supplemented by local and instantaneous constitutive equations. Such constitutive functions are determined through the validity requirement of universal physical principles, such as the entropy principle (existence of an entropy inequality and concavity of the entropy density) and the principle of relativity.

Müller and Ruggeri (2004) proved that, if one considers the stationary heat conduction problem in a monatomic classical ideal gas in radial symmetry, even the simplest model of extended thermodynamics (the 13-moment theory) predicts results different from classical thermodynamics. This fact has motivated many investigations of the heat conduction problem using 13-moment extended thermodynamics: the stationary heat conduction problem was studied in the case of a gas between two coaxial rotating cylinders (Barbera and Müller 2006) or between two confocal elliptical cylinders at rest (Barbera and Müller 2008). Then, the analysis of stationary heat transfer problems was extended to general 3D symmetric domains (Barbera and Brini 2010; Barbera *et al.* 2012). All these studies lead to the general conjecture that, the differences between the stationary solutions of classical and extended thermodynamics increase when the geometry of the problem becomes more complex and further from the planar one.

Meanwhile, other authors studied the solution of 13-moment extended thermodynamics when a flow is introduced. In particular, Marques Jr. and Kremer (2001) investigated the planar Couette flow, whereas Gramani Cumin *et al.* (2002) (see also the references therein) investigated the non-isotermal cylindrical Couette flow with a tangential velocity. It was shown that the nonlinear equations of 13-moment extended thermodynamics are already able to predict some differences from the classical thermodynamics which are in agreement with the expectation of the kinetic theory.

In this article, we consider the field equation of 13-moment extended thermodynamics in order to describe a non-isotermal axial flow in a rarefied monatomic gas. More precisely, we assume that the gas is enclosed in the gap between two coaxial cylinders. The external and the internal cylinders are kept at two different constant temperatures T_e and T_i. In this way a radial temperature field is imposed. Moreover, we assume that the external cylinder is at rest, while the internal one translates with a constant axial velocity V, parallel to its axis. To our knowledge this is the first time that this problem is studied in the literature for rarefied gases in the context of 13-moments extended thermodynamics. We show that extended thermodynamics is able to predict a stationary solution which depends only of the radial coordinate r; the heat flux admits an axial non-vanishing heat flux; moreover, the stress tensor presents more non-vanishing components with respect to those predicted by the classical theory.

The article is organized as following. The field equations appropriate to our problem are derived in Section 2, while in Section 3 the boundary conditions are introduced. Then, the solutions of the problems are shown, commented and compared with the results of the classical theory in Section 4. Finally, in the last section, some final remarks are addressed.

2. Field equations

The field variables of extended thermodynamics (Müller and Ruggeri 1998) for an ideal monatomic gas are the moments of the distribution function $f(\mathbf{x},\mathbf{c},t)$ of the gas, where $f(\mathbf{x},\mathbf{c},t)\mathrm{d}\mathbf{c}$ represents the number density of the atoms at the time t, position $\mathbf{x}$ with velocity $\mathbf{c}$. The moments are defined as

$$F_{i_1 i_2 \dots i_N} = m \int c_{i_1} c_{i_2} \dots c_{i_N} f \mathrm{d}\mathbf{c}, \tag{1}$$

where m is the atomic mass of the gas. The equations for these quantities are obtained from the Boltzmann equation which, in the BGK approximation (Bhatnagar *et al.* 1954), reads

$$\frac{\partial f}{\partial t} + c_k \frac{\partial f}{\partial x_k} = -\frac{f - f_\mathrm{E}}{\tau} \quad \text{with} \quad f_\mathrm{E} = \frac{\rho}{m}\sqrt{\frac{m}{2\pi k_\mathrm{B} T}}\, e^{-\frac{m(c-v)^2}{2k_\mathrm{B}T}} \,. \tag{2}$$

if τ denotes a constant relaxation time, k_B the Boltzmann constant, T the temperature, ρ the density, and $\mathbf{v}$ the velocity of the gas, while the symbol "E" indicates an equilibrium state and the equilibrium distribution function is the Maxwellian one $(2)_2$.

Multiplying the Boltzmann equation $(2)_1$ by $c_{i_1} c_{i_2} \dots c_{i_N}$ and integrating over the whole range of $\mathbf{c}$, one obtains the balance equations

$$\frac{\partial F_{i_1 i_2 \dots i_N}}{\partial t} + \frac{\partial F_{i_1 i_2 \dots i_N k}}{\partial x_k} = -\frac{F_{i_1 i_2 \dots i_N} - F^\mathrm{E}_{i_1 i_2 \dots i_N}}{\tau}, \tag{3}$$

that, for $N = 0, 1, ..$, represents the infinite hierarchy of moment balance equations.

Usually, the quantities $F_{i_1 i_2 \dots i_N}$ are decomposed into internal parts, that is velocity independent parts, and the remaining parts which depend explicitly on the velocity. The internal moments are defined in terms of the peculiar velocity, $\mathbf{C} = \mathbf{c} - \mathbf{v}$, as

$$\rho_{i_1 i_2 \dots i_N} = m \int C_{i_1} C_{i_2} \dots C_{i_N} f \mathrm{d}\mathbf{c}. \tag{4}$$

The first thirteen internal moments can be expressed in terms of the most common thermodynamic variables: ρ is the density of the gas, $\rho_i = 0$, $\rho_{ll} = 3p = 3\frac{k_\mathrm{B}}{m}\rho T$, where p represents the pressure. Furthermore, we have $\rho_{<ij>} = -t_{<ij>}$ and $\rho_{ill} = 2q_i$, where t_{ij} is the stress tensor, the square brackets indicate the traceless part of a symmetric tensor, and q_i represents the heat flux.

Requiring the Galilean invariance, it can be proved (Ruggeri 1989; Müller and Ruggeri 1998) that the relation between $F_{i_1 i_2 \dots i_N}$ and $\rho_{i_1 i_2 \dots i_N}$ reads

$$F_{i_1 i_2 \dots i_N} = \sum_{k=1}^{N} \begin{pmatrix} N \\ k \end{pmatrix} \rho_{(i_1 i_2 \dots i_{N-k}} v_{i_{N-k+1}} \dots v_{i_N)}, \tag{5}$$

where round brackets denote the symmetrization.

Then, inserting the decomposition (5) into the balance equations (3) one gets the following equations for the internal moments

$$\begin{aligned} &\frac{\mathrm{d}\rho_{i_1 i_2 \dots i_N}}{\mathrm{d}t} + \frac{\partial \rho_{i_1 i_2 \dots i_N k}}{\partial x_k} + \rho_{i_1 i_2 \dots i_N} \frac{\partial v_k}{\partial x_k} + N\rho_{(i_1 i_2 \dots i_{N-1}} \frac{\mathrm{d}v_{i_N)}}{\mathrm{d}t} + \\ &\qquad + N\rho_{k(i_1 i_2 \dots i_{N-1}} \frac{\partial v_{i_N)}}{\partial x_k} = -\frac{\rho_{i_1 i_2 \dots i_N} - \rho^\mathrm{E}_{i_1 i_2 \dots i_N}}{\tau}, \end{aligned} \tag{6}$$

where the symbol d/dt denotes the material time derivative, that is $\mathrm{d}/\mathrm{d}t = \partial/\partial t + v_k \partial/\partial x_k$, and $N = 0, 1, \dots$.

The infinite hierarchy of balance equations for the internal moments (6) must be truncated at a finite number of equations. A common choice is to consider the first thirteen moments, which indeed as said before have a clear physical meaning, then we have explicitly

$$\begin{aligned}
&\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\frac{\partial v_k}{\partial x_k} = 0,\\
&\rho\frac{\mathrm{d}v_i}{\mathrm{d}t} + \frac{\partial \rho_{ik}}{\partial x_k} = 0,\\
&\frac{\mathrm{d}\rho_{ij}}{\mathrm{d}t} + \rho_{ij}\frac{\partial v_k}{\partial x_k} + \frac{\partial \rho_{ijk}}{\partial x_k} + 2\rho_{k(i}\frac{\partial v_{j)}}{\partial x_k} = -\frac{\rho_{<ij>}}{\tau},\\
&\frac{\mathrm{d}q_i}{\mathrm{d}t} + \frac{1}{2}\frac{\partial \rho_{ikll}}{\partial x_k} + q_i\frac{\partial v_k}{\partial x_k} + \frac{3}{2}\rho_{(il}\frac{\mathrm{d}v_{il)}}{\mathrm{d}t} + \frac{3}{2}\rho_{k(il}\frac{\partial v_{l)}}{\partial x_k} = -\frac{q_i}{\tau}.
\end{aligned} \tag{7}$$

The first two equations and the trace of the third one represent respectively the conservation laws of mass, momentum and energy. The remaining equations are the balance laws for the traceless parts of the stress tensor and for the heat flux and are peculiar of extended thermodynamics.

Unfortunately, these last equations are not closed due to the occurrence of the third and four rank moments, $\rho_{<ijk>}$ and ρ_{ikll}, which are not a priori related to the field variables. They are determined through the Grad distribution function (Grad 1949; Müller and Ruggeri 1998), that represents an expansion of the distribution function f in the neighborhood of its equilibrium value, f_E, in terms of the Hermite polynomials, i.e.

$$f_\mathrm{G} = f_\mathrm{E}\left[1 + \frac{m}{2pk_\mathrm{B}T}\rho_{<ij>}C_iC_j - \frac{m^2}{pk_\mathrm{B}^2T^2}q_iC_i\left(1 - \frac{m}{5k_\mathrm{B}T}C^2\right)\right]. \tag{8}$$

Inserting the Grad distribution f_G into the expressions of the moments $\rho_{<ijk>}$ and ρ_{ikll} (4), one gets the constitutive relations

$$\rho_{<ijk>} = 0, \quad \rho_{ikll} = 5p\frac{k_\mathrm{B}}{m}T\delta_{ij} + 7\frac{k_\mathrm{B}}{m}T\rho_{<ij>}, \tag{9}$$

where δ_{ij} denotes the Kronecker tensor, as usually.

So, substituting the constitutive relations (9) into the balance equations (7), one obtains the closed system of field equations, which consists of thirteen equations in the thirteen field variables ρ, v_i, T, $\rho_{<ij>}$ and q_i. For convenience, in the following we will use $p = \frac{k_\mathrm{B}}{m}\rho T$ as field variables instead of ρ.

In order to study the physical problem presented in the introduction, it is more suitable to use cylindrical coordinates (r, ϑ, z), surely more appropriate to the description of the gas in the gap between two cylinders. So, we rewrite the thirteen field equations (7,9) in terms of these coordinates, assuming that the fields depend only on the radial coordinate r, and that the radial and the tangential components of the gas velocity identically[1] vanish. Furthermore, we consider the physical components of the tensors involved, instead of the contravariant and the covariant components. In fact, in some cases the contravariant and the covariant components of the tensors do not have an immediate physical meaning and usually their physical dimensions do not coincide with those of the tensors to which they refer. We recall that the physical components (Truesdell 1953; Müller 1985) of a tensor are defined as

$$\hat{f}_{i_1 i_2 \dots i_n} = \sqrt{g_{\underline{i_1 i_1}}}\sqrt{g_{\underline{i_2 i_2}}}\cdots\sqrt{g_{\underline{i_n i_n}}}f^{i_1 i_2 \dots i_n}, \tag{10}$$

[1]These assumptions are not necessary, since they follow directly from the field equations if we suppose that they vanish at the boundary.

where $f^{i_1 i_2 \dots i_n}$ represent the contravariant components of the tensor f, while g_{ij} are the covariant components of the metric tensor and the underlined indices indicate that they are unsummed.

The dimensionless physical quantities and the parameter Kn are defined as

$$\begin{gathered}\tilde{r}=\frac{r}{r_\mathrm{e}}, \qquad \tilde{p}=\frac{p}{p_0}, \qquad \tilde{T}=\frac{T}{T_\mathrm{e}}, \qquad \tilde{v}_i=\frac{v_i}{\sqrt{\frac{k_\mathrm{B}}{m}T_\mathrm{e}}} \\ \tilde{\rho}_{<ij>}=\frac{\hat{\rho}_{<ij>}}{p_0}, \quad \tilde{q}_i=\frac{\hat{q}_i}{p_0\sqrt{\frac{k_\mathrm{B}}{m}T_\mathrm{e}}}, \quad \mathrm{Kn}=\frac{\tau\sqrt{\frac{k_\mathrm{B}}{m}T_\mathrm{e}}}{r_\mathrm{e}},\end{gathered} \tag{11}$$

where r_e is the radius of the external cylinder, T_e its temperature, while p_0 denotes a suitable value of the pressure. Such a quantity will be defined in the following. The parameter Kn in (11) is related to the Knudsen number and represents a good measure of the gas rarefaction. In particular, Kn$\ll 1$ corresponds to a dense gas, while when the gas is rarefied, Kn is closer to 1.

Therefore, after dropping the tilde for notation simplicity, the system of field equations, in terms of the physical dimensionless variables (11) reads [2]

$$\rho_{<r\vartheta>}=0, \quad \rho_{<\vartheta z>}=0, \quad q_\vartheta=0, \tag{12}$$

and

$$\begin{aligned}
&\frac{\mathrm{d}(p+\rho_{<rr>})}{\mathrm{d}r}+\frac{\rho_{<rr>}-\rho_{<\vartheta\vartheta>}}{r}=0,\\
&\frac{\mathrm{d}\rho_{<rz>}}{\mathrm{d}r}+\frac{1}{r}\rho_{<rz>}=0,\\
&\frac{\mathrm{d}q_r}{\mathrm{d}r}+\frac{q_r}{r}+\rho_{<rz>}\frac{\mathrm{d}v_z}{\mathrm{d}r}=0,\\
&\frac{6}{5}\frac{\mathrm{d}q_r}{\mathrm{d}r}+\frac{2}{5}\frac{q_r}{r}=\underline{-\frac{\rho_{<rr>}}{\mathrm{Kn}}},\\
&\frac{2}{5}\frac{\mathrm{d}q_z}{\mathrm{d}r}+\left(\underline{p}+\rho_{<rr>}\right)\underline{\frac{\mathrm{d}v_z}{\mathrm{d}r}}=\underline{-\frac{\rho_{<rz>}}{\mathrm{Kn}}},\\
&\frac{2}{5}\frac{\mathrm{d}q_r}{\mathrm{d}r}+\frac{6}{5}\frac{q_r}{r}=\underline{-\frac{\rho_{<\vartheta\vartheta>}}{\mathrm{Kn}}},\\
&\frac{7}{2}\left(\underline{p}+\rho_{<rr>}\right)\underline{\frac{\mathrm{d}T}{\mathrm{d}r}}-\underline{\frac{\mathrm{d}(pT)}{\mathrm{d}r}}-\frac{2}{5}q_z\frac{\mathrm{d}v_z}{\mathrm{d}r}=\underline{-\frac{q_r}{\mathrm{Kn}}},\\
&\frac{7}{2}\rho_{<rz>}\frac{\mathrm{d}T}{\mathrm{d}r}+\frac{7}{5}q_r\frac{\mathrm{d}v_z}{\mathrm{d}r}=\underline{-\frac{q_z}{\mathrm{Kn}}}.
\end{aligned} \tag{13}$$

The differences between the classical Navier-Stokes-Fourier and extended thermodynamics theories are evident here. Indeed, in classical thermodynamics the field equations are the balance laws for mass, momentum and energy closed by the Navier-Stokes and Fourier constitutive relations, which assume the stress tensor and the heat flux proportional to gradient of the velocity and the one of temperature, respectively. Now, under the assumption that the variables depend only on r and $v_r=v_\vartheta=0$, the conservation laws coincide with $(13)_{1-3}$, while the constitutive relations differ from the remaining equations $(13)_{4-8}$. In particular, the classical constitutive relations can be deduced from equations $(13)_{4-8}$ imposing that the non-equilibrium variables in their left-hand side are equal to zero. So, the classical

[2]The conservation law of mass is identically satisfied with the assumption $v_r=0$.

constitutive relations coincide with the underlined terms in $(13)_{4-8}$, and explicitly read

$$\begin{aligned} \rho_{<rz>} &= -\mathrm{Kn}\, p \tfrac{\mathrm{d}v_z}{\mathrm{d}r}, \quad & \rho_{<rr>} &= 0, \\ q_r &= -\tfrac{5}{2}\mathrm{Kn}\, p \tfrac{\mathrm{d}T}{\mathrm{d}r} & \rho_{<\vartheta\vartheta>} &= 0. \end{aligned} \tag{14}$$

In this way, $(13)_{4-8}$ contain also the correction to the constitutive relations (14). In particular, the field equations of extended thermodynamics, obtained from the Boltzmann equation, predict non-vanishing components of the heat flux orthogonal to the temperature gradient and non-vanishing components of the components the tress tensor $\rho_{<rr>}$, $\rho_{<\vartheta\vartheta>}$ and $\rho_{<rz>}$. The result is in agreement with the kinetic theory of gas.

3. Boundary conditions

As already mentioned, in many cases extended thermodynamics models are able to catch and predict more details of the physical phenomena with respect to the classical Navier-Stokes-Fourier theory. Unfortunately, from the mathematical point of view, they often introduce a further problem in the determination of the solutions in bounded domains related to the prescription of the boundary data (the so-called non-controllable boundary data). This is due to the higher number of independent field variables taken into account by the Extended Thermodynamics. The problem of the boundary conditions is surely not new in the context of extended thermodynamics and several different methods and principles have been proposed in the literature to overcome it, but it must be said that, at this stage, the problem of boundary conditions is not definitely solved.

Also in the present case we deal with such a problem. Infact, the classical equations $(13)_{1-3}$ and (14) form a set ordinary differential equations (ODE) of order 5. Therefore, for this classical case we need only 5 boundary conditions to get the solution and exactly 5 are the quantities that we can prescribe in the corresponding experiment, as we will recall later in this section. On the contrary, the equations of extended thermodynamics (13) form a system of the seventh order. As a matter of fact, system (13) contains 8 equations in 8 fields, but the variable $\rho_{<\vartheta\vartheta>}$ can be algebraically determined in terms of the others. So, only 7 boundary conditions are necessary for the integration.

In this article we will integrate the set of the field equations of extended thermodynamics (13) in a way similar to the ones already considered by Barbera and Müller (2006), Arima *et al.* (2014), and Barbera *et al.* (2014). We firstly introduce the 5 boundary conditions that can be assigned in an experiment, then we leave the remaining 2 conditions free and integrate the full set of equations for different arbitrary values of these 2 remaining quantities.

Changing these two values arbirarily, we obtain solutions, that after a very steep gradient ath the boundary (very close to a vertical line), sweep into the same single functions. Therefore, although there is no proof of that, we will take as "correct"boundary values the ones for which no boundary layers are observed and that surely represents the appropriate prediction at least far from the boundaries.

More precisely in order to determine the solutions of (13), we consider as integration domain for the dimensionless radius r, the range between $r_\mathrm{i} = 0.2$ and $r_\mathrm{e} = 1$ and assume as numerical value for the Knudsen number Kn$= 0.1$, which is appropriate for a moderate rarefied gas.

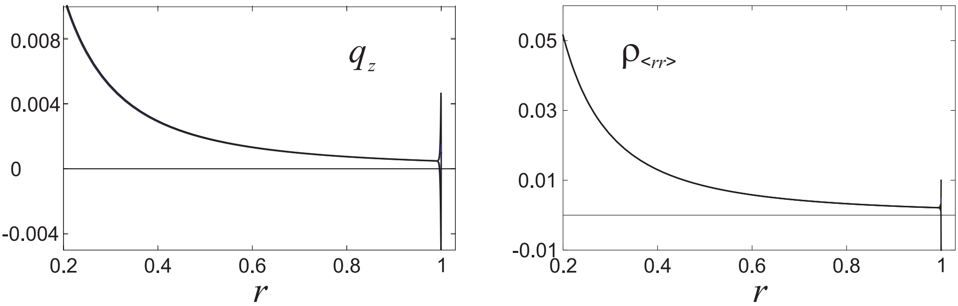

Figure 1. Solutions for the axial heat flux q_z and $\rho_{<rr>}$ obtained from equations (13) with the boundary conditions (15) for different values of Q and H.

The internal and external cylinders are kept at two different temperatures T_i and T_e, therefore the two dimensionless boundary temperatures are $T(r_\mathrm{i}) = 1.15$ and $T(r_\mathrm{e}) = 1$. We prescribe the pressure p_0 at the external radius, then the dimensionless pressure at r_e reads $p(r_\mathrm{e}) = 1$. Furthermore, the no-slip boundary conditions for the boundary axial velocities are assumed, so $v_z(r_\mathrm{i}) = V = 0.1$ and $v_z(r_\mathrm{e}) = 0$. Summarizing, we associate to our ODE system the following 5 boundary conditions

$$\begin{aligned} &T(r_\mathrm{i}) = 1.15, \quad T(r_\mathrm{e}) = 1, \quad p(r_\mathrm{e}) = 1, \\ &v_z(r_\mathrm{i}) = 0.1, \quad\; v_z(r_\mathrm{e}) = 0. \end{aligned} \tag{15}$$

As already said, these 5 conditions which can be prescribed in experiments are sufficient for the integration of the classical field equations $(13)_{1-3}$ and (14) but not for our system (13). Then, we integrate system (13) with these 5 boundary values and arbitrary values of

$$q_z(r_\mathrm{e}) = Q \quad \text{and} \quad \rho_{<rr>}(r_\mathrm{e}) = H. \tag{16}$$

As it can be easily seen in Fig. 1, the solutions for q_z and $\rho_{<rr>}$ differ only in the steep boundary layer. So, as explained before, we choose as values of the two arbitrary constants Q and H, the ones for which no boundary layers can be observed, which are

$$Q \simeq 0.00049 \quad \text{and} \quad H \simeq 0.0021. \tag{17}$$

The solutions for all field variables are shown and discussed in the next section.

4. Solutions and discussion of the results

In this section we analyse the solution of the equation system (13-17), starting off with the velocity component v_z and the rz-component of the deviatoric part of the stress tensor, $\rho_{<rz>}$. These quantities are shown with continuous lines in Fig. 2, while the classical solutions is plotted in dashed line, in order to show the differences between the predictions of these two theories.

It can be easily seen that the solutions for the z-component of the velocity and for $\rho_{<rz>}$ are almost coincident. The small differences are due to the occurrence of some terms in equation $(13)_5$, that are not present in the corresponding equation of classical thermodynamics $(14)_1$.

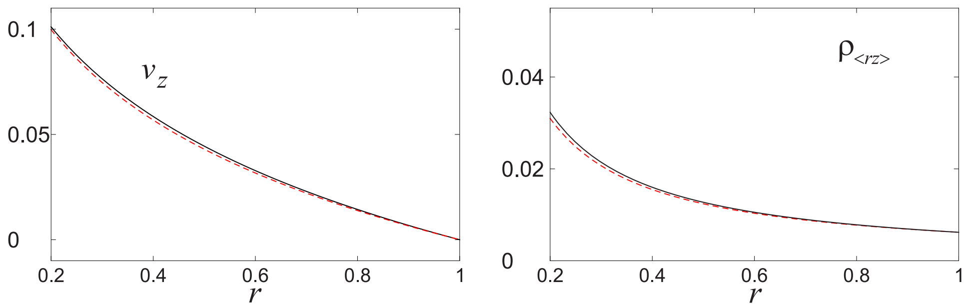

Figure 2. Solutions for the axial velocity v_z and $\rho_{<rz>}$ obtained from equations (13-17). The continuous lines refer to extended thermodynamics, while the dashed lines refer to the classical theory.

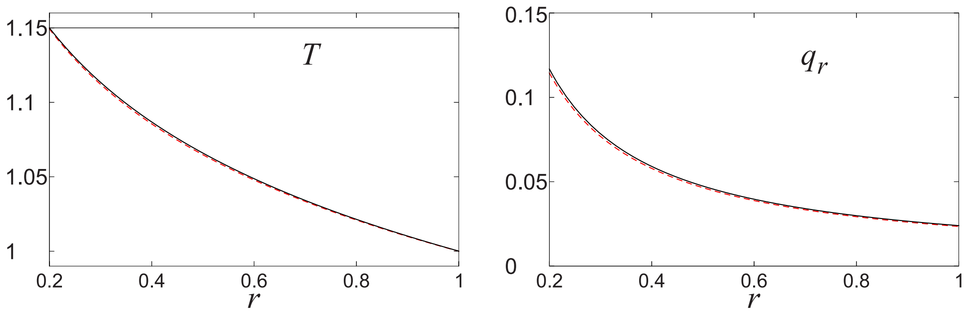

Figure 3. Solutions for the temperature T and the radial component of the heat flux q_r obtained from equations (13-17). The continuous lines refer to extended thermodynamics, the dashed lines refer to the classical theory.

In Fig. 3 we consider the temperature together with the radial component of the heat flux. Also here, no relevant differences between the predictions of the two theories are observable. In fact, the presence of the non-underlined terms in $(13)_7$, which are not contained in the Fourier law $(14)_3$, gives rise to very small discrepancies.

The first significant differences between the results of the two theories are shown in Fig. 4. As a matter of fact, the classical thermodynamics predict a constant pressure while, the pressure described by equations (13) increases when r decreases. This is due to the presence of the two non-vanishing components of the stress tensor $\rho_{<rr>}$ and $\rho_{<\vartheta\vartheta>}$, drawn in the same figure.

As already said in the introduction, the presence of these two non-vanishing quantities was firstly predicted in bounded domains with cylindrical and spherical symmetries by Müller and Ruggeri (2004), as an effect of the radial geometry. In particular, in the article of Müller and Ruggeri (2004), where $v_z = 0$, one had $\rho_{<rr>} \neq 0$ and $\rho_{<\vartheta\vartheta>} \neq 0$, but the combinations $\frac{\mathrm{d}\rho_{<rr>}}{\mathrm{d}r} + \frac{\rho_{<rr>} - \rho_{<\vartheta\vartheta>}}{r}$ and $\rho_{<zz>} = -\rho_{<rr>} - \rho_{<\vartheta\vartheta>}$ vanished identically, so that the pressure remained constant and no-normal stress is observable.

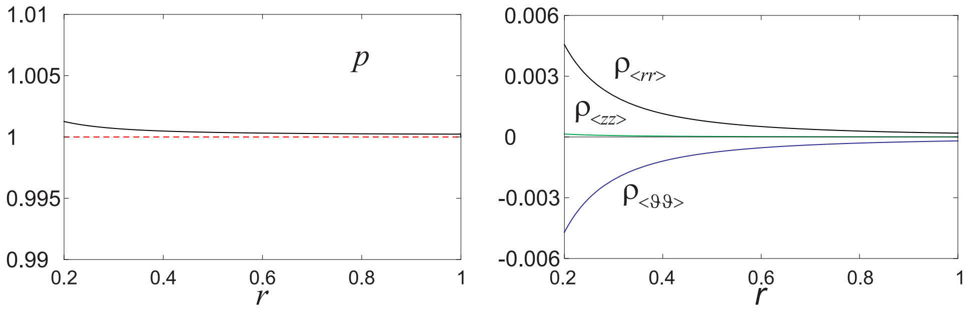

Figure 4. Solutions for pressure p and for three components of the traceless part of the stress tensor obtained from equations (13-17). The continuous lines refer to extended thermodynamics, while the dashed lines refer to the classical theory.

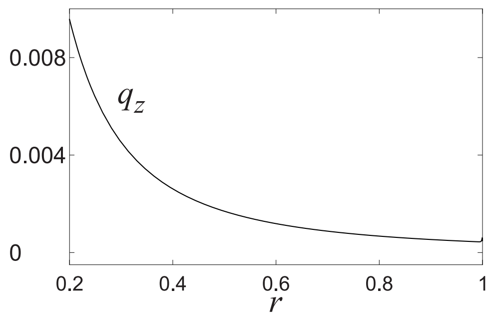

Figure 5. Solutions for the axial component of the heat flux q_z obtained from equations (13-17).

Here the behavior of such variables is also related to the effect of the velocity field v_z, which causes a non-constant pressure and a non-vanishing $\rho_{<zz>}$.

On the contrary, in the case of axial velocity $v_z \neq 0$ for a planar domain, the pressure remains constant and $\rho_{<zz>} = 0$.

The effects in Fig. 4 are quite small, since we supposed a small axial velocity of the internal cylinder owing to stability reasons. Obviously, for increasing boundary velocity V, the order of magnitude of the deviation of the pressure and of $\rho_{<zz>}$ increases, as shown later.

Finally, Fig. 5 presents the most interesting result of this article, that is the presence of an axial component of the heat flux, contrary to the assumption of the classical Fourier law, which suppose the heat flux parallel to the temperature gradient.

It can be proved, see equation $(13)_8$, that the presence of this non-vanishing q_z in extended thermodynamics is due to the simultaneous presence of the temperature field and of the axial velocity of the gas. The field q_z vanishes if v_z vanishes and/or if an isothermal flow is taken into account. Its presence is predicted also in the planar case.

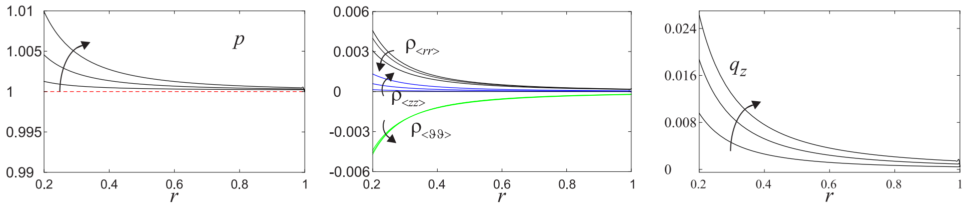

Figure 6. Dependence of the pressure, stress tensor and axial heat flux on V.

Figure 6 shows the dependence of the results on the values of the velocity of the internal boundary, V. The solutions are plotted for $V = 0.1,\ 0.2,\ 0.3$. In particular, the pressure deviation grows for increasing values of V, as expected. Instead, the stress tensor component $\rho_{<rr>}$ decreases, when V grows. This fact can be easily understood from the relation between $\rho_{<rr>}$, q_r and $\frac{\mathrm{d}v_z}{\mathrm{d}r}$, which follows from combinations of equations $(13)_{3,4}$, that is

$$\rho_{<rr>} = \mathrm{Kn}\left(\frac{4}{5}\frac{q_r}{r} + \frac{6}{5}\rho_{<rz>}\frac{\mathrm{d}v_z}{\mathrm{d}r}\right). \tag{18}$$

When v_z increases, the second term $\frac{\mathrm{d}v_z}{\mathrm{d}r}$ increases in modulus, but it is negative. So, at the end, $\rho_{<rr>}$ becomes smaller.

On the contrary $\rho_{<\vartheta\vartheta>}$, which from $(13)_{3,6}$ is given by

$$\rho_{<\vartheta\vartheta>} = \mathrm{Kn}\left(-\frac{4}{5}\frac{q_r}{r} + \frac{4}{5}\rho_{<rz>}\frac{\mathrm{d}v_z}{\mathrm{d}r}\right) \tag{19}$$

decreases when V increases. The third component $\rho_{<zz>} = -2\mathrm{Kn}\rho_{<rz>}\frac{\mathrm{d}v_z}{\mathrm{d}r}$ depends only on the gradient of the velocity and not on the heat flux. Finally, Fig. 6 shows clearly that the axial component of the heat flux increases proportionally to increasing axial boundary velocities. It is clear the the order of magnitude of q_z cannot be so high, since it is related to the magnitude of q_r and V, by a dependence of the kind $\|q_z\| \approx 0.8V\,\|q_r\|$.

5. Conclusions and final remarks

In this article the 13-moment extended thermodynamics equations are used in order to study the non-isothermal axial flow in the gap between two coaxial cylinders at different temperatures and under the assumption that the inner cylinder is moving along its axis, while the outer one is at rest. To our knowledge the problem was never studied before in the literature for rarefied gases.

The solutions for this problem are analysed and compared with those obtained in the framework of classical thermodynamics.

Moreover, the role of the non-zero axial velocity in the 13-moment equations is investigated comparing the predictions herein obtained with the ones for the planar geometry and also with the cylindrical ones when both boundary walls are kept at rest. In addition, the dependence on the boundary velocity of the gas is discussed.

The article can be viewed as the last one of a series (Marques Jr. and Kremer 2001; Gramani Cumin *et al.* 2002; Müller and Ruggeri 2004; Barbera and Müller 2006, 2008; Barbera and Brini 2010; Barbera *et al.* 2012) dedicated to stationary heat transfer phenomena in monatomic simple gases. The results obtained here prove once more that extended thermodynamics with 13 moments is already able to predict more general and elaborate behaviors than the classical Navier-Stokes-Fourier theory. The qualitative differences between classical and extended thermodynamics increases, when the phenomena are more complex and/or far from equilibrium.

The physical effects that we have predicted here are observable in experiments, at least for what concerns the mass density and the heat flux behaviors. For this reason, qualitative and quantitative comparisons with experimental data would be crucial to validate the predictions and to better understand the range of validity of different theories. To this goal, we wish that new experiments will be carried on in the future.

Moreover, it would be very interesting to analyse the same problem for dense gases, whose extended thermodynamics theory was recently proposed by Ruggeri and Sugiyama and studied by several authors (Ruggeri and Sugiyama 2014; Carrisi and Pennisi 2015; Carrisi *et al.* 2015).

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