Addendum to: "The Bolzano–Weierstrass theorem is the jump of weak Kőnig's lemma" [Ann. Pure Appl. Logic 163 (6) (2012) 623–655]

Vasco Brattka [a,b,*], Andrea Cettolo [c], Guido Gherardi [d], Alberto Marcone [c], Matthias Schröder [e]

[a] *Department of Mathematics & Applied Mathematics, University of Cape Town, South Africa*
[b] *Faculty of Computer Science, Universität der Bundeswehr München, Germany*
[c] *Dipartimento di Scienze Matematiche, Informatiche e Fisiche, Università di Udine, Italy*
[d] *Dipartimento di Filosofia e Comunicazione, Università di Bologna, Italy*
[e] *Fachbereich Mathematik, Universität Darmstadt, Germany*

A B S T R A C T

The purpose of this addendum is to close a gap in the proof of [1, Theorem 11.2], which characterizes the computational content of the Bolzano–Weierstraß Theorem for arbitrary computable metric spaces.

In [1, Theorem 11.2] it is stated that $\mathsf{BWT}_X \equiv_{\mathrm{sW}} \mathsf{K}'_X$ holds for all computable metric spaces X. Here BWT_X denotes the Bolzano–Weierstraß Theorem, K'_X denotes the jump of compact choice and $\equiv_{\mathrm{sW}}$ stands for strong Weihrauch equivalence. We refer the reader to [1] for the definition of all notions that are not defined here.

While the reduction $\mathsf{BWT}_X \leq_{\mathrm{sW}} \mathsf{K}'_X$ was proved correctly in [1], the proof provided for $\mathsf{K}'_X \leq_{\mathrm{sW}} \mathsf{BWT}_X$ contains a gap and is only correct for the special case of compact X as it stands. This fact was pointed out by one of us (M. Schröder) and is due to the fact that in general the closure of $\mathsf{L}_X^{-1}(K)$ is not compact. We close this gap in this addendum.

We start with a lemma that shows that compact sets given in $\mathcal{K}'_-(X)$ are effectively totally bounded in a particular sense. By $\mathcal{O}(X)$ we denote the set of open subsets of X, represented as complements of elements

E-mail addresses: Vasco.Brattka@cca-net.de (V. Brattka), Andrea.Cettolo@spes.uniud.it (A. Cettolo), Guido.Gherardi@unibo.it (G. Gherardi), Alberto.Marcone@uniud.it (A. Marcone), Matthias.Schroeder@cca-net.de (M. Schröder).

of $\mathcal{A}_-(X)$, i.e., p is a name of an open set U if and only if it is a ψ_--name of the closed set $X \setminus U$. We call an open ball $B(a,r)$ *rational*, if a is a point of the dense subset of X (that is used to define the computable metric space X) and $r \geq 0$ is a rational number.

Lemma 1. *Let X be a computable metric space. Consider the multivalued function $F_X :\subseteq \mathcal{K}'_-(X) \rightrightarrows \mathcal{O}(X)^{\mathbb{N}}$ with $\mathrm{dom}(F_X) = \{K \in \mathcal{K}'_-(X) : K \neq \emptyset\}$ and such that, for each $K \neq \emptyset$, we have $(U_n)_n \in F_X(K)$ if and only if the following conditions hold for each $n \in \mathbb{N}$:*

(1) U_n is a union of finitely many rational open balls of radius $\leq 2^{-n}$,
(2) $K \subseteq U_n$.

Then F_X is computable.

Proof. Let X be a computable metric space and let $K \subseteq X$ be a nonempty compact set. Let $\langle p_i \rangle_i$ be a κ'_--name of K. This means that $p := \lim_{i\to\infty} p_i$ is a κ_--name for K and, in particular, for each $n \in \mathbb{N}$:

- $p_i(n)$ is a name for a finite set of rational open balls for each $i \in \mathbb{N}$,
- there exists $k \in \mathbb{N}$ such that the finite set of rational balls given by $p_k(n)$ covers K and $p_k(n) = p_i(n)$ for all $i \geq k$.

We also have that $\{p(n) : n \in \mathbb{N}\}$ is a set of names of all finite covers of K by rational open balls. We want to build a sequence of open sets $(U_n)_n$ such that (1) and (2) hold. We describe how to construct a name of a generic open set U_n for $n \in \mathbb{N}$. We start at stage 0 with $U_n = \emptyset$. At each stage $s = \langle m, i \rangle$ that the computation reaches, we focus on the balls $B(a_0, r_0), \ldots, B(a_l, r_l)$ given by $p_i(m)$ and we check whether $r_0, \ldots, r_l \leq 2^{-n}$. If this is not true, then we go to stage $s+1$. Otherwise, if the condition is met, we add these balls to the name of U_n and we check whether $p_i(m) = p_{i+1}(m)$. If this is the case we add again $B(a_0, r_0), \ldots, B(a_l, r_l)$ to the name of U_n. We repeat this operation as long as we find the same open balls given by $p_j(m)$ for $j > i$. If we find $p_i(m) \neq p_j(m)$ for some $j > i$, then the computation goes to stage $s+1$.

We claim that, for each n, there exists a stage in which the computation goes on indefinitely. Consider, in fact, $\{B(a_0, r_0), \ldots, B(a_l, r_l)\}$, a finite rational cover of K with $r_0, \ldots, r_l \leq 2^{-n}$, which exists by a simple argument using the compactness of K. Since $\langle p_i \rangle_i$ is a κ'_--name of K, there exists a minimum $\langle m, i \rangle$ such that:

- $p_i(m)$ is a name for the cover $\{B(a_0, r_0), \ldots, B(a_l, r_l)\}$,
- $p_i(m) = p_j(m)$ for each $j > i$.

If the algorithm reaches stage $s = \langle m, i \rangle$, then it is clear that the computation goes on indefinitely within this stage. If the algorithm never reaches stage s, then necessarily it already stopped at a previous stage. In both cases our claim is true.

Finally, since we built the name of U_n by adding only balls of radius $\leq 2^{-n}$ and since the computation stabilizes at a finite stage, it is clear that conditions (1) and (2) are met. □

We note that even though the open sets U_n constructed in the previous proof are finite unions of rational open balls, the algorithm does not provide a corresponding rational cover in a finitary way. It rather provides an infinite list of rational open balls that is guaranteed to contain only finitely many distinct rational balls. This is a weak form of effective total boundedness and the best one can hope for, given that the input is represented by the jump of κ_-.

The following lemma shows that sequences that we choose in range(F_X) in a particular way give rise to totally bounded sets.

Lemma 2. *Let X be a metric space and let $U_n \subseteq X$ be a finite union of balls of radius $\leq 2^{-n}$ for each $n \in \mathbb{N}$. Let $(x_n)_n$ be a sequence in X with $x_n \in \bigcap_{i=0}^{n} U_i$. Then $\overline{\{x_n : n \in \mathbb{N}\}}$ is totally bounded.*

Proof. We obtain $\{x_n : n \in \mathbb{N}\} \subseteq \bigcap_{i=0}^{\infty} \left(U_i \cup \bigcup_{n=0}^{i-1} B(x_n, 2^{-i}) \right)$ and the set on the right-hand side is clearly totally bounded. Hence the set on the left-hand side is totally bounded and so is its closure. □

We mention that it is well known that a subset of a metric space is totally bounded if and only if any sequence in it has a Cauchy subsequence [2, Exercise 4.3.A (a)].

Now we use the previous two lemmas to complete the proof of [1, Theorem 11.2]. Within the proof we use the canonical completion $\hat{X}$ of a computable metric space. It is known that this completion is a computable metric space again and that the canonical embedding $X \hookrightarrow \hat{X}$ is a computable isometry that preserves the dense sequence [3, Lemma 8.1.6]. We will identify X with a subset of $\hat{X}$ via this embedding.

Theorem 3 *([1, Theorem 11.2]).* $\mathsf{BWT}_X \equiv_{\mathrm{sW}} \mathsf{K}'_X$ *for all computable metric spaces X.*

Proof. The reduction $\mathsf{BWT}_X \leq_{\mathrm{sW}} \mathsf{K}'_X$ has been proved in [1], so we focus on the reduction $\mathsf{K}'_X \leq_{\mathrm{sW}} \mathsf{BWT}_X$. Let (X, d, α) be a computable metric space and let $K \subseteq X$ be a nonempty compact set given by a κ'_{-}-name $\langle p_i \rangle_i$. We want to compute a point of K using BWT_X. The idea is to define a sequence $(x_n)_n$ in X, working within the completion $\hat{X}$ of X and using the open sets built in Lemma 1, such that $\overline{\{x_n : n \in \mathbb{N}\}}$ is compact in X.

It is clear that K is a compact subset of $\hat{X}$ and that $\langle p_i \rangle_i$ can be considered as a κ'_{-}-name for K in $\hat{X}$. We consider the map

$$\mathsf{L}_{\hat{X}} : \hat{X}^{\mathbb{N}} \to \mathcal{A}'_{-}(\hat{X}), (x_n)_n \mapsto \{x \in \hat{X} : x \text{ is a cluster point of } (x_n)_n\}.$$

By [1, Corollary 9.5] $\mathsf{L}_{\hat{X}}^{-1}$ is computable and hence $\mathsf{L}_{\hat{X}}^{-1}(K)$ yields a sequence $(z_m)_m$ in $\hat{X}$ whose cluster points are exactly the elements of K.

Let $F_{\hat{X}}$ be the multivalued function defined in Lemma 1. We can compute a sequence $(U_n)_n \in F_{\hat{X}}(K)$. Since $\overline{\{z_m : m \in \mathbb{N}\}}$ is not compact (and hence not in dom(BWT_X)) in general, we refine it recursively to a sequence $(y_n)_n$ using $(U_n)_n$ in the following way: for each $n \in \mathbb{N}$, $y_n := z_{m_n}$ for the first m_n that we find with $z_{m_n} \in U_0 \cap \cdots \cap U_n$ and such that $m_i < m_n$ for all $i < n$. Note that we can always find such a y_n, since $U_0 \cap \cdots \cap U_n$ covers K which is the set of cluster points of $(z_m)_m$. Clearly every cluster point of $(y_n)_n$ is also a cluster point of $(z_m)_m$, hence it belongs to K.

Recall now that $(y_n)_n$ is a sequence of points in $\hat{X}$ and that we want a sequence $(x_n)_n$ in X in order to apply BWT_X. We compute $(x_n)_n$ as follows: for each $n \in \mathbb{N}$, x_n is the first element that we find in the dense subset range(α) such that $d(x_n, y_n) < 2^{-n}$ and $x_n \in U_0 \cap \cdots \cap U_n$, where d also denotes the extension of the metric to $\hat{X}$. By density of X in $\hat{X}$ such an x_n always exists and it is clear that the cluster points of $(x_n)_n$ and those of $(y_n)_n$ are the same in $\hat{X}$.

Now $A := \overline{\{x_n : n \in \mathbb{N}\}}$ is totally bounded in X by Lemma 2 and hence every sequence in A has a Cauchy subsequence, which has a limit in $\hat{X}$, since $\hat{X}$ is complete. By construction of $(x_n)_n$ the limit of such a subsequence is in K and hence in X. Thus every sequence in A has a subsequence that converges in X and hence A is compact in X.

Finally, we can obtain an element of K by applying BWT_X to $(x_n)_n$. □

References

[1] Vasco Brattka, Guido Gherardi, Alberto Marcone, The Bolzano–Weierstrass theorem is the jump of weak Kőnig's lemma, Ann. Pure Appl. Logic 163 (2012) 623–655.

[2] Ryszard Engelking, General Topology, Sigma Ser. Pure Math., vol. 6, Heldermann, Berlin, 1989.

[3] Klaus Weihrauch, Computable Analysis, Springer, Berlin, 2000.