On the subRiemannian cut locus in a model of free two-step Carnot group

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Annamaria Montanari   Daniele Morbidelli

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Abstract

We characterize the subRiemannian cut locus of the origin in the free Carnot group of step two with three generators, giving a new, independent proof of a result by Myasnichenko [Mya02]. We also calculate explicitly the cut time of any extremal path and the distance from the origin of all points of the cut locus. Furthermore, by using the Hamiltonian approach, we show that the cut time of strictly normal extremal paths is a smooth explicit function of the initial velocity covector. Finally, using our previous results, we show that at any cut point the distance has a corner-like singularity.

1.   Introduction and statement of the results

One of the most interesting aspects of subRiemannian analysis is the study of the cut locus of a given distance. It is well known that a correct understanding of the cut locus is crucial in problems concerning subRiemannian optimal transport (see [AR04, AL09, FR10] and analysis of the subelliptic heat kernel (see [BBN12]).

In this paper, we introduce a very explicit technique which provides the calculation of the subRiemannian cut locus of the origin in the free Carnot group of step two with

three generators. We are also able to explicitly calculate the cut time of any extremal path and the distance from the origin of all points of the cut locus. Furthermore, we show the presence of Lipschitz singularities at any point of the cut locus.

To state our results, let us introduce the free Carnot group of step two with three generators. Let $\mathbb{V}$ be a three dimensional vector space equipped with an inner product $\langle \cdot, \cdot \rangle$. Consider the six-dimensional linear space $\mathbb{F}_3 := \mathbb{V} \times \wedge^2 \mathbb{V}$ with the Lie group law

$$(x, t) \cdot (\xi, \tau) = \left(x + \xi, t + \tau + \frac{1}{2} x \wedge \xi \right), \tag{1.1}$$

where (x, t) and $(\xi, \tau) \in \mathbb{V} \times \wedge^2 \mathbb{V}$. Let d be the left invariant subRiemannian distance defined on the Lie group $(\mathbb{F}_3, \cdot)$ by the data $(\mathbb{V}, \langle \cdot, \cdot \rangle)$ and denote by Cut_0 the cut locus of the origin (see Section 2 for detailed explanations).

The first result of this paper can be stated as follows:

Theorem 1.1. *Let $\mathbb{F}_3$ be the free two step Carnot group with three generators. Then*

$$\text{Cut}_0 = \left\{ (x, t) : t \neq 0 \in \wedge^2 \mathbb{V} \text{ and } x \perp \text{supp } t \right\}. \tag{1.2}$$

Here, if $y, y' \in \mathbb{V}$, given the bivector $t = y \wedge y' \in \wedge^2 \mathbb{V}$, we denote by $\text{supp}(t) = \text{span}\{y, y'\}$ its support (see the discussion in Section 2.1). The set Cut_0 is a smooth four dimensional submanifold of $\mathbb{F}_3$ (see Remark 4.3). The cut locus of any point $(x, t) \neq (0, 0)$ can be easily obtained from Cut_0 by group translation.

Remark 1.2. *Some comments on this theorem are in order.*
1. *The cut locus Cut_0 has been already described by Myasnichenko [Mya02] with a technique based on the analysis of conjugate points. Our technique is completely independent from the analysis of the conjugate locus and it allows us to get several more information on the subRiemannian distance (see the theorems below).*
2. *In [RT09, Lemma 2.11], Rifford and Trélat prove that $\text{Cut}_0 \cup \{0\}$ is closed in absence of abnormal minimizers. In our model, which contains abnormal minimizers, it turns out that $\text{Cut}_0 \cup \{0\}$ is not closed. Indeed, points of the form $(x, 0) \in \mathbb{V} \times \wedge^2 \mathbb{V}$ (which belong to the abnormal set, see [DMO⁺16]) are never cut points, but they belong to the closure of Cut_0.*
3. *If we think of the Heisenberg group as the free two step Carnot group with two generators $\mathbb{F}_{2,2}$ (i.e. $\dim V = 2$), then formula (1.2) reduces as expected to the known fact $\text{Cut}_0 = \{(0, t) \in \mathbb{V} \times \wedge^2 \mathbb{V} \sim \mathbb{V} \times \mathbb{R}\}$. Formula (1.2) is dimension free and one could conjecture that such structure of the cut locus holds in any dimension. We plan to come back to these generalizations in a subsequent project.[1]*

Fine properties of subRiemannian length-minimizing curves and of the related distance in Carnot groups of step two are studied by [ABB12] and [BBG12] in the corank one and two. The case of h-type groups is discussed in [AM16]. Analysis in step three examples has been performed in [ABCK97], for the Martinet case, and in [AS11, AS15] for the Engel group. We also mention the very recent paper [BBN16], where a detailed discussion of the cut locus in the biHeisenberg group is performed. Outside of the setting of Carnot groups,

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[1]After the submission of this paper, a precise conjecture for the cut locus on the free group in higher dimension has been formulated by Rizzi and Serres [RS16].

we mention [Sac11] and [PS16]. Finally, we quote the results in [AF07] on the cut locus from surfaces in the Heisenberg group.

Our further results involve a calculation of the cut time of any extremal curve and an explicit computation of the distance from the origin of points $(x, t) \in \text{Cut}_0$. To describe such result, we recall that, up to a reparametrization, length-minimizers for the subRiemannian length have the form $s \mapsto \gamma(s) = (x(s), t(s))$, where $\dot{x}(s) =: u(s)$ has the form

$$u(s) = a \cos(2\varphi s) + b \sin(2\varphi s) + z \tag{1.3}$$

and $\dot{t}(s) = \frac{1}{2} x(s) \wedge u(s)$. Here the vectors $a, b, z \in \mathbb{R}^3$ must be pairwise perpendicular, $|a| = |b| > 0$ and φ can be assumed to be nonnegative. The vector z can possibly vanish (this corresponds to a curve which lives in a "Heisenberg subgroup"). See the discussion in Section 2.2 for a precise description. We will calculate the cut time of any such extremal curve in terms of the following explicit functions:

$$S(\theta) := \frac{\sin \theta}{\theta}, \qquad U(\theta) := \frac{\theta - \sin \theta \cos \theta}{4\theta^2}, \qquad V(\theta) := \frac{\sin \theta - \theta \cos \theta}{2\theta^2}. \tag{1.4}$$

More precisely, letting

$$Q(\theta) = -\frac{U(\theta) S(\theta)}{V(\theta)}, \tag{1.5}$$

and denoting by $\varphi_1 \in]\pi, \frac{3}{2}\pi[$ the first positive zero of the function V in (1.4), we will show that the function Q is a strictly increasing bijection from $[\pi, \varphi_1[$ to $[Q(\pi), Q(\varphi_1-)[= [0, +\infty[$ (see Lemma 4.1 and the plot in figure 4.1). Then we will prove the following theorem.

Theorem 1.3. *Let $a, b, z \in \mathbb{V}$ be orthogonal vectors with $|a| = |b| > 0$ and let $\varphi > 0$. Consider the control*

$$u(s) = a \cos(2\varphi s) + b \sin(2\varphi s) + z$$

and let $s \mapsto \gamma(s, a, b, z, 2\varphi)$ be the corresponding curve. Then $\gamma(\cdot, a, b, z, 2\varphi)$ minimizes length up to the time

$$t_{\text{cut}}(a, b, z, \varphi) = \frac{Q^{-1}(|z|^2/|a|^2)}{\varphi}.$$

Observe that, if we choose $z = 0$ we get $t_{\text{cut}}(a, b, 0, \varphi) = \frac{\pi}{\varphi}$, which is the familiar case of the Heisenberg group. However, it is interesting to remark that the cut time displays the following "discontinuous" behavior. If a, b, z, φ are fixed with $a, b, z \neq 0$ pairwise orthogonal, $|a| = |b|$ and $\varphi > 0$, it turns out that

$$t_{\text{cut}}(\varepsilon a, \varepsilon b, z, \varphi) = \frac{1}{\varphi} Q^{-1}\left(\frac{|z|^2}{\varepsilon^2 |a|^2}\right) \to \frac{Q^{-1}(+\infty)}{\varphi} = \frac{\varphi_1}{\varphi},$$

as $\varepsilon \to 0^+$. However, the cut time of the limit curve $\gamma(s) = (sz, 0)$ is $+\infty$. Note that such limit curve is abnormal (see Section 2). Using the Hamiltonian approach, we will express the cut time as an explicit function of the initial velocity covector (see Section 5).

Our further result involves the explicit computation of the distance from the origin and an arbitrary point of the cut locus. To state our result we introduce the real valued functions

$$P(\theta) := -\frac{S(\theta)}{V(\theta)} \sqrt{\frac{W(\theta)}{U(\theta)}} \quad \text{and} \quad R(\theta) := \frac{1 - S(\theta)^2}{\sqrt{U(\theta) W(\theta)}},$$

where $W(\theta) := U(\theta) - S(\theta)V(\theta)$. The function R is well defined and positive for $\theta > 0$, because $W(\theta) > 0$ for all $\theta > 0$ (see Lemma 3.1). Furthermore, we will show that if $\varphi_1 \in]\pi, \frac{3}{2}\pi[$ denotes the first positive zero of the function V in (1.4), then $P : [\pi, \varphi_1[\to [0, +\infty[$ is an increasing bijection. We denote by P^{-1} the inverse function of $P\big|_{[\pi,\varphi_1[}$. Then we will prove the following formula:

Theorem 1.4. *Let $(x, t) \in \text{Cut}_0$. Then*

$$d\big((0,0),(x,t)\big)^2 = |x|^2 + R(\theta)|t|, \quad \text{where } \theta = P^{-1}\Big(\frac{|x|^2}{|t|}\Big) \in [\pi, \varphi_1[. \quad (1.6)$$

We will see that the equation $P(\theta) = \frac{|x|^2}{|t|}$ has a countable family of solutions $\theta_k \in]k\pi, \varphi_k[$, where $\varphi_k \in]k\pi, (k+\frac{1}{2})\pi[$ is the k-th positive solution of $\tan\theta = \theta$. See the discussion after (3.3). In Theorem 1.4 we prove that the minimizing choice is the smallest one θ_1.

If we specialize formula (1.6) to the case $x = 0$, we must take $\theta = P^{-1}(0) = \pi$ and then $S(\theta) = 0$, $U(\theta) = \frac{1}{4\pi}$ and we find

$$d((0,0),(0,t))^2 = 4\pi|t|,$$

which is the familiar formula for the distance between the origin and a point in the center in the Heisenberg group (for a complete description of geodesics in Heisenberg group see [Mon00]). The value $\theta = \varphi_1$ is never achieved.

Our final result concerns the regularity of the subRiemannian distance from the origin in the cut locus. Observe that weak regularity properties (like semiconcavity) have been discussed in [CR08, FR10, MM16, LN15]. Concerning smooth regularity properties, it is well known that the subRiemannian distance from a point 0 cannot be smooth in any pointed neighborhood of 0. However, it is known that the subRiemannian distance is smooth outside of the *abnormal set* and of the cut locus. In our model, it turns out that the abnormal set has the form $\text{Abn}_0 = \mathbb{V} \times 0_{\wedge^2\mathbb{V}}$ (see Section 2.3) and it has been proved that the distance from the origin d is not differentiable at any abnormal point (see [MM16] for precise estimates). As a byproduct of our Theorem 3.2 it is easy to see that d is not differentiable at any point of Cut_0 (see Remark 3.7). In the present paper we obtain more precise estimates using the description of the profile for the subRiemannian spheres (Corollary 4.4). Indeed, we are able to exhibit the presence of a kind of two-dimensional corner-like singularities at cut points. Our estimates also ensure that the distance cannot be semiconvex at any point of the cut locus.

Theorem 1.5. *Let $(\overline{x}, \overline{t}) \in \text{Cut}_0$. Then, there is $C > 0$ and a C^1-smooth two-dimensional submanifold Σ containing $(\overline{x}, \overline{t})$ such that*

$$d(x, t) \leq d(\overline{x}, \overline{t}) - C|(x, t) - (\overline{x}, \overline{t})| \quad \text{for all } (x, t) \in \Sigma. \quad (1.7)$$

Furthermore, for any open neighborhood Ω of $(\overline{x}, \overline{t})$ we have the estimate

$$\inf_{(x,t),(x+h,t+k),(x-h,t-k)\in\Omega} \frac{d(x+h, t+k) + d(x-h, t-k) - 2d(x,t)}{h^2 + k^2} = -\infty. \quad (1.8)$$

Estimate (1.8) gives a positive answer in our model to a question raised by Figalli and Rifford (see the *Open problem* at p 145-146 in [FR10]).

The paper is structured as follows. Section 2 contains preliminaries. In Section 3 we prove the inclusion $\Sigma \subset \mathrm{Cut}_0$ and Theorem 1.4. In Subsection 4.1 we prove Theorem 1.3 and we conclude the proof of Theorem 1.1 (inclusion $\mathrm{Cut}_0 \subset \Sigma$). In Subsection 4.2 we describe the profile of subRiemannian spheres. Section 5 contains some remarks on the Hamiltonian point of view. Using such approach we show that in the strictly normal case, the cut time of extremal paths is a smooth explicit function of the initial covector. In Section 6 we state and prove Theorem 1.5, on the singularity of the subRiemannian distance at cut points.

2.   Preliminaries

2.1.   Bivectors

Let $\mathbb{V} = \mathrm{span}\{e_1, \ldots, e_m\}$ be a linear space. Define $\wedge^2 \mathbb{V} := \mathrm{span}\{e_j \wedge e_k : 1 \leq j < k \leq m\}$. Given two vectors $x, y \in \mathbb{V}$, the elementary bivector $t = x \wedge y \in \wedge^2 \mathbb{V}$ can be expanded as

$$x \wedge y = \sum_j (x_j e_j) \wedge \sum_k (y_k e_k) = \sum_{1 \leq j < k \leq m} (x_j y_k - x_k y_j) e_j \wedge e_k =: \sum_{1 \leq j < k \leq m} (x \wedge y)_{jk} e_j \wedge e_k.$$

If $e_1, \ldots, e_m$ is an orthonormal basis with respect to some inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{V}$, we define the related product on $\wedge^2 \mathbb{V}$ by requiring that the basis $e_j \wedge e_k$ with $1 \leq j < k \leq m$ is orthonormal. It turns out that, on elementary bivectors,

$$\langle x \wedge y, \xi \wedge \eta \rangle = \langle x, \xi \rangle \langle y, \eta \rangle - \langle x, \eta \rangle \langle y, \xi \rangle \quad \text{for all } x, y, \xi, \eta \in \mathbb{R}^m. \tag{2.1}$$

The inner product $\langle z, \zeta \rangle$ can be extended linearly to general bivectors.

Concerning the three dimensional case, $\dim \mathbb{V} = 3$, it is well known that any bivector $t \in \wedge^2 \mathbb{V}$ can be written in the elementary form $t = x \wedge y$, for suitable $x, y \in \mathbb{V}$. Although the choice of x and y is not unique, the subspace $\mathrm{span}\{x, y\}$ does not depend on such choice and it is called *support* of the bivector. See the discussion in [MM16], where the higher dimensional case is treated.

Finally, it is easy to check that if $\mathbb{V}$ is a finite dimensional vector space and if x_0 and y_0 are independent in $\mathbb{V}$, then $x \wedge y = x_0 \wedge y_0$ implies that $x, y \in \mathrm{span}\{x_0, y_0\}$.

2.2.   The free group $\mathbb{F}_3$

Let $\mathbb{V}$ be a three-dimensional vector space with an inner product $\langle \cdot, \cdot \rangle$. Denote by (x, t) variables in $\mathbb{V} \times \wedge^2 \mathbb{V}$. If e_1, e_2, e_3 is an orthonormal basis of $\mathbb{V}$, then $x = x_1 e_1 + x_2 e_2 + x_3 e_3 \sim (x_1, x_2, x_3)$ and $t = t_{12} e_1 \wedge e_2 + t_{13} e_1 \wedge e_3 + t_{23} e_2 \wedge e_3 \sim (t_{12}, t_{13}, t_{23}) \in \mathbb{R}^3$. Introduce the law

$$(x, t) \cdot (\xi, \tau) = \left(x + \xi, t + \tau + \frac{1}{2} x \wedge \xi \right), \tag{2.2}$$

Denote by $\mathbb{F}_3$ the Lie group $(\mathbb{V} \times \wedge^2 \mathbb{V}, \cdot)$.

Fix an orthonormal basis e_1, e_2, e_3 of V and corresponding coordinates $(x_1, x_2, x_3, t_{12}, t_{13}, t_{23})$ in $\mathbb{V} \times \wedge^2 \mathbb{V}$. Define the family of three vector fields

$$X_1 = \partial_1 - \frac{x_2}{2} \partial_{12} - \frac{x_3}{2} \partial_{13}, \quad X_2 = \partial_2 + \frac{x_1}{2} \partial_{12} - \frac{x_3}{2} \partial_{23}, \quad X_3 = \partial_3 + \frac{x_1}{2} \partial_{13} + \frac{x_2}{2} \partial_{23}.$$

Note the commutation relations $[X_j, X_k] = \partial_{jk}$ for all j, k such that $1 \leq j < k \leq 3$. The vector fields X_1, X_2, X_3 are homogeneous of degree 1 with respect to the family of dilations $(\delta_r)_{r>0}$ defined by

$$\delta_r(x, t) = (rx, r^2 t) \qquad \forall \quad (x, t) \in V \times \wedge^2 V. \tag{2.3}$$

Namely, we have $X_j(f \circ \delta_r)(x, t) = rX_j f(\delta_r(x, t))$ for all function f.

A path $\gamma \in W^{1,2}((0, T), \mathbb{F}_3)$ is said to be *horizontal* if there is a *control* $u \in L^2((0, T), \mathbb{R}^3)$ such that we can write $\dot{\gamma}(s) = \sum_{j=1}^{3} u_j(s) X_j(\gamma(s))$ for a.e. $s \in (0, T)$. The length of the horizontal path γ is length$(\gamma) := \int_0^T |u(t)| dt$. If γ is arclength, then we have length$(\gamma) = \sqrt{T}(\int_0^T |u(t)|^2 dt)^{1/2}$. Given points $(\widehat{x}, \widehat{t})$ and (x, t), define $d((\widehat{x}, \widehat{t}), (x, t)) = \inf\{\text{length}(\gamma)\}$, where the infimum is taken among all horizontal curves γ such that $\gamma(0) = (\widehat{x}, \widehat{t})$ and $\gamma(T) = (x, t)$. It is well known that for any pair of points $(x, t), (\widehat{x}, \widehat{t}) \in \mathbb{F}_3$ the infimum is a minimum, i.e. there is a length-minimizing path. Constant speed length-minimizing paths in $\mathbb{F}_3$ are curves γ associated with controls (which we call *extremal*) of the form

$$u(s) = a\cos(\lambda s) + b\sin(\lambda s) + z, \tag{2.4}$$

where $a, b, z \in \mathbb{V}$ is an *admissible triple*. By *admissible triple* we mean an ordered triple of vectors $a, b, z \in \mathbb{R}^3$, where the vectors are pairwise orthogonal and such that $|a| = |b| \gneq 0$. These facts are well known (see [AGL15, DMO$^+$16]). Here we refer to the self-contained discussion in [MM16, Section 3.1.3]. Without loss of generality, we can always assume that $\lambda \geq 0$. The case $\lambda < 0$ can be recovered by an adjustment of the sign of b. In many computations below, controls will be written in the form $a\cos(2\varphi s) + b\sin(2\varphi s) + z$, and we will use the variable φ.

The curve $s \mapsto \gamma(s) = (x(s), t(s))$ corresponding to the control u in (2.4) is obtained integrating the ODE

$$\dot{x} = u, \qquad \dot{t} = \frac{1}{2} x \wedge u \tag{2.5}$$

with initial data $(x(0), t(0)) = (0, 0)$. An elementary computation gives the general form of extremal curves

$$\begin{aligned}
\gamma(s, a, b, z, \lambda) &= (x(s, a, b, z, \lambda), t(s, a, b, z, \lambda)) \\
&= \Big(\frac{\sin(\lambda s)}{\lambda} a + \frac{1 - \cos(\lambda s)}{\lambda} b + sz, \\
&\quad \frac{\lambda s - \sin(\lambda s)}{2\lambda^2} a \wedge b + \frac{2(1 - \cos(\lambda s)) - \lambda s \sin(\lambda s)}{2\lambda^2} a \wedge z \\
&\quad + \frac{\lambda s(1 + \cos(\lambda s)) - 2\sin(\lambda s)}{2\lambda^2} b \wedge z \Big).
\end{aligned} \tag{2.6}$$

For integration of similar systems in higher dimension, see [MPAM06]. On sufficiently small intervals, γ is a length-minimizer (see [LS95], [Rif14] and [ABB16]). Under linear change of parameter and dilation in (2.3), γ behaves as follows

$$\gamma(\mu s, a, b, z, \lambda) = \gamma(s, \mu a, \mu b, \mu z, \mu \lambda) \qquad \text{for all } \mu > 0, \tag{2.7}$$

$$\gamma(s, ra, rb, rz, \lambda) = \delta_r \gamma(s, a, b, z, \lambda) \qquad \text{for any } r > 0. \tag{2.8}$$

Observe that $\gamma(s, a, b, z, \lambda)$ tends smoothly to $\gamma(s, a, b, z, 0) = (a + z, 0)$, as $\lambda \to 0$. Finally, observe the following rotation invariance property of the distance: if $M \in O(3)$, then $d(x, y \wedge z) = d(Mx, My \wedge Mz)$ for all $x, y, z \in \mathbb{R}^3$.

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2.3.   Abnormal curves in $\mathbb{F}_3$

Let $u \in L^2((0,1), \mathbb{R}^3)$. It is easy to see that the solution $s \mapsto \gamma_u(s) =: (x_u(s), t_u(s))$ of the ODE (2.5) with initial data $\gamma_u(0) = (0,0)$ is defined globally on $[0,1]$ for any choice of $u \in L^2((0,1), \mathbb{R}^3)$. Denote by $E(u)$ such solution at time $s = 1$. The map $E : L^2 \to \mathbb{F}_3$ is called *end point* map.

   Let $u(s)$ be an extremal control of the form (2.4). By definition, the control u is abnormal if the differential $dE(u) : L^2((0,1), \mathbb{R}^3) \to \mathbb{F}_3$ is not onto. The corresponding curve $\gamma(\cdot, a, b, z, \lambda)$ is called *abnormal extremal curve*. Denote by Abn_0 the set of points $(x,t) = \gamma(1)$, where $\gamma : [0,1] \to \mathbb{R}^3$ is an abnormal extremal curve with $\gamma(0) = (0,0)$. From the discussion in [DMO$^+$16] (see also [MM16, Section 3.1.3]), we know that

$$\mathrm{Abn}_0 = \{(x,0) \in \mathbb{V} \times \wedge^2 \mathbb{V}\} = \mathbb{V} \times 0_{\wedge^2 \mathbb{V}}.$$

2.4.   Cut time

Definition 2.1. *Let a, b, z be an admissible triple, fix $\varphi > 0$ and let $s \mapsto \gamma(s, a, b, z, 2\varphi)$ be the curve defined in (2.6). Define*

$$
\begin{aligned}
t_{\mathrm{cut}}(a, b, z, \varphi) &= \inf\Big\{\overline{s} \geq 0 : \ \gamma(\cdot, a, b, z, 2\varphi) \ \text{does not minimize length on } [0, \overline{s}] \ \Big\} \\
&= \sup\Big\{\overline{s} > 0 : \ \gamma(\cdot, a, b, z, 2\varphi) \ \text{is a length minimizer on } [0, \overline{s}]\Big\}.
\end{aligned}
\tag{2.9}
$$

   It is well known that $t_{\mathrm{cut}} > 0$ for all such curve. Moreover, if $t_{\mathrm{cut}} < \infty$, then the supremum is a maximum.

   Standard invariance properties give the following general form for the cut time.

Proposition 2.2. *There exists a function $h_{\mathrm{cut}} : [0, +\infty[\to]0, +\infty]$ such that*

$$t_{\mathrm{cut}}(a, b, z, \varphi) = \frac{h_{\mathrm{cut}}(|z|/|a|)}{\varphi} \tag{2.10}$$

for all admissible triple a, b, z and $\varphi > 0$.

   Note that we do not need to define $h_{\mathrm{cut}}(\infty)$ because the case $z \neq 0$ and $a = b = 0$ corresponds to a constant control $u(s) = z$. Such kind of control is included in the class of admissible controls by choosing $\varphi = 0$. In such case it is easy to see that the corresponding curve minimizes globally, i.e. $t_{\mathrm{cut}} = +\infty$. Observe finally that h_{cut} can never vanish, by a classical result in control theory (see [LS95], [Rif14] and [ABB16])

   One of the main result of our paper is the calculation of the function h_{cut}. Interestingly, it turns out that the function h_{cut} is bounded.

Proof. In order to show that the cut time of the curve corresponding to the control $u(s) = a \cos(2\varphi s) + b \sin(2\varphi s) + z$ depends on $|a|, |z|$ and φ, it suffices to identify $\mathbb{V}$ with $\mathbb{R}^3 = \mathrm{span}\{e_1, e_2, e_3\}$ where e_1, e_2, e_3 are orthonormal and $a = re_1$, $b = re_2$ and $z = \rho e_3$. Then the control u becomes $u(s) = \cos(2\varphi s)re_1 + \sin(2\varphi s)re_2 + \rho e_3$, and it is clear that the cut time depends on r, ρ and φ.

Let now $\varphi = 1$, i.e. consider the path $\sigma \mapsto \gamma(\sigma, a, b, z, 2)$. Let $\overline{\sigma}(r, \rho)$ its cut time, where $r = |a|$ and $\rho = |z|$. Taking an arbitrary $\varphi > 0$, the riparametrization invariance (2.7) shows that the path

$$s \mapsto \gamma(s, a, b, z, 2\varphi) = \gamma\Big(\varphi s, \frac{a}{\varphi}, \frac{b}{\varphi}, \frac{z}{\varphi}, 2\Big)$$

is length minimizing up to the time $s = \dfrac{\overline{\sigma}(r/\varphi, \rho/\varphi)}{\varphi}$. By the dilation invariance (2.8), we also have

$$\overline{\sigma}(r/\varphi, \rho/\varphi) = \overline{\sigma}(r, \rho) = \overline{\sigma}(1, \rho/r) =: h_{\text{cut}}(\rho/r) \in \,]0, +\infty]\,.$$

Thus the cut time has the form (2.10), as required. $\qquad\qquad\qquad\qquad\qquad\square$

We know that $h(0) = \pi$, because in such case we are in Heisenberg subgroups of the form $\text{span}\{(a,0), (b,0), (0, a \wedge b)\}$. We will show that $h_{\text{cut}}(\mu) \to \varphi_1$, as $\mu \to +\infty$, where $\varphi_1 \in \,]\pi, \frac{3}{2}\pi[$ denotes the first positive solution of the equation $\tan \varphi = \varphi$.

Define for $\varphi = \lambda/2 \geq 0$ and a, b, z admissible triple

$$F(a, b, z, \varphi) := \gamma(1, a, b, z, 2\varphi). \qquad\qquad\qquad (2.11)$$

Recall the definition of the functions S, U, V introduced in (1.4)

$$S(\varphi) := \frac{\sin \varphi}{\varphi}, \qquad U(\varphi) := \frac{\varphi - \sin \varphi \cos \varphi}{4\varphi^2}, \qquad V(\varphi) := \frac{\sin \varphi - \varphi \cos \varphi}{2\varphi^2}.$$

After some elementary calculations, it turns out that

$$F(a, b, z, \varphi) = \Big(S(\varphi)(a \cos \varphi + b \sin \varphi) + z,\ U(\varphi) a \wedge b + V(\varphi)(a \sin \varphi - b \cos \varphi) \wedge z\Big). \quad (2.12)$$

Remark 2.3. *After the orthogonal change of variable*

$$a' = a \sin \varphi - b \cos \varphi \quad and \quad b' = a \cos \varphi + b \sin \varphi \qquad\qquad (2.13)$$

we have

$$F(a, b, z, \varphi) = G(a', b', z, \varphi), \qquad where$$
$$G(\alpha, \beta, \zeta, \varphi) := \big(S(\varphi)\beta + \zeta,\ \alpha \wedge (U(\varphi)\beta + V(\varphi)\zeta)\big)$$

for all admissible α, β, z and $\varphi > 0$. Note that a', b', z are admissible if and only if a, b, z are admissible. Moreover, $|a'| = |a|$, $|b'| = |b|$ and $a' \wedge b' = a \wedge b$.

3. Extremal paths and the set Σ

Define the set $\Sigma \subset \mathbb{R}^3 \times \wedge^2 \mathbb{R}^3$ as

$$\Sigma := \Big\{(x, t) \in \mathbb{R}^3 \times \wedge^2 \mathbb{R}^3 : x \perp \text{supp}\, t \text{ and } t \neq 0 \Big\}$$
$$= \{(x, y \wedge y') : x, y, y' \in \mathbb{R}^3 \text{ where } y \text{ and } y' \text{ are independent and } x \perp \{y, y'\}\}. \qquad (3.1)$$

Here and hereafter, without loss of generality, we may assume that $\mathbb{V} = \mathbb{R}^3$ and $\langle \cdot, \cdot \rangle$ is the Euclidean inner product. We will show in this section that $\Sigma \subset \text{Cut}_0$. The proof of the opposite inclusion will be achieved at the end of Subsection 4.1.

Define the function

$$W(\theta) = U(\theta) - S(\theta)V(\theta) = \frac{\theta^2 + \theta\sin\theta\cos\theta - 2\sin^2\theta}{4\theta^3}, \qquad (3.2)$$

where S, U and V are defined in (1.4).

Lemma 3.1. $W(\theta) > 0$ *for all* $\theta > 0$.

Proof. Let $q(\theta) = \theta^2 + \theta\sin\theta\cos\theta - 2\sin^2\theta$, for all $\theta > 0$ we get $q(\theta) > \theta^2 - \theta - 2$ which is positive if $\theta > 2$. For $\theta \in (0, 2)$, write $2\theta = x$ and $q(\theta) = \frac{1}{4}(x^2 + x\sin(x) - 4 + 4\cos(x)) =: \frac{1}{4}p(x)$. Differentiating we find $p(0) = p'(0) = p''(0) = 0$ and $p'''(x) = \sin(x) - x\cos x$, which is positive for all $x \in (0, \varphi_1)$, where $\varphi_1 > 4$ is the first positive solution of $\tan(x) = x$. Thus the function is increasing in $\theta \in (0, \varphi_1/2) \supset (0, 2)$ and the proof is concluded. $\qquad\square$

Let $P :]0, +\infty[\setminus \bigcup_{k\in\mathbb{N}}\{\varphi_k\}$

$$P(\theta) := -\frac{S(\theta)}{V(\theta)}\sqrt{\frac{W(\theta)}{U(\theta)}} \qquad (3.3)$$

where $\varphi_k \in]k\pi, (k + \frac{1}{2})\pi[$ denotes the k-th positive solution of $V(\theta) = 0$ (i.e. $\tan(\theta) = \theta$). Note that $P(k\pi) = 0$ and $P(\varphi_k-) = +\infty$ for all $k \in \mathbb{N}$. We will show that P is strictly increasing in $]\pi, \varphi_1[$. Actually the same happens in any interval $]k\pi, \varphi_k[$ for $k \geq 2$, but we do not need such property. A plot of P is exhibited in Figure 3.1.

3.1.   Characterization of extremal controls connecting the origin with Σ

Next we find all extremal curves (minimizing or not) which connect the origin with a point $(x, t) \in \Sigma$. Without loss of generality we always write a vector $t \in \wedge^2\mathbb{R}^3$ in the form $t = y \wedge y^\perp$ for a suitable pair of orthogonal vectors with equal length. In such case, it turns out that $|t| = |y|^2$ (see (2.1)).

Theorem 3.2 (Characterization of extremal paths passing for a given point of Σ)**.** *Let* $(x, y \wedge y^\perp) \in \Sigma$, *where* $x, y, y^\perp$ *are pairwise orthogonal and* $|y| = |y^\perp| > 0$. *Then the following facts hold.*

(i) Let $\theta > 0$ *be any solution of the equation*

$$P(\theta) = \frac{|x|^2}{|y|^2}. \qquad (3.4)$$

Let $\sigma \in \mathbb{R}$ *be an arbitrary parameter and choose the corresponding vectors*

$$\begin{cases} \alpha' := \alpha'_\sigma := \dfrac{1}{(U_\theta W_\theta)^{1/4}}(y\sin\sigma - y^\perp\cos\sigma) \\[2mm] \beta' := \beta'_\sigma := \dfrac{(U_\theta W_\theta)^{1/4}}{W_\theta}(y\cos\sigma + y^\perp\sin\sigma) - \dfrac{V_\theta}{W_\theta}x \\[2mm] \zeta := \zeta_\sigma := \dfrac{U_\theta}{W_\theta}x - \dfrac{S_\theta}{W_\theta}(U_\theta W_\theta)^{1/4}(y\cos\sigma + y^\perp\sin\sigma) \end{cases} \qquad (3.5)$$

Then the triple $\alpha'_\sigma, \beta'_\sigma, \zeta_\sigma$ *is admissible and the path*

$$s \mapsto \gamma(s, \alpha'_\sigma\sin\theta + \beta'_\sigma\cos\theta, \beta'_\sigma\sin\theta - \alpha'_\sigma\cos\theta, \zeta_\sigma, 2\theta) =: \gamma_\sigma(s) \qquad (3.6)$$

satisfies $\gamma_\sigma(1) = (x, y \wedge y^\perp)$ *(the function* $\gamma(s, a, b, z, 2\varphi)$ *is defined in (2.6)).*

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(ii) If $\theta, \alpha', \beta', \zeta$ satisfy (3.4) and (3.5), for all $\sigma \in \mathbb{R}$ the length of the path γ_σ in (3.6) does not depend on σ and is

$$\text{length}^2(\gamma_\sigma|_{[0,1]}) = |x|^2 + \frac{1 - S(\theta)^2}{\sqrt{U(\theta)W(\theta)}}|y|^2.$$

(iii) If $\theta, \alpha', \beta', \zeta$ satisfy (3.4) and (3.5), for all $\sigma \in \mathbb{R}$ we have

$$\frac{|\zeta|^2}{|\beta'|^2} = \frac{|\zeta|^2}{|\alpha'|^2} = -\frac{U(\theta)S(\theta)}{V(\theta)} =: Q(\theta). \tag{3.7}$$

(iv) Let γ be an extremal path on $[0,1]$ which satisfies $\gamma(0) = 0$ and $\gamma(1) = (x, y \wedge y^\perp)$. Then there is $\theta > 0$ satisfying (3.4), there is $\sigma \in \mathbb{R}$ such that γ has the form (3.6), where $\alpha'_\sigma, \beta'_\sigma$ and ζ_σ are the vectors appearing in (3.5).

Item (i) and (iv) give the characterization of all extremal paths connecting the origin with points of Σ. The length in (ii) is needed to discuss their minimizing properties and ratio $Q(\theta)$ appearing in (iii) is crucial in the calculation of the cut time.

Figure 3.1: A plot of the positive part of the function P for $\theta < \varphi_2$ with a representation of θ_1 and θ_2, the first two (of the infinitely many) solutions of the equation $P(\theta) = \frac{|x|^2}{|t|}$.

Remark 3.3. *If we take a curve $\gamma_\sigma(s) = (x_\sigma(s), t_\sigma(s))$ and we rearrange the choice of $y, y^\perp$ to ensure that α' and β' take the simple form*

$$\alpha' = \frac{y}{(U_\theta W_\theta)^{1/4}} \qquad \beta' = \frac{(U_\theta W_\theta)^{1/4}}{W_\theta}y^\perp - \frac{V_\theta}{W_\theta}x \qquad \zeta = \frac{U_\theta}{W_\theta}x - \frac{S_\theta}{W_\theta}(U_\theta W_\theta)^{1/4}y^\perp, \quad (3.8)$$

we discover that the corresponding control $u(s)$ takes the simple form

$$\begin{aligned} u(s) &= (\alpha' \sin\theta + \beta' \cos\theta)\cos(2\theta s) + (\beta' \sin\theta - \alpha' \cos\theta)\sin(2\theta s) + \zeta \\ &= \alpha' \sin(\theta(1 - 2s)) + \beta' \cos(\theta(1 - 2s)) + \zeta \end{aligned} \tag{3.9}$$

where α', β', ζ are the vectors in (3.8) and $s \in [0, 1]$.

Proof of Theorem 3.2. We start with the proof of part (iv) of the statement. Let $(x, y \wedge y^\perp) \in \Sigma$ and assume that $x, y, y^\perp$ are pairwise orthogonal, $|y| = |y^\perp| > 0$, while x may possibly vanish. Let $s \mapsto \gamma(s, \alpha, \beta, \zeta, 2\theta)$ be the extremal path defined in (2.6) and assume that $\gamma(1) = (x, y \wedge y^\perp)$. We must solve the system $F(\alpha, \beta, \zeta, \theta) = (x, y \wedge y^\perp)$, where F appears in (2.11) and (2.12) and the triple α, β, ζ must be admissible. Write for brevity $U_\theta, V_\theta, S_\theta$ instead of $U(\theta), V(\theta), S(\theta)$. Observe also that θ cannot be 0, because $F(\alpha, \beta, \zeta, 0) = (a + z, 0)$ cannot belong to Σ. Thus assume that θ is strictly positive (we can always exclude the case $\theta < 0$, see the discussion after (2.4)). Using the explicit form of F we find the system

$$\begin{cases} S_\theta(\alpha \cos \theta + \beta \sin \theta) + \zeta = x \\ U_\theta \alpha \wedge \beta + V_\theta(\alpha \sin \theta - \beta \cos \theta) \wedge \zeta = y \wedge y^\perp. \end{cases}$$

Making the change of variable

$$\alpha' = \alpha \sin \theta - \beta \cos \theta \quad \text{and} \quad \beta' = \alpha \cos \theta + \beta \sin \theta, \tag{3.10}$$

(see Remark 2.3), we get

$$\begin{cases} S_\theta \beta' + \zeta = x \\ \alpha' \wedge (U_\theta \beta' + V_\theta \zeta) = y \wedge y^\perp. \end{cases} \tag{3.11}$$

Therefore α' and $U_\theta \beta' + V_\theta \zeta$ belong to span$\{y, y^\perp\}$. Furthermore they must be orthogonal, by admissibility. This means that we can write

$$U_\theta \beta' + V_\theta \zeta = \xi y + \eta y^\perp \tag{3.12}$$

$$\alpha' = q\eta y - q\xi y^\perp, \tag{3.13}$$

where $q \neq 0$ and $\xi, \eta \in \mathbb{R}$ satisfy $\xi^2 + \eta^2 \neq 0$. The first line of (3.11) and (3.12) are linear in β' and ζ. Solving, we get

$$\beta' = \frac{\xi}{W_\theta} y + \frac{\eta}{W_\theta} y^\perp - \frac{V_\theta}{W_\theta} x \quad \text{and} \quad \zeta = \frac{U_\theta}{W_\theta} x - \frac{S_\theta \xi}{W_\theta} y - \frac{S_\theta \eta}{W_\theta} y^\perp, \tag{3.14}$$

where $W_\theta := U_\theta - S_\theta V_\theta$ cannot vanish, by Lemma 3.1. On the vectors α' in (3.11) and β', ζ in (3.14), we must require that $|\alpha'| = |\beta'|$. Moreover it must be $\langle \beta', \zeta \rangle = 0$ and finally $\alpha' \wedge (U_\theta \beta' + V_\theta \zeta) = y \wedge y^\perp$. These three conditions become

$$W_\theta^2 q^2 (\xi^2 + \eta^2)|y|^2 = (\xi^2 + \eta^2)|y|^2 + V_\theta^2 |x|^2$$
$$S_\theta(\xi^2 + \eta^2)|y|^2 + U_\theta V_\theta |x|^2 = 0$$
$$q(\xi^2 + \eta^2) = 1.$$

We used the fact that $x, y, y^\perp$ are pairwise orthogonal and $|y| = |y^\perp|$. Observe that $q > 0$ and that the unknown ξ and η appear in the form $(\xi^2 + \eta^2)$. Therefore we will find an infinite family of curves.

Eliminating $(\xi^2 + \eta^2)$, the first two equations become

$$W_\theta^2 q |y|^2 = \frac{|y|^2}{q} + V_\theta^2 |x|^2 \quad \text{and} \quad \frac{S_\theta}{q} |y|^2 + U_\theta V_\theta |x|^2 = 0. \tag{3.15}$$

Note that $U_\theta > 0$ for all $\theta > 0$. Moreover, it cannot be $V_\theta = 0$, because in such case it should be $S_\theta = 0$ too, but this is impossible because V ans S do not have common zeros on $]0, +\infty[$. Therefore, $q = -\dfrac{S_\theta |y|^2}{U_\theta V_\theta |x|^2}$, and since $q > 0$, we discover that it must be $S(\theta)V(\theta) < 0$. Eliminating q from the first equation of (3.15), we obtain

$$-W_\theta^2 \frac{S_\theta |y|^4}{U_\theta V_\theta |x|^2} = -\frac{U_\theta V_\theta |x|^2}{S_\theta} + V_\theta^2 |x|^2 = -\frac{W_\theta V_\theta}{S_\theta}|x|^2,$$

because $W_\theta = U_\theta - S_\theta V_\theta$. Then we have

$$\frac{W(\theta)S(\theta)^2}{U(\theta)V(\theta)^2} = \frac{|x|^4}{|y|^4}.$$

Taking the square root, and remembering that $U_\theta, W_\theta > 0$ for all $\theta > 0$, while, as we already observed, the product $S_\theta V_\theta$ must be negative, we conclude that (3.4) must hold.

To complete the proof of (iv), we obtain the explicit form of α', β' and ζ. We start from

$$\xi^2 + \eta^2 = \frac{1}{q} = -\frac{U_\theta V_\theta}{S_\theta} \frac{|x|^2}{|y|^2} = (U_\theta W_\theta)^{1/2},$$

where we used (3.4) in the last equality. Thus $q = \dfrac{1}{(U_\theta W_\theta)^{1/2}}$, $\xi = (U_\theta W_\theta)^{1/4} \cos\sigma$ and $\eta = (U_\theta W_\theta)^{1/4} \sin\sigma$ for some $\sigma \in \mathbb{R}$. Therefore, using (3.13) and (3.14), it turns out that

$$\begin{aligned}
\alpha' &= \frac{1}{(U_\theta W_\theta)^{1/4}}(y\sin\sigma - y^\perp \cos\sigma) \\
\beta' &= \frac{(U_\theta W_\theta)^{1/4}}{W_\theta}(y\cos\sigma + y^\perp \sin\sigma) - \frac{V_\theta}{W_\theta}x \qquad\qquad (3.16) \\
\zeta &= \frac{U_\theta}{W_\theta}x - \frac{S_\theta}{W_\theta}(U_\theta W_\theta)^{1/4}(y\cos\sigma + y^\perp \sin\sigma).
\end{aligned}$$

Inverting the change of variables (3.10), $\alpha = \alpha'\sin\theta + \beta'\cos\theta$ and $\beta = \beta'\sin\theta - \alpha'\cos\theta$, we conclude the proof of part (iv).

Proof of (i). It suffices to check that, given $(x, y \wedge y^\perp) \in \Sigma$, if θ satisfies (3.4), then for all $\sigma \in \mathbb{R}$ the triple in (3.5) is admissible and the path γ in (3.6) satisfies

$$\gamma(1) =: F(\alpha'\sin\theta + \beta'\cos\theta, \beta'\sin\theta - \alpha'\cos\theta, \zeta, \theta) = (x, y \wedge y^\perp).$$

But we have $F(\alpha'\sin\theta + \beta'\cos\theta, \beta'\sin\theta - \alpha'\cos\theta, \zeta, \theta) = G(\alpha', \beta', \zeta, \theta)$ by Remark 2.3. Thus we must check first that α', β', ζ is admissible and second that $G(\alpha', \beta', \zeta, \theta) = (x, y \wedge y^\perp)$, i.e.

$$S_\theta \beta' + \zeta = x \qquad \text{and} \qquad \alpha' \wedge (U_\theta \beta' + V_\theta \zeta) = y \wedge y^\perp. \qquad (3.17)$$

To check admissibility let us start from a triple as in (3.5) and calculate $|\alpha'|^2 = \dfrac{|y|^2}{(U_\theta W_\theta)^{1/2}}$

$$\begin{aligned}
|\beta'|^2 &= \frac{(U_\theta W_\theta)^{1/2}}{W_\theta^2}|y|^2 + \frac{V_\theta^2}{W_\theta^2}|x|^2 = \text{(eliminating } |y|^2 \text{ by (3.4))} \\
&= \Big(\frac{-U_\theta V_\theta}{S_\theta W_\theta^2} + \frac{V_\theta^2}{W_\theta^2}\Big)|x|^2 = -\frac{V_\theta}{S_\theta W_\theta}|x|^2.
\end{aligned} \qquad (3.18)$$

Then $|\alpha'|^2 = |\beta'|^2$ if (3.4) holds.

From (3.5) it is obvious that $\langle \alpha', \zeta \rangle = 0 = \langle \beta', \alpha' \rangle$. To check that $\beta' \perp \zeta$, it suffices to observe that

$$\langle \beta', \zeta \rangle = -\frac{S_\theta (U_\theta W_\theta)^{1/2}}{W_\theta^2}|y|^2 - \frac{U_\theta V_\theta}{W_\theta^2}|x|^2 = 0,$$

as soon as (3.4) holds. Next we check that the x-component takes the correct value, i.e. the first equality in (3.17).

$$S_\theta \beta' + \zeta = S_\theta \left[\frac{(U_\theta W_\theta)^{1/4}}{W_\theta}(y \cos \sigma + y^\perp \sin \sigma) - \frac{V_\theta}{W_\theta}x \right]$$
$$+ \left[\frac{U_\theta}{W_\theta}x - \frac{S_\theta}{W_\theta}(U_\theta W_\theta)^{1/4}(y \cos \sigma + y^\perp \sin \sigma) \right] = \left(\frac{-S_\theta V_\theta}{W_\theta} + \frac{U_\theta}{W_\theta} \right)x = x,$$

as required. To check the t-component (second equality in (3.17)), note that, taking β' and ζ' as in (3.5), we have

$$U_\theta \beta' + V_\theta \zeta = \frac{U_\theta - S_\theta V_\theta}{W_\theta}(U_\theta W_\theta)^{1/4}(y \cos \sigma + y^\perp \sin \sigma) = (U_\theta W_\theta)^{1/4}(y \cos \sigma + y^\perp \sin \sigma).$$

Then

$$\alpha' \wedge (U_\theta \beta' + V_\theta \zeta) = \frac{1}{(U_\theta W_\theta)^{1/4}}(y \sin \sigma - y^\perp \cos \sigma) \wedge (U_\theta W_\theta)^{1/4}(y \cos \sigma + y^\perp \sin \sigma) = y \wedge y^\perp,$$

as desired, and the proof of (i) is concluded.

Proof of (ii). From (3.18) we already know that $|\beta'|^2 = -\dfrac{V_\theta}{S_\theta W_\theta}|x|^2$. Moreover,

$$|\zeta|^2 = \frac{U_\theta^2}{W_\theta^2}|x|^2 + \frac{S_\theta^2}{W_\theta^2}(U_\theta W_\theta)^{1/2}|y|^2 = (\text{eliminating } |y|^2 \text{ by } (3.4)) = \frac{U_\theta}{W_\theta}|x|^2. \qquad (3.19)$$

Summing up, we find

$$\text{length}^2(\gamma|_{[0,1]}) = \frac{|x|^2}{W_\theta}\left(U_\theta - \frac{V_\theta}{S_\theta} \right) = \frac{|x|^2}{W_\theta}\left(W_\theta + S_\theta V_\theta - \frac{V_\theta}{S_\theta} \right)$$
$$= |x|^2 - \frac{V_\theta(1 - S_\theta^2)}{S_\theta W_\theta}|x|^2 = |x|^2 + \frac{1 - S_\theta^2}{\sqrt{U_\theta W_\theta}}|y|^2,$$

where the last equality follows again from (3.4).

Proof of (iii). It follows immediately from (3.19) and (3.18). $\qquad \square$

3.2.   Minimizing curves and distance on Σ

Here, among the controls described in Theorem 3.2, we will find the minimizing ones. Observe that $\sin(\theta)V(\theta) < 0$ if and only if $\theta \in \bigcup_{k=1}^{\infty}]k\pi, \varphi_k[$, where $\varphi_k \in \left]k\pi, (k + \frac{1}{2})\pi\right[$ is the k-th positive zero of the function v.

Differentiating the functions S, U, V we find the useful formulas

$$S'(\varphi) = -2V(\varphi), \qquad U'(\varphi) = \frac{\cos \varphi}{\varphi}V(\varphi), \qquad V'(\varphi) = \frac{S(\varphi)}{2} - \frac{2}{\varphi}V(\varphi). \qquad (3.20)$$

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Lemma 3.4. *The function*

$$R(\theta) := \frac{1 - S(\theta)^2}{\sqrt{U(\theta)W(\theta)}}$$

is strictly increasing on $A := \bigcup_{k=1}^{\infty}]k\pi, \varphi_k[$.

The function is actually increasing on the whole half-line $\theta > 0$, but the statement of the lemma is easier to prove and it is enough for our purposes. Observe the limit behaviour $R(\theta) \sim 4\theta \to +\infty$, as $\theta \to +\infty$.

Proof. It suffices to prove that the following function is strictly decreasing on A.

$$f = \frac{UW}{(1 - S^2)^2} = \frac{U^2}{(1 - S^2)^2} + \frac{U}{1 - S^2} \cdot \frac{(-SV)}{1 - S^2} =: F^2 + FG. \tag{3.21}$$

We are omitting the variable θ in U, V, W. On the functions F and G observe that

$$F(\theta) := \frac{U}{1 - S^2} = \frac{\theta - \sin\theta\cos\theta}{4(\theta^2 - \sin^2\theta)} = \frac{1}{8}\frac{d}{d\theta}\log(\theta^2 - \sin^2\theta) = \frac{1}{8}\frac{d}{d\theta}\log(\theta^2(1 - S^2))$$

By formula $S' = -2V$, we have

$$G(\theta) := \frac{-SV}{1 - S^2} = \frac{SS'}{2(1 - S^2)} = -\frac{1}{4}\frac{d}{d\theta}\log(1 - S^2).$$

Thus,

$$G + 2F = \frac{1}{4}\frac{d}{d\theta}\log(\theta^2) = \frac{1}{2\theta}.$$

We will show that $f' < 0$ on any interval $]k\pi, \varphi_k[$ and that $f(\varphi_k) < f((k+1)\pi)$ for all $k \in \mathbb{N}$.

$$f' = (F^2 + FG)' = F(2F' + G') + F'G = -\frac{F(\theta)}{2\theta^2} + F'(\theta)G(\theta). \tag{3.22}$$

A computation gives

$$F'(\theta) = \frac{\sin^2\theta(\theta^2 - \sin^2\theta) - (\theta - \sin\theta\cos\theta)^2}{2(\theta^2 - \sin^2\theta)^2}$$

$$= -\frac{(\theta^2\cos^2\theta + \sin^2\theta - 2\theta\sin\theta\cos\theta)}{2(\theta^2 - \sin^2\theta)^2} = -\frac{(\theta\cos\theta - \sin\theta)^2}{2(\theta^2 - \sin^2\theta)^2} < 0,$$

on $]0, +\infty[\setminus \bigcup_{k \geq 1}\{\varphi_k\}$. Thus F is strictly decreasing on $]0, +\infty[$. Since G and F are positive on each interval $]k\pi, \varphi_k[$, looking at (3.22), we conclude that $f' < 0$ on each $]k\pi, \varphi_k[$.

To conclude the proof, observe that $G(k\pi) = G(\varphi_k) = 0$ for all $k \in \mathbb{N}$. Thus we get

$$f(\varphi_k) = F^2(\varphi_k) > F^2((k+1)\pi) = f((k+1)\pi)$$

for each $k \geq 1$, because F is strictly decreasing on $]0, +\infty[$. $\square$

Lemma 3.5. *The function P defined in (3.4) is strictly increasing on $]\pi, \varphi_1[$ and $P(\pi) = 0$, $P(\varphi_1-) = +\infty$.*

Proof of Lemma 3.5. Since $P > 0$ on $]\pi, \varphi_1[$, we shall prove that P^2 is increasing on $]\pi, \varphi_1[$.

$$P^2 = \frac{WS^2}{UV^2} = \frac{(U - SV)S^2}{UV^2} = \left(-\frac{S}{V} + \frac{S^2}{U}\right) \cdot \left(-\frac{S}{V}\right).$$

that $V > 0$ and $S < 0$ in $]\pi, \varphi_1[$. Thus $SV < 0$.

To show the statement it suffices to prove that both the positive functions $-S/V$ and S^2/U are increasing on $]\pi, \varphi_1[$. Keeping (3.20) into account, we find

$$V^2\left(\frac{-S}{V}\right)' = -S'V + SV' = 2V^2 + \frac{S^2}{2} - \frac{2SV}{\theta} > 0,$$

because $VS < 0$. Furthermore

$$U^2\left(\frac{S^2}{U}\right)' = 2SS'U - S^2U' = -4VSU - S^2\frac{\cos\theta}{\theta}V > 0,$$

because $\cos\theta < 0$ in $]\pi, \varphi_1[$. The proof is concluded. $\qquad\square$

Next we are ready to prove Theorem 1.4. We will denote by $P^{-1} : [0, +\infty[\to [\pi, \varphi_1[$ the inverse function of $P\big|_{[\pi,\varphi_1[}$, which is well defined by Lemma 3.5.

Proof of Theorem 1.4. Let $(x, t) \in \Sigma$. Let $\theta \in]\pi, \frac{3}{2}\pi[$ be the smallest solution of $P(\theta) = \frac{|x|^2}{|t|}$. Write $t = y \wedge y^{\perp}$ for some $y, y^{\perp}$ orthogonal and of equal length. By Theorem 3.2, item (i), we know that for all $\sigma \in \mathbb{R}$ the curve γ_σ in (3.6) satisfies $\gamma_\sigma(0) = (0,0)$, $\gamma_\sigma(1) = (x, t)$. By item (ii) of the same theorem, its length is

$$\text{length}^2(\gamma_\sigma|_{[0,1]}) = |x|^2 + \frac{1 - S(\theta)^2}{\sqrt{U(\theta)W(\theta)}}|t|.$$

Finally, by Lemma 3.4, γ_σ is length minimizing on $[0, 1]$. The proof is concluded. $\qquad\square$

We are now ready to prove the following inclusion.

Proposition 3.6. *We have the inclusion $\Sigma \subset \text{Cut}_0$.*

The proof of the opposite inclusion will be achieved at the end of Subsection 4.1.

Proof of Proposition 3.6. Assume by contradiction that a point $(x, t) \in \Sigma$ does not belong to Cut_0. We will show by a standard nonuniqueness argument that we can find a constant-speed path Γ which minimizes for $s \in [0, 1+\varepsilon]$, such that $\Gamma(0) = 0, \Gamma(1) = (x, t)$ and Γ is not differentiable at $s = 1$. This contradicts the known fact that arclength length-minimizers in two step Carnot groups are normal and then smooth.

To prove the claim, observe first that, since $(x, t) \in \Sigma$ does not belong to Cut_0, we can find $y, y^{\perp}$ and a minimizer $s \mapsto \gamma_{\sigma_1}(s)$ of the form (3.6), i.e.

$$s \mapsto \gamma(s, \alpha'_{\sigma_1}\sin\theta + \beta'_{\sigma_1}\cos\theta, \beta'_{\sigma_1}\sin\theta - \alpha'_{\sigma_1}\cos\theta, \zeta, 2\theta) =: \gamma(s) =: \gamma_{\sigma_1}(s)$$

such that $\gamma_{\sigma_1}(1) = (x, t)$ and γ_{σ_1} minimizes for $s \in [0, 1+\varepsilon]$ for some $\varepsilon > 0$. The minimizing choice of $\theta \in]\pi, \varphi_1[$ is uniquely determined, by Lemma 3.4 and Lemma 3.5.

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Let us consider $\sigma_2 \neq \sigma_1$ and the corresponding path $\gamma_{\sigma_2}(s)$ in (3.6) corresponding to the same choice of θ. We know that this path minimizes on $[0, 1]$ for any choice of $\sigma_2 \in \mathbb{R}$. Therefore, the new path Γ defined as

$$\Gamma(s) = \begin{cases} \gamma_{\sigma_2}(s) & \text{if } s \in [0, 1] \\ \gamma_{\sigma_1}(s) & \text{if } s \in [1, 1 + \varepsilon] \end{cases}$$

is a constant-speed length minimizer on $[0, 1 + \varepsilon]$. To prove the nondifferentiability of Γ, it suffices to show that $\dot\gamma_1(1) \neq \dot\gamma_2(1)$. Since $\gamma_{\sigma_1}(1) = \gamma_{\sigma_2}(1)$, this is equivalent to saying that $u_{\sigma_1}(1) \neq u_{\sigma_2}(1)$ (see (2.5)). But, letting $\gamma_\sigma = (x_\sigma, t_\sigma)$ and $\dot x_\sigma =: u_\sigma$, a computation shows that for $\sigma \in \mathbb{R}$ we have

$$u_\sigma(1) = \cos(2\theta)(\alpha'_\sigma \sin\theta + \beta'_\sigma \cos\theta) + \sin(2\theta)(\beta'_\sigma \sin\theta - \alpha'_\sigma \cos\theta) + \zeta_\sigma$$
$$= -\alpha'_\sigma \sin\theta + \beta'_\sigma \cos\theta + \zeta_\sigma$$
$$= -\frac{1}{(U_\theta W_\theta)^{1/4}}(y\sin\sigma - y^\perp\cos\sigma)\sin\theta + \Big(\frac{(U_\theta W_\theta)^{1/4}}{W_\theta}(y\cos\sigma + y^\perp\sin\sigma) - \frac{V_\theta}{W_\theta}x\Big)\cos\theta$$
$$+ \Big(\frac{U_\theta}{W_\theta}x - \frac{S_\theta}{W_\theta}(U_\theta W_\theta)^{1/4}(y\cos\sigma + y^\perp\sin\sigma)\Big).$$

The proof will be concluded as soon as we show that $\sigma \mapsto u_\sigma(1)$ is a nonconstant function.

To prove the claim, write

$$u_\sigma(1) = \Big(\frac{\sin\theta}{(U_\theta W_\theta)^{1/4}}y^\perp + \frac{\cos\theta}{W_\theta}(U_\theta W_\theta)^{1/4}y - \frac{S_\theta}{W_\theta}(U_\theta W_\theta)^{1/4}y\Big)\cos\sigma$$
$$+ \Big(\frac{\cos\theta}{W_\theta}(U_\theta W_\theta)^{1/4}y^\perp - \frac{\sin\theta}{(U_\theta W_\theta)^{1/4}}y - \frac{S_\theta}{W_\theta}(U_\theta W_\theta)^{1/4}y^\perp\Big)\sin\sigma$$
$$- \cos\theta\frac{V_\theta}{W_\theta}x + \frac{U_\theta}{W_\theta}x$$
$$=: A\cos\sigma + B\sin\sigma + C.$$

It is easy to see that a function $H(\sigma) = A\cos\sigma + B\sin\sigma + C$ where A, B, C are given vectors is constant if and only if $A = B = 0$. Therefore, since $y \perp y^\perp$, the function $\sigma \mapsto u_\sigma(1)$ is constant if and only if

$$\frac{(\cos\theta - S(\theta))}{W_\theta}(U_\theta W_\theta)^{1/4} = 0 \qquad \text{and} \qquad \frac{\sin\theta}{(U_\theta W_\theta)^{1/4}} = 0.$$

But this is impossible, because these condition would imply that $\cos\theta = \sin\theta = 0$ (recall that $U_\theta > 0$ and $W_\theta > 0$ for all $\theta > 0$). $\qquad\square$

Remark 3.7. *Observe that in the argument of the proof above we have shown by explicit computation that for any point $(x, t) \in \Sigma$, we can connect the origin with (x, t) with multiple minimizing paths with different tangent vectors at (x, t). Thus, by a standard argument, one can conclude that the distance from the origin cannot be differentiable at any such (x, t). In Section 6, we will prove some very precise corner-like estimates at cut points.*

4.   Calculation of $h_{\mathbf{cut}}$ and consequences

In the first part of this section (Subsection 4.1) we calculate the cut time for any given extremal and we describe the cut locus. In Subsection 4.2 we describe the profile of subRiemannian spheres.

4.1.   Cut time and cut locus

Lemma 4.1. *The function*

$$Q(\theta) := -\frac{U(\theta)S(\theta)}{V(\theta)}$$

is positive and strictly increasing on $[\pi, \varphi_1[$. *Furthermore,* $Q(\pi+) = 0$ *and* $Q(\varphi_1-) = +\infty$.
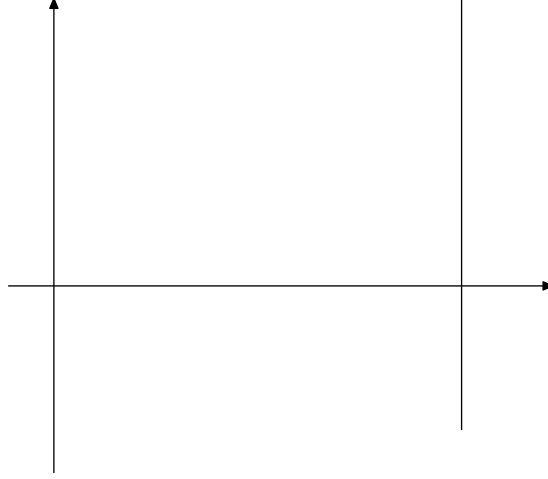

Figure 4.1: Plot of the function Q for $\theta \in\ ]0, \varphi_1[$.

The function Q is actually increasing on the whole interval $[0, \varphi_1[$, but we need (and prove) such property only in the sub-interval $[\pi, \varphi_1[$. See the plot in Figure 4.1.

Proof of Lemma 4.1. Write

$$2Q(\theta) = (-S(\theta)(\theta - \sin\theta\cos\theta)) \cdot \frac{1}{\sin\theta - \theta\cos\theta} =: f(\theta) \cdot \frac{1}{g(\theta)}.$$

The function g is positive and decreasing on $]\pi, \varphi_1[\ \subset\ ]\pi, 3/2\pi[$, because $g' = \theta\sin\theta < 0$. The function f is instead positive and increasing. To check this property, observe that

$$f'(\theta) = -S'(\theta)(\theta - \sin\theta\cos\theta) - S(\theta)(1 - \cos^2\theta + \sin^2\theta) > 0,$$

because $-S'(\theta) = 2V(\theta) > 0$. This concludes the proof. $\square$

We restate Theorem 1.3 as follows.

Theorem 4.2 (Calculation of the cut time). *Let Q be the function defined in Lemma 4.1 and in (1.5). Let h_{cut} be the function in Proposition 2.2. Then we have*

$$h_{\text{cut}}(\mu) = Q^{-1}(\mu^2) \qquad \forall \mu \in \,]0, +\infty[. \tag{4.1}$$

As a consequence, if a, b, z is any admissible triple, $\varphi > 0$ and we consider the corresponding path $s \mapsto \gamma(s, a, b, z, 2\varphi) =: \gamma(s)$, the path γ minimizes up to the time

$$t_{\text{cut}}(a, b, z, \varphi) = \frac{h_{\text{cut}}(|z|/|a|)}{\varphi} = \frac{Q^{-1}(|z|^2/|a|^2)}{\varphi} \tag{4.2}$$

and not further. Finally

$$\gamma(t_{\text{cut}}, a, b, z, 2\varphi) \in \Sigma \quad \textit{for all admissible } a, b, z \textit{ and } \varphi > 0. \tag{4.3}$$

Remark 4.3. Cut_0 *is a four dimensional smooth submanifold of the six-dimensional $\mathbb{F}_3$. Indeed it can be parametrized as follows*

$$\text{Cut}_0 = \left\{ (\lambda x e_1 + \lambda y e_2 + \lambda z e_3, z e_1 \wedge e_2 - y e_1 \wedge e_3 + x e_2 \wedge e_3) : \lambda \in \mathbb{R} \quad x^2 + y^2 + z^2 > 0 \right\}$$

and it is easy to check that the map $\Phi(\lambda, x, y, z) = (\lambda x, \lambda y, \lambda z, x, y, z) \in \mathbb{R}^6$ has full rank on the open set $\{(\lambda, x, y, z) : \lambda \in \mathbb{R} \quad x^2 + y^2 + z^2 > 0\} \subset \mathbb{R}^4$. By Theorem 1.4, we see that the restriction of the distance from the origin to the cut locus is smooth.

Proof of Theorem 4.2. We first prove that $h_{\text{cut}}(\mu) = Q^{-1}(\mu^2)$. Start with an arbitrary $\theta \in \,]\pi, \varphi_1[$. Choose any $(x, y \wedge y^\perp) \in \Sigma$ such that $P(\theta) = \frac{|x|^2}{|y|^2}$. By Theorem 3.2 and Lemma 3.4, we may choose (infinitely many) admissible triples α', β', ζ such that the path γ_σ in (3.6) satisfies $\gamma_\sigma(1) = (x, y \wedge y^\perp)$, minimizes length up to time $t_{\text{cut}} = 1$ and not further (we already know from Proposition 3.6 that $\Sigma \subseteq \text{Cut}_0$). Thus,

$$1 = t_{\text{cut}}(\alpha' \sin\theta + \beta' \cos\theta, \beta' \sin\theta - \alpha' \cos\theta, \zeta, \theta) = \frac{h_{\text{cut}}(|\zeta|/|\alpha'|)}{\theta},$$

where last equality follows from Proposition 2.2. Then $h_{\text{cut}}(|\zeta|/|\alpha'|) = \theta$. By item (iii) of Theorem 3.2 we get $h_{\text{cut}}(\sqrt{Q(\theta)}) = \theta$. Since $\theta \in \,]\pi, \varphi_1[$ is arbitrary, Lemma 4.1 gives the conclusion $h_{\text{cut}}(\mu) = Q^{-1}(\mu^2)$ for all $\mu \in \,]0, +\infty[$. Thus we have proved (4.1) and (4.2).

Next we prove (4.3). Let a, b, z be any admissible triple and let $\varphi > 0$. Recall that the admissibility condition ensures that $|a| = |b| > 0$. By the first part of the theorem, we know that

$$t_{\text{cut}}(a, b, z, \varphi) = \frac{h_{\text{cut}}(|z|/|a|)}{\varphi} = \frac{Q^{-1}(|z|^2/|a|^2)}{\varphi}.$$

To prove (4.3), we claim that, under the choice

$$\theta := Q^{-1}(|z|^2/|a|^2), \tag{4.4}$$

(i.e. $t_{\text{cut}}(a, b, z, \varphi) =: \frac{\theta}{\varphi}$) we have $\gamma\left(\frac{\theta}{\varphi}, a, b, z, 2\varphi\right) \in \Sigma$.

To prove the claim, note that the invariance property (2.7) gives

$$\gamma\left(\frac{\theta}{\varphi}, a, b, z, 2\varphi\right) = \gamma\left(1, \frac{\theta}{\varphi}a, \frac{\theta}{\varphi}b, \frac{\theta}{\varphi}z, 2\theta\right) = F\left(\frac{\theta}{\varphi}a, \frac{\theta}{\varphi}b, \frac{\theta}{\varphi}z, \theta\right),$$

by the definition of F. Since the set Σ is dilation invariant, the latter point belongs to Σ if and only if its $\frac{\varphi}{\theta}$-dilated $F(a, b, z, \theta)$ belongs to Σ. Here we need the dilation invariance property of F. [2] But

$$F(a, b, z, \theta) = \Big(S_\theta(a\cos\theta + b\sin\theta) + z, \; U_\theta a \wedge b + V_\theta(a\sin\theta - b\cos\theta) \wedge z\Big)$$

$$=: \Big(S_\theta b' + z, \; a' \wedge \{U_\theta b' + V_\theta z\}\Big),$$

where a', b' are defined (2.13) and we keep Remark 2.3 into account. Since a', b', z is an admissible triple, we have for free that $S_\theta b' + z \perp a'$, and ultimately the point $F(a, b, z, \theta)$ belongs to Σ if and only if $S_\theta b' + z \perp U_\theta b' + V_\theta z$, i.e. if

$$U_\theta S_\theta |b|^2 + V_\theta |z|^2 = 0,$$

which holds true because our choice of θ in (4.4) $\square$

Theorem 4.2 shows that, if a, b, z is admissible and $\varphi > 0$, then

$$d(0, \gamma(s, a, b, z, 2\varphi)) = s\sqrt{|a|^2 + |z|^2} \quad \text{if} \quad 0 \le s \le \frac{1}{\varphi} h_{\text{cut}}\left(\frac{|z|}{|a|}\right). \tag{4.5}$$

Finally we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Theorem 4.2 and in particular formula (4.3) imply easily the inclusion $\text{Cut}_0 \subset \Sigma$. The opposite inclusion has been proved in Proposition 3.6. $\square$

4.2. Profile of the unit sphere

Let for all admissible α, β, ζ and for any $\theta > 0$

$$G(\alpha, \beta, \zeta, \theta) := \Big(S(\theta)\beta + \zeta, \; \alpha \wedge (U(\theta)\beta + V(\theta)\zeta)\Big).$$

The profile of the subRiemannian sphere can be described as follows

Corollary 4.4 (Profile of the subRiemannian sphere). *Let $S(0, r) := \{(x, t) \in \mathbb{F}_3 : d(x, t) = r\}$ be the subRiemannian sphere of radius r. Then*

$$S(0, r) := \Big\{G(\alpha, \beta, \zeta, \theta) : \alpha, \beta, \zeta \text{ admissible triple with } |\alpha|^2 + |\zeta|^2 = r^2$$

$$\text{and} \quad 0 \le \theta \le h_{\text{cut}}\big(|\zeta|/|\alpha|\big)\Big\}.$$

Proof. By dilation invariance we may assume $r = 1$. Start from (4.5) and observe that

$$S(0, 1) = \big\{\gamma(s, a, b, z, 2\varphi) \mid s^2(|a|^2 + |z|^2) = 1 \quad \text{and} \quad 0 < s \le \varphi^{-1} h_{\text{cut}}\big(|z|/|a|\big)\big\}.$$

$$= \big\{F(sa, sb, sz, s\varphi) \mid s^2(|a|^2 + |z|^2) = 1 \quad \text{and} \quad 0 < s \le \varphi^{-1} h_{\text{cut}}\big(|z|/|a|\big)\big\}.$$

$$= \big\{F(\alpha, \beta, \zeta, \theta) \mid |\alpha|^2 + |\zeta|^2 = 1 \quad \text{and} \quad 0 \le \theta \le h_{\text{cut}}\big(|\zeta|/|\alpha|\big)\big\}$$

and the required thesis follows. In the last chain of equalities, a, b, z and α, β, ζ are always admissible triples. $\square$

---

[2] That is, the property $F(r\alpha, r\beta, r\zeta, \varphi) = \delta_r F(\alpha, \beta, \zeta, \varphi)$ for all $r > 0$, α, β, ζ admissible and $\varphi > 0$.

5.   Some remarks on the Hamiltonian point of view

In this section we consider the Hamiltonian point of view. In particular we will calculate the cut time as an explicit function of the initial covector. In order to look at the Hamiltonian point of view, it is convenient to represent the group $\mathbb{F}_3$ in the form $\mathbb{R}^3_x \times \mathbb{R}^3_t$ with the group law

$$(x,t) \circ (x',t') = \left(x + x', t + t' + \frac{1}{2} x \times x' \right), \tag{5.1}$$

where $\times$ denotes the standard cross product in $\mathbb{R}^3$. This law is not scalable to higher dimensional cases, but it is rather convenient in the rank three case (compare [MM13]).

The standard basis of horizontal vector fields is $Y_j = \left(e_j, \frac{1}{2} x \times e_j \right) \in \mathbb{R}^3 \times \mathbb{R}^3$, for $j = 1, 2, 3$. Namely

$$Y_1 = \partial_{x_1} + \frac{1}{2} x_3 \partial_{t_2} - \frac{1}{2} x_2 \partial_{t_3}, \quad Y_2 = \partial_{x_2} - \frac{1}{2} x_3 \partial_{t_1} + \frac{1}{2} x_1 \partial_{t_3}, \quad Y_3 = \partial_{x_3} + \frac{1}{2} x_2 \partial_{t_1} - \frac{1}{2} x_1 \partial_{t_2}.$$

Commutation relations take the form $[Y_1, Y_2] = \partial_{t_3}$, $[Y_1, Y_3] = -\partial_{t_2}$, and $[Y_2, Y_3] = \partial_{t_1}$. If we denote by d the distance generated by Y_1, Y_2, Y_3 and by Cut_0 the cut locus of the origin of the distance d, then it turns out that

$$\text{Cut}_0 = \{(x,t) \in \mathbb{R}^3 \times \mathbb{R}^3 : t \neq 0 \quad \text{and } x = \lambda t \text{ for some } \lambda \in \mathbb{R}\}.$$

Observe also the invariance property $d(Mx, My \times Mz) = d(x, y \times z)$ for all $x, y, z \in \mathbb{R}^3$ and $M \in O(3)$.

Denote by $q = (x,t) \in \mathbb{R}^6$ and $p = (\xi, \tau) \in \mathbb{R}^6$ coordinates in $T^*\mathbb{R}^6$ and define the subRiemannian Hamiltonian

$$H(q,p) := \frac{1}{2} \sum_{j=1}^3 \langle p, Y_j(q) \rangle^2 = \frac{1}{2} \sum_j \langle (\xi, \tau), Y_j(x,t) \rangle^2 =: \frac{1}{2} \sum_j u_j^2(x,t,\xi,\tau). \tag{5.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{R}^6$.

Recall the following definition of subRiemannian exponential (see [ABB16]. Let $p \in T^*_{(0.0)}$. Denote by $s \mapsto (q(s, p_0), p(s, p_0))$ the solution of the Hamiltonian system associated to (5.2) with initial data $q(0) = (0,0)$ and $p(0) = p_0 = (\xi_0, \tau_0)$. Define

$$\mathcal{E}(p_0) = \mathcal{E}(\xi_0, \tau_0) = q(1, p_0). \tag{5.3}$$

Note that in our case solutions are defined globally in time for all initial datum p_0. Furthermore, by the property $(q(s, \alpha p_0), p(s, \alpha p_0)) = (q(\alpha s, p_0), \alpha p(\alpha s, p_0))$, we have $\mathcal{E}(sp) = q(s, p)$ for all p, s. The curve $s \mapsto q(s, p) = q(s, (\xi, \tau))$ is minimizing on some interval $[0, T_{\text{cut}}(\xi, \tau)]$, where

$$T_{\text{cut}}(\xi, \tau) = \text{the cut time of the path } s \mapsto q(s, (\xi, \tau)). \tag{5.4}$$

As usual in this setting, instead of $(x_j, t_j, \xi_j, \tau_j)$ we may use coordinates (x_j, t_j, u_j, τ_j), where

$$u_j(x,t,\xi,\tau) = \langle Y_j(x,t), (\xi,\tau) \rangle = \xi_j + \frac{1}{2} \langle \tau, x \times e_j \rangle = \xi_j + \sum_{\sigma \in S_3} \sigma(j,k,\ell) \tau_k x_\ell,$$

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where $\sigma(jk\ell)$ denotes the sign of the permutation. Hamilton's equation, see [BBG12], take the form

$$\begin{cases} \dot{u}_j = \sum_k \langle (\xi, \tau), [Y_k, Y_j](x,t) \rangle u_k = (\tau \times u)_j \\ \dot{\tau}_j = 0 \\ \dot{x} = u \\ \dot{t} = \dfrac{1}{2} x \times u \end{cases}$$

with initial condition $x(0) = 0$, $t(0) = 0$, $u(0) = \xi(0) = u$ and $\tau(0) = \tau$.

By the Rodrigues' rotation formula we get that if we consider the skew-symmetric map

$$\mathbb{R}^3 \ni x \mapsto \tau \times x = \begin{bmatrix} 0 & -\tau_3 & \tau_2 \\ \tau_3 & 0 & -\tau_1 \\ -\tau_2 & \tau_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =: A_\tau x$$

then we have

$$e^{sA_\tau} u = a \cos(\lambda s) + b \sin(\lambda s) + z, \tag{5.5}$$

where

$$\lambda = |\tau|, \qquad a = -\frac{\tau}{|\tau|} \times \left(\frac{\tau}{|\tau|} \times u \right) = u - \left\langle u, \frac{\tau}{|\tau|} \right\rangle \frac{\tau}{|\tau|}$$
$$b = \frac{\tau}{|\tau|} \times u, \qquad z = \left\langle u, \frac{\tau}{|\tau|} \right\rangle \frac{\tau}{|\tau|}. \tag{5.6}$$

(These relations can be obtained easily with a purely analitic argument comparing derivatives of order $0, 1, 2$ at $s = 0$ of the two sides of (5.5)).

Thus we have proved the following facts.

Proposition 5.1. *Let $s \mapsto \mathcal{E}(s(\xi, \tau)) = \mathcal{E}(s\xi, s\tau)$ be the (projection of the) solution of the Hamiltonian associated with (5.2) with $(x(0), t(0)) = (0,0)$ and $(\xi(0), \tau(0)) = (\xi, \tau) = (\xi_1, \xi_2, \xi_3, \tau_1, \tau_2, \tau_3)$. Then the corresponding control $u(s)$ has the form*

$$u(s) = \left(\xi - \left\langle \xi, \frac{\tau}{|\tau|} \right\rangle \frac{\tau}{|\tau|} \right) \cos(|\tau|s) + \left(\frac{\tau}{|\tau|} \times \xi \right) \sin(|\tau|s) + \left\langle \xi, \frac{\tau}{|\tau|} \right\rangle \frac{\tau}{|\tau|}. \tag{5.7}$$

In particular

$$T_{\text{cut}}(\xi, \tau) = \frac{2}{|\tau|} h_{\text{cut}} \left(\frac{|\langle \xi, \tau \rangle|}{|\tau \times \xi|} \right), \tag{5.8}$$

and the function T_{cut} is C^∞ smooth on

$$\Omega := \{(\xi, \tau) : \tau \times \xi \neq 0\}. \tag{5.9}$$

Remark 5.2. *The function T_{cut} is singular if ξ is parallel to τ. Namely, $T_{\text{cut}}(\xi, \tau) \in [\frac{2\pi}{|\tau|}, \frac{2\varphi_1}{|\tau|}[$ for all $(\xi, \tau) \in \Omega$, while $T_{\text{cut}}(\mu\tau, \tau) = +\infty$ for all $\tau \neq 0$ and $\mu \in \mathbb{R} \setminus \{0\}$. The corresponding final points $\mathcal{E}(\mu\tau, \tau) = (\mu\tau, 0)$ belong to the abnormal set Abn_0.*

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6. Corner-like estimates at cut points

In this section we show that at any point of the cut locus we can construct a two dimensional C^1-smooth surface such that on such surface the distance has a corner-like singularity. Our estimates give also an affirmative answer in our model to a question raised by Figalli and Rifford [FR10] on whether the distance fails to be semiconvex at the cut locus.

To show our construction, we work in the model used in the previous section, with the group law (5.1) and we consider a point $(\overline{x}, \overline{t}) \in \mathrm{Cut}_0$. Then we may write

$$(\overline{x}, \overline{t}) = (S_\varphi \beta + \zeta, \alpha \times (U_\varphi \beta + V_\varphi \zeta)),$$

where the triple α, β, ζ is admissible and belonging to the cut locus means that $\varphi \in [\pi, \varphi_1[$ satisfies $-\frac{U_\varphi S_\varphi}{V_\varphi} = \frac{|\zeta|^2}{|\beta|^2}$. In other words, $S_\varphi \beta + \zeta$ and $U_\varphi \beta + V_\varphi \zeta$ are perpendicular, which is equivalent to the fact that $\overline{x}$ and $\overline{t}$ are parallel. We may also write $\overline{t} = \overline{y} \times \overline{y}^\perp$, where $\overline{y}, \overline{y}^\perp \in \mathrm{span}\{\overline{x}\}^\perp$ are orthogonal and have the same length. Finally, recall the formula $d(\overline{x}, \overline{t})^2 = |\beta|^2 + |\zeta|^2 = |\alpha|^2 + |\zeta|^2$.

Let $\sigma_0 > 0$ and let x and $t :] -\sigma_0, \sigma_0[\to \mathbb{R}^3$ be smooth functions of the form

$$(x(\sigma), t(\sigma)) = \left(\overline{x} + \sigma(\mu_1 y + \mu_2 y^\perp) + \sigma^2 r(\sigma), \overline{t} + \sigma(\nu_1 y + \nu_2 y^\perp) + \sigma^2 \rho(\sigma) \right) \qquad (6.1)$$

where r and $\rho :] -\sigma_0, \sigma_0[\to \mathbb{R}^3$ are C^∞ vector-valued functions. We always assume that $(\mu_1, \mu_2) \neq (0,0)$. The vector (ν_1, ν_2) can possibly vanish. Let $R(\theta)$ be the rotation of an angle θ which leaves fixed $\overline{x}$ (we can fix the sign by requiring for example that for any vector $\xi \perp \overline{x}$, the vector $R(\theta)\xi \times (R'(\theta)\xi)$ points in the same direction of $\overline{x}$). Then define the parametrization

$$
\begin{aligned}
H(\sigma, \theta) &:= (R(\theta)x(\sigma), R(\theta)t(\sigma)) \\
&= \left(\overline{x} + \sigma R(\theta)\{\mu_1 y + \mu_2 y^\perp\} + \sigma^2 R(\theta)r(\sigma), \overline{t} + \sigma R(\theta)\{\nu_1 y + \nu_2 y^\perp\} + \sigma^2 R(\theta)\rho(\sigma) \right).
\end{aligned}
\qquad (6.2)
$$

Since $(\mu_1, \mu_2) \neq (0,0)$, for small $\sigma_0 > 0$, the set $\Gamma = \{H(\sigma, \theta) : 0 \leq \sigma < \sigma_0, \text{ and } 0 \leq \theta < 2\pi\}$ is a C^1-smooth two dimensional surface containing $(\overline{x}, \overline{t})$. The expansion (6.1) shows that $T_{(\overline{x}, \overline{t})}\Gamma = \mathrm{span}\{(\mu_1 y + \mu_2 y^\perp, \nu_1 y + \nu_2 y^\perp), (-\mu_2 y + \mu_1 y^\perp, -\nu_2 y + \nu_1 y^\perp)\}$. Furthermore the distance from the origin of points on Γ enjoys the rotational invariance property $d(H(\sigma, \theta_1)) = d(H(\sigma, \theta_2))$ for all θ_1, θ_2 and σ.

We will show that

Theorem 6.1 (Corner-like estimate for the distance at cut points). *For any point $(\overline{x}, \overline{t}) = (S_\varphi \beta + \zeta, \alpha \times (U_\varphi \beta + V_\varphi \zeta)) \in \mathrm{Cut}_0$ there is $\sigma_0 > 0$, there are smooth functions x, t as in (6.1) with $(\mu_1, \mu_2) \neq (0,0)$ and there is $C > 0$ such that*

$$d(H(\sigma, \theta)) \leq d(\overline{x}, \overline{t}) - C\sigma, \quad \text{for all } \sigma \in [0, \sigma_0[\text{ and } \theta \in [0, 2\pi[. \qquad (6.3)$$

This gives a generalization of the well known estimates at cut points $(0, t) \neq (0,0) \in \mathbb{C} \times \mathbb{R}$ of the Heisenberg group $\mathbb{H}^1$, namely $d(z, t) \leq d(0, t) - C|z|$, for $|z|$ small. One can recover a similar estimate as a limiting case of Theorem 6.1, as $\varphi = \pi$. See Remark 6.3.

As a corollary we get an answer in this setting to the question raised by Figalli and Rifford (see the *Open problem* at p 145-146 in [FR10]).

Corollary 6.2 (Failure of semiconvexity at cut points). *At any cut point $(\overline{x}, \overline{t}) \in \text{Cut}_0$, for any neighborhood Ω of $(\overline{x}, \overline{t})$, we have*

$$\inf_{p,q,(p+q)/2 \in \Omega} \frac{d(p) + d(q) - 2d((p+q)/2)}{|p-q|^2} = -\infty. \tag{6.4}$$

Strictly speaking, to get a complete analogous of the Riemannian estimate (2.6) in Proposition 2.5 of [?], our infimum should be calculated with $\frac{p+q}{2} = (\overline{x}, \overline{t})$. Corollary 6.2 will be proved at the end of the section.

We also remark that in [FR10] the authors define the cut locus Cut_0^{FR} as the closure of the set of points $(x, t) \neq (0, 0)$ where the distance is not continuously differentiable. This set is strictly larger than our set Cut_0. Namely, we have $\text{Cut}_0^{\text{FR}} = \text{Cut}_0 \cup \text{Abn}_0$, where the union is disjoint and Abn_0 is the set of final points of length minimizing abnormal curves and it is known that $\text{Abn}_0 = \{(x, 0) \in \mathbb{R}^3 \times \mathbb{R}^3\}$ (see [DMO$^+$16, MM16]). At points $(x, 0)$, in [MM16] we proved that

$$d(x, \sigma x) \geq d(x, 0) + C|\sigma|, \quad \text{for } \sigma \text{ close to } 0.$$

Roughly speaking, the presence of abnormals gives rise to the existence of directions where the Hessian of the distance is $+\infty$, i.e. semiconcavity fails. On the contrary, we are not aware of the existence of directions where the Hessian is $-\infty$ at abnormal points $(x, 0)$.

Proof of Theorem 6.1. Let $(\overline{x}, \overline{t}) = (S_\varphi \beta + \zeta, \alpha \times (U_\varphi \beta + V_\varphi \zeta)) \in \text{Cut}_0$. We first construct a smooth curve $\sigma \mapsto (x(\sigma), t(\sigma))$ for σ in a neighborhood of the origin defined as

$$(x(\sigma), t(\sigma)) = \Big(S_{\varphi-\sigma}(1-c_1\sigma)\beta + (1-c_2\sigma)\zeta, (1-c_1\sigma)\alpha \times \{U_{\varphi-\sigma}(1-c_1\sigma)\beta + V_{\varphi-\sigma}(1-c_2\sigma)\zeta\} \Big).$$

Observe that, given $\varphi \in [\pi, \varphi_1[$, the triple $\alpha(\sigma) = (1 - c_1\sigma)\alpha$, $\beta(\sigma) = (1 - c_1\sigma)\beta$ and $\zeta(\sigma) = (1 - c_2\sigma)\zeta$ is admissible for all σ close to 0. Therefore we have the upper estimate $d(x(\sigma), t(\sigma))^2 \leq (1 - c_1\sigma)^2|\alpha|^2 + (1 - c_2\sigma)^2|\zeta|^2$ (which, by our previous results, becomes an equality if and only if $-\frac{U_{\varphi-\sigma}S_{\varphi-\sigma}}{V_{\varphi-\sigma}} \leq \frac{(1-c_2\sigma)^2|\zeta|^2}{(1-c_1\sigma)^2|\alpha|^2}$).

Step 1. We show that there is a (actually unique) choice of c_1, c_2 such that $x'(0)$ and $t'(0) \in \text{span}\{y, y^\perp\}$. We start with $x'(0)$. A calculation shows that

$$\begin{aligned} x'(0) &= \frac{d}{d\sigma}\Big|_{\sigma=0} S_{\overline{\varphi}-\sigma}(1 - c_1\sigma)\beta + (1 - c_2\sigma)\zeta = -S'_\varphi\beta - c_1 S_\varphi\beta - c_2\zeta \\ &= (2V_\varphi - c_1 S_\varphi)\beta - c_2\zeta, \end{aligned}$$

by formula $S'_\varphi = -2V_\varphi$, where we write $S'_\varphi = \frac{dS}{d\varphi}$. We must require that $\langle x'(0), \overline{x} \rangle = 0$, where $\overline{x} = S_\varphi\beta + \zeta$. Thus

$$0 = \langle x'(0), \overline{x} \rangle = S_\varphi(2V_\varphi - c_1 S_\varphi)|\beta|^2 - c_2|\zeta|^2 = \frac{S_\varphi}{V_\varphi}|\beta|^2(2V_\varphi^2 - c_1 V_\varphi S_\varphi + c_2 U_\varphi),$$

where we used the identity $\frac{|\zeta|^2}{|\beta|^2} = -\frac{U_\varphi S_\varphi}{V_\varphi}$ which holds on the cut locus. Thus we have found a first linear equation in c_1, c_2.

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Next we calculate t'.

$$t'(0) = -c_1\alpha \times (U_\varphi\beta + V_\varphi\zeta) + \alpha \times [-(U_\varphi'\beta + c_1 U_\varphi\beta + V_\varphi'\zeta + V_\varphi c_2\zeta)]$$
$$= -\alpha \times [(2c_1 U_\varphi + U_\varphi')\beta + ((c_1 + c_2)V_\varphi + V_\varphi')\zeta].$$

Since $x = S_\varphi\beta + \zeta$ is orthogonal to $U_\varphi\beta + V_\varphi\zeta$ on the cut locus, condition $\langle t'(0), \overline{x}\rangle = 0$ is equivalent to $\langle (2c_1 U_\varphi + U_\varphi')\beta + ((c_1 + c_2)V_\varphi + V_\varphi')\zeta, U_\varphi\beta + V_\varphi\zeta\rangle = 0$. A calculation of the inner product gives

$$(2c_1 U_\varphi + U_\varphi')U_\varphi|\beta|^2 + ((c_1 + c_2)V_\varphi + V_\varphi')V_\varphi|\zeta|^2 = 0 \quad \text{i.e.}$$
$$(2U_\varphi - S_\varphi V_\varphi)c_1 - S_\varphi V_\varphi c_2 = S_\varphi V_\varphi' - U_\varphi',$$

which is again a linear equation in c_1, c_2. Ultimately we have the linear system

$$\begin{cases} V_\varphi S_\varphi c_1 - U_\varphi c_2 = 2V_\varphi^2 \\ (2U_\varphi - S_\varphi V_\varphi)c_1 - S_\varphi V_\varphi c_2 = S_\varphi V_\varphi' - U_\varphi'. \end{cases}$$

The solution has the form

$$c_1 = c_1(\varphi) = \frac{-2S_\varphi V_\varphi^3 - U_\varphi U_\varphi' + U_\varphi S_\varphi V_\varphi'}{2U_\varphi^2 - U_\varphi S_\varphi V_\varphi - V_\varphi^2 S_\varphi^2} \quad \text{and} \quad c_2 = c_2(\varphi) = \frac{V_\varphi}{U_\varphi}(S_\varphi c_1 - 2V_\varphi). \quad (6.5)$$

Note that $2U_\varphi^2 - U_\varphi S_\varphi V_\varphi - V_\varphi^2 S_\varphi^2 > 0$ for all $\varphi \in [\pi, \varphi_1[$, because $2U_\varphi^2 > 0$, $S_\varphi V_\varphi \leq 0$ and $U_\varphi + S_\varphi V_\varphi > 0$, as it can be easily seen using the very definition of U, V, S.

Observe that, if we write $x'(0)$ and $t'(0)$ in terms of c_1 only, it turns out that

$$x'(0) = \frac{2V_\varphi - c_1 S_\varphi}{U_\varphi}(U_\varphi\beta + V_\varphi\zeta) \quad \text{and} \quad t'(0) = -\frac{2c_1 U_\varphi + U_\varphi'}{S_\varphi}\alpha \times (S_\varphi\beta + \zeta),$$

and ultimately, since $U_\varphi\beta + V_\varphi\zeta$ and $S_\varphi\beta + \zeta$ are orthogonal, $x'(0)$ and $t'(0)$ are parallel.

Step 2. Concerning the curve constructed above, we show that there is $C > 0$ and $\sigma_0 > 0$ such that

$$d(x(\sigma), t(\sigma)) \leq d(\overline{x}, \overline{t}) - C\sigma \quad \text{for all } \sigma \in [0, \sigma_0[.$$

To prove such estimate, start from the upper estimate

$$d(x(\sigma), t(\sigma))^2 \leq (1 - c_1\sigma)^2|\alpha|^2 + (1 - c_2\sigma)^2|\zeta|^2 = |\alpha|^2 + |\zeta|^2 - 2\sigma(c_1|\alpha|^2 + c_2|\zeta|^2) + O(\sigma^2).$$

Thus, we must require that

$$c_1|\alpha|^2 + c_2|\zeta|^2 > 0, \quad (6.6)$$

where c_1, c_2 have been found in the previous step. This inequality is non trivial, because we do not know the sign of c_1 and c_2 (until Step 3 below). Inserting again the cut relation $\frac{|\zeta|^2}{|\alpha|^2} = -\frac{U_\varphi S_\varphi}{V_\varphi}$, we obtain $V_\varphi c_1 - U_\varphi S_\varphi c_2 > 0$. Inserting the expression of $c_2(\varphi)$ obtained in (6.5), we find

$$(1 - S_\varphi^2)c_1 + 2S_\varphi V_\varphi > 0 \qquad \text{which gives}$$
$$(1 - S_\varphi^2)\frac{-2S_\varphi V_\varphi^3 - U_\varphi U_\varphi' + U_\varphi S_\varphi V_\varphi'}{2U_\varphi^2 - U_\varphi S_\varphi V_\varphi - V_\varphi^2 S_\varphi^2} + 2S_\varphi V_\varphi > 0. \qquad (6.7)$$

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We check that the latter (strict) inequality holds on the whole interval $[\pi, \varphi_1[$. By formulas (3.20) for the derivatives U', V', we get

$$(1 - S_\varphi^2)\Big(-2S_\varphi V_\varphi^3 - \frac{\cos\varphi}{\varphi}U_\varphi V_\varphi + \frac{U_\varphi S_\varphi^2}{2} - \frac{2U_\varphi S_\varphi V_\varphi}{\varphi}\Big)$$
$$+ 2V_\varphi S_\varphi\Big(2U_\varphi^2 - U_\varphi V_\varphi S_\varphi - V_\varphi^2 S_\varphi^2\Big) > 0.$$

(Recall again that $-1 < S \leq 0$, $\cos\varphi < 0$, while U and V are positive on $[\pi, \varphi_1[$). We simplify the term with $V_\varphi^3 S_\varphi^3$ and we observe that

$$-2S_\varphi V_\varphi^3 + (1 - S_\varphi^2)\Big(-\frac{\cos\varphi}{\varphi}U_\varphi V_\varphi + \frac{U_\varphi S_\varphi^2}{2} - \frac{2U_\varphi S_\varphi V_\varphi}{\varphi}\Big) + 2V_\varphi S_\varphi\Big(2U_\varphi^2 - U_\varphi V_\varphi S_\varphi\Big)$$
$$\gneqq (1 - S_\varphi^2)\Big(-\frac{2U_\varphi S_\varphi V_\varphi}{\varphi}\Big) + 2V_\varphi S_\varphi\Big(2U_\varphi^2 - U_\varphi V_\varphi S_\varphi\Big)$$
$$= -2S_\varphi U_\varphi V_\varphi\Big(\frac{(1 - S_\varphi^2)}{\varphi} + V_\varphi S_\varphi - 2U_\varphi\Big),$$

where in the first inequality we deleted some positive terms (note that $-\frac{\cos\varphi}{\varphi}U_\varphi V_\varphi + \frac{U_\varphi S_\varphi^2}{2} > 0$ strictly on the interval $[\pi, \varphi_1[$). Next, using the definition of U, V, S, we show that the last parenthesis multiplied by φ^3 is positive.

$$\varphi^2\Big((1 - S_\varphi^2) + \varphi V_\varphi S_\varphi - 2\varphi U_\varphi\Big) = \varphi^2 - \sin^2\varphi + \frac{1}{2}\Big(\sin\varphi(\sin\varphi - \varphi\cos\varphi) - \varphi(\varphi - \sin\varphi\cos\varphi)\Big)$$
$$= \varphi^2 - \sin^2\varphi + \frac{1}{2}\Big(\sin^2\varphi - \varphi^2\Big) = \frac{1}{2}\Big(\varphi^2 - \sin^2\varphi\Big) > 0,$$

as required.

Step 3. We show that $x'(0) \neq 0$. From the first line of (6.7), which has been already proved in Step 2 and from the fact that $S_\varphi V_\varphi \leq 0$, we conclude that $c_1(\varphi) > 0$ (this and (6.5) tell us that $c_2 < 0$). Then, since $x'(0) = (2V_\varphi - c_1 S_\varphi)\beta - c_2\zeta$, this vector can never vanish, because the coefficient $(2V_\varphi - c_1(\varphi)S_\varphi)$ is strictly positive for all $\varphi \in [\pi, \varphi_1[$. $\qquad\square$

Remark 6.3. *If we choose $\varphi = \pi$ in the proof above, the condition $-\frac{u_\varphi s_\varphi}{v_\varphi} = \frac{|\zeta|^2}{|\beta|^2}$ shows that $\zeta = 0$ and therefore $(\overline{x}, \overline{t}) = (0, U_\pi \alpha \times \beta)$, a point on the "$t$-axis" of the (Heisenberg) subgroup $\operatorname{span}\{(\alpha, 0), (\beta, 0), (0, \alpha \times \beta)\}$. Since $\zeta = 0$, the constant c_2 plays no role, the condition on $x'(0) = 2V_\pi\beta = \frac{1}{\pi}\beta$ becomes empty and the condition on $t'(0) = -(2c_1 U_\pi + U'_\pi)\alpha \times \beta$ becomes $2c_1 U_\pi + U'_\pi = 0$, i.e. $c_1 = \frac{1}{\pi}$. Ultimately, the curve has the form*

$$(x(\sigma), t(\sigma)) = \Big(S_{\pi-\sigma}(1 - c_1\sigma)\beta, (1 - c_1\sigma)^2 U_{\pi-\sigma}\alpha \times \beta\Big)$$
$$= \Big(\frac{\sin\sigma}{\pi}\beta, \frac{1}{4\pi^2}(\pi - \sigma + \sin\sigma\cos\sigma)\alpha \times \beta\Big) = \Big(\frac{\beta}{\pi}\sigma + O(\sigma^2), \frac{1}{4\pi}(1 + O(\sigma^2))\alpha \times \beta\Big)$$

which gives the corner estimate on the two dimensional surface obtained by rotating the curve $\sigma \mapsto (x(\sigma), t(\sigma))$ around the vertical axis $\operatorname{span}\{\alpha \times \beta\}$ of the Heisenberg subgroup $\operatorname{span}\{(\alpha, 0), (\beta, 0), (0, \alpha \times \beta)\}$.

Proof of Corollary 6.2. Let $(\overline{x}, \overline{t}) = (S_\varphi \beta + \zeta, \alpha \times (U_\varphi \beta + V_\varphi \zeta)) \in \text{Cut}_0$ and let $\sigma \mapsto (x(\sigma), t(\sigma)) =: (x_\sigma, t_\sigma)$ be the curve constructed in the proof of the theorem above for $\sigma \geq 0$. Let $M \in O(3)$ be the rotation of 180 degrees leaving fixed $\overline{x}$. By invariance of the distance, we have $d(Mx_\sigma, Mt_\sigma)) = d(x_\sigma, t_\sigma)$. Denote by $(\xi_\sigma, \tau_\sigma) = \frac{1}{2}((Mx_\sigma, Mt_\sigma) + (x_\sigma, t_\sigma))$. Such point belongs to $\text{span}\{(\overline{x}, 0), (0, \overline{x})\}$ (i.e. to Cut_0) and has coordinates

$$(\xi_\sigma, \tau_\sigma) = \big(\overline{x}(1 + O(\sigma^2)), \overline{t}(1 + O(\sigma^2))\big),$$

as $\sigma \to 0+$. This follows from the property $\langle x'(0), \overline{x} \rangle = \langle t'(0), \overline{t} \rangle = 0$. By formula (1.6) for the distance on Cut_0, we have

$$d(\xi_\sigma, \tau_\sigma)^2 = |\xi_\sigma|^2 + R(\varphi_\sigma)|\tau_\sigma|, \quad \text{where } \varphi_\sigma = P^{-1}\Big(\frac{|\xi_\sigma|^2}{|\tau_\sigma|}\Big).$$

Since the function P^{-1} and R are smooth, we have $\varphi_\sigma = \varphi + O(\sigma^2)$ and $R(\varphi_\sigma) = R(\varphi) + O(\sigma^2)$, which implies $d(\xi_\sigma, \tau_\sigma) = d(\overline{x}, \overline{t}) + O(\sigma^2)$. Ultimately, we have

$$d(x_\sigma, t_\sigma) + d(Mx_\sigma, Mt_\sigma) - 2d\Big(\frac{(x_\sigma, t_\sigma) + (Mx_\sigma, Mt_\sigma)}{2}\Big) \leq -C\sigma + O(\sigma^2)$$

and the theorem is proved by letting $\sigma \to 0+$. $\qquad\square$

Acknowledgements

7.   References

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ANNAMARIA MONTANARI, DANIELE MORBIDELLI
DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI BOLOGNA (ITALY)
EMAIL: annamaria.montanari@unibo.it, daniele.morbidelli@unibo.it