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Gudrun Albrecht[1], Carolina Vittoria Beccari[2], Jean-Charles Canonne[3], Lucia Romani[4]

1. Escuela de Matematicas, Universidad Nacional de Colombia, Sede Medellin, Calle 59 A No 63-20, Medellin, Colombia.

2. Department of Mathematics, University of Bologna, P.zza Porta San Donato 5, 40127 Bologna, Italy.

3.Univ Lille Nord de France, UVHC, LAMAV, FR CNRS 2956, F-59313 Valenciennes, France.

4. Department of Mathematics and Applications, University of Milano-Bicocca, Via R. Cozzi 55, 20125 Milano, Italy.

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Gudrun Albrecht[a,*], Carolina Vittoria Beccari[b], Jean-Charles Canonne[c], Lucia Romani[d]

[a] *Escuela de Matemáticas, Universidad Nacional de Colombia, Sede Medellín, Calle 59 A No 63-20, Medellín, Colombia*
[b] *Department of Mathematics, University of Bologna, P.zza Porta San Donato 5, 40127 Bologna, Italy*
[c] *Univ Lille Nord de France, UVHC, LAMAV, FR CNRS 2956, F-59313 Valenciennes, France.*
[d] *Department of Mathematics and Applications, University of Milano-Bicocca, Via R. Cozzi 55, 20125 Milano, Italy*

Abstract

We introduce a new class of planar Pythagorean-Hodograph (PH) B–Spline curves. They can be seen as a generalization of the well-known class of planar Pythagorean-Hodograph (PH) Bézier curves, presented by R. Farouki and T. Sakkalis in 1990, including the latter ones as special cases. Pythagorean-Hodograph B–Spline curves are non-uniform parametric B–Spline curves whose arc-length is a B–Spline function as well. An important consequence of this special property is that the offsets of Pythagorean-Hodograph B–Spline curves are non-uniform rational B–Spline (NURBS) curves. Thus, although Pythagorean-Hodograph B–Spline curves have fewer degrees of freedom than general B–Spline curves of the same degree, they offer unique advantages for computer-aided design and manufacturing, robotics, motion control, path planning, computer graphics, animation, and related fields. After providing a general definition for this new class of planar parametric curves, we present useful formulae for their construction and discuss their remarkable attractive properties. Then we solve the reverse engineering problem consisting of determining the complex pre-image spline of a given PH B–Spline, and we also provide a method to determine within the set of all PH B–Splines the one that is closest to a given reference spline having the same degree and knot partition.

Keywords: Plane curve; Non-uniform B–Spline; Pythagorean-Hodograph; Arc-length; Offset; Reverse engineering

1. Introduction

The purpose of the present article is to introduce the general concept of Pythagorean–Hodograph (PH) B–Spline curves. On the one hand, B–Spline curves, since their introduction by Schoenberg [19] in 1946 have become the standard for curve representation in all areas where curve design is an issue, see, e.g., [4, 11, 12]. On the other hand, the concept of polynomial PH curves has widely been studied since its introduction by Farouki and Sakkalis in [10]. The essential characteristic of these curves is that the Euclidean norm of their hodograph is also polynomial, thus yielding the useful properties of admitting a closed–form polynomial representation of their arc–length as well as exact rational parameterizations of their offset curves. These polynomial curves are defined over the space of polynomials using its Bernstein basis thus yielding a control polygon or so–called Bézier representation for them. Rational and spatial counterparts of polynomial PH curves have as well been proposed, and most recently an algebraic–trigonometric counterpart, so–called Algebraic–Trigonometric Pythagorean–Hodograph (ATPH) curves have been introduced [18].

So far a general theory for *B–Spline curves* having the *PH property* is missing. To the best of the authors' knowledge the only partial attempt in this direction has been made in [7], where the problem of determining a B–Spline form of a C^2 PH quintic spline curve interpolating given points is addressed. Prior to this, based on [2, 8] in [17] a relation between a planar C^2 PH quintic spline curve and the control polygon of a related C^2 cubic B–Spline curve is presented.

The present article shows how to construct a general PH B–Spline curve of arbitrary degree, over an arbitrary knot sequence. To this end, we start by defining the complex variable model of a B–Spline curve $\mathbf{z}(t)$ of degree n, defined

*Corresponding author.
Email addresses: `galbrecht@unal.edu.co` (Gudrun Albrecht), `carolina.beccari2@unibo.it` (Carolina Vittoria Beccari), `jean-charles.canonne@univ-valenciennes.fr` (Jean-Charles Canonne), `lucia.romani@unimib.it` (Lucia Romani)

over a knot partition $\boldsymbol{\mu}$. We then square $\mathbf{z}(t)$ by using results for the product of normalized B–Spline basis functions from [3, 16]. Here, the determination of the required coefficients involves the solution of linear systems of equations. Finally, the result is integrated in order to obtain the general expression of the PH B–Spline curve, i.e., its B–Spline control points and its knot partition $\boldsymbol{\rho}$. General formulae are derived also for the parametric speed, the arc length and the offsets of the resulting curves. The interesting subclasses of *clamped* and *closed* PH B–spline curves are discussed in great detail.

The remainder of the paper is organized as follows. In section 2 we recall the basic definition of B–Spline curves as well as the Pythagorean Hodograph (PH) concept, thus defining the notion of a PH B–Spline curve. In section 3, the general construction of PH B–Spline curves is developed (section 3.1), and then adapted to the important particular cases of clamped and closed PH B–Spline curves (section 3.2). In section 4, general formulae for their parametric speed, arc length and offsets are given. Section 5 is devoted to accurately illustrate the subcase of clamped and closed PH B–Spline curves of degree 3 and 5. Finally, in section 6 we solve the reverse engineering problem consisting of determining the complex pre-image spline of a given PH B–Spline, and in section 7 we provide a method to determine within the set of all PH B–Splines the one that is closest to a given reference spline having the same degree and knot partition. Conclusions are drawn in section 8.

2. Preliminary notions and notation

While a Bézier curve is unequivocally identified by its degree, a B–Spline curve involves more information, namely an arbitrary number of control points, a knot vector and a degree, which are related by the formula *number of knots - number of control points = degree +1*. For readers not familiar with B–Spline curves, we first recall the definition of normalized B–Spline basis functions and successively the one of planar B–Spline curve (see, e.g., [12]).

Definition 1. *Let* $\boldsymbol{\mu} = \{t_i \in \mathbb{R} \mid t_i \leq t_{i+1}\}_{i=0,\ldots,m+n+1}$ *be a sequence of non-decreasing real numbers called* knots*, and let* $n, m \in \mathbb{N}$ *with* $m \geq n$*. The i-th* normalized B–Spline basis function *of degree n defined over the* knot partition $\boldsymbol{\mu}$ *is the function* $N^n_{i,\boldsymbol{\mu}}(t)$ *having support* $[t_i, t_{i+n+1}]$ *and for* $i = 0, \ldots, m$ *defined recursively as*

$$N^n_{i,\boldsymbol{\mu}}(t) = \frac{t - t_i}{t_{i+n} - t_i} N^{n-1}_{i,\boldsymbol{\mu}}(t) + \frac{t_{i+n+1} - t}{t_{i+n+1} - t_{i+1}} N^{n-1}_{i+1,\boldsymbol{\mu}}(t),$$

where

$$N^0_{i,\boldsymbol{\mu}}(t) = \begin{cases} 1, & \text{if } t \in [t_i, t_{i+1}) \\ 0, & \text{otherwise} \end{cases}$$

and "$\frac{0}{0} = 0$".

Definition 2. *Let* $m, n \in \mathbb{N}$ *with* $m \geq n$, $\boldsymbol{\mu} = \{t_i\}_{i=0,\ldots,m+n+1}$ *be a finite knot partition, and* $\mathbf{s}_0, \ldots, \mathbf{s}_m \in \mathbb{R}^2$*. Then, the planar parametric curve*

$$\mathbf{s}(t) = \sum_{i=0}^{m} \mathbf{s}_i\, N^n_{i,\boldsymbol{\mu}}(t)\,, \quad t \in [t_n, t_{m+1}],$$

is called a planar B–Spline curve *(of degree n associated with the knot partition* $\boldsymbol{\mu}$*) with de Boor points or control points* $\mathbf{s}_0, \ldots, \mathbf{s}_m$*.*

Remark 1. *If* M_i *denotes the multiplicity of the knot* t_i*, then* $\mathbf{s}(t)$ *is of continuity class* $C^{n-max_i(M_i)}(t_n, t_{m+1})$*.*

If the knot vector $\boldsymbol{\mu}$ does not have any particular structure, the B–Spline curve $\mathbf{s}(t)$ will not pass through the first and last control points neither will be tangent to the first and last legs of the control polygon. In this case $\mathbf{s}(t)$ is simply called *open* B–Spline curve. In order to clamp $\mathbf{s}(t)$ so that it is tangent to the first and the last legs at the first and last control points, respectively (as a Bézier curve does), the multiplicity of the first and the last knot must be adapted. For later use, the precise conditions we use for identifying a clamped B–Spline curve are the following.

Remark 2. *If* $t_0 = t_1 = \ldots = t_n$ *and* $t_{m+1} = \ldots = t_{m+n+1}$*, then the B–Spline curve* $\mathbf{s}(t)$ *given in Definition 2 satisfies*

$$\mathbf{s}(t_n) = \mathbf{s}_0, \quad \mathbf{s}(t_{m+1}) = \mathbf{s}_m$$

as well as

$$\mathbf{s}'(t_n) = \frac{n}{t_{n+1} - t_1}\,(\mathbf{s}_1 - \mathbf{s}_0), \quad \mathbf{s}'(t_{m+1}) = \frac{n}{t_{m+n} - t_m}\,(\mathbf{s}_m - \mathbf{s}_{m-1}),$$

i.e., $\mathbf{s}(t)$ *is a* clamped *B–Spline curve. Moreover, if all the knots* $t_{n+1}, \ldots, t_m$ *are simple, then* $\mathbf{s}(t) \in C^{n-1}(t_n, t_{m+1})$.

On the other hand, to make the B–Spline curve $\mathbf{s}(t)$ closed, some knot intervals and control points must be repeated such that the start and the end of the generated curve join together forming a closed loop. The precise conditions to be satisfied by knot intervals and control points in order to get a closed B–Spline curve are recalled in the following.

Remark 3. *If, in Definition 2 we replace* m *by* $m+n$*, consider the knot partition* $\boldsymbol{\mu} = \{t_i\}_{i=0,\ldots,m+2n+1}$ *with* $t_{m+1+k} - t_{m+k} = t_k - t_{k-1}$ *for* $k = 2, \ldots, 2n-1$*, assume* $\mathbf{s}_0, \ldots, \mathbf{s}_m \in \mathbb{R}^2$ *to be distinct control points and* $\mathbf{s}_{m+1} = \mathbf{s}_0, \ldots, \mathbf{s}_{m+n} = \mathbf{s}_{n-1}$*, then the B–Spline curve*

$$\mathbf{s}(t) = \sum_{i=0}^{m+n} \mathbf{s}_i N^n_{i,\boldsymbol{\mu}}(t), \quad t \in [t_n, t_{m+n+1}],$$

has the additional property

$$\mathbf{s}(t_n) = \mathbf{s}(t_{m+n+1}),$$

i.e., $\mathbf{s}(t)$ *is a* closed *B–Spline curve. Moreover, if all the knots* $t_n, \ldots, t_{m+n+1}$ *are simple, then* $\mathbf{s}(t) \in C^{n-1}[t_n, t_{m+n+1}]$.

At this point, we have all the required preliminary notions to generalize the definition of Pythagorean-Hodograph Bézier curves (see [10]) to *Pythagorean-Hodograph B–Spline curves*.

Definition 3. *For* $p, n \in \mathbb{N}, p \geq n$*, let* $u(t)$*,* $v(t)$ *and* $w(t)$ *be non-zero degree-*n *spline functions over the knot partition* $\boldsymbol{\mu} = \{t_i\}_{i=0,\ldots,p+n+1}$*, i.e., let*

$$u(t) = \sum_{i=0}^{p} u_i N^n_{i,\boldsymbol{\mu}}(t), \quad v(t) = \sum_{i=0}^{p} v_i N^n_{i,\boldsymbol{\mu}}(t), \quad w(t) = \sum_{i=0}^{p} w_i N^n_{i,\boldsymbol{\mu}}(t), \qquad t \in [t_n, t_{p+1}],$$

with $u_i, v_i, w_i \in \mathbb{R}$ *for all* $i = 0, \ldots, p$*, such that* $u(t)$ *and* $v(t)$ *are non-constant and do not have a non–constant spline function over the partition* $\boldsymbol{\mu}$ *as common factor. Then, the planar parametric curve* $(x(t), y(t))$ *whose coordinate components have first derivatives of the form*

$$x'(t) = w(t)\big(u^2(t) - v^2(t)\big) \qquad \text{and} \qquad y'(t) = 2w(t)u(t)v(t), \tag{1}$$

is called a planar Pythagorean-Hodograph B–Spline curve *or* a planar PH B–Spline curve *of degree* $2n+1$.

Indeed, as in the case of PH polynomial Bézier curves [5, 9], the parametric speed of the plane curve $(x(t), y(t))$ is given by

$$\sigma(t) := \sqrt{(x'(t))^2 + (y'(t))^2} = |w(t)|\,(u^2(t) + v^2(t)), \tag{2}$$

and its unit tangent, unit normal and (signed) curvature are given respectively by

$$\mathbf{t} = \frac{(u^2 - v^2, 2uv)}{u^2 + v^2}, \quad \mathbf{n} = \frac{(2uv, v^2 - u^2)}{u^2 + v^2}, \quad \kappa = \frac{2(uv' - u'v)}{w(u^2 + v^2)^2}, \tag{3}$$

where, for conciseness, in (3) the parameter t is omitted.

Remark 4. *The above defining property of PH B–Spline curves is inspired by the condition*

$$\sigma(t)^2 = (x'(t))^2 + (y'(t))^2\,. \tag{4}$$

As Kubota [14] proved in 1972, this equation has the unique solution given by (1) and (2) in a unique factorization domain. This fact has been exploited in the case of the ring of polynomials $\mathbb{R}[t]$*, which is a unique factorization domain, for constructing planar polynomial Bézier curves.*

The situation for B-Splines with respect to polynomials is different as the same set of polynomials (up to degree n) at the same time form a ring and a vector space. In the case of normalized B-Splines we have to distinguish between normalized B-Splines of fixed degree and over a fixed partition on the one hand, which form a vector space, and the set of normalized B-Splines of arbitrary degree and over arbitrary partition on the other hand, which form a ring. While the vector space of B-Splines has been widely studied we here consider the ring of normalized B-Splines of arbitrary degree and over arbitrary partition. The natural question now is whether this set of normalized B–Splines of arbitrary degree and over arbitrary partition also forms a unique factorization domain. A unique factorization domain is defined to be a commutative ring, which is an integral domain and in which every non-zero non-unit element can be written as a product of prime elements (or irreducible elements), uniquely up to order and units. An integral domain is a non-zero commutative ring, in which the product of any two non-zero elements is non-zero (see, e.g., [13]). The principal question thus is whether a non-zero non-unit normalized B–Spline allows a unique factorization. To this end we consider the following example. By introducing the abbreviations $h_0 = 2(-1 + 4\mathrm{i} - (\mathrm{i} + 1)\sqrt{-6 + 6\mathrm{i}})$, $h_1 = 3 - 3\sqrt{3\sqrt{2} + 3} + 3\sqrt{3\sqrt{2} - 3}$, $h_2 = 3\sqrt{12\sqrt{2} + 17 - 10\sqrt{3\sqrt{2} + 3} - 6\sqrt{3\sqrt{2} - 3}}$ *we consider the quadratic B–Spline curve*

$$\mathbf{p}(t) = \sum_{k=0}^{6} \mathbf{p}_k N_{k,\boldsymbol{\nu}}^2(t)$$

over the partition $\boldsymbol{\nu} = \{0, 0, 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, 1, 1, 1\}$ *with control points* $\mathbf{p}_0 = -2\mathrm{i}$, $\mathbf{p}_1 = -1 - \mathrm{i}$, $\mathbf{p}_2 = -1$, $\mathbf{p}_3 = -1 + \mathrm{i}$, $\mathbf{p}_4 = 2\mathrm{i}$, $\mathbf{p}_5 = -\sqrt{h_2 + h_1} + \sqrt{h_2 - h_1} + \mathrm{i}(\sqrt{h_2 - h_1} + \sqrt{h_2 + h_1})$, $\mathbf{p}_6 = -2h_1 + 2\mathrm{i}\sqrt{h_2 + h_1}\sqrt{h_2 - h_1}$.

With the tools we will develop in this article it is easy to show that

$$\mathbf{p}(t) = \mathbf{z}^2(t) = \mathbf{b}(t)\mathbf{c}(t)\,,$$

where

$$\mathbf{z}(t) = \sum_{i=0}^{3} \mathbf{z}_i N_{i,\boldsymbol{\tau}}^1(t)$$

over the partition $\boldsymbol{\tau} = \{0, 0, \frac{1}{4}, \frac{1}{2}, 1, 1\}$ *with control points* $\mathbf{z}_0 = -1 + \mathrm{i}$, $\mathbf{z}_1 = \mathrm{i}$, $\mathbf{z}_2 = 1 + \mathrm{i}$, $\mathbf{z}_3 = \sqrt{h_2 - h_1} + \mathrm{i}\sqrt{h_2 + h_1}$, *and*

$$\mathbf{b}(t) = \sum_{i=0}^{2} \mathbf{b}_i N_{i,\boldsymbol{\rho}_1}^1(t)$$

over the partition $\boldsymbol{\rho}_1 = \{0, 0, \frac{1}{4}, 1, 1\}$ *with (co-linear) control points* $\mathbf{b}_0 = -\frac{3h_0}{50}(7+\mathrm{i})$, $\mathbf{b}_1 = \frac{3h_0}{50}(-4+3\mathrm{i})$, $\mathbf{b}_2 = \frac{3h_0}{10}(1+3\mathrm{i})$, *as well as*

$$\mathbf{c}(t) = \sum_{i=0}^{2} \mathbf{c}_i N_{i,\boldsymbol{\rho}_2}^1(t)$$

over the partition $\boldsymbol{\rho}_2 = \{0, 0, \frac{1}{2}, 1, 1\}$ *with control points* $\mathbf{c}_0 = \frac{2(1+7\mathrm{i})}{3h_0}$, $\mathbf{c}_1 = \frac{2(7-\mathrm{i})}{3h_0}$, $\mathbf{c}_2 = 1 - 3\mathrm{i}$. *That means that non-zero non-unit normalized B–Splines do not allow for unique factorization. Consequently, over the set of normalized B–Splines equation (4) does not have a unique solution, but solution (1) is a valid solution of (4). The fact that there might be other solutions does not affect the results we are presenting in this article.*

In the following we will restrict our attention to the so-called *primitive* case $w(t) = 1$. Since in this case equation (2) simplifies as $\sigma(t) = u^2(t) + v^2(t)$, the primitive case coincides with the *regular* case. In this case, the representation (1) may be obtained by squaring the complex function $\mathbf{z}(t) = u(t) + \mathrm{i}v(t)$ yielding $\mathbf{z}^2(t) = u^2(t) - v^2(t) + \mathrm{i}2u(t)v(t)$. The coordinate components $x'(t), y'(t)$ of the hodograph $\mathbf{r}'(t)$ of the parametric curve $\mathbf{r}(t) = (x(t), y(t))$ are thus given by the real and imaginary part of $\mathbf{z}^2(t)$, respectively. In the remainder of the paper we will exclusively use this complex notation, and we will thus write

$$\mathbf{r}'(t) = x'(t) + \mathrm{i}y'(t) = u^2(t) - v^2(t) + \mathrm{i}2u(t)v(t) = \mathbf{z}^2(t)\,, \tag{5}$$

as also previously done for planar PH quintics [5, 9]. Since, by construction, $\mathbf{r}'(t)$ is a degree-$2n$ B–Spline curve, then the PH B–Spline curve $\mathbf{r}(t) = \int \mathbf{r}'(t)dt$ has degree $2n + 1$. In the next section we will construct the corresponding knot vector and thus know the continuity class of the resulting PH B–Spline curve $\mathbf{r}(t)$.

3. Construction of Pythagorean–Hodograph B–Spline curves

3.1. The general approach

We start with a knot partition of the form

$$\boldsymbol{\mu} = \{t_i\}_{i=0,\ldots,p+n+1} \tag{6}$$

over which a degree-n B–Spline curve

$$\mathbf{z}(t) = u(t) + \mathrm{i}v(t)$$

is defined for $t \in [t_n, t_{p+1}]$. Thus, according to Definition 2, the planar parametric curve $\mathbf{z}(t)$ can be written as

$$\mathbf{z}(t) = \sum_{i=0}^{p} \mathbf{z}_i N^n_{i,\boldsymbol{\mu}}(t)\,, \quad t \in [t_n, t_{p+1}]\,, \tag{7}$$

where $\mathbf{z}_i = u_i + \mathrm{i}v_i$, $i = 0, \ldots, p$. To express the product $\mathbf{z}^2(t)$ as a B–Spline curve, according to [3, 16], we have to augment the multiplicity of each single knot t_i to $n + 1$. We thus obtain the knot partition

$$\boldsymbol{\nu} = \{s_i\}_{i=0,\ldots,(p+n+2)(n+1)-1} = \{< t_i >^{n+1}\}_{i=0,\ldots,p+n+1}\,, \tag{8}$$

where $< t_i >^k$ denotes a knot t_i of multiplicity k. The product $\mathbf{z}^2(t)$ is thus a degree-$2n$ B–Spline curve over the knot partition $\boldsymbol{\nu}$, which can be written in the form

$$\mathbf{p}(t) = \mathbf{z}^2(t) = \sum_{i=0}^{p}\sum_{j=0}^{p} \mathbf{z}_i\mathbf{z}_j N^n_{i,\boldsymbol{\mu}}(t) N^n_{j,\boldsymbol{\mu}}(t) = \sum_{k=0}^{q} \mathbf{p}_k N^{2n}_{k,\boldsymbol{\nu}}(t)\,, \tag{9}$$

with $q = (n + 1)(p + n)$, according to Definition 2. Our goal is thus to obtain the explicit expressions of the coefficients $\mathbf{p}_k$, for $k = 0, \ldots, q$. To this end we set $f_{i,j}(t) := N^n_{i,\boldsymbol{\mu}}(t)N^n_{j,\boldsymbol{\mu}}(t)$ and look for the unknown coefficients $\boldsymbol{\chi}^{i,j} := (\chi^{i,j}_0, \chi^{i,j}_1, ..., \chi^{i,j}_q)^T$, $i, j = 0, ..., p$ such that

$$f_{i,j}(t) = \sum_{k=0}^{q} \chi^{i,j}_k N^{2n}_{k,\boldsymbol{\nu}}(t)\,.$$

For accomplishing this we apply the method from [3] as follows. Let $< \cdot, \cdot >$ be an inner product of the linear space of B–Splines of degree $2n$ with knot vector $\boldsymbol{\nu}$. According to [3], for any pair of functions $a(t)$, $b(t)$ defined over the interval $[t_0, t_{p+n+1}]$, we use $< a(t), b(t) >= \int_{t_0}^{t_{p+n+1}} a(t)b(t)\,dt$ to construct the $(q + 1) \times (q + 1)$ linear equation system

$$\mathbf{A}\boldsymbol{\chi}^{i,j} = \mathbf{b}^{i,j}, \tag{10}$$

with

$$\mathbf{A} = (a_{k,l})_{k,l=0,\ldots,q}, \quad a_{k,l} := \langle N^{2n}_{k,\boldsymbol{\nu}}, N^{2n}_{l,\boldsymbol{\nu}} \rangle = \int_{t_0}^{t_{p+n+1}} N^{2n}_{k,\boldsymbol{\nu}}(t)\, N^{2n}_{l,\boldsymbol{\nu}}(t)\, dt$$

and

$$\mathbf{b}^{i,j} = (b^{i,j}_l)_{l=0,\ldots,q}, \quad b^{i,j}_l := \langle f_{i,j}, N^{2n}_{l,\boldsymbol{\nu}} \rangle = \int_{t_0}^{t_{p+n+1}} f_{i,j}(t)\, N^{2n}_{l,\boldsymbol{\nu}}(t)\, dt.$$

Since $\{N^{2n}_{k,\boldsymbol{\nu}}(t)\}_{k=0,\ldots,q}$ are linearly independent, the matrix $\mathbf{A}$ is a Gramian and therefore nonsingular. This allows us to work out the unknown coefficients $\{\chi^{i,j}_k\}_{k=0,\ldots,q}$ by solving the linear system in (10).

Remark 5. *Since $\mathbf{b}^{i,j} = \mathbf{b}^{j,i}$ for all $i, j = 0, ..., p$, then $\boldsymbol{\chi}^{i,j} = \boldsymbol{\chi}^{j,i}$ for all $i, j = 0, ..., p$. Therefore, the unknown vectors to be obtained from* (10) *are indeed $\boldsymbol{\chi}^{i,j}$, $i = 0, ..., p$, $j = 0, ..., i$. Since $\mathbf{A}$ is a non-singular Gramian matrix, all the corresponding linear systems always have a unique solution. Moreover, since $a_{h,k} = 0$ if $|h - k| > 2n$, $\mathbf{A}$ is not only symmetric and positive definite, but also of band form. Thus, by applying the Cholesky decomposition algorithm one* *can compute the factorization* $\mathbf{A} = \mathbf{L}\mathbf{L}^T$*, where* $\mathbf{L}$ *is a lower triangular matrix of the same band form of* $\mathbf{A}$ *(i.e. such that* $l_{h,k} = 0$ *if* $h - k > 2n$*). The non-zero elements of* $\mathbf{L}$ *may be determined row by row by the formulas*

$$\begin{aligned} l_{h,k} &= \left(a_{h,k} - \textstyle\sum_{s=h-2n}^{h-1} l_{h,s}\, l_{k,s}\right)/l_{k,k}, \quad k = h-2n, ..., h-1 \\ l_{h,h} &= \left(a_{h,h} - \textstyle\sum_{s=h-2n}^{h-1} l_{h,s}^2\right)^{\frac{1}{2}}, \end{aligned}$$

with the convention that $l_{r,c} = 0$ *if* $c \le 0$ *or* $c > r$*. Hence, the solution of each linear system in* (10) *can be easily obtained by solving the two triangular linear systems* $\mathbf{L}\mathbf{y}^{i,j} = \mathbf{b}^{i,j}$ *and* $\mathbf{L}^T\boldsymbol{\chi}^{i,j} = \mathbf{y}^{i,j}$ *via the formulas*

$$\begin{aligned} y_h^{i,j} &= \left(b_h - \textstyle\sum_{k=h-2n}^{h-1} l_{h,k}\, y_k^{i,j}\right)/l_{h,h}, \qquad h = 0, ..., q \\ \chi_h^{i,j} &= \left(y_h^{i,j} - \textstyle\sum_{k=h+1}^{h+2n} l_{k,h}\, \chi_k^{i,j}\right)/l_{h,h}, \qquad h = 0, ..., q \end{aligned}$$

where a similar convention as above is adopted with respect to suffices outside the permitted ranges (see [15]).

From the computed expressions of $\chi_k^{i,j}$, $k = 0, ..., q$, $0 \le i, j \le p$, we thus get

$$\mathbf{p}(t) = \mathbf{z}^2(t) = \sum_{k=0}^{q}\sum_{i=0}^{p}\sum_{j=0}^{p} \chi_k^{i,j}\mathbf{z}_i\mathbf{z}_j N_{k,\boldsymbol{\nu}}^{2n}(t) \qquad \text{and} \qquad \mathbf{p}_k = \sum_{i=0}^{p}\sum_{j=0}^{p} \chi_k^{i,j}\mathbf{z}_i\mathbf{z}_j\,, \quad k = 0, \ldots, q. \tag{11}$$

The resulting PH B–Spline curve $\mathbf{r}(t)$ is now obtained by integrating $\mathbf{p}(t)$ as:

$$\mathbf{r}(t) = \int \mathbf{p}(t)dt = \sum_{i=0}^{q+1} \mathbf{r}_i N_{i,\boldsymbol{\rho}}^{2n+1}(t)\,, \quad t \in [t_n, t_{p+1}]\,, \tag{12}$$

where $\boldsymbol{\rho} = \{s'_i\}_{i=0,...,(p+n+2)(n+1)+1}$ with $s'_i = s_{i-1}$ for $i = 1, \ldots, (p+n+2)(n+1)$, $t_{-1} = s'_0 \le s'_1$ and $s'_{(p+n+2)(n+1)} \le s'_{(p+n+2)(n+1)+1} = t_{p+n+2}$, i.e.,

$$\boldsymbol{\rho} = \{t_{-1},\ \{< t_k >^{n+1}\}_{k=0,...,p+n+1},\ t_{p+n+2}\} \tag{13}$$

with the additional knots t_{-1}, t_{p+n+2}, as well as

$$\mathbf{r}_{i+1} = \mathbf{r}_i + \frac{s'_{i+2n+2} - s'_{i+1}}{2n+1}\,\mathbf{p}_i = \mathbf{r}_i + \frac{s_{i+2n+1} - s_i}{2n+1}\,\mathbf{p}_i\,, \tag{14}$$

for $i = 0, \ldots, q$ and arbitrary $\mathbf{r}_0$.

Remark 6. *Note that, by construction,* $s_{2n} = s'_{2n+1} = t_n$ *as well as* $s_{q+1} = s'_{q+2} = t_{p+1}$*, namely the B–Spline curves* $\mathbf{z}(t)$*,* $\mathbf{p}(t)$ *and* $\mathbf{r}(t)$ *are defined on the same domain. If the knot partition* $\boldsymbol{\mu}$ *contains simple inner knots* $t_{n+1}, \ldots, t_p$*, then the degree-n spline* $\mathbf{z}(t) \in C^{n-1}(t_n, t_{p+1})$*. As a consequence, the degree-*$2n$ *spline* $\mathbf{r}'(t) \in C^{n-1}(t_n, t_{p+1})$ *and the degree-*$(2n+1)$ *spline* $\mathbf{r}(t) \in C^n(t_n, t_{p+1})$*.*

3.2. Construction of clamped and closed PH B–Spline curves

We now consider the conditions for obtaining a clamped, respectively closed, PH B–Spline curve $\mathbf{r}(t)$.

Proposition 1. *Let* $\mathbf{r}(t)$ *be the PH B–Spline curve in (12) defined over the knot partition* $\boldsymbol{\rho}$ *in (13), where for a clamped, respectively, closed PH B–Spline curve* $\mathbf{r}(t)$ *we assume* $p = m$*, respectively,* $p = m + n$*.*

a) For $\mathbf{r}(t)$ *to be clamped, i.e., satisfying*

$$\begin{aligned} \mathbf{r}(t_n) = \mathbf{r}_0\,, \quad \mathbf{r}(t_{m+1}) = \mathbf{r}_{q+1}, \\ \mathbf{r}'(t_n) = \tfrac{2n+1}{s'_{2n+2}-s'_1}\,(\mathbf{r}_1 - \mathbf{r}_0), \quad \mathbf{r}'(t_{m+1}) = \tfrac{2n+1}{s'_{q+2n+2}-s'_{q+1}}\,(\mathbf{r}_{q+1} - \mathbf{r}_q), \end{aligned} \tag{15}$$

the following conditions have to be fulfilled

$$\sum_{k=0}^{n} \mathbf{r}_{(n-1)(n+1)+k+1}\, B_k^n(\alpha) = \mathbf{r}_0 \ \ \textit{and} \ \ \sum_{k=0}^{n-1} \mathbf{p}_{(n-1)(n+1)+k+1}\, B_k^{n-1}(\alpha) = \mathbf{p}_0 \ \ \textit{with} \ \ \alpha = \frac{t_n - t_{n-1}}{t_{n+1} - t_{n-1}}\,, \tag{16}$$

$$\sum_{k=0}^{n} \mathbf{r}_{m(n+1)+k+1}\, B_k^n(\beta) = \mathbf{r}_{q+1} \quad \textit{and} \quad \sum_{k=0}^{n-1} \mathbf{p}_{m(n+1)+k+1}\, B_k^{n-1}(\beta) = \mathbf{p}_q \quad \textit{with} \quad \beta = \frac{t_{m+1} - t_m}{t_{m+2} - t_m}\,, \tag{17}$$

where $B_k^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$*,* $k = 0, ..., n$ *denote the Bernstein polynomials of degree n.*

b) *For* $\mathbf{r}(t)$ *to be closed and of continuity class* C^n *at the junction point* $\mathbf{r}(t_n) = \mathbf{r}(t_{m+n+1})$*, we require the fulfillment of the conditions*

$$\sum_{j=n(n+1)-k}^{(m+n+1)(n+1)-k-1} (s_{j+2n+1} - s_j)\, \mathbf{p}_j = \mathbf{0}\,, \quad \textit{for} \quad k = 0, \ldots, n, \tag{18}$$

and

$$t_{m+1+k} - t_{m+k} = t_k - t_{k-1}\,, \quad \textit{for} \quad k = n, n+1. \tag{19}$$

Proof. According to [11] for every degree-n B-Spline curve $\mathbf{x}(u) = \sum_{i=0}^{q} \mathbf{x}_i N_{i,\boldsymbol{\mu}}^n(u) \in \mathbb{R}^d$ over the knot partition $\boldsymbol{\mu} = \{t_i\}_{i=0,\ldots,n+q+1}$ there exists a unique multi-affine, symmetric function or *blossom* $X : \mathbb{R}^n \longrightarrow \mathbb{R}^d, (u_1, \ldots, u_n) \mapsto X(u_1, \ldots, u_n)$ such that $\mathbf{x}(u) = X(u, \ldots, u) = X(< u >^n)$. Its control points are $\mathbf{x}_i = X(t_{i+1}, \ldots, t_{n+i})$, $i = 0, \ldots, q$. Thus, denoting $P(u_1, \ldots, u_{2n})$ the blossom of the curve $\mathbf{p}(u)$ from (9) over the knot partition $\boldsymbol{\nu}$ from (8), the control points $\mathbf{p}_i$ may be written as:

$$\begin{aligned}
\mathbf{p}_{k-1} &= P(< t_0 >^{n+1-k}, < t_1 >^{\min\{n-1+k,n+1\}}, < t_2 >^{\max\{k-2,0\}}), && \text{for } k = 1, \ldots, n,\\
\mathbf{p}_{j(n+1)+k-1} &= P(< t_j >^{n+1-k}, < t_{j+1} >^{\min\{n-1+k,n+1\}}, < t_{j+2} >^{\max\{k-2,0\}}), && \text{for } j = 1, \ldots, p+n-1 \text{ and } k = 0, \ldots, n,\\
\mathbf{p}_{(p+n)(n+1)+k-1} &= P(< t_{p+n} >^{n+1-k}, < t_{p+n+1} >^{\min\{n-1+k,n+1\}}), && \text{for } k = 0, 1.
\end{aligned}$$

Analogously, denoting by $R(u_1, \ldots, u_{2n+1})$ the blossom of the curve $\mathbf{r}(u)$ from (12) over the knot partition $\boldsymbol{\rho}$ from (13), the control points $\mathbf{r}_i$ may be written as

$$\begin{aligned}
\mathbf{r}_{j(n+1)+k} &= R(< t_j >^{n+1-k}, < t_{j+1} >^{\min\{n+k,n+1\}}, < t_{j+2} >^{\max\{k-1,0\}}), && \text{for } j = 0, \ldots, p+n-1 \text{ and } k = 0, \ldots, n,\\
\mathbf{r}_{(p+n)(n+1)+k} &= R(< t_{p+n} >^{n+1-k}, < t_{p+n+1} >^{\min\{n+k,n+1\}}), && \text{for } k = 0, 1.
\end{aligned}$$

Recalling de Boor's algorithm and the properties of blossoms, the control points involved for calculating a point $\mathbf{r}(t_j) = R(< t_j >^{2n+1})$ are the following:

$$R(< t_{j-1} >^n, < t_j >^{n+1}) = \mathbf{r}_{j(n+1)-n}, \ldots, R(< t_j >^{n+1}, < t_{j+1} >^n) = \mathbf{r}_{j(n+1)}. \tag{20}$$

a) We wish to obtain a clamped curve satisfying conditions (15). In order to satisfy the positional constraints we thus apply de Boor's algorithm for calculating $\mathbf{r}(t_n)$, respectively, $\mathbf{r}(t_{m+1})$ to the control points (20) for $j = n$, respectively, $j = m+1$. For $j = n$ we obtain

$$\begin{aligned}
&R(< t_{n-1} >^{n-l}, < t_n >^{n+1+k}, < t_{n+1} >^{l-k}) = (1-\alpha)\, R(< t_{n-1} >^{n+1-l}, < t_n >^{n+k}, < t_{n+1} >^{l-k})\\
&+\alpha\, R(< t_{n-1} >^{n-l}, < t_n >^{n+k}, < t_{n+1} >^{l+1-k}) \quad \text{for} \quad k, l = 1, \ldots, n.
\end{aligned} \tag{21}$$

This yields the following condition for α, which is independent of the indices k, l:

$$(1-\alpha)\, t_{n-1} + \alpha\, t_{n+1} = t_n.$$

De Boor's algorithm thus degenerates to de Casteljau's algorithm yielding the first equation of condition (16). The first equation of condition (17) is obtained analogously.

In order to satisfy the tangential constraints of (15), we first note that they are equivalent to the following positional constraints for $\mathbf{p}(t)$: $\mathbf{p}(t_n) = \mathbf{p}_0$, $\mathbf{p}(t_{m+1}) = \mathbf{p}_q$. We thus apply the same reasoning as above to $\mathbf{p}(t)$, and obtain the second equations in (16) and (17).

b) We wish to obtain a closed curve $\mathbf{r}(t)$ with

$$\mathbf{r}(t_n) = \mathbf{r}(t_{m+n+1})\,. \tag{22}$$

Recalling de Boor's algorithm and the properties of blossoms the control points involved for calculating a point $\mathbf{r}(t_j) = R(< t_j >^{2n+1})$ are the following:

$$R(< t_{j-1} >^n, < t_j >^{n+1}) = \mathbf{r}_{j(n+1)-n}, \ldots, R(< t_j >^{n+1}, < t_{j+1} >^n) = \mathbf{r}_{j(n+1)}$$

Condition (22) holds true if the following points and their corresponding knot intervals coincide:

$$\mathbf{r}_{n(n+1)-k} = \mathbf{r}_{(m+n+1)(n+1)-k}\,, \quad \text{for} \quad k = 0, \dots, n, \tag{23}$$

as well as

$$t_{m+1+k} - t_{m+k} = t_k - t_{k-1}\,, \quad \text{for} \quad k = n, n+1.$$

Setting $f_j = \frac{s_{j+2n+1} - s_j}{2n+1}$ for $j = 0, \dots, q$ and considering condition (14), condition (23) is equivalent to

$$\sum_{j=0}^{n(n+1)-k-1} f_j \mathbf{p}_j = \sum_{j=0}^{(m+n+1)(n+1)-k-1} f_j \mathbf{p}_j, \quad \text{for} \quad k = 0, \dots, n,$$

or equivalently

$$\sum_{j=n(n+1)-k}^{(m+n+1)(n+1)-k-1} f_j \mathbf{p}_j = \mathbf{0}, \quad \text{for} \quad k = 0, \dots, n.$$

These conditions also guarantee the maximum possible continuity class at the junction point.

□

In the clamped case of the above proposition we notice that if $t_{n-1} = t_n$ then $\alpha = 0$, which yields $\mathbf{r}(t_n) = \mathbf{r}_{n(n+1)-n}$ and $\mathbf{p}(t_n) = \mathbf{p}_{n(n+1)-n}$, and if $t_{m+1} = t_{m+2}$ then $\beta = 1$, which yields $\mathbf{r}(t_{m+1}) = \mathbf{r}_{(m+1)(n+1)}$ and $\mathbf{p}(t_{m+1}) = \mathbf{p}_{(m+1)(n+1)-1}$. If in the knot partition $\boldsymbol{\mu}$ from (6) we have $t_0 = \dots = t_n$ and $t_{m+1} = \dots = t_{m+n+1}$, i.e., if $\mathbf{z}(t)$ from (7) is a clamped curve itself, the first respectively last $(n-1)(n+1)+2$ control points of $\mathbf{r}(t)$ and $\mathbf{p}(t)$ coincide, i.e.,

$$\mathbf{r}_0 = \dots = \mathbf{r}_{n(n+1)-n} \text{ and } \mathbf{r}_{(m+1)(n+1)} = \dots = \mathbf{r}_{(m+n)(n+1)+1}\,, \tag{24}$$

as well as

$$\mathbf{p}_0 = \dots = \mathbf{p}_{n(n+1)-n} \text{ and } \mathbf{p}_{(m+1)(n+1)-1} = \dots = \mathbf{p}_{(m+n)(n+1)}\,.$$

In this case condition (15) from Proposition 1 a) is automatically satisfied yielding a more intuitive way of obtaining a clamped PH B–Spline curve. In order to simplify the notation we remove redundant knots in the knot partition $\boldsymbol{\nu}$ from (8) together with the control point multiplicities from (24) and summarize the result in the following Corollary.

Corollary 1. *Let* $\mathbf{z}(t) = \sum_{i=0}^{m} \mathbf{z}_i N_{i,\boldsymbol{\mu}}^{n}(t)\,,\ t \in [t_n, t_{m+1}]$ *be a clamped B–Spline curve over the knot partition*

$$\boldsymbol{\mu} = \{< t_n >^{n+1},\ \{t_i\}_{i=n+1,\dots,m},\ < t_{m+1} >^{n+1}\}, \tag{25}$$

as in Remark 2. Then,

$$\mathbf{p}(t) = \mathbf{z}^2(t) = \sum_{k=0}^{q} \mathbf{p}_k N_{k,\boldsymbol{\nu}}^{2n}(t)\,,$$

where $q = 2n + (n+1)(m-n)$ *and*

$$\boldsymbol{\nu} = \{s_i\}_{i=0,\dots,4n+1+(n+1)(m-n)} = \{< t_n >^{2n+1},\ \{< t_i >^{n+1}\}_{i=n+1,\dots,m},\ < t_{m+1} >^{2n+1}\}\,,$$

as well as

$$\mathbf{r}(t) = \int \mathbf{p}(t)dt = \sum_{i=0}^{q+1} \mathbf{r}_i N_{i,\boldsymbol{\rho}}^{2n+1}(t)\,, \quad t \in [t_n, t_{m+1}],$$

where $\boldsymbol{\rho} = \{s'_i\}_{i=0,\dots,4n+3+(n+1)(m-n)}$ *with* $s'_i = s_{i-1}$ *for* $i = 1, \dots, 4n+2+(n+1)(m-n)$*,* $s'_0 = s'_1$ *and* $s'_{4n+3+(n+1)(m-n)} = s'_{4n+2+(n+1)(m-n)}$*, i.e.,*

$$\boldsymbol{\rho} = \{< t_n >^{2n+2},\ \{< t_i >^{n+1}\}_{i=n+1,\dots,m},\ < t_{m+1} >^{2n+2}\}\,, \tag{26}$$

and the control points $\mathbf{r}_i$ *satisfy (14).*

Proof. The result is obtained by removing $(n-1)(n+1)+1$ of the multiple control points from (24) together with n^2 of the multiple knots of $\boldsymbol{\nu}$ from (8) at the beginning and at the end. The same result is obtained by proceeding with the general construction of the PH B–Spline curve starting with a clamped B–Spline curve over the knot partition $\boldsymbol{\mu}$ from (25). □

In this way, in both cases (the clamped and the closed one), the resulting PH B–Spline curve is of degree $2n+1$ and of continuity class C^n. For example, for $n=1$ we obtain PH B–Spline curves of degree 3 and continuity class C^1, while for $n=2$ we have PH B–Spline curves of degree 5 and continuity class C^2.

4. Parametric speed, arc length and offsets

According to (2), the parametric speed of the regular PH curve $\mathbf{r}(t) = x(t) + \mathrm{i}y(t)$ is given by

$$\sigma(t) = |\mathbf{r}'(t)| = |\mathbf{z}^2(t)| = \mathbf{z}(t)\,\bar{\mathbf{z}}(t)\,.$$

Exploiting (7) we thus obtain

$$\sigma(t) = \sum_{i=0}^{p}\sum_{j=0}^{p} \mathbf{z}_i \bar{\mathbf{z}}_j N_{i,\boldsymbol{\mu}}^{n}(t) N_{j,\boldsymbol{\mu}}^{n}(t) = \sum_{k=0}^{q} \sigma_k N_{k,\boldsymbol{\nu}}^{2n}(t)\,,$$

where

$$\sigma_k = \sum_{i=0}^{p}\sum_{j=0}^{p} \chi_k^{i,j} \mathbf{z}_i \bar{\mathbf{z}}_j \tag{27}$$

in analogy to (11).

4.1. Arc-length

The arc length of the PH B–Spline curve is thus obtained as

$$\int \sigma(t)dt = \sum_{i=0}^{q+1} l_i N_{i,\boldsymbol{\rho}}^{2n+1}(t)\,, \quad t \in [t_n, t_{p+1}]\,,$$

where

$$l_{i+1} = l_i + \frac{s'_{i+2n+2} - s'_{i+1}}{2n+1}\,\sigma_i = l_i + \frac{s_{i+2n+1} - s_i}{2n+1}\,\sigma_i\,, \tag{28}$$

with $l_0 = 0$. The cumulative arc length is given by

$$\ell(\xi) = \int_{t_n}^{\xi} \sigma(t)dt = \sum_{i=0}^{q+1} l_i \left(N_{i,\boldsymbol{\rho}}^{2n+1}(\xi) - N_{i,\boldsymbol{\rho}}^{2n+1}(t_n) \right),$$

and the curve's total arc length is thus

$$L = \ell(t_{p+1}) = \int_{t_n}^{t_{p+1}} \sigma(t)dt = \sum_{i=0}^{q+1} l_i \left(N_{i,\boldsymbol{\rho}}^{2n+1}(t_{p+1}) - N_{i,\boldsymbol{\rho}}^{2n+1}(t_n) \right)\,. \tag{29}$$

This general formula simplifies in the clamped and closed cases as follows. In the clamped case for the knot partition $\boldsymbol{\rho}$ from (26) we notice that

$$N_{i,\boldsymbol{\rho}}^{2n+1}(t_n) = \begin{cases} 1\,, & \text{if} \quad i = 0 \\ 0\,, & \text{else} \end{cases}$$

and

$$N_{i,\boldsymbol{\rho}}^{2n+1}(t_{m+1}) = \begin{cases} 1\,, & \text{if} \quad i = 2n+1+(n+1)(m-n) \\ 0\,, & \text{else.} \end{cases}$$

Considering $l_0 = 0$ and recalling Corollary 1, in the clamped case the total arc length L in (29) thus becomes

$$L = l_{2n+1+(n+1)(m-n)} \,. \tag{30}$$

Due to the structure of the knot partition $\boldsymbol{\rho}$ in (13), we notice that

$$N^{2n+1}_{i,\boldsymbol{\rho}}(t_n) \begin{cases} \neq 0\,, & \text{if} \quad (n-1)(n+1) < i < n(n+1)+1 \\ = 0\,, & \text{else} \end{cases}$$

and

$$N^{2n+1}_{i,\boldsymbol{\rho}}(t_{p+1}) \begin{cases} \neq 0\,, & \text{if} \quad p(n+1) < i < (p+1)(n+1)+1 \\ = 0\,, & \text{else.} \end{cases}$$

In this case the total arc length L from (29) thus becomes

$$\begin{aligned} L &= \sum_{i=p(n+1)+1}^{(p+1)(n+1)} l_i N^{2n+1}_{i,\boldsymbol{\rho}}(t_{p+1}) - \sum_{i=(n-1)(n+1)+1}^{n(n+1)} l_i N^{2n+1}_{i,\boldsymbol{\rho}}(t_n) \\ &= \sum_{k=0}^{n} \Big(l_{p(n+1)+1+k}\, N^{2n+1}_{p(n+1)+1+k,\boldsymbol{\rho}}(t_{p+1}) - l_{(n-1)(n+1)+1+k}\, N^{2n+1}_{(n-1)(n+1)+1+k,\boldsymbol{\rho}}(t_n) \Big) \,. \end{aligned} \tag{31}$$

In the case of a closed curve from Proposition 1 b) we have $p = m + n$ and conditions (19). On the knot partition $\boldsymbol{\rho}$ from (13) the normalized B–Spline basis functions having as support $[t_{n-1}, t_{n+1}]$ are $N^{2n+1}_{i,\boldsymbol{\rho}}(t)$ for $i = (n-1)(n+1) + 1, \ldots, n(n+1)$, and those having as support $[t_{m+n}, t_{m+n+2}]$ are $N^{2n+1}_{i,\boldsymbol{\rho}}(t)$ for $i = (m+n)(n+1)+1, \ldots, (m+n+1)(n+1)$. With conditions (19) this means

$$N^{2n+1}_{(n-1)(n+1)+1+k,\boldsymbol{\rho}}(t_n) = N^{2n+1}_{(m+n)(n+1)+1+k,\boldsymbol{\rho}}(t_{m+n+1})\,, \quad \text{for} \quad k = 0, \ldots, n \,.$$

In the case of a closed curve its total arc length L from (31) thus reads

$$L = \sum_{k=0}^{n} \Big(l_{p(n+1)+1+k} - l_{(n-1)(n+1)+1+k} \Big) N^{2n+1}_{(n-1)(n+1)+1+k,\boldsymbol{\rho}}(t_n)\,,$$

which, by taking into account (28), becomes

$$L = \sum_{k=0}^{n} \left(\sum_{j=(n-1)(n+1)+k+1}^{(m+n)(n+1)+k} \frac{s_{j+2n+1} - s_j}{2n+1}\, \sigma_j \right) N^{2n+1}_{(n-1)(n+1)+1+k,\boldsymbol{\rho}}(t_n) \,. \tag{32}$$

4.2. Offsets

The offset curve $\mathbf{r}_h(t)$ at (signed) distance h of a PH B–Spline curve $\mathbf{r}(t)$ is the locus defined by

$$\mathbf{r}_h(t) = \mathbf{r}(t) + h\,\mathbf{n}(t)$$

where

$$\mathbf{n}(t) = \frac{(y'(t), -x'(t))}{\sqrt{(x'(t))^2 + (y'(t))^2}} = \frac{-\mathrm{i}\mathbf{r}'(t)}{\sigma(t)} = \frac{-\mathrm{i}\mathbf{z}^2(t)}{\sigma(t)} \,.$$

(Note that, since we are dealing with the regular case, $\sigma(t) \neq 0$ and the offset curve is always well defined.) Thus

$$\mathbf{r}_h(t) = \frac{\sigma(t)\mathbf{r}(t) - \mathrm{i}\,h\,\mathbf{z}^2(t)}{\sigma(t)} \,.$$

Herein the product $\sigma(t)\mathbf{r}(t)$ reads as

$$\sigma(t)\mathbf{r}(t) = \sum_{i=0}^{q+1} \sum_{j=0}^{q} \sigma_j \mathbf{r}_i N^{2n+1}_{i,\boldsymbol{\rho}}(t) N^{2n}_{j,\boldsymbol{\nu}}(t) \,.$$

Again, according to [3, 16], we can write

$$N_{i,\boldsymbol{\rho}}^{2n+1}(t)N_{j,\boldsymbol{\nu}}^{2n}(t) = \sum_{k=0}^{w} \zeta_k^{i,j} N_{k,\boldsymbol{\tau}}^{4n+1}(t) \tag{33}$$

with the knot partition

$$\boldsymbol{\tau} = \{< t_{-1} >^{2n+1},\ \{< t_k >^{3n+2}\}_{k=0,\dots,p+n+1},\ < t_{p+n+2} >^{2n+1}\}$$

and

$$w = (3n+2)(p+n+2) - 1\,.$$

Remark 7. *In the clamped case, from Corollary 1 we obtain*

$$\boldsymbol{\tau} = \{< t_n >^{4n+2},\ \{< t_k >^{3n+2}\}_{k=n+1,\dots,m},\ < t_{m+1} >^{4n+2}\}$$

and

$$w = 4n + 1 + (m-n)(3n+2)\,.$$

Differently, in the closed case, we have

$$\boldsymbol{\tau} = \{< t_{-1} >^{2n+1},\ \{< t_k >^{3n+2}\}_{k=0,\dots,m+2n+1},\ < t_{m+2n+2} >^{2n+1}\}$$

and

$$w = (3n+2)(m+2n+2) - 1\,.$$

To work out the unknown coefficients $\boldsymbol{\zeta}^{i,j} := (\zeta_0^{i,j}, \zeta_1^{i,j}, \dots, \zeta_w^{i,j})^T$ in (33) we solve the linear system

$$\mathbf{C}\boldsymbol{\zeta}^{i,j} = \mathbf{e}^{i,j}, \tag{34}$$

with

$$\mathbf{C} = (c_{k,h})_{k,h=0,\dots,w}, \qquad c_{k,h} := \langle N_{k,\boldsymbol{\tau}}^{4n+1}, N_{h,\boldsymbol{\tau}}^{4n+1} \rangle = \int_{t_0}^{t_{p+n+1}} N_{k,\boldsymbol{\tau}}^{4n+1}(t)\, N_{h,\boldsymbol{\tau}}^{4n+1}(t)\, dt$$

and

$$\mathbf{e}^{i,j} = (e_h^{i,j})_{h=0,\dots,w}, \qquad e_h^{i,j} := \langle g_{i,j}, N_{h,\boldsymbol{\tau}}^{4n+1} \rangle = \int_{t_0}^{t_{p+n+1}} g_{i,j}(t)\, N_{h,\boldsymbol{\tau}}^{4n+1}(t)\, dt \quad \text{where} \quad g_{i,j}(t) := N_{i,\boldsymbol{\rho}}^{2n+1}(t)\, N_{j,\boldsymbol{\nu}}^{2n}(t).$$

Like in the previous case, $\mathbf{C}$ is a banded Gramian, and thus nonsingular. This guarantees that each of the linear systems in (34) has a unique solution that can be efficiently computed by means of the Cholesky decomposition algorithm for symmetric positive definite band matrices.
The computed expressions of $\zeta_k^{i,j}$, $k = 0, \dots, w$, $i = 0, \dots, q+1$, $j = 0, \dots, q$ thus yield

$$\sigma(t)\mathbf{r}(t) = \sum_{k=0}^{w} \sum_{i=0}^{q+1} \sum_{j=0}^{q} \zeta_k^{i,j} \sigma_j \mathbf{r}_i N_{k,\boldsymbol{\tau}}^{4n+1}(t)\,.$$

By writing

$$\mathbf{p}(t) = \mathbf{p}(t) \cdot 1 = \Big(\sum_{j=0}^{q} \mathbf{p}_j N_{j,\boldsymbol{\nu}}^{2n}(t) \Big) \Big(\sum_{i=0}^{q+1} N_{i,\boldsymbol{\rho}}^{2n+1}(t) \Big)$$

and

$$\sigma(t) = \sigma(t) \cdot 1 = \Big(\sum_{j=0}^{q} \sigma_j N_{j,\boldsymbol{\nu}}^{2n}(t) \Big) \Big(\sum_{i=0}^{q+1} N_{i,\boldsymbol{\rho}}^{2n+1}(t) \Big),$$

we thus obtain

$$\mathbf{p}(t) = \sum_{k=0}^{w} \sum_{i=0}^{q+1} \sum_{j=0}^{q} \zeta_k^{i,j} \mathbf{p}_j N_{k,\boldsymbol{\tau}}^{4n+1}(t),$$

$$\sigma(t) = \sum_{k=0}^{w} \sum_{i=0}^{q+1} \sum_{j=0}^{q} \zeta_k^{i,j} \sigma_j N_{k,\boldsymbol{\tau}}^{4n+1}(t)\,.$$

The offset curve $\mathbf{r}_h(t)$ finally has the form

$$\mathbf{r}_h(t) = \frac{\sum_{k=0}^{w} \gamma_k \tilde{\mathbf{q}}_k N_{k,\boldsymbol{\tau}}^{4n+1}(t)}{\sum_{k=0}^{w} \gamma_k N_{k,\boldsymbol{\tau}}^{4n+1}(t)}\,, \tag{35}$$

where, for $k = 0, \dots, w$,

$$\tilde{\mathbf{q}}_k = \frac{\mathbf{q}_k}{\gamma_k} \quad \text{with} \quad \mathbf{q}_k = \sum_{i=0}^{q+1} \sum_{j=0}^{q} (\sigma_j \mathbf{r}_i - \mathrm{i}\, h\, \mathbf{p}_j) \zeta_k^{i,j} \quad \text{and} \quad \gamma_k = \sum_{i=0}^{q+1} \sum_{j=0}^{q} \sigma_j \zeta_k^{i,j}\,.$$

5. Illustration of the cubic and quintic case

5.1. Clamped cubic and quintic PH B–Splines (cases $n = 1, 2$)

Let $n = 1$ and $m \in \mathbb{N}$, $m \geq 1$. For a general knot vector $\boldsymbol{\mu} = \{\langle 0 \rangle^2 < t_2 < \ldots < t_m < \langle t_{m+1} \rangle^2\}$ satisfying the constraints $t_0 = t_1 = 0$, $t_{m+1} = t_{m+2}$ and defining the knot intervals

$$d_0 = 0, \quad d_1 = t_2, \quad d_i = t_{i+1} - t_i,\ i = 2, ..., m, \quad d_{m+1} = 0,$$

we apply the above method to construct the knot partitions

$$\begin{aligned}
\boldsymbol{\nu} &= \{\langle 0 \rangle^3 < \langle t_2 \rangle^2 < ... < \langle t_m \rangle^2 < \langle t_{m+1} \rangle^3\},\\
\boldsymbol{\rho} &= \{\langle 0 \rangle^4 < \langle t_2 \rangle^2 < ... < \langle t_m \rangle^2 < \langle t_{m+1} \rangle^4\},\\
\boldsymbol{\tau} &= \{\langle 0 \rangle^6 < \langle t_2 \rangle^5 < ... < \langle t_m \rangle^5 < \langle t_{m+1} \rangle^6\}.
\end{aligned}$$

Then, by solving the linear systems (10) we calculate the coefficients $\chi_k^{i,j}$, $0 \leq i, j \leq m$, $0 \leq k \leq 2m$. All of them turn out to be zero with the exception of

$$\begin{aligned}
&\chi_{2k}^{k,k} = 1\,, \quad k = 0, \dots, m\,,\\
&\chi_{2k+1}^{k,k+1} = \chi_{2k+1}^{k+1,k} = \tfrac{1}{2}\,, \quad k = 0, \dots, m-1.
\end{aligned} \tag{36}$$

By means of the computed coefficients $\{\chi_k^{i,j}\}_{0 \leq k \leq 2m}^{0 \leq i,j \leq m}$ we can thus shortly write the control points of $\mathbf{r}'(t)$ as

$$\begin{aligned}
&\mathbf{p}_{2k} = \mathbf{z}_k^2, \quad k = 0, ..., m\,,\\
&\mathbf{p}_{2k+1} = \mathbf{z}_k \mathbf{z}_{k+1}, \quad k = 0, ..., m-1\,.
\end{aligned}$$

Thus, according to (12), the clamped cubic PH B–Spline curve defined over the knot partition $\boldsymbol{\rho}$ is given by

$$\mathbf{r}(t) = \sum_{i=0}^{2m+1} \mathbf{r}_i N_{i,\boldsymbol{\rho}}^3(t)\,, \quad t \in [t_1, t_{m+1}] \qquad (t_1 = 0),$$

with control points

$$\begin{aligned}
&\mathbf{r}_1 = \mathbf{r}_0 + \tfrac{d_1}{3}\, \mathbf{z}_0^2,\\
&\mathbf{r}_{2i+2} = \mathbf{r}_{2i+1} + \tfrac{d_{i+1}}{3}\, \mathbf{z}_i \mathbf{z}_{i+1}, \quad i = 0, ..., m-1\,,\\
&\mathbf{r}_{2i+3} = \mathbf{r}_{2i+2} + \tfrac{d_{i+1}+d_{i+2}}{3}\, \mathbf{z}_{i+1}^2, \quad i = 0, ..., m-2\,,\\
&\mathbf{r}_{2m+1} = \mathbf{r}_{2m} + \tfrac{d_m}{3}\, \mathbf{z}_m^2,
\end{aligned} \tag{37}$$

and arbitrary $\mathbf{r}_0$.

Remark 8. *Note that, when $m = 1$ and $t_2 = t_3 = 1$, the expressions of the control points in* (37) *coincide with those of Farouki's PH Bézier cubic from [5, 10].*

Clamped quintic PH B-Splines can be derived in a similar way by considering $n = 2$, $m \geq 2$ and a general knot vector $\boldsymbol{\mu} = \{\langle 0\rangle^3 < t_3 < \ldots < t_m < \langle t_{m+1}\rangle^3\}$ satisfying the constraints $t_0 = t_1 = t_2 = 0$, $t_{m+1} = t_{m+2} = t_{m+3}$ and defining the knot intervals

$$d_0 = 0, \quad d_1 = t_3, \quad d_i = t_{i+2} - t_{i+1},\ i = 2, ..., m-1, \quad d_m = 0.$$

After having calculated the solutions $\chi_k^{i,j}$, $0 \leq i, j \leq m$, $0 \leq k \leq 3m-2$ to the linear systems (10) and having computed the control points $\mathbf{p}_i$, $i = 0, ..., 3m-2$ of $\mathbf{r}'(t)$, the control points of the clamped quintic PH B–Spline curve

$$\mathbf{r}(t) = \sum_{i=0}^{3m-1} \mathbf{r}_i N^5_{i,\boldsymbol{\rho}}(t), \quad t \in [t_2, t_{m+1}] \qquad (t_2 = 0),$$

defined over the knot partition $\boldsymbol{\rho}$, assume the expressions

$$\begin{aligned}
&\mathbf{r}_1 = \mathbf{r}_0 + \tfrac{d_1}{5}\,\mathbf{z}_0^2,\\
&\mathbf{r}_2 = \mathbf{r}_1 + \tfrac{d_1}{5}\,\mathbf{z}_0\mathbf{z}_1,\\
&\mathbf{r}_{3i} = \mathbf{r}_{3i-1} + \tfrac{d_i}{5}\left(\tfrac{2}{3}\mathbf{z}_i^2 + \tfrac{1}{3}\left(\tfrac{d_i\mathbf{z}_{i-1}+d_{i-1}\mathbf{z}_i}{d_{i-1}+d_i}\right)\left(\tfrac{d_{i+1}\mathbf{z}_i+d_i\mathbf{z}_{i+1}}{d_i+d_{i+1}}\right)\right), \quad i = 1, ..., m-1\,,\\
&\mathbf{r}_{3i+1} = \mathbf{r}_{3i} + \tfrac{\mathbf{z}_i}{5}\,(d_{i+1}\mathbf{z}_i + d_i\mathbf{z}_{i+1})\,, \quad i = 1, ..., m-2\,,\\
&\mathbf{r}_{3i+2} = \mathbf{r}_{3i+1} + \tfrac{\mathbf{z}_{i+1}}{5}\,(d_{i+1}\mathbf{z}_i + d_i\mathbf{z}_{i+1})\,, \quad i = 1, ..., m-2\,,\\
&\mathbf{r}_{3m-2} = \mathbf{r}_{3m-3} + \tfrac{d_{m-1}}{5}\,\mathbf{z}_{m-1}\mathbf{z}_m,\\
&\mathbf{r}_{3m-1} = \mathbf{r}_{3m-2} + \tfrac{d_{m-1}}{5}\,\mathbf{z}_m^2\,,
\end{aligned} \tag{38}$$

for an arbitrary $\mathbf{r}_0$.

Remark 9. *Note that, when $m = 2$ and $t_3 = t_4 = t_5 = 1$, the expressions of the control points in* (38) *coincide with those of Farouki's PH Bézier quintic from [5, 10].*

For both cubic and quintic clamped PH B–Spline curves we can obtain from (27) the coefficients of the parametric speed $\sigma(t)$, and subsequently from (30) their total arc length.
Explicit formulae for the coefficients $\zeta_k^{i,j}$, $0 \leq i \leq q+1$, $0 \leq j \leq q$, $0 \leq k \leq w$ (with $q = 2m$, $w = 5m$ if $n = 1$ and $q = 3m-2$, $w = 8m-7$ if $n = 2$) are provided by the solutions to the linear systems (34) and can be exploited to work out from (35) the corresponding offset curve $\mathbf{r}_h(t)$, whose weights and control points can be found in [1].
Some examples of clamped cubic and quintic PH B–Spline curves and their offsets are shown in Figure 1.

5.2. *Closed cubic and quintic PH B–Splines (cases $n = 1, 2$)*

Let $n = 1$ and $m \in \mathbb{N}$, $m \geq 1$. For a general knot vector $\boldsymbol{\mu} = \{0 = t_0 < t_1 < ... < t_{m+3}\}$ characterized by the knot intervals $d_1 = t_1$, $d_i = t_i - t_{i-1}$, $i = 2, ..., m+3$, by applying the above method we construct the knot partitions

$$\begin{aligned}
&\boldsymbol{\nu} = \{\langle 0\rangle^2 < \langle t_1\rangle^2 < ... < \langle t_{m+3}\rangle^2\},\\
&\boldsymbol{\rho} = \{t_{-1} < \langle 0\rangle^2 < \langle t_1\rangle^2 < ... < \langle t_{m+3}\rangle^2 < t_{m+4}\},\\
&\boldsymbol{\tau} = \{\langle t_{-1}\rangle^3 < \langle 0\rangle^5 < \langle t_1\rangle^5 < ... < \langle t_{m+3}\rangle^5 < \langle t_{m+4}.\rangle^3\},
\end{aligned}$$

Then, by solving the linear systems (10) we calculate the coefficients $\chi_k^{i,j}$, $0 \leq i, j \leq m+1$, $0 \leq k \leq 2m+4$. All of them turn out to be zero with the exception of

$$\begin{aligned}
&\chi_{2k+1}^{k,k} = 1\,, \quad k = 0, \ldots, m+1\,,\\
&\chi_{2k+2}^{k,k+1} = \chi_{2k+2}^{k+1,k} = \tfrac{1}{2}\,, \quad k = 0, \ldots, m.
\end{aligned} \tag{39}$$

By means of the computed coefficients $\{\chi_k^{i,j}\}_{0\leq k\leq 2m+4}^{0\leq i,j\leq m+1}$ we can shortly write the control points of $\mathbf{r}'(t)$ as

$$\begin{aligned}
&\mathbf{p}_0 = 0,\\
&\mathbf{p}_{2k+1} = \mathbf{z}_k^2, \quad k = 0, ..., m+1\,,\\
&\mathbf{p}_{2k+2} = \mathbf{z}_k\mathbf{z}_{k+1}, \quad k = 0, ..., m\,,\\
&\mathbf{p}_{2m+4} = 0.
\end{aligned}$$

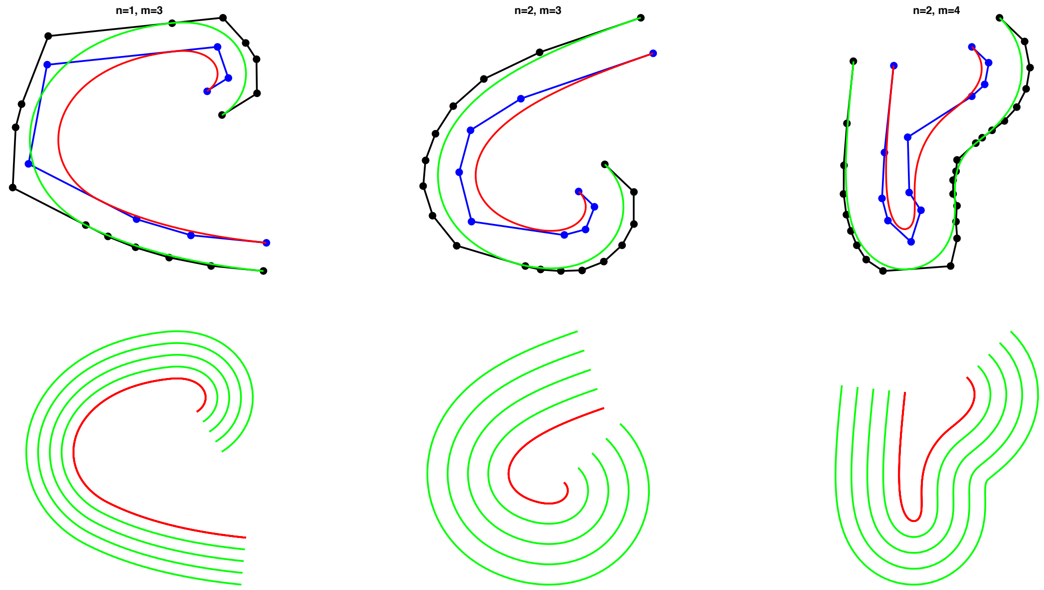

Figure 1: Offsets of clamped cubic and quintic PH B–Spline curves with (first row) and without (second row) control polygon: case $n = 1, m = 3$ (left column), case $n = 2, m = 3$ (center column), case $n = 2, m = 4$ (right column).

Thus, according to (12), the closed cubic PH B–Spline curve defined over the knot partition $\boldsymbol{\rho}$ is given by

$$\mathbf{r}(t) = \sum_{i=0}^{2m+5} \mathbf{r}_i N^3_{i,\boldsymbol{\rho}}(t), \quad t \in [t_1, t_{m+2}] \quad (t_0 = 0),$$

with control points

$$\begin{aligned}
\mathbf{r}_1 &= \mathbf{r}_0, \\
\mathbf{r}_{2i+2} &= \mathbf{r}_{2i+1} + \tfrac{d_{i+1}+d_{i+2}}{3}\, \mathbf{z}_i^2, \quad i = 0, ..., m+1\,, \\
\mathbf{r}_{2i+3} &= \mathbf{r}_{2i+2} + \tfrac{d_{i+2}}{3}\, \mathbf{z}_i \mathbf{z}_{i+1}, \quad i = 0, ..., m\,, \\
\mathbf{r}_{2m+5} &= \mathbf{r}_{2m+4}\,,
\end{aligned}$$

and arbitrary $\mathbf{r}_0$. Note that, due to condition (19), $d_{m+2} = d_1$, $d_{m+3} = d_2$. Moreover, $\mathbf{z}_m$ and $\mathbf{z}_{m+1}$ must be suitably fixed in order to satisfy condition (18).

Closed quintic PH B-Splines can be derived in a similar way by considering $n = 2$, $m \geq 2$. For a general knot vector $\boldsymbol{\mu} = \{0 = t_0 < t_1 < ... < t_{m+5}\}$ characterized by the knot intervals $d_1 = t_1$, $d_i = t_i - t_{i-1}$, $i = 2, ..., m+5$, by applying the above method we first construct the knot partitions

$$\begin{aligned}
\boldsymbol{\nu} &= \{\langle 0\rangle^3 < \langle t_1\rangle^3 < ... < \langle t_{m+5}\rangle^3\}, \\
\boldsymbol{\rho} &= \{t_{-1} < \langle 0\rangle^3 < \langle t_1\rangle^3 < ... < \langle t_{m+5}\rangle^3 < t_{m+6}\}, \\
\boldsymbol{\tau} &= \{\langle t_{-1}\rangle^5 < \langle 0\rangle^8 < \langle t_1\rangle^8 < ... < \langle t_{m+5}\rangle^8 < \langle t_{m+6}\rangle^5\}.
\end{aligned}$$

Then, by solving the linear systems (10) we calculate the coefficients $\chi_k^{i,j}$, $0 \leq i, j \leq m+2$, $0 \leq k \leq 3m+12$, and by means of the computed coefficients we obtain the control points $\mathbf{p}_i$, $i = 0, ..., 3m+12$ of $\mathbf{r}'(t)$. According to (12), the closed quintic PH B–Spline curve defined over the knot partition $\boldsymbol{\rho}$ is given by

$$\mathbf{r}(t) = \sum_{i=0}^{3m+13} \mathbf{r}_i N^5_{i,\boldsymbol{\rho}}(t)\,, \quad t \in [t_2, t_{m+3}] \quad (t_0 = 0)$$

with control points

$$
\begin{aligned}
\mathbf{r}_2 &= \mathbf{r}_1 = \mathbf{r}_0,\\
\mathbf{r}_3 &= \mathbf{r}_2 + \tfrac{d_1}{5}\,\mathbf{z}_0^2,\\
\mathbf{r}_{3i+1} &= \mathbf{r}_{3i} + \tfrac{d_{i+1}}{5}\left(\tfrac{2}{3}\mathbf{z}_{i-1}^2 + \tfrac{1}{3}\left(\tfrac{d_{i+1}\mathbf{z}_{i-2}+d_i\mathbf{z}_{i-1}}{d_i+d_{i+1}}\right)\left(\tfrac{d_{i+2}\mathbf{z}_{i-1}+d_{i+1}\mathbf{z}_i}{d_{i+1}+d_{i+2}}\right)\right), \quad i = 1, ..., m+3\,,\\
\mathbf{r}_{3i+2} &= \mathbf{r}_{3i+1} + \tfrac{\mathbf{z}_{i-1}}{5}\,(d_{i+2}\mathbf{z}_{i-1} + d_{i+1}\mathbf{z}_i), \quad i = 1, ..., m+2\,,\\
\mathbf{r}_{3i+3} &= \mathbf{r}_{3i+2} + \tfrac{\mathbf{z}_i}{5}\,(d_{i+2}\mathbf{z}_{i-1} + d_{i+1}\mathbf{z}_i), \quad i = 1, ..., m+2\,,\\
\mathbf{r}_{3m+11} &= \mathbf{r}_{3m+10} + \tfrac{d_{m+5}}{5}\,\mathbf{z}_{m+2}^2,\\
\mathbf{r}_{3m+13} &= \mathbf{r}_{3m+12} = \mathbf{r}_{3m+11}\,,
\end{aligned}
$$

and arbitrary $\mathbf{r}_0$. Note that, due to condition (19), $d_{m+3} = d_2$, $d_{m+4} = d_3$. Moreover, we use $\mathbf{z}_{-1} = \mathbf{z}_{m+3} = 0$, and $\mathbf{z}_m$, $\mathbf{z}_{m+1}$, $\mathbf{z}_{m+2}$ must be suitably fixed in order to satisfy condition (18).

For both cubic and quintic closed PH B–Spline curves we can obtain from (27) the coefficients of the parametric speed $\sigma(t)$, and subsequently from (32) their total arc length.

Explicit formulae for the coefficients $\zeta_k^{i,j}$, $0 \le i \le q+1$, $0 \le j \le q$, $0 \le k \le w$ (with $q = 2m+4$, $w = 5m+19$ if $n = 1$ and $q = 3m+12$, $w = 8m+47$ if $n = 2$) can be obtained by solving the linear systems (34) and exploited to work out from (35) the associated offset curve $\mathbf{r}_h(t)$, whose weights and control points can be found in [1].

We conclude this section by providing in Figure 2 some examples of closed cubic PH B–Spline curves obtained from the data in Table 1 where

$$
r_\pm = -(4d_1d_2 + 4d_1d_3 + 4d_2d_3 + 3d_1^2)\mathbf{z}_0^2 - (4d_1d_2 \pm 2d_1d_3 + 4d_2d_3)\mathbf{z}_0\mathbf{z}_1 - (4d_1d_2 + 4d_1d_3 + 4d_2d_3 + 3d_3^2)\mathbf{z}_1^2\,,
$$

and

$$
\begin{aligned}
R_\pm \;=\;& -(4d_1d_2 + 4d_1d_4 + 4d_2d_4 + 3d_1^2)\mathbf{z}_0^2 - (4d_1d_2 + 4d_2d_4)\mathbf{z}_0\mathbf{z}_1 \pm 2d_1d_4\mathbf{z}_0\mathbf{z}_2 - (4d_1d_2 + 4d_1d_3 + 4d_2d_4 + 4d_3d_4)\mathbf{z}_1^2\\
&- (4d_1d_3 + 4d_3d_4)\mathbf{z}_1\mathbf{z}_2 - (4d_1d_3 + 4d_1d_4 + 4d_3d_4 + 3d_4^2)\mathbf{z}_2^2\,.
\end{aligned}
$$

m	$\mathbf{z}(t)$ closed/open	Conditions on $\mathbf{z}(t)$ in order to satisfy (18)	Illustration
2	closed	$\mathbf{z}_2 = -\frac{d_1\mathbf{z}_0+d_3\mathbf{z}_1+\sqrt{r_-}}{2(d_1+d_3)}, \quad \mathbf{z}_3 = \mathbf{z}_0,$	Figure 2, first column
2	open	$\mathbf{z}_2 = \frac{d_1\mathbf{z}_0-d_3\mathbf{z}_1+\sqrt{r_+}}{2(d_1+d_3)}, \quad \mathbf{z}_3 = -\mathbf{z}_0,$	Figure 2, second column
3	closed	$\mathbf{z}_3 = -\frac{d_1\mathbf{z}_0+d_4\mathbf{z}_2+\sqrt{R_+}}{2(d_1+d_4)}, \quad \mathbf{z}_4 = \mathbf{z}_0$	Figure 2, third column
3	open	$\mathbf{z}_3 = \frac{d_1\mathbf{z}_0-d_4\mathbf{z}_2+\sqrt{R_-}}{2(d_1+d_4)}, \quad \mathbf{z}_4 = -\mathbf{z}_0$	Figure 2, fourth column

Table 1: Data for examples of closed cubic PH B–Spline curves.

Finally, examples of closed quintic PH B–Spline curves are shown together with their offsets in Figure 4.

6. Reverse engineering of PH B–Spline curves

The goal of this section is to determine the complex coefficients $\mathbf{z}_0, ..., \mathbf{z}_p$ defining the preimage $\mathbf{z}(t)$ of a PH B–Spline curve of degree-$(2n+1)$, from its control points $\mathbf{r}_0, ..., \mathbf{r}_{q+1}$. To this purpose, the PH B–Spline control points are regarded as complex values $\mathbf{r}_k = x_k + \mathrm{i}y_k$, $k = 0, ..., q+1$, and by recalling equation (14), the hodograph control points are written in the form

$$
\mathbf{p}_k = (2n+1)\,\frac{\mathbf{r}_{k+1} - \mathbf{r}_k}{s'_{k+2n+2} - s'_{k+1}} = \frac{2n+1}{s'_{k+2n+2} - s'_{k+1}}\Big((x_{k+1} - x_k) + \mathrm{i}(y_{k+1} - y_k)\Big), \quad k = 0, ..., q.
$$

6.1. Clamped PH B–Spline curves

In the clamped case, $p = m$ and $q = 2n + (n+1)(m-n)$. Thus, the complex coefficients defining the preimage $\mathbf{z}(t)$ are $\mathbf{z}_0, ..., \mathbf{z}_m$, and they have to be determined from the hodograph control points $\mathbf{p}_0, ..., \mathbf{p}_q$ with $q = 2n + (n+1)(m-n)$.

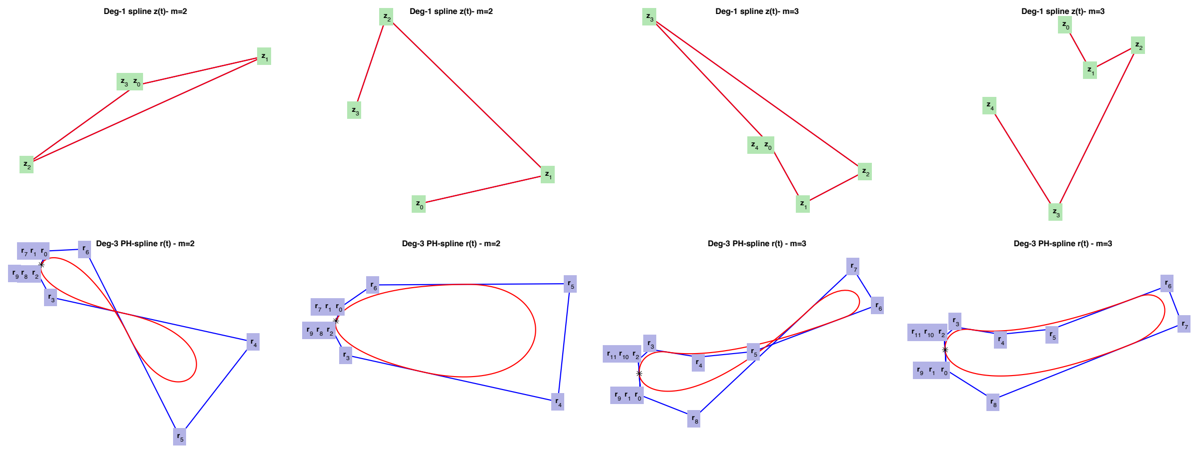

Figure 2: Closed cubic PH B–Spline curves with $m = 2$ (first/second column) and $m = 3$ (third/fourth column), respectively originated from closed/open degree-1 splines $\mathbf{z}(t)$.

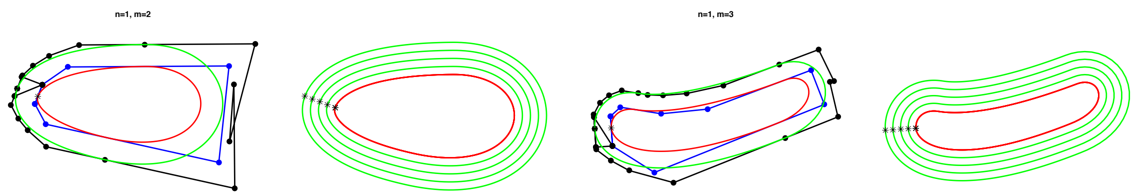

Figure 3: Offsets of the non self-intersecting cubic PH B–Spline curves displayed in Figure 2, with and without control polygon.

6.1.1. Clamped PH B–Spline curves of degree 3 ($n = 1$)

When $n = 1$, from the $2m + 1$ nonlinear equations

$$\begin{aligned} \mathbf{p}_{2k} &= \mathbf{z}_k^2, \quad k = 0, ..., m\,, \\ \mathbf{p}_{2k+1} &= \mathbf{z}_k \mathbf{z}_{k+1}, \quad k = 0, ..., m-1\,, \end{aligned}$$

the $m + 1$ unknowns $\mathbf{z}_0, ..., \mathbf{z}_m$ can be determined by the formulae

$$\begin{aligned} \mathbf{z}_k &= \pm\sqrt{\mathbf{p}_{2k}}, \quad k = 0 : 2 : m - mod(m, 2)\,, \\ \mathbf{z}_{k+1} &= \frac{\mathbf{p}_{2k+1}}{\pm\sqrt{\mathbf{p}_{2k}}}, \quad k = 0 : 2 : (m-1) - mod(m-1, 2)\,, \end{aligned} \tag{40}$$

where the hodograph control points are assumed to satisfy the m constraints

$$\mathbf{p}_{2k+1}^2 = \mathbf{p}_{2k}\mathbf{p}_{2k+2}, \quad k = 0, ..., m-1.$$

Note that, for $m = 1$, the formulae in (40) coincide with the results in [6, Section 4.1]. Figure 5 shows an example of application of formulae (40) in the case $m > 1$.

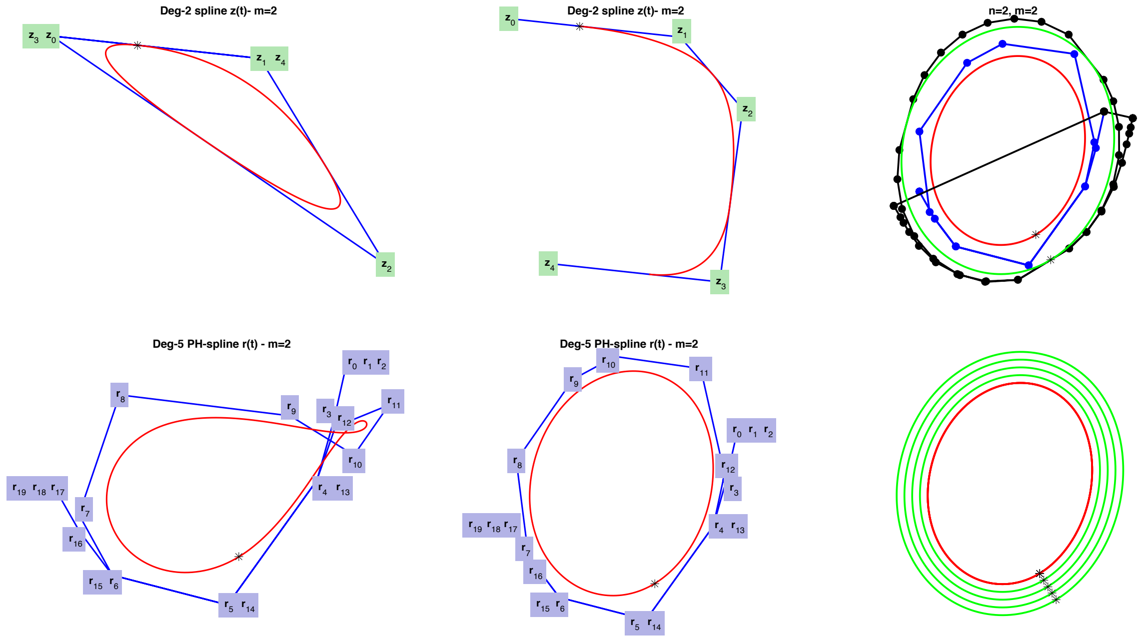

Figure 4: Closed quintic PH B–Spline curve with $m = 2$ originated from a closed/open degree-2 spline $\mathbf{z}(t)$ (left/center column) and offsets of the non self-intersecting curve displayed in center column, with and without control polygon (right column).

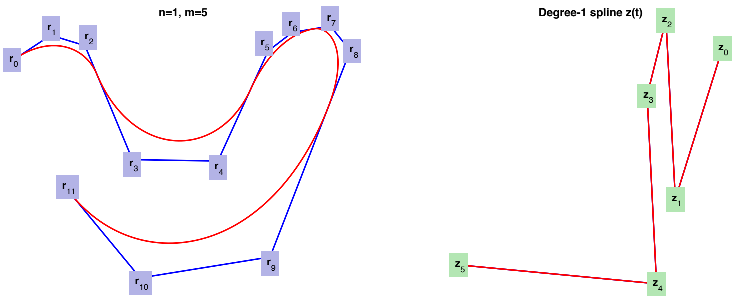

Figure 5: Reverse engineering of a clamped cubic PH B–Spline curve with $m = 5$: the given PH- Bspline $\mathbf{r}(t)$ (left) and one of the open degree-1 splines $\mathbf{z}(t)$ obtained from formulae (40) (right).

6.1.2. Clamped PH B–Spline curves of degree 5 ($n = 2$)

When $n = 2$, from the $3m - 1$ nonlinear equations

$$
\begin{aligned}
\mathbf{p}_0 &= \mathbf{z}_0^2, \\
\mathbf{p}_1 &= \mathbf{z}_0 \mathbf{z}_1, \\
\mathbf{p}_{3k-1} &= \frac{2}{3}\mathbf{z}_k^2 + \frac{1}{3}\left(\frac{d_k \mathbf{z}_{k-1} + d_{k-1}\mathbf{z}_k}{d_{k-1}+d_k}\right)\left(\frac{d_{k+1}\mathbf{z}_k + d_k \mathbf{z}_{k+1}}{d_k + d_{k+1}}\right), \quad k = 1, ..., m-1, \\
\mathbf{p}_{3k} &= \mathbf{z}_k \frac{d_{k+1}\mathbf{z}_k + d_k \mathbf{z}_{k+1}}{d_k + d_{k+1}}, \quad k = 1, ..., m-2, \\
\mathbf{p}_{3k+1} &= \mathbf{z}_{k+1} \frac{d_{k+1}\mathbf{z}_k + d_k \mathbf{z}_{k+1}}{d_k + d_{k+1}}, \quad k = 1, ..., m-2, \\
\mathbf{p}_{3m-3} &= \mathbf{z}_{m-1}\mathbf{z}_m, \\
\mathbf{p}_{3m-2} &= \mathbf{z}_m^2,
\end{aligned}
$$

the $m+1$ unknowns $\mathbf{z}_0, ..., \mathbf{z}_m$ can be determined by the formulae

$$\begin{aligned}
\mathbf{z}_0 &= \pm\sqrt{\mathbf{p}_0}, \\
\mathbf{z}_1 &= \frac{\mathbf{p}_1}{\pm\sqrt{\mathbf{p}_0}}, \\
\mathbf{z}_k &= \pm\mathbf{p}_{3k}\sqrt{\frac{d_k+d_{k+1}}{d_k\mathbf{p}_{3k+1}+d_{k+1}\mathbf{p}_{3k}}}, \quad k = 3:2:m-1-mod(m-1,2) \\
\mathbf{z}_{k+1} &= \pm\mathbf{p}_{3k+1}\sqrt{\frac{d_k+d_{k+1}}{d_k\mathbf{p}_{3k+1}+d_{k+1}\mathbf{p}_{3k}}}, \quad k = 1:2:m-2-mod(m-2,2) \\
\mathbf{z}_{m-1} &= \frac{\mathbf{p}_{3m-3}}{\pm\sqrt{\mathbf{p}_{3m-2}}}, \\
\mathbf{z}_m &= \pm\sqrt{\mathbf{p}_{3m-2}},
\end{aligned} \tag{41}$$

where the hodograph control points have now to satisfy more complicated constraints.
Note that, for $m = 2$, the formulae in (41) coincide with the results in [6, Section 4.1]. Figure 6 shows an example of application of formulae (41) in the case $m > 2$.

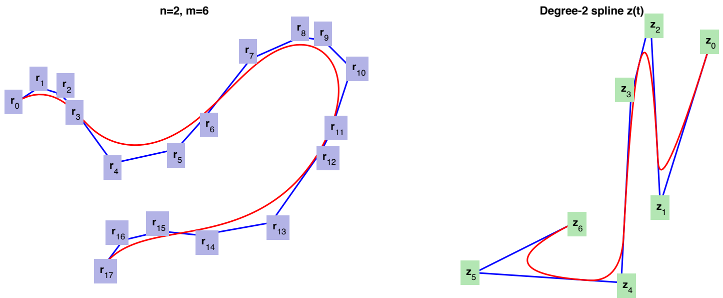

Figure 6: Reverse engineering of a clamped quintic PH B–Spline curve with $m = 6$: the given PH- Bspline $\mathbf{r}(t)$ (left) and one of the open degree-2 splines $\mathbf{z}(t)$ obtained from formulae (41) (right).

6.2. *Closed PH B–Spline curves*

In the closed case, $p = m + n$ and $q = (n + 1)(m + 2n)$. Thus, the complex coefficients defining the preimage $\mathbf{z}(t)$ are $\mathbf{z}_0, ..., \mathbf{z}_{m+n}$, and they have to be determined from the hodograph control points $\mathbf{p}_0, ..., \mathbf{p}_q$ with $q = (n + 1)(m + 2n)$.

6.2.1. *Closed PH B–Spline curves of degree 3 ($n = 1$)*

When $n = 1$, from the $2m + 3$ nonlinear equations

$$\begin{aligned}
\mathbf{p}_{2k+1} &= \mathbf{z}_k^2, \quad k = 0, ..., m+1\,, \\
\mathbf{p}_{2k+2} &= \mathbf{z}_k\mathbf{z}_{k+1}, \quad k = 0, ..., m\,,
\end{aligned}$$

the $m + 2$ unknowns $\mathbf{z}_0, ..., \mathbf{z}_{m+1}$ can be determined by the formulae

$$\begin{aligned}
\mathbf{z}_k &= \pm\sqrt{\mathbf{p}_{2k+1}}, \quad k = 0:2:(m+1)-mod(m+1,2)\,, \\
\mathbf{z}_{k+1} &= \frac{\mathbf{p}_{2k+2}}{\pm\sqrt{\mathbf{p}_{2k+1}}}, \quad k = 0:2:m-mod(m,2)\,,
\end{aligned} \tag{42}$$

where the hodograph control points are assumed to satisfy the $m + 1$ constraints

$$\mathbf{p}_{2k+2}^2 = \mathbf{p}_{2k+1}\mathbf{p}_{2k+3}, \quad k = 0, ..., m.$$

Figure 7 shows an example of application of formulae (42).

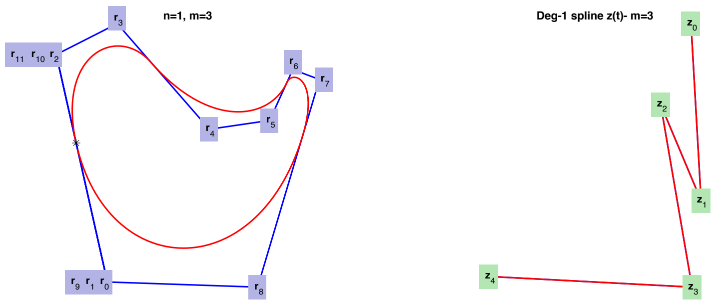

Figure 7: Reverse engineering of a closed cubic PH B–Spline curve with $m = 3$: the given PH- Bspline $\mathbf{r}(t)$ (left) and one of the open degree-1 splines $\mathbf{z}(t)$ obtained from formulae (42) (right).

6.2.2. Closed PH B–Spline curves of degree 5 (n = 2)

When $n = 2$, from the $3m + 9$ nonlinear equations

$$\begin{aligned}
\mathbf{p}_2 &= \tfrac{d_1}{d_1+d_2}\,\mathbf{z}_0^2,\\
\mathbf{p}_{3k} &= \tfrac{2}{3}\mathbf{z}_{k-1}^2 + \tfrac{1}{3}\left(\tfrac{d_{k+1}\mathbf{z}_{k-2}+d_k\mathbf{z}_{k-1}}{d_k+d_{k+1}}\right)\left(\tfrac{d_{k+2}\mathbf{z}_{k-1}+d_{k+1}\mathbf{z}_k}{d_{k+1}+d_{k+2}}\right), \qquad k = 1, ..., m+3\,,\\
\mathbf{p}_{3k+1} &= \mathbf{z}_{k-1}\,\tfrac{d_{k+2}\mathbf{z}_{k-1}+d_{k+1}\mathbf{z}_k}{d_{k+1}+d_{k+2}}, \qquad k = 1, ..., m+2\,,\\
\mathbf{p}_{3k+2} &= \mathbf{z}_k\,\tfrac{d_{k+2}\mathbf{z}_{k-1}+d_{k+1}\mathbf{z}_k}{d_{k+1}+d_{k+2}}, \qquad k = 1, ..., m+2\,,\\
\mathbf{p}_{3m+10} &= \tfrac{d_{m+5}}{d_{m+4}+d_{m+5}}\,\mathbf{z}_{m+2}^2,
\end{aligned}$$

the $m + 3$ unknowns $\mathbf{z}_0, ..., \mathbf{z}_{m+2}$ can be determined by the formulae

$$\begin{aligned}
\mathbf{z}_0 &= \pm\sqrt{\tfrac{d_1+d_2}{d_1}\mathbf{p}_2},\\
\mathbf{z}_{k-1} &= \pm\mathbf{p}_{3k+1}\sqrt{\tfrac{d_{k+1}+d_{k+2}}{d_{k+1}\mathbf{p}_{3k+2}+d_{k+2}\mathbf{p}_{3k+1}}}, \qquad k = 3:2:m+3-mod(m+3,2)\\
\mathbf{z}_k &= \pm\mathbf{p}_{3k+2}\sqrt{\tfrac{d_{k+1}+d_{k+2}}{d_{k+1}\mathbf{p}_{3k+2}+d_{k+2}\mathbf{p}_{3k+1}}}, \qquad k = 1:2:m+2-mod(m+2,2)\\
\mathbf{z}_{m+2} &= \pm\sqrt{\tfrac{d_{m+4}+d_{m+5}}{d_{m+5}}\mathbf{p}_{3m+10}},
\end{aligned} \tag{43}$$

where the hodograph control points have now to satisfy more complicated constraints. Figure 8 shows an example of application of formulae (43).

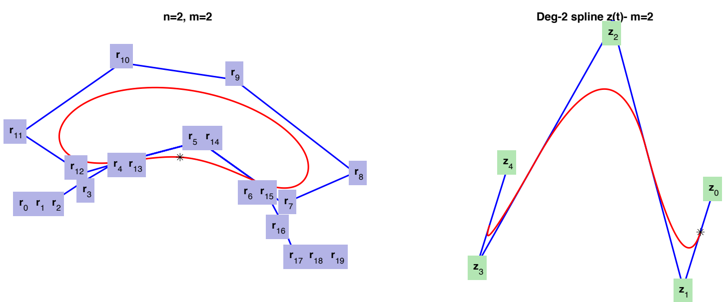

Figure 8: Reverse engineering of a closed quintic PH B–Spline curve with $m = 2$: the given PH- Bspline $\mathbf{r}(t)$ (left) and one of the open degree-2 splines $\mathbf{z}(t)$ obtained from formulae (43) (right).

7. Design of PH B–Splines

In this section we explore the problem of constructing a PH B–Spline curve starting from a user-defined control polygon. We will content ourselves to demonstrating by examples that this is possible, postponing to a future work a deeper investigation on the topic.

Suppose that a clamped or closed B–Spline curve of degree $2n+1$, $n \geq 1$ and continuity class C^n is given. We call this curve the *target B–Spline curve* or simply *target curve*. The degree and continuity are chosen compatibly with the definition of a PH B–Spline. Therefore both the target curve and the sought PH B-spline curve will have the same knot partition $\boldsymbol{\rho}$, defined as in (13). The target curve $\mathbf{r}^T(t)$ will be associated with a control polygon having vertices $\mathbf{r}_0^T, \ldots, \mathbf{r}_{q+1}^T$, with $q = (n+1)(p+n)$, where we take $p = m$ or $p = m+n$ in the clamped or closed case, respectively (see Sections 2 and 3). We refer to $\mathbf{r}_i^T$, $i = 0, \ldots, q+1$, as the *target control points*.

The leading principle is choosing the points $\mathbf{r}_i$ of the sought PH B-spline in such a way that their distance to the points $\mathbf{r}_i^T$ is minimized. To this aim, for clamped curves we minimize the functional

$$F(\mathbf{z}_0, \ldots, \mathbf{z}_p) = \sum_{i=1}^{q} \| \mathbf{r}_i - \mathbf{r}_i^T \|^2, \tag{44}$$

subject to the constraint on interpolation of the first and last control points, namely requiring that

$$\mathbf{r}_{q+1} - \mathbf{r}_0 = \mathbf{r}_{q+1}^T - \mathbf{r}_0^T. \tag{45}$$

For closed curves, we minimize

$$F(\mathbf{z}_0, \ldots, \mathbf{z}_p) = \sum_{i=n^2}^{(n+1)(m+n)} \| \mathbf{r}_i - \mathbf{r}_i^T \|^2 . \tag{46}$$

The above summation considers distances of distinct control points only. This is why the points $\mathbf{r}_{(m+n+1)(n+1)-k}$, $k = 0, \ldots, n$, do not appear at the right-hand side of (46) (see relation (23)). In addition, the points $\mathbf{r}_k$, $k < n^2$ and $k > (m+n+1)(n+1)$ are not taken into account, since their location has no effect on the shape of the spline in its interval of definition $[t_n, t_{n+m+1}]$. To obtain a closed spline, we additionally need to impose that certain control points coincide (see case b) in the proof of Proposition 1). This yields the following constraints

$$\mathbf{r}_{n(n+1)-k} = \mathbf{r}_{(m+n+1)(n+1)-k}, \quad \text{for} \quad k = 0, \ldots, n. \tag{47}$$

The two constrained minimization problems *minimize* (44) *subject to* (45) and *minimize* (46) *subject to* (47) can be solved by introducing standard Lagrange multipliers. This allows us to translate each of the two minimization problems into a system of nonlinear equations, of size $p+2$ and $p+n+2$ in the clamped and closed case respectively.

In our experiments the obtained systems were solved using Newton-Raphson iterative method. The initial approximation $\mathbf{z}^{(0)} = (\mathbf{z}_0^{(0)}, \ldots, \mathbf{z}_p^{(0)})$ is chosen following a similar principle as in [8]. In analogy with [8], we shall assume that the target control points form a "reasonable sequence", namely that they provide an unambiguous characterization of the shape, by enforcing the conditions

$$\Delta\mathbf{r}_i^T \cdot \Delta\mathbf{r}_{i+1}^T > 0 \qquad \text{and} \qquad \frac{1}{2} \leq \frac{|\Delta\mathbf{r}_{i+1}^T|}{|\Delta\mathbf{r}_i^T|} \leq 2, \tag{48}$$

where $\Delta\mathbf{r}_i^T := \mathbf{r}_i^T - \mathbf{r}_{i-1}^T$. The left-hand side inequality signifies requiring that $\mathbf{r}_{i+1}^T$ must lie in the general direction of $\Delta\mathbf{r}_i^T$, whereas the other precludes an excessively uneven distribution of the control points. For closed curves, the above conditions should hold for the closed polygon formed by the points $\mathbf{r}_k^T$, $k = n^2, \ldots, (n+1)(m+n)$.

We then impose the fulfillment of equation (5) at $p+1$ parameter values τ_i, $i = 0, \ldots, p$, in the interval $[t_n, t_{p+1}]$, with $\tau_0 = t_n$ and $\tau_p = t_{p+1}$, namely

$$\left(\mathbf{z}^{(0)}(\tau_i)\right)^2 = \left(\mathbf{r}^T\right)'(\tau_i), \qquad i = 0, \ldots, p. \tag{49}$$

Taking the square root of the left- and right-hand sides yields the linear system

$$\mathbf{z}^{(0)}(\tau_i) = \pm\sqrt{(\mathbf{r}^T)'(\tau_i)}, \qquad i = 0, \ldots, p. \tag{50}$$

To select the sign + or − at the right-hand side of the above equation, we choose one particular sign for the first equation (corresponding to $i = 0$), and for the subsequent equations, corresponding to $i = 1, \ldots, p$, we take the sign which guarantees that $\mathbf{z}^{(0)}(\tau_{i-1}) \cdot \mathbf{z}^{(0)}(\tau_i) > 0$ (interpreting complex numbers as vectors in $\mathbb{R}^2$). Note that, to guarantee that the system (50) has a solution, the τ_i need to satisfy the Schoenberg–Whitney conditions for the spline space having degree n and knots $\boldsymbol{\mu}$.

Examples of PH B–Splines curves obtained from different target control polygons are given in Figs. 9–10. A uniform knot partition is used in all the following examples and the parameters τ_i are equally spaced. All the illustrations demonstrate that the target curve and the corresponding PH B–Spline have very similar shapes. Whether this agreement can further be improved by a different method for computation of the initial guess or by a different choice of the τ_i is another topic worthy of future investigation.

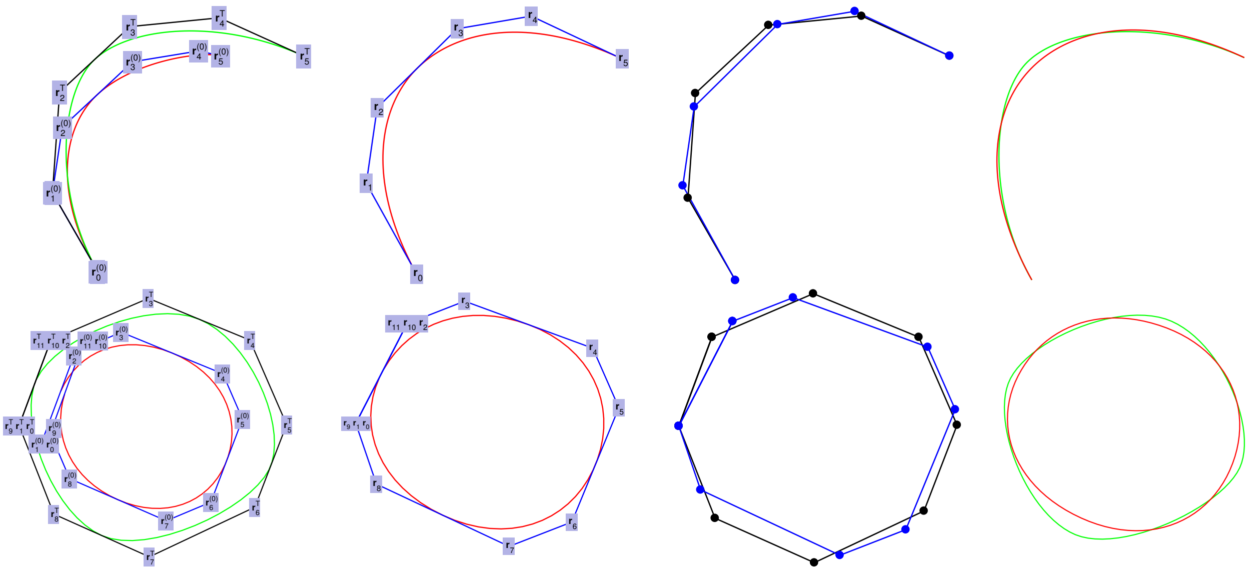

Figure 9: Clamped and closed cubic PH B–Spline corresponding to $m = 2$ (first row) and $m = 3$ (second row). From left to right: target B–Spline $\mathbf{r}^T(t)$ and initial approximation $\mathbf{r}^{(0)}(t)$, PH B–Spline $\mathbf{r}(t)$, control polygons and spline curves superimposed.

We finally observe that the described method can also be used for editing the control points of a given PH B–Spline. To this aim we shall simply take as a starting point $\mathbf{z}^{(0)}$ the preimage of the given PH B–Spline curve and as a target control polygon its polygon modified by the displacement of one or more control points. Note however that the displacement of one point entails updating the entire control polygon.

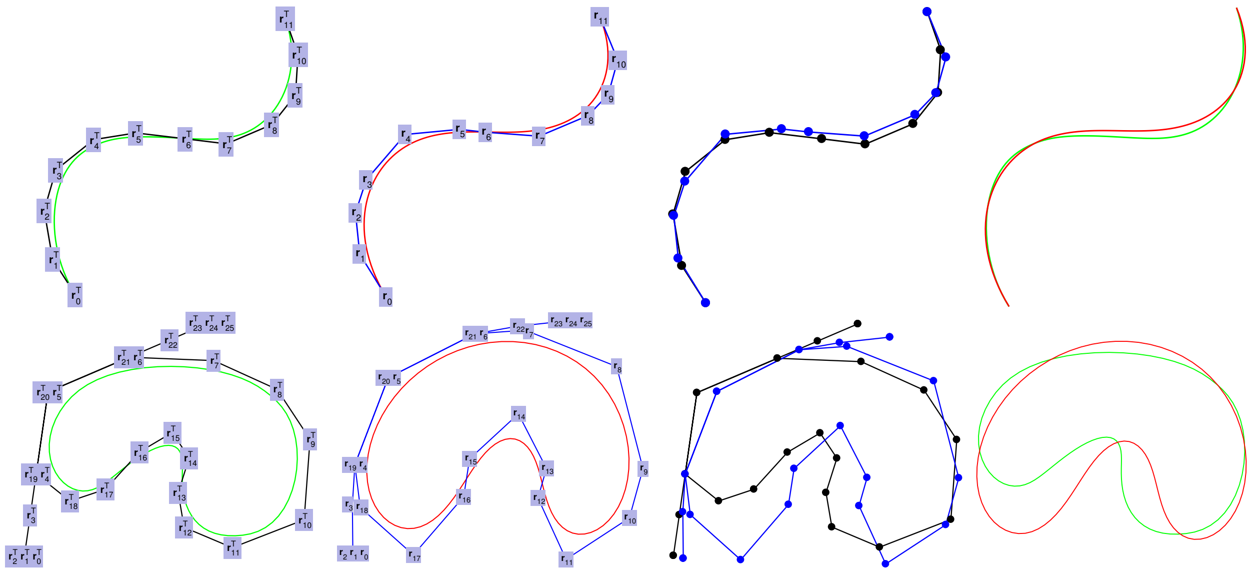

Figure 10: Clamped and closed quintic PH B–Splines corresponding to $m = 4$. From left to right: target B–Spline $\mathbf{r}^T(t)$, PH B–Spline $\mathbf{r}(t)$, control polygons and spline curves superimposed.

8. Conclusions and future work

We have presented the construction of the very general class of Pythagorean-Hodograph (PH) B–Spline curves and we have proposed a computational strategy for efficiently calculating their control points as well as their arc-length and offset curves. In virtue of its high generality and its unique advantages, this new class of curves has a great potential for applications in computer-aided design and manufacturing, robotics, motion control, path planning, computer graphics, animation, and related fields.
The generalization of these planar PH B–Spline curves to 3D-space is currently under investigation. It is also our intention to propose practical applications of this new class of curves dealing with the solution of different interpolation and approximation problems as well as with the problem of interactive local shape modification by control point manipulations.

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