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METRIC PROJECTIVE GEOMETRY, BGG DETOUR COMPLEXES AND PARTIALLY MASSLESS GAUGE THEORIES

A. ROD GOVER[♢], EMANUELE LATINI[♣] & ANDREW WALDRON[♠]

Abstract. A projective geometry is an equivalence class of torsion free connections sharing the same unparametrised geodesics; this is a basic structure for understanding physical systems. Metric projective geometry is concerned with the interaction of projective and pseudo-Riemannian geometry. We show that the BGG machinery of projective geometry combines with structures known as Yang-Mills detour complexes to produce a general tool for generating invariant pseudo-Riemannian gauge theories. This produces (detour) complexes of differential operators corresponding to gauge invariances and dynamics. We show, as an application, that curved versions of these sequences give geometric characterizations of the obstructions to propagation of higher spins in Einstein spaces. Further, we show that projective BGG detour complexes generate both gauge invariances and gauge invariant constraint systems for partially massless models: the input for this machinery is a projectively invariant gauge operator corresponding to the first operator of a certain BGG sequence. We also connect this technology to the log-radial reduction method and extend the latter to Einstein backgrounds.

Keywords: Projective geometry, BGG sequences, gauge theories, higher spin theories, detour complex.

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1. Introduction

A fundamental problem in physics, mathematics, and their interface is that of finding the right way to describe and treat the natural differential equations describing fields. Naturality here refers to equations determined by the underlying geometry, so this is in essence a geometric problem. For the case of particle theories in space-time physics it appears, on the surface, that pseudo-Riemannian geometry should be the central structure. However pseudo-Riemannian invariance is a rather weak condition, in the sense that by far more equations are invariant in this sense than are interesting or important. For field equations of motion, other principles can be brought to bear, and in particular the requirement that theories exhibit suitable gauge invariance, and corresponding integrability conditions, plays a critical *rôle* in determining systems with, for example, the correct propagating degrees of freedom (DoF). It is then reasonable to ask if there is a more fundamental geometric structure, underlying pseudo-Riemmannian geometry, that includes the entire picture. The payoff for a positive answer can be significant. Apart from developing theory for the unification and extension of gauge theories, this can give insight into how to treat fields at infinity and thus understand decay, scattering, and possible holographic features.

It has long been realised that, in dimension four, the equations governing massless fields exhibit conformal invariance. This suggests conformal geometry (a manifold M and a conformal class of metrics $\boldsymbol{c}$) as a central organizing principle for field theories [52, 53]. However four dimensional physics also requires fields that are not massless, and so one may consider alternative basic structures, such as a projective class of connections: Given an affine connection ∇, the collection of its geodesics as unparametrised curves is the corresponding *projective structure* $\boldsymbol{p}$; or $\boldsymbol{p}$ may be viewed as the equivalence class of torsion free connections sharing the same unparametrised geodesics. To each pseudo-Riemannian metric, there is associated a unique projective structure via the metric's Levi-connection (or its geodesics). Then, *metric projective geometry* studies the interaction between these geometries. Building partly on earlier works, *e.g.* [63, 65], there has been a recent surge of interest in the links between metric and projective geometry with powerful results obtained, see *e.g.* [58, 62]. Through its projective structure a metric determines a projective Cartan connection or, equivalently, tractor connection [2, 19, 68]. This is a higher order geometric structure which encodes geometry, at each point, not just in the tangent space but also in higher order Taylor series data of the manifold. Furthermore it exposes deeper links between metrics and projective geometry [42]. In particular, there is a striking connection between projective geometry and Einstein metrics [1, 17, 49]. The latter are generalised through suitable Cartan holonomy reductions [15, 16]. These new insights have led to the development of a metric-projective analogue of conformal compactification [14].

Geodesics are basic geometric structures; physically they encode how particles interact with background geometries. Projective geometry is also intimately related to massless spin two dynamics: On constant curvature backgrounds, the differential complex controlling deformations of Riemannian structure (metric fluctuations) is a projectively invariant Bernstein-Gelfand-Gelfand (BGG) complex–see Eastwood's interpretation [37] of the Calabi complex [9].

BGG complexes have their origins in representation theory [6, 74], but in geometry the dual structures that go by the same name (or *BGG sequences* more generally) arise naturally from a tractor connection twisting of the de Rham complex and algebraic tools of Kostant [59, 3, 10, 18]. These are extremely powerful tools for the organisation and interpretation of invariant differential operators but, as we shall explain in Section 4.1, are not the right objects for dynamical field theories. In the setting of even dimensional conformal geometry, *detour complexes* were introduced in [7, 8] as complexes which are linked to the BGG sequences but which, importantly, involve weaker integrability conditions. It was quickly realised that these fit into a wider framework which fits closely with variational principles. In particular, in [54] it is shown that, for each linear vector bundle connection, there is a corresponding differential (detour) sequence that forms a complex if and only if the given connection satisfies the Yang-Mills equations; for the vector bundle corresponding to the adjoint representation of the gauge group the sequence governs second variations of the Yang-Mills action. These *Yang-Mills detour complexes* can be linked to the first operators in BGG sequences (and their adjoints) via differential splitting operators to yield further complexes that may be called *BGG detour complexes*.

In this article, we show that bringing together the BGG machinery and the Yang-Mills detour theory, in the setting of metric projective geometry, produces a general tool for generating invariant pseudo-Riemannian gauge theories. This yields the underlying geometric picture we were seeking. In particular in Theorem 5.1, for fields of any integral spin on constant curvature backgrounds, we construct equations of motion that are

invariant with respect to maximal depth (*i.e.*, highest possible derivative order) gauge transformations. The detour complexes involved also give the corresponding Bianchi identities. In fact, more than the complex arises: in Theorem 5.3 we show that the detour machinery also produces the gauge invariant constraint systems and corresponding relations between the constraints. Thus one obtains a rather complete picture for what are called, following [28], (maximal depth) *partially massless fields*.

To illustrate concretely by example the link between partially massless (PM) theories and projective BGG sequences, consider the formula

$$\delta\varphi_{ab} = \big(\nabla_a\nabla_b + \frac{\Lambda}{3}\, g_{ab}\big)\alpha\,.$$

Physically the above is the gauge invariance of a PM spin two field. From a geometry perspective, the terms in brackets are a projectively invariant BGG operator, but specialised to an Einstein scale. There have also been indications of the importance of projective geometry for PM systems in the physics literature: It was first observed in [56] that PM models have as a geometric origin massless models in a flat space of one higher dimension. This was achieved by a log-radial reduction [4] which projectivised this flat ambient space. This also suggests an intimate link between projective geometry and PM models. Further evidence for this, especially in light of the conformal-projective link on Einstein manifolds [49], is that these models can also be described by conformally invariant equations coupled to a parallel, conformal, scale tractor [52, 53, 55]. Since PM models are our running example of a physical system whose underpinning is a projective structure, let us briefly review those models: The first PM theory was discovered by Deser and Nepomechie who were searching for a modification of the spin two equations of motion that supported lightcone propagation in conformally flat spacetimes [24]. Subsequently Higuchi realized that this gives a unitarity bound on massive spin two excitations [57]. Later still, it was understood that PM theories existed for all spins and also Fermi fields [27]. Their novel, higher derivative, gauge invariances implied they described the lightlike propagation [30] of sets of helicity states that were intermediate between the usual massless and massive models [27, 28, 29, 31].

In more detail, massless higher spin systems are described by second derivative order, gauge invariant equations of motion. These gauge symmetries guarantee that only physical DoF propagate. Massive higher spin systems take a different route. Again, their equations of motion are second derivative order, but are no longer gauge invariant. Instead, a set of integrability conditions of the equations of motion imply constraints that are required for propagation of only physical DoF. The PM system's route is an intermediate one: Their second derivative field equations enjoy both gauge invariances and integrability conditions implying *gauge invariant* constraints. As mentioned, the beauty of our BGG detour complex approach to these systems is that it automatically produces a system of gauge invariant equations, for a minimal field content, that includes both equations of motion and constraints.

The BGG and detour apparatus is also linked to other approaches in the physics literature such as the higher spin unfolding programme [64] whose σ_- cohomology, see for example [70, 66], is really the homology of the Kostant differential [59]. Also, BRST machinery (which is intimately related to Lie algebra cohomology) applied to parabolic Lie superalgebras represented by differential operators acting on higher rank tensor bundles, has been used to construct detour operators for massless higher spin models and related systems describing supersymmetric black hole dynamics in [20, 21, 22].

Despite its being a fundamental geometric structure, projective geometry is still rarely utilized in physical settings, so we briefly review its key ingredients, and those of metric projective geometry, in Section 2. Our key tool for handling projective geometries is the tractor calculus of [2]. This is detailed in Section 3. Detour complexes and BGG sequences are powerful technologies, these are described in generality in Section 4. There we also explain their relationship to physical systems. Armed with all the above machinery, we finally turn to explicit physical models in Section 5. In Theorem 5.1, using the BGG detour complex apparatus, we establish the existence of a complex describing the equations of motion and gauge invariances of a broad class of models. Then in Theorem 5.3, we show that the BGG complex also encodes a gauge invariant system of constraints for those models. In Section 5.2 we spell out the case of PM spin three on constant curvature backgrounds. In Section 6 we study the extension of our higher spin results to general Einstein backgrounds. For spin two there is no obstruction here, but already for spin three, the BGG detour technology neatly characterizes an obstruction to propagation in these spaces. There we also present a new complex coupling spin three to a mixed symmetry field giving, at least, gauge invariant dynamics in an Einstein background. Section 7 discusses action principles within the BGG formulation; here we focus on the spin two case. In Appendix A, we relate the log-radial reduction technique (which is a popular method for studying higher spin systems and their interactions) to projective tractor calculus.

2. Metric projective differential geometry

At the level of underlying geometry we will exploit the interaction between metric and projective geometry. The discussion of projective geometry here follows [2, 39, 50] while [17, 42, 49] provide background theory for metric projective theory.

2.1. **Projective Geometry.** Projective geometry is one of the simplest examples of a parabolic geometry; it can be defined as a Cartan geometry modeled on the homogeneous space $Sl(n+1,\mathbb{R})/P$ where P is the parabolic subgroup stabilizing a ray in $\mathbb{R}^{n+1}$. A projective manifold is the structure $(M,\boldsymbol{p})$ where M is a smooth n-dimensional manifold and $\boldsymbol{p}$ is an equivalence class of torsion-free affine connections, where $\widehat{\nabla}\sim\nabla$ if, acting on any one-form field ω, or vector field v, they are related by

$$\widehat{\nabla}_a\omega_b=\nabla_a\omega_b-\Upsilon_a\omega_b-\Upsilon_b\omega_a \;\Leftrightarrow\; \widehat{\nabla}_a v^b=\nabla_a v^b+\Upsilon_a v^b+\delta^b_a\Upsilon_c v^c\,, \tag{2.1}$$

for some one-form $\boldsymbol{\Upsilon}$. This definition is derived from the fact that the relationship given in (2.1) is exactly the condition that $\widehat{\nabla}$ and ∇ share the same geodesics as unparametrised curves. This is a classical result; see [39] for a modern treatment. Extending the above by linearity to a p-form $\omega_{bc\cdots d}$ we have,

$$\widehat{\nabla}_a\omega_{bc\cdots d}=\nabla_a\omega_{bc\cdots d}-(p+1)\Upsilon_a\omega_{bc\cdots d}-(p+1)\Upsilon_{[a}\omega_{bc\cdots d]}\,,$$

and in particular, for a top form $\widehat{\nabla}_a\omega^{\rm top}_{bc\cdots d}=\nabla_a\omega^{\rm top}_{bc\cdots d}-(n+1)\Upsilon_a\omega^{\rm top}_{bc\cdots d}$. Taking powers of the volume density bundle gives the projective density bundle

$$\mathcal{E}(w):=\left((\wedge^n T^*M)^2\right)^{-w/2(n+1)}\,;$$

and thus for a section σ of this we have:

$$\widehat{\nabla}_a\sigma=\nabla_a\sigma+w\Upsilon_a\sigma\,.$$

As a point of notation, in the following, given any vector bundle $\mathcal{B}$, we will write $\mathcal{B}(w)$ as a shorthand for $\mathcal{B}\otimes\mathcal{E}(w)$, and we say the vector bundle $\mathcal{B}(w)$ (and any section thereof)

has *projective weight* w. Quite generally, we also use the same notation for a bundle and its section space.

Note that, at this juncture, there is no notion of a Riemannian metric, nevertheless given a torsion-free affine connection ∇, its curvature tensor is given by

$$[\nabla_a, \nabla_b]\omega_c = -R_{ab}{}^d{}_c\omega_d\,,$$

and $R_{[ab}{}^d{}_{c]} = 0$. Moreover, we can decompose the curvature into its trace-free and trace pieces as [2]

$$R_{ab}{}^d{}_c = W_{ab}{}^d{}_c + 2\delta^d_{[a}Q_{b]c} - 2Q_{[ab]}\delta^d_c\,, \tag{2.2}$$

where $W_{ab}{}^d{}_c$ and Q_{ab} are the *projective Weyl* and *Schouten tensors*. The former obeys $W_{[ab}{}^d{}_{c]} = 0 = W_{ad}{}^d{}_c$ and is projectively invariant while the projective Schouten tensor has no definite symmetry and transforms as

$$\widehat{Q}_{ab} = Q_{ab} - \nabla_a\Upsilon_b + \Upsilon_b\Upsilon_a\,.$$

It should not be confused with its conformal geometry counterpart denoted P_{ab}. The curl of this, $\nabla_aQ_{bc} - \nabla_bQ_{ac} =: C_{abc}$, defines the projective Cotton tensor.

2.2. **Scales.** Any $\nabla \in \boldsymbol{p}$ also gives a connection on any tensor bundle, and in particular on the line bundle $(\wedge^nT^*M)^2$. Conversely a connection $\nabla \in \boldsymbol{p}$ is determined by a choice of connection on $(\wedge^nT^*M)^2$. The skew part of the Schouten tensor, $Q_{[ab]}$, is (up to a non-zero constant multiple) the curvature of ∇ on that line bundle. As already used above, the line bundle $(\wedge^nT^*M)^2$ is trivial. In fact it is also canonically oriented and the positive square root is the *volume density bundle*. A nonvanishing section of $(\wedge^nT^*M)^2$, or equivalently any of its roots $\mathcal{E}(w)$ with $w \neq 0$, is called a *choice of scale* and in an obvious way determines a line bundle connection preserving the given section (which may be thought of as a global frame). Thus a choice of scale determines a connection $\nabla \in \boldsymbol{p}$ which, by slight abuse of terminology, we shall also call a choice of scale. (Such a connection determines a section of $\mathcal{E}(w)$, $w \neq 0$, up to multiplication by a non-zero constant.) It is clear that for any such connection we have

$$Q_{[ab]} = 0,$$

and different choices of scale yield a transformation on the form (2.1) where $\boldsymbol{\Upsilon}$ is exact, *cf.* [17, 49, 50].

In our subsequent discussions we restrict to connections $\nabla \in \boldsymbol{p}$ which correspond to a choice of scale; such connections form a distinguished class so this results in no loss of generality. Moreover, for simplicity, we will assume M is orientable.

2.3. **Connecting with pseudo-Riemannian geometry.** Given a projective manifold $(M, \boldsymbol{p})$, a natural question is whether there is a Levi-Civita connection in the projective class $\boldsymbol{p}$, and if so, what the consequences are. A route towards answering these questions is provided by the following result due to Mikes [63] and Sinjukov [65].

Proposition 2.1. *If $\widehat{\nabla}_a$ preserves some volume density, and there exists a metric tensor $\sigma^{ab} \in \odot^2TM(-2)$ satisfying the projectively invariant condition*

$$\text{trace-free } (\widehat{\nabla}_a\sigma^{bc}) = 0\,, \tag{2.3}$$

then there is a projectively related connection ∇ which is the Levi-Civita connection of the metric $g^{ab} = \tau\sigma^{ab}$, for some nonvanishing smooth density $\tau \in \mathcal{E}(2)$, where $\boldsymbol{\Upsilon}$ (as defined in (2.1)*) is given by $\boldsymbol{\Upsilon} = -\frac{1}{2}\boldsymbol{\nabla}\log\tau$.*

The formulation here follows [42].

Remark 2.2. When the projective class of connections has a Levi-Civita connection ∇^g, the antisymmetric part of the projective Schouten tensor vanishes. Then, in this scale, we can decompose the projective Weyl tensor into SO-irreducible parts, see [42]:

$$W_{ab}{}^d{}_c = \mathring{W}_{ab}{}^d{}_c + \frac{2}{(n-1)(n-2)}\,\delta^d_{[a}\mathring{R}_{b]c} + \frac{2}{n-2}\,\mathring{R}_{[a}{}^d g_{b]c}\,,$$

where $\mathring{R}_{bd}$ and $\mathring{W}_{ab}{}^d{}_c$ are the totally trace-free parts of, respectively, the Ricci and projective Weyl tensors. By construction the latter is the usual, conformally invariant, Weyl tensor W of Riemannian geometry (so $\mathring{W} = \mathsf{W}$). It follows immediately that, for an Einstein metric g, we have

$$W_{ab}{}^d{}_c = \mathsf{W}_{ab}{}^d{}_c.$$

It turns out that many simplifications occur when dealing with an Einstein metric. First we make a definition.

Definition 2.3. If there exists $\nabla^g \in \boldsymbol{p}$ where ∇^g is the Levi-Civita connection for a metric, we call $(M, \boldsymbol{p})$ a *metric projective structure*. If in addition, g is an Einstein metric, we say that this is *Einstein projective*.

When the structure is Einstein projective, computations performed using the Einstein metric g and its Levi-Civita connection ∇^g will be referred to as being done in an *Einstein scale*.

The following observation of [49] is an easy consequence of equation (2.2) and the subsequent discussion.

Proposition 2.4. *When g is an Einstein metric, the conformal Schouten tensor P for g and the projective Schouten tensor Q for ∇^g obey*

$$\mathsf{P}_{ab} = \frac{\Lambda}{(n-1)(n-2)} g_{ab} = \frac{1}{2}\,Q_{ab}\,.$$

Moreover the conformal Weyl curvature W of g equals the projective Weyl curvature W of ∇^g.

Remark 2.5. Note that the scalar curvature is $\frac{2n\Lambda}{n-2}$ and Λ in the above has been defined to coincide with the usual cosmological constant appearing in physics applications for which the Einstein tensor obeys $G_{ab} + \Lambda g_{ab} = 0$.

3. Tractor calculus for projective geometries

On a general projective manifold $(M, \boldsymbol{p})$ there is no distinguished (*i.e.*, canonical) connection on the tangent bundle. However the projective structure $\boldsymbol{p}$ does determine a distinguished connection on a related vector bundle of rank $(n+1)$ called the standard tractor bundle; this connection is known as the (standard) projective tractor connection and is due to Thomas [68]. The modern treatment was initiated in [2], and is equivalent to the projective Cartan connection of Cartan [19], see [13]. Other developments relevant to our treatment can be found in [17, 49].

The *(projective) tractor bundle* $\mathcal{T}$, or $\mathcal{T}^A$ in an abstract index notation, can be defined as follows. For each choice of a connection in the projective class we identify the tractor bundle with the direct sum

$$\mathcal{T}^A \cong TM(-1) \oplus \mathcal{E}(-1) \ni \begin{pmatrix} v^a \\ \rho \end{pmatrix},$$

where on the right we indicate how elements will be denoted. Equivalently for its dual, namely the *(projective) cotractor bundle* $\mathcal{T}^*$, or $\mathcal{T}_A$, we have

$$\mathcal{T}_A \cong T^*M(1) \oplus \mathcal{E}(1) \ni \begin{pmatrix} w_a & \sigma \end{pmatrix} .$$

Changing the connection in the projective class according to (2.1), these transform as

$$\begin{pmatrix} v^a \\ \rho \end{pmatrix} \to \begin{pmatrix} v^a \\ \rho + \Upsilon_a v^a \end{pmatrix} , \qquad \begin{pmatrix} \omega_a & \sigma \end{pmatrix} \to \begin{pmatrix} \omega_a + \Upsilon_a \sigma & \sigma \end{pmatrix} , \tag{3.1}$$

and it is easily verified that this leads to well defined vector bundles on $(M, \boldsymbol{p})$. The transformations (3.1) mean that these bundles are filtered with composition series, respectively,

$$\mathcal{T} = TM(-1) \dot{+} \mathcal{E}(-1), \quad \text{and} \quad \mathcal{T}^* = T^*M(1) \dot{+} \mathcal{E}(1). \tag{3.2}$$

For the second of these, for example, this means that there is a canonical (so projectively invariant) short exact sequence of bundles

$$0 \to \mathcal{E}_a(1) \xrightarrow{Z^a_A} \mathcal{T}_A \xrightarrow{X^A} \mathcal{E}(1) \to 0, \tag{3.3}$$

where we have written $\mathcal{E}_a$ as an abstract index notation for T^*M. The canonical homomorphism X^A may be viewed as a section of $\mathcal{T}^A(1)$ and is often called the *canonical tractor*. This also gives the canonical bundle inclusion $X^A : \mathcal{E}(-1) \to \mathcal{T}^A$ indicated by the dual composition the series for $\mathcal{T}$.

In a given scale $\nabla \in \boldsymbol{p}$, we may define a covariant derivative on $\mathcal{T}$ (and its dual on $\mathcal{T}^*$) as follows:

$$\nabla^{\mathcal{T}}_a \begin{pmatrix} v^b \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla_a v^b + \delta^b_a \sigma \\ \nabla_a \rho - Q_{ab} v^b \end{pmatrix} , \qquad \nabla^{\mathcal{T}}_a \begin{pmatrix} \omega_b & \sigma \end{pmatrix} = \begin{pmatrix} \nabla_a \omega_b + Q_{ab}\sigma & \nabla_a \sigma - \omega_a \end{pmatrix} . \tag{3.4}$$

Upon changing to a different connection in $\boldsymbol{p}$, as in (2.1), it is easily verified that the right-hand-sides here transform according to (3.1), signaling that $\nabla^{\mathcal{T}}$ descends to a projectively invariant connection on $\mathcal{T}$. This is the *projective tractor connection* and we denote it also by $\nabla^{\mathcal{T}}$.

The tractor connection determines a connection on all tensor products of $\mathcal{T}$ and its dual, in an abstract index notation such a bundle may be denoted $\mathcal{T}_{A_1 \dots A_p}{}^{B_1 \dots B_q}$. The connection above also induces connections on "$Sl(n+1)$-irreducible" tensor parts of these bundles. An alternative perspective on this is via a principal bundle picture as follows.

It is straightforward to construct an adapted frame bundle $\mathcal{G}$ for $\mathcal{T}$, with frame transformations that respect the filtration structure (3.2) and give $\mathcal{G}$ a typical fibre isomorphic to the parabolic P. The tractor connection then determines a Cartan connection ω on $\mathcal{G}$, and this is the normal Cartan connection for projective geometry, see [13]. We do not need the details of this here, but the point is that the tractor bundle and connection may then be viewed as induced by the Cartan connection through the standard representation of P on $\mathbb{R}^{n+1}$ with

$$\mathcal{T} = \mathcal{G} \times_P \mathbb{R}^{n+1}.$$

From the properties of Cartan connections it follows at once that the Cartan connection similarly gives a connection on any associated bundle

$$\mathcal{G} \times_P \mathbb{V}, \tag{3.5}$$

where $\mathbb{V}$ is an $Sl(n+1)$-representation space viewed as a P-representation by restriction. This perspective is developed fully, in the setting of general parabolic geometries, in [13]. Here we denote any such *tractor connection* by $\nabla^{\mathcal{T}}$.

3.1. **Projective curvature.** The curvature of the tractor connection is given by

$$\Omega_{ab}{}^{C}{}_{D} = \begin{pmatrix} W_{ab}{}^{c}{}_{d} & 0 \\ C_{bad} & 0 \end{pmatrix},$$

where $W_{ab}{}^{c}{}_{d}$ and C_{abc} are the projective Weyl and Cotton tensors. This is the invariant curvature associated with projective geometry and a projective structure is flat if and only if $\Omega_{ab}{}^{C}{}_{D} = 0$.

3.2. **The Thomas D-operator.** Given any tractor bundle $\mathcal{V}$ there is a projectively invariant operator

$$D : \mathcal{V}(w) \to \mathcal{T}^* \otimes \mathcal{V}(w-1)$$

known as the Thomas D-operator, see [2]. In a choice of scale this is given by

$$V \mapsto D_A V = \begin{pmatrix} wV \\ \nabla_a V \end{pmatrix}, \tag{3.6}$$

where ∇_a is a coupling on the tractor connection with the scale connection on $\mathcal{E}(w)$, and the indices of $\mathcal{V}$ are omitted. It is an elementary exercise to verify that this transforms according to (3.1), and so D is projectively invariant.

3.3. **Einstein projective structures.** Recall that a metric is said to be *Einstein* if $R_{ab} = \Lambda g_{ab}$ for some constant Λ. The Einstein condition has a striking interpretation in projective geometry, and one that plays a key *rôle* in our developments below. The following result is established from different perspectives in [17] and [49]. (The non-degenerate case was first due to Armstrong [1].)

Proposition 3.1. *Let $(M, \boldsymbol{p})$ be a projective manifold. There is an Einstein metric g with Levi-Civita connection $\boldsymbol{p}$ if and only if there is a symmetric tractor field*

$$H^{AB} \in \odot^2 \mathcal{T}$$

that is parallel for the tractor connection and of rank at least n. If H^{AB} is non-degenerate, then it is equivalent to a non-Ricci flat metric g. If H^{AB} is degenerate (of rank n) then it is equivalent to Ricci flat metric.

In the case that H^{AB} is non-degenerate there is an easy and conceptual way to reconstruct the metric g from H^{AB}, as follows. Denote by H_{AB} its inverse. Then this determines a scale

$$\tau := H_{AB} X^A X^B \in \mathcal{E}(2), \tag{3.7}$$

where X^A is the canonical tractor from (3.3). This or its negative is a positive section (we assume M connected) and so may be used to trivialise density bundles. Thus from the sequence (3.3) we may view T^*M as a subbundle on $\mathcal{T}^*$ and so we obtain a metric on T^*M by the restriction of H to T^*M. This is g^{-1}. It is a straightforward use of the formula for the tractor connection to verify that the metric g has its Levi-Civita connection in $\boldsymbol{p}$.

In fact, a slight variant of this construction recovers g in the case that H^{AB} has rank n, as shown in [17] and [49], but this is less obvious. Some insight is gained by recalling from Proposition 2.1 that if we have a connection and tensor σ^{ab} in $\odot^2 TM(-2)$ satisfying (2.3), then we have a metric connection in the projective class. In fact this

equation is projectively invariant and the operator on the left-hand-side is a first BGG operator. In this case the BGG splitting operator $\mathcal{L}_0$ applied to σ^{bc} is given explicitly by

$$\mathcal{L}_0(\sigma^{bc}) = \begin{pmatrix} \sigma^{ab} & \frac{1}{n+1}\nabla_c\sigma^{cb} \\ \frac{1}{n+1}\nabla_c\sigma^{ca} & \frac{1}{n(n+1)}\nabla_c\nabla_d\sigma^{cd} + Q_{cd}\sigma^{cd} \end{pmatrix}.$$

If σ^{ab} is non-degenerate is clear that this has rank at least n and σ^{ab} is said to be a *normal solution* to (2.3) if and only if $H^{AB} := \mathcal{L}_0(\sigma^{bc})$ is parallel [17]. By taking its determinant in an obvious way, the solution σ^{ab} determines a scale $\tau \in \mathcal{E}(2)$ and $\tau\sigma^{ab} = g^{ab}$ is the inverse of the Einstein metric. Further details from this perspective may be found in [14, 17].

Computed in the Einstein scale τ, the previous display reduces to

$$H^{AB} = \begin{pmatrix} g^{ab} & 0 \\ 0 & \frac{1}{(n-1)}\Lambda \end{pmatrix}. \tag{3.8}$$

Note that on an Einstein manifold the projective Cotton tensor $\nabla_{[a}Q_{b]c}$ is zero. Moreover $\nabla^a W_{ab}{}^c{}_d = 0$ by dint of the contracted Bianchi identity. From these observations it follows easily that

$$\nabla^a \Omega_{ab}{}^C{}_D = 0\,,$$

and so the tractor connection is Yang-Mills.

4. BGG and detour complexes

Algebraic BGG resolutions appear naturally in the representation theory of semi-simple Lie algebras, and sequences of Verma modules associated to representations of Borel subalgebras [6]. An extension to parabolic representation theory was developed with Verma modules replaced by generalised Verma modules where the *rôle* of Borel subalgebras is replaced by parabolic subalgebras [60].

These constructions are, in a suitable sense, dual to complexes of invariant differential operators on the corresponding G/P, where G is a semi-simple Lie group and P a parabolic subgroup. Such *homogeneous parabolic geometries* G/P are the flat models for parabolic geometries and the complexes so obtained are called BGG complexes. There are canonical curved analogues of these sequences due to Baston, Čap et al, and others [3, 10, 41], but in general these sequences, do not form complexes.

4.1. **Gauge theory and differential complexes.** It has been known for some time that BGG complexes are related to gauge theories in physics, see *e.g.* [35]. The latter are described in terms of gauge fields, curvatures and their Bianchi identities. Their kinematical gauge structure is captured by a complex:

$$\cdots \longrightarrow \begin{matrix}\text{gauge}\\ \text{parameters}\end{matrix} \longrightarrow \begin{matrix}\text{gauge}\\ \text{potentials}\end{matrix} \longrightarrow \text{curvatures} \longrightarrow \begin{matrix}\text{Bianchi}\\ \text{identities}\end{matrix} \longrightarrow \cdots$$

However, for the dynamics, a different sort of complex is required.

It was observed, first in the setting of even dimensional conformal geometry, that as well as BGG complexes there are *detour complexes* that, in addition to part of the BGG sequence, also use conformally invariant "long operators" [7, 8]. See also [45, 5], where a non-conformal but related construction is developed and studied from a very different perspective. Part of the interest in these detour complexes stems from the fact that they can be complexes in curved settings where the BGG sequences fail to be a complex. This idea was further extended in [54], where classes of detour complexes are constructed for

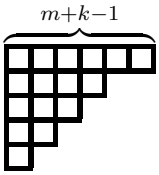
each solution of the Yang-Mills equations; these are conformally invariant in dimension 4. Both [8] and [54] link to variational constructions. Quantisation of the latter was taken up in [48].

For our current purposes, detour complexes are important because they provide a tool for generating gauge theories with dynamics where the diagram we seek takes the form:

$$\cdots \longrightarrow \begin{array}{c}\text{gauge}\\ \text{parameters}\end{array} \longrightarrow \begin{array}{c}\text{gauge}\\ \text{potentials}\end{array} \xrightarrow{\text{detour operator}} \begin{array}{c}\text{equations}\\ \text{of}\\ \text{motion}\end{array} \longrightarrow \begin{array}{c}\text{Noether}\\ \text{identities}\end{array} \longrightarrow \cdots$$

It is this approach that we follow below. First we sketch the construction of BGG sequences focusing on projective geometries.

4.2. **BGG sequences.** In this section we mostly follow the notation and conventions of [38]. On a projective manifold $(M, \boldsymbol{p})$ there is a BGG sequence for every irreducible representation of $Sl(n+1)$,

$\mathbb{V} =$ 

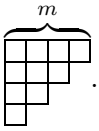
which we indicate schematically here with a Young diagram. Let us fix $\mathbb{V}$. The corresponding tractor bundle is associated to the Cartan bundle via this representation, as in (3.5).

The *first operator* in the BGG sequence is then determined by the tractor connection acting on $\mathcal{V}$. This is understood in terms of the general tools developed in [18, 10, 12]. We sketch the key ideas.

The operator $\mathcal{D} : \mathcal{B}^0 \to \mathcal{B}^1$ is projectively invariant and acts between weighted irreducible tensor bundles $\mathcal{B}^0$ and $\mathcal{B}^1$. Let us say this has order k. Then its symbol is obtained by a composition of the form

$$\mathcal{B}^0 \to (\odot^k T^*M) \otimes \mathcal{B}^0 \to (\odot^k T^*M) \circledcirc \mathcal{B}^0 \cong \mathcal{B}^1 \,,$$

where $\circledcirc$ denotes the Cartan product. Ignoring the projective weight, the bundle $\mathcal{B}^0$ is associated to the Cartan bundle by an irreducible $Sl(n)$ representation (extended trivially to a P-representation)

$\mathcal{B}^0 \simeq$ 

.

The projective weight of $\mathcal{B}^0$ is determined easily from $\mathcal{V}$, but this detail is not important for this general discussion. In the above we used unbolded and bolded Young tableaux for $Sl(n)$ and $Sl(n+1)$ representations, respectively.

Before describing the construction of $\mathcal{D}$, we introduce some algebraic ingredients. There is a grading operator $h \in \mathfrak{sl}(n+1)$ which (identifying $Sl(n+1)$ with its standard linear representation) can be given in the form

$$h = \frac{1}{n+1} \begin{pmatrix} \mathbf{1}_{n\times n} & 0 \\ 0 & -n \end{pmatrix} .$$

This induces a decomposition of $\mathfrak{g} := \mathfrak{sl}(n+1)$

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1} \,,$$

with $[h, \mathfrak{g}_\ell] = \ell\, \mathfrak{g}_\ell$. This is a $|1|$-grading, so $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$. On any $\mathfrak{g}$-irreducible representation $\mathbb{V}$, the grading element can be diagonalized thus producing a natural splitting into eigenspaces for the action of h

$$\mathbb{V}_0 \oplus \mathbb{V}_1 \oplus ... \oplus \mathbb{V}_N\,,$$

with $\mathbb{V}_0$ (or $\mathbb{V}_N$) corresponding to the eigenspace with the lowest (or highest) eigenvalue; moreover, for any eigenvector v with eigenvalue i, we have $\mathfrak{g}_j \mathbb{V}_i \subset \mathbb{V}_{i+j}$. The parabolic subgroup P has Lie algebra $\mathfrak{g}_0 \oplus \mathfrak{g}_1$, and so we thus obtain a natural filtration on the corresponding projective tractor bundle

$$\mathcal{V} = \mathcal{B}^0 \oplus \cdots\,.$$

We now introduce the Kostant codifferential

$$\boldsymbol{\partial}^* : \wedge^{p+1}\mathfrak{g}_{+1} \otimes \mathbb{V} \quad \longrightarrow \quad \wedge^{p}\mathfrak{g}_{+1} \otimes \mathbb{V}\,, \tag{4.1}$$

defined by

$$\boldsymbol{\partial}^*(Z_1 \wedge \cdots \wedge Z_{p+1} \otimes v) = \sum_{i=1}^{p+1} Z_0 \wedge \cdots \hat{Z}_i \cdots \wedge Z_{p+1} \otimes Z_i v\,,$$

when $\mathfrak{g}_{+1}$ is abelian. The Kostant codifferential is P-equivariant and nilpotent (meaning $\boldsymbol{\partial}^* \circ \boldsymbol{\partial}^* = 0$), and its homology is denoted $H_p(\mathfrak{g}_{+1}, \mathbb{V})$. We use now the fact that T^*M can be canonically identified with $\mathcal{G} \times_P \mathfrak{g}_{+1}$, and so the Kostant codifferential induces a projectively invariant map of tractor valued differential forms:

$$\boldsymbol{\partial}^* : \wedge^{p+1}\mathcal{V} \quad \longrightarrow \quad \wedge^{p}\mathcal{V}\,. \tag{4.2}$$

We denote by $H_p(\mathcal{V})$ the corresponding holomology bundle at degree p and note that $H_p(\mathcal{V}) = \mathcal{G} \times_P H_p(\mathfrak{g}_{+1}, \mathbb{V})$. We denote by π the natural bundle map $\pi : \ker(\boldsymbol{\partial}^*) \to H_p(\mathcal{V})$. We are finally ready to construct the splitting operator and the BGG operators.

By construction the bundle $\mathcal{B}^0 = H_0(\mathcal{V})$ and $\mathcal{B}^1 = H_1(\mathcal{V})$ and the BGG sequence continues in this way. The BGG machinery constructs for each p a projectively invariant differential operator called a *splitting operator* $\mathcal{L}_p : H_p(\mathcal{V}) \to \wedge^p\mathcal{V}$. This is characterised as follows.

Proposition 4.1. *The following conditions determine a splitting operator:*

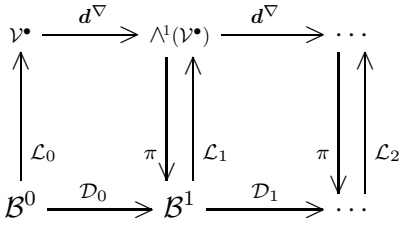
- $\boldsymbol{\partial}^* \mathcal{L}_p(\alpha) = 0$
- $\pi\, \mathcal{L}_p(\alpha) = \alpha$
- $\boldsymbol{\partial}^* \boldsymbol{d}^\nabla \mathcal{L}_p(\alpha) = 0$

for any section α of $H_p(\mathcal{V})$.

We then obtain, by construction, the invariant differential operators

$$\mathcal{D}_p := \pi \circ \boldsymbol{d}^\nabla \circ \mathcal{L}_p : H_p(\mathcal{V}) \to H_{p+1}(\mathcal{V})\,,$$

and these operators form the BGG sequence:



When the geometry is flat, in the Cartan sense, this sequence becomes a complex.

In this section we sketched the construction of the BGG sequences for projective geometries; the construction is quite similar for general parabolic geometries.

4.3. **Yang–Mills detour complexes.** On a pseudo-Riemannian manifold, a basic detour complex arises from the de Rham complex and the Maxwell operator $\boldsymbol{\delta d}$, where $\boldsymbol{\delta}$ is the formal adjoint of the exterior derivative $\boldsymbol{d}$.

$$0 \xrightarrow{\boldsymbol{d}} \bigwedge^0 M \xrightarrow{\boldsymbol{d}} \cdots \xrightarrow{\boldsymbol{d}} \bigwedge^p(M) \xrightarrow{\boldsymbol{\delta d}} \bigwedge^p(M) \xrightarrow{\boldsymbol{\delta}} \cdots \xrightarrow{\boldsymbol{\delta}} \bigwedge^0 M \xrightarrow{\boldsymbol{\delta}} 0. \tag{4.3}$$

In particular if n is even and $p = n/2 - 1$ then this complex is conformally invariant (at least after the introduction of conformally weighted bundles for the right hand side of the complex). This is the simplest case of the family of (in general higher order) complexes found in [7].

If we drop the requirement of conformal invariance then the complex (4.3) is available in both dimension parities and for all p-forms. In particular, if $p = 1$, we obtain the complex

$$0 \to \bigwedge^0(M) \xrightarrow{\boldsymbol{d}} \bigwedge^1(M) \xrightarrow{\boldsymbol{\delta d}} \bigwedge^1(M) \xrightarrow{\boldsymbol{\delta}} \bigwedge^0(M) \to 0\,, \tag{4.4}$$

which, in the spirit of discussion above, encodes the gauge theory of classical source-free electromagnestism.

Now consider twisting the Maxwell detour (4.4) with a connection ∇ on some vector bundle $\mathcal{V}$ and replacing the exterior and interior derivatives by $\boldsymbol{d}^\nabla$, $\boldsymbol{\delta}^\nabla$ twisted by the connection on $\mathcal{V}$. This is in general doomed to failure as curvature means that $\boldsymbol{\delta}^\nabla \circ \boldsymbol{d}^\nabla \circ \boldsymbol{d}^\nabla$ is not zero. However, as shown in [54], there is a useful variant on this first idea as follows.

Suppose that the connection has curvature $\mathcal{F}$. We define its action on $\mathcal{V}$-valued differential forms by

$$\begin{array}{rccc} \mathrm{End}(\mathcal{F}^\sharp): & \bigwedge^1(\mathcal{V}) & \longrightarrow & \bigwedge^1(\mathcal{V}) \\ & \cup & & \cup \\ & \Xi_a{}^{\mathcal{C}} & \longmapsto & \mathcal{F}_a{}^{b\mathcal{C}}{}_{\mathcal{D}}\, \Xi_b{}^{\mathcal{D}} \end{array}$$

where $\mathcal{C}$, $\mathcal{D}$ are abstract indices for the bundle $\mathcal{V}$. We will often simply write $\mathrm{End}(\mathcal{F}^\sharp)\Xi_a{}^{\mathcal{C}}$ for image of this map acting on $\Xi_a{}^{\mathcal{C}}$.

We construct now the operator

$$\begin{array}{rccc} \boldsymbol{M}^\nabla: & \bigwedge^1(\mathcal{V}) & \longrightarrow & \bigwedge^1(\mathcal{V}) \\ & \cup & & \cup \\ & \Xi_a{}^{C} & \longmapsto & \left(\boldsymbol{\delta}^\nabla \boldsymbol{d}^\nabla - \mathrm{End}(\mathcal{F}^\sharp)\right) \Xi_a{}^{C}\,, \end{array}$$

which leads to the following theorem:

Theorem 4.2. *(See* [54]*.) The sequence of operators*

$$0 \longrightarrow \bigwedge^0(\mathcal{V}) \xrightarrow{\boldsymbol{d}^\nabla} \bigwedge^1(\mathcal{V}) \xrightarrow{\boldsymbol{\delta}^\nabla \boldsymbol{d}^\nabla - \mathrm{End}(\mathcal{F}^\sharp)} \bigwedge^1(\mathcal{V}) \xrightarrow{\boldsymbol{\delta}^\nabla} \bigwedge^0(\mathcal{V}) \longrightarrow 0\,,$$

is a complex if and only if the curvature $\mathcal{F}$ satisfies the Yang–Mills equation

$$\boldsymbol{\delta}^\nabla \mathcal{F} = 0\,. \tag{4.5}$$

If ∇ preserves a metric on $\mathcal{V}$ then the complex is formally self-adjoint.

Proof. A short calculation shows that

$$\boldsymbol{M}^{\nabla}\boldsymbol{d}^{\nabla} = \varepsilon(\boldsymbol{\delta}^{\nabla}\mathcal{F}) \quad \text{and} \quad \boldsymbol{\delta}^{\nabla}\boldsymbol{M}^{\nabla} = -\iota(\boldsymbol{\delta}^{\nabla}\mathcal{F})\,, \tag{4.6}$$

where ι and ε denote interior and exterior multiplication. Finally, in the case that ∇ preserves a metric, the operator $\boldsymbol{M}^{\nabla}$ is formally self-adjoint by construction. Then, also by construction, the sequence is formally self-adjoint. □

Following [54], we call the sequence of the theorem the *Yang-Mills detour complex* and say a connection is *Yang–Mills* when it obeys the source-free Yang-Mils equation (4.5). There is such a complex for every Yang-Mills connection, but our interest here is that some of these can be used to generate other interesting complexes. A first observation in this direction is that if a connection on a vector bundle $\mathcal{U}$ is Yang-Mills, then so are the induced connections on the dual of $\mathcal{U}$, on tensor powers of these and on tensor parts thereof. So in each case there is such a complex. We wish to combine this observation with the next less obvious construction.

4.4. **Translating.** Consider the general situation

$$\begin{array}{ccccccc} \wedge^0(\mathcal{V}) & \xrightarrow{\boldsymbol{d}^{\nabla}} & \wedge^1(\mathcal{V}) & \xrightarrow{\boldsymbol{M}^{\nabla}} & \wedge^1(\mathcal{V}) & \xrightarrow{\boldsymbol{\delta}^{\nabla}} & \wedge^0(\mathcal{V}) \\ \uparrow{\scriptstyle \mathcal{L}_0} & & \uparrow{\scriptstyle \mathcal{L}_1} & & \downarrow{\scriptstyle \mathcal{L}^1} & & \downarrow{\scriptstyle \mathcal{L}^0} \\ E & \xrightarrow{\mathcal{D}} & F & \xrightarrow{M} & F^\star & \xrightarrow{\mathcal{D}^\star} & E^\star \end{array} \tag{4.7}$$

where $\mathcal{D} : E \to F$ and $\mathcal{D}^\star : F^\star \to E^\star$, $\mathcal{L}_0 : E \to \wedge^0(\mathcal{V})$, $\mathcal{L}_1 : F \to \wedge^1(\mathcal{V})$, $\mathcal{L}^1 : \wedge^1(\mathcal{V}) \to F^\star$, and $\mathcal{L}^0 : \wedge^0(\mathcal{V}) \to E^\star$ are differential operators and M is defined to be the composition

$$\mathcal{L}^1\,\boldsymbol{M}^{\nabla}\mathcal{L}_1 : F \to F^\star.$$

Suppose now that the left and right squares are commutative:

$$\boldsymbol{d}^{\nabla}\mathcal{L}_0 = \mathcal{L}_1\mathcal{D} \quad \text{and} \quad \mathcal{D}^\star\mathcal{L}^1 = \mathcal{L}^0\boldsymbol{\delta}^{\nabla}\,.$$

Then it follows that

$$\begin{aligned} M\mathcal{D} &= \mathcal{L}^1\epsilon(\boldsymbol{\delta}^{\nabla}\Omega)\mathcal{L}_0\,, \\ \mathcal{D}^\star M &= -\mathcal{L}^0\iota(\boldsymbol{\delta}^{\nabla}\Omega)\mathcal{L}_1\,. \end{aligned}$$

Thus, if the connection is Yang–Mills (*i.e.*, $\boldsymbol{\delta}^{\nabla}\mathcal{F} = 0$), it follows at once from Theorem 4.2 that the lower differential sequence

$$E \xrightarrow{\mathcal{D}} F \xrightarrow{M} F^\star \xrightarrow{\mathcal{D}^\star} E^\star\,,$$

is a complex. Furthermore, observe that if E and F are tensor bundles and also the connection ∇ preserves a metric on $\mathcal{V}$, then we can construct the last square by taking adjoints of all the differential operators from the first square. In this case the commutativity of the last square is immediate from commutativity of the first square, and by using once more Theorem 4.2 it follows that the lower sequence is formally self-adjoint.

In summary we have recovered the following paraphrasing of a result from [54]:

Theorem 4.3. *Suppose that the connection ∇ on $\mathcal{V}$ is metric and Yang-Mills. Suppose also that the first square of the diagram (4.7) commutes. Then we obtain a formally self-adjoint detour complex*

$$0 \longrightarrow E \xrightarrow{\mathcal{D}} F \xrightarrow{M} F^\star \xrightarrow{\mathcal{D}^\star} E^\star \longrightarrow 0. \tag{4.8}$$

In the spirit of the discussion in Section 4.1, we can use the construction leading to Theorem 4.3 as a tool for generating equations on motion M operators on potentials in F. Then, also by construction, these are invariant with respect to the gauge transformations $\mathcal{D} : E \to F$, and satisfy Bianchi identities $\mathcal{D}^\star : F^\star \to E^\star$.

4.5. **BGG detour complexes.** We use the term *BGG detour complex* to mean detour complexes which use, in part, operators from the BGG complexes.

In the following we will be, in particular, interested in using Theorem 4.3 to generate BGG detour complexes of the form (4.8), where $\mathcal{D}$ is a first BGG operator. In these constructions, suitable projective tractor bundles will play the *rôle* of $\mathcal{V}$, and we recall that, in the case the projective structure is Einstein and non Ricci-flat, these have a metric preserved by the tractor connection.

5. Partially massless models of maximal depth

Here, on non-flat constant curvature backgrounds, we show, using the Yang-Mills detour theory, that for any spin $k \in \mathbb{Z}_{\geq 2}$, there is a PM gauge invariant equation of motion and gauge invariant constraint system for fields in $\odot^k T^*M$. The gauge operators are given by the restriction to constant curvature manifolds of the projectively invariant first BGG operators

$$\mathcal{D} : \mathcal{E}(k-1) \to \odot^k T^*M(k-1)\,.$$

These linear operators take the form

$$\mathcal{D}(\sigma) = \nabla_{(a} \cdots \nabla_{c)}\sigma + \text{ lower order terms.}$$

General algorithms for the explicit formulae for these are available in [40]. In fact these formulae are members of general families that take the same form on all parabolic geometries [11, 47]. The examples of order three and two are given in, respectively, (5.8) and (6.4) below.

Theorem 5.1. *Let k be a positive integer. On a constant curvature manifold the Yang-Mills detour complex associated with the projective tractor connection determines a canonical formally self-adjoint detour complex*

$$0 \to \mathcal{E} \xrightarrow{\mathcal{D}} \odot^k T^*M \xrightarrow{M} \odot^k T^*M \xrightarrow{\mathcal{D}^\star} \mathcal{E} \to 0,$$

where $\mathcal{D}^\star$ is the (order k) adjoint of the operator $\mathcal{D}$ and the equation of motion operator M is second order.

Proof. We work first on any projectively flat structure $(M, \boldsymbol{p})$.

There is a projectively invariant BGG splitting operator

$$\mathcal{L}_0 : \mathcal{E}(k-1) \to \odot^{k-1}\mathcal{T}^*.$$

Thus the composition $\boldsymbol{d}^\nabla \mathcal{L}_0 : \mathcal{E}(k-1) \to \wedge^1(\odot^{k-1}\mathcal{T}^*)$ is also projectively invariant. Next, by the standard BGG theory $H_1(\odot^{k-1}\mathcal{T}^*) \cong \odot^k T^*M(k-1)$ (and we identify these spaces), $\boldsymbol{\partial}^* \circ \boldsymbol{d}^\nabla \circ \mathcal{L}_0 = 0$, and the first BGG operator above is given by $\pi \circ \boldsymbol{d}^\nabla \circ \mathcal{L}_0$, where π is the bundle map arising from the map from $\ker(\boldsymbol{\partial}^*)$ to homology.

In fact we can say more. First we observe that, from the composition series for the tractor bundle, there is a projectively invariant bundle inclusion

$$\imath : \odot^k T^*M(k-1) \to \wedge^1(\odot^{k-1}\mathcal{T}^*), \tag{5.1}$$

and so it follows from the characterising properties of $\mathcal{L}_1$ that $\mathcal{L}_1 = \imath$. But then it follows that

$$\boldsymbol{d}^\nabla \circ \mathcal{L}_0 = \mathcal{L}_1 \circ \mathcal{D}. \tag{5.2}$$

This last result holds because, from the classification of projectively invariant differential operators on the sphere, there is only one, up to multiplication by a non-zero constant, projectively invariant operator on $\mathcal{E}(k-1)$, and this is the operator $\mathcal{D}$. On the other hand, the bundle $\odot^k T^*M(k-1)$ occurs only once in the composition series for $\wedge^1(\odot^{k-1}\mathcal{T}^*)$, and this is realised by $\imath$.

This establishes a commuting first square, as in the diagram (4.7) leading to Theorem 4.3. Now we restrict to a constant curvature manifold (M, g), with $\Lambda \neq 0$. This has the projective structure $\boldsymbol{p} = [\nabla^g]$ and we note that $\boldsymbol{p}$ is, in particular, projectively flat, so the above results are available. Furthermore, in this setting the manifold is Einstein so we may use τ (see (3.7)) to trivialise density bundles and there is a parallel tractor metric given by (3.8). This metric induces a metric on $\odot^{k-1}\mathcal{T}^*$. Thus we may write adjoints for all of the operators in the first square and so obtain a commuting last square, as in the diagram (4.7). Furthermore using this metric, the operator $\boldsymbol{M}^\nabla$ is formally self adjoint and so the claimed system follows from Theorem 4.3. □

By construction then, the operator $M : \odot^k T^*M \to \odot^k T^*M$ is gauge invariant with respect to the order k gauge operator $\mathcal{D}$, while $\mathcal{D}^\star : \odot^k T^*M \to \mathcal{E}$ provides the integrability conditions (*i.e.* "Bianchi identities") for this.

Remark 5.2. For those familiar with tractor calculus, there is an alternative approach to aspects of the proof above. For example, in the projectively flat setting (as in the theorem) it is easily verified that the splitting operator $\mathcal{L}_0 : \mathcal{E}(k-1) \to \odot^{k-1}\mathcal{T}^*$, is given explicitly by

$$\sigma \mapsto D_{A_1} \cdots D_{A_{k-1}} \sigma \,,$$

where D_A is the projective Thomas D-operator. The right-hand-side here is symmetric since on projectively flat manifolds the D-operators mutually commute. Then

$$X^{A_1} D_{A_1} \cdots D_{A_{k-1}} D_{A_k} \sigma = 0 \,,$$

and from this it follows that (5.2) holds.

5.1. **Constraints.** In our current context, the detour construction leading to Theorem 4.3 encodes more than the detour complex of Theorem 5.1. As well as the Bianchi identity on the equation of motion operator M of the Yang-Mills detour complex, it also captures gauge invariant constraints on the potential.

Theorem 5.3. *Applied to give equations on the potential, the differential operator*

$$\boldsymbol{M}^\nabla \circ \mathcal{L}_1 : \odot^k T^*M \to \wedge^1(\odot^k \mathcal{T}^*), \tag{5.3}$$

extends the equation of motion of the BGG detour operator $M : \odot^k T^*M \xrightarrow{M} \odot^k T^*M$ (of Theorem 5.1) by a system of gauge invariant constraints.

There are relations between these constraints and the equations of motion captured by the fact that image of $\boldsymbol{M}^\nabla \circ \mathcal{L}_1$ lies in the kernel of $\boldsymbol{\delta}^\nabla$.

Proof. The first statement follows at once from the fact that

$$\boldsymbol{M}^\nabla \circ \mathcal{L}_1 \circ \mathcal{D} = 0,$$

since the diagram (4.7) is commutative. The last statement is adjoint of this. □

We are now ready to take up examples. There is no strictly partially massless system for spin one. We will treat spin two in more general setting below, so we begin with spin three.

5.2. **Spin three.** Theorem 5.1 shows that the BGG detour complex produces a system of gauge invariant equations of motion and constraints for higher spin fields in non-flat, constant curvature backgrounds. These spaces arise from the intersection of projectively flat and Einstein projective structures. In the following we show, as a special case, that this recovers the standard, maximal depth, PM spin three theory in four dimensions of [27, 28]. This theory is described in terms of a totally symmetric rank three tensor φ_{abc} and a scalar auxiliary χ, introduced in order that the equations of motion are Lagrangian. These read

$$\begin{cases} \Delta\varphi_{abc} - \frac{5\Lambda}{3}\varphi_{abc} - 3\nabla_{(a}\nabla.\varphi_{bc)} + 3\nabla_{(a}\nabla_b\bar{\varphi}_{c)} - 3g_{(ab}\Delta\bar{\varphi}_{c)} \\ \qquad + 3g_{(ab}\nabla^d\nabla^e\varphi_{c)de} - \frac{3}{2}g_{(ab}\nabla_{c)}\nabla.\bar{\varphi} + \frac{3}{4}g_{(ab}\nabla_{c)}\chi = 0\,, \\ \\ \Delta\chi + \frac{\Lambda}{3}\nabla.\bar{\varphi} = 0\,, \end{cases} \tag{5.4}$$

where $\bar{\varphi}_a := \varphi_{ab}{}^b$ and $\nabla.\varphi_{bc} := \nabla_a\varphi^a{}_{bc}$. These equations imply, as integrability conditions, the constraints

$$\begin{cases} \nabla_{(a}\nabla_{b)_\circ}\chi - \Lambda\nabla.\varphi_{(ab)_\circ} + 2\Lambda\nabla_{(a}\bar{\varphi}_{b)_\circ} = 0\,, \\ \nabla_a\chi + \frac{\Lambda}{3}\bar{\varphi}_a = 0\,. \end{cases} \tag{5.5}$$

We use the notation $(\cdots)_\circ$ to indicate the trace-free, symmetrized part of a group of indices. In addition to these constraints, this system enjoys a higher derivative, scalar, gauge invariance

$$\begin{cases} \delta\varphi_{abc} = \big(\nabla_{(a}\nabla_{(b}\nabla_{c)_\circ)} + \frac{\Lambda}{2}\,g_{(ab}\nabla_{c)}\big)\,\varsigma\,, \\ \delta\chi = -\frac{\Lambda}{3}\big(\Delta + \frac{10\Lambda}{3}\big)\,\varsigma\,. \end{cases}$$

In the above, the key ingredient linking the PM model to the BGG machinery is the gauge operator $\nabla_{(a}\nabla_{(b}\nabla_{c)_\circ)} + \frac{\Lambda}{2}g_{(ab}\nabla_{c)}$ acting on the scalar gauge parameter ς. For that, we must relate it to a projectively invariant operator. To facilitate this we remove all instances of the inverse metric; this can be achieved by defining the trace-adjusted field

$$\psi_{abc} := \varphi_{abc} + \frac{1}{2}g_{(ab}\bar{\varphi}_{c)}\,, \tag{5.6}$$

whose gauge transformation is

$$\delta\psi_{abc} = \Big(\nabla_{(a}\nabla_b\nabla_{c)} + \frac{4\Lambda}{3}g_{(ab}\nabla_{c)}\Big)\,\varsigma\,. \tag{5.7}$$

Remarkably, the projectively invariant operator

$$\begin{array}{ccc} \mathcal{D} : \mathcal{E}(2) & \longrightarrow & \odot^3T^*M(2) \\ \cup & & \cup \\ \sigma & \mapsto & \frac{1}{2}\big[\nabla_{(a}\nabla_b\nabla_{c)} + 4Q_{(ab}\nabla_{c)} + 2(\nabla_{(a}Q_{bc)})\big]\sigma \end{array} \tag{5.8}$$

matches the gauge operator appearing in (5.7) when computed in the Einstein scale (for which $Q_{ab} = \frac{\Lambda}{3}g_{ab}$).

As a further happy consequence of the field redefinition (5.6), the *divergence constraint* on the first line of (5.5), using the second line of that display to eliminate χ, simplifies to

$$\nabla.\psi_{ab} = \nabla_{(a}\bar{\psi}_{b)}\,.$$

According to the BGG construction described in the previous section, the $Sl(5)$ module required for maximal depth, spin three is given by the Young diagram $\square\square$, and hence sections of the symmetric product of the cotractor bundle, denoted by $\mathcal{T}_{(AB)}$. In a given scale, a general section of this bundle splits as

$$\begin{pmatrix} w_{ab} & v_a \\ v_b & \tau \end{pmatrix} =: \begin{pmatrix} \tau \\ v_a \\ w_{ab} \end{pmatrix},$$

on which the tractor connection acts as follows:

$$\nabla^{\mathcal{T}}_c \begin{pmatrix} \tau \\ v_a \\ w_{ab} \end{pmatrix} = \begin{pmatrix} \nabla_c\tau - 2v_c \\ \nabla_c v_a + Q_{ca}\tau - w_{ca} \\ \nabla_c w_{ab} + 2Q_{c(a}v_{b)} \end{pmatrix}.$$

The first part of the diagram (4.7) (encoding the gauge transformation) is easily computed:

$$\begin{array}{ccccc}
\mathcal{T}_{(AB)} \ni \begin{pmatrix} \sigma \\ \frac{1}{2}\nabla_a\sigma \\ (\frac{1}{2}\nabla_a\nabla_b + Q_{ab})\sigma \end{pmatrix} & \overset{\boldsymbol{d}^\nabla}{\longmapsto} & \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2}\,\boldsymbol{\nabla}\nabla_a\nabla_b\sigma + \boldsymbol{Q}_{(a}\nabla_{b)}\sigma + \boldsymbol{\nabla}(Q_{ab}\sigma) \end{pmatrix} \in \bigwedge^1(\mathcal{T}_{(AB)}) & \ni & \begin{pmatrix} 0 \\ 0 \\ \boldsymbol{\psi}_{ab} \end{pmatrix} \\
\uparrow \mathcal{L}_0 & & \downarrow \pi & & \uparrow \mathcal{L}_1 \\
\mathcal{E}(2) \ni \sigma & \overset{\mathcal{D}}{\longmapsto} & \big(\tfrac{1}{2}\nabla_{(a}\nabla_b\nabla_{c)} + 2Q_{(ab}\nabla_{c)} + (\nabla_{(a}Q_{bc)})\big)\sigma \in \odot^3 T^*M(2) & \ni & \psi_{abc}
\end{array}$$

Here $\boldsymbol{\psi}_{ab}$ and $\boldsymbol{Q}_a$ denote the one-forms made from ψ_{abc} and Q_{ab} by soldering (the soldering form is denoted $\boldsymbol{e}^a$) and ∇ is any connection in $\boldsymbol{p}$.

We now calculate in the Einstein scale and use the parallel tractor metric:

$$H_{AB} = \begin{pmatrix} g_{ab} & 0 \\ 0 & \frac{3}{\Lambda} \end{pmatrix}.$$

This allows us to construct the detour long operator $\boldsymbol{M}^\nabla \equiv \boldsymbol{\delta}^\nabla\boldsymbol{d}^\nabla - \mathrm{End}(\Omega^\#)$:

$$\begin{array}{ccccc}
\bigwedge^1(\mathcal{T}_{(AB)}) \ni \begin{pmatrix} 0 \\ 0 \\ \boldsymbol{\psi}_{ab} \end{pmatrix} & \overset{\boldsymbol{M}^\nabla}{\longmapsto} & \begin{pmatrix} 0 \\ \nabla.\boldsymbol{\psi}_a - \boldsymbol{\nabla}\bar{\psi}_a \\ \Delta\boldsymbol{\psi}_{ab} - \boldsymbol{\nabla}\nabla.\psi_{ab} + \frac{2\Lambda}{3}\boldsymbol{e}_{(a}\bar{\psi}_{b)} - \frac{5\Lambda}{3}\boldsymbol{\psi}_{ab} \end{pmatrix} \in \bigwedge^1(\mathcal{T}_{(AB)}) \\
\uparrow \mathcal{L}_1 & & \downarrow \mathcal{L}^1 \\
\odot^3 T^*M \ni \psi_{abc} & \overset{M}{\longmapsto} & \Delta\psi_{abc} - \nabla_{(a}\nabla.\psi_{bc)} + \frac{2\Lambda}{3}g_{(ab}\bar{\psi}_{c)} - \frac{5\Lambda}{3}\psi_{abc} \in \odot^3 T^*M
\end{array}$$

The operator M on the bottom line of this diagram should be compared with the equations of motion of the PM system. By construction, it annihilates the gauge operator of (5.7). Before making this comparison, we first note that the detour complex on the top line of the diagram encodes further information. First, the second line in the image of $\boldsymbol{M}^\nabla$ reads (making the form index explicit)

$$\nabla.\psi_{ab} - \nabla_a\bar{\psi}_b = 0\,.$$

This equation has both a symmetric and antisymmetric piece. The latter implies that $\bar{\psi}_a$ is a closed one-form, and thus locally the gradient of some scalar, which we identify with the auxiliary field χ:

$$\bar{\psi}_a = -\frac{6}{\Lambda}\nabla_a\chi\,.$$

This reproduces the second constraint in (5.5). Turning to the totally symmetric piece, it immediately reproduces the divergence constraint as expressed in (5.2).

It remains to analyse the bottom slot of the image of $\boldsymbol{M}^{\nabla}$. It is not difficult to verify that its mixed symmetry part vanishes, modulo the divergence constraint (5.2); here one also uses that the structure is Weyl-flat. The totally symmetric part of the bottom slot in the image of $\boldsymbol{M}^{\nabla}$ matches exactly the equation of motion (5.4) for φ_{abc}, upon employing (5.6), again modulo (5.2). Since the operator $\mathcal{L}^1$ projects the bottom slot onto its totally symmetric part, this establishes our claim.

Finally, we note, that for any projectively flat structure, $\boldsymbol{d}^{\nabla}\mathcal{L}_1(\varphi_{abc})$ produces a tractor for which only the bottom slot, $\boldsymbol{\nabla\varphi}_{bc}$, is nonvanishing, and hence both projectively and gauge invariant. This quantity is the spin three generalization of the PM curvature found in [33].

6. Einstein Backgrounds

Einstein metrics play a special *rôle* in projective geometry. In particular, recall from Section 3.3, that a metric g which is Einstein but not Ricci flat, is equivalent to a non-degenerate parallel metric H on the projective tractor bundle $\mathcal{T}$ [17]. We now study projective BGG sequences and possible detour complexes in an Einstein setting. This allows us to address the physical question of higher spin propagation on Einstein manifolds.

6.1. **Partially massless spin two.** We begin our Einstein investigation with the propitious, spin two case. As mentioned in the introduction, PM spin two was first discovered in studies of constant curvature, lightcone propagation [24] (see as well [27, 30]). However, it also has an interesting geometric, conformal gravity, origin [61, 25, 26]: Consider the four-dimensional conformal gravity action,

$$S = \int \epsilon_{abcd}\, \boldsymbol{W}^{ab} \wedge \boldsymbol{W}^{cd}\,.$$

This action is extremized by vanishing of the Bach tensor B_{cd}. The latter can be defined by a differential operator acting on the Schouten tensor

$$\mathsf{B}_{ab} = \Big(-\delta^c_a\, \Delta\, \delta^d_b + \nabla^c \delta^d_{(a} \nabla_{b)} + \mathsf{W}^c{}_{ab}{}^d \Big) \mathsf{P}_{cd}\,.$$

Defining $\varphi_{ab} = \frac{\Lambda}{6} g_{ab} - \mathsf{P}_{ab}$, then the Bach flat condition of the above display, dropping terms nonlinear in φ, becomes (denoting the trace by $\bar\varphi := \varphi^a{}_a$ and $\nabla.\varphi_a := \nabla^b \varphi_{ab}$)

$$\begin{aligned} &\Delta\varphi_{ab} - 2\nabla_{(b}\nabla.\varphi_{a)} + g_{ab}\nabla.\nabla.\varphi + \nabla_a\nabla_b\bar\varphi - g_{ab}\Delta\bar\varphi \\ &\qquad - 2\mathsf{W}_a{}^{cd}{}_b\varphi_{cd} - \frac{4}{3}\Lambda(\varphi_{ab} - \frac{1}{4} g_{ab}\bar\varphi) = 0\,. \end{aligned} \tag{6.1}$$

The above display is exactly the PM equation of motion in an Einstein background [36] while the quantity φ_{ab} measures the failure of g_{ab} to be Einstein. The divergence constraint

$$\nabla^b \varphi_{ab} = \nabla_a \bar\varphi\,, \tag{6.2}$$

follows as an integrability condition of (6.1). Moreover, the PM equation of motion is variational and enjoys a higher derivative gauge invariance

$$\delta\varphi_{ab} = \big(\nabla_a\nabla_b + \frac{\Lambda}{3} g_{ab}\big)\varepsilon\,. \tag{6.3}$$

The gauge operator here is intimately related to projective geometry and BGG sequences.

6.2. **Spin two BGG.** We now return to projective geometry and consider the operator

$$\begin{array}{rccc} \mathcal{D}: & \mathcal{E}(1) & \longrightarrow & \odot^2 T^*M(1) \\ & \cup & & \cup \\ & \sigma & \mapsto & \big(\nabla_{(a}\nabla_{b)} + Q_{(ab)}\big)\,\sigma \end{array} \tag{6.4}$$

For Einstein projective structures, computing in the Einstein scale, the above reproduces the gauge operator appearing in 6.3.

Using the parallel tractor metric we construct a detour complex which describes the spin two PM system on Einstein backgrounds. The first BGG operator is the one displayed in (6.4). We have summarized the other ingredients of this complex in the table below:

$\mathcal{D}$:	$\mathcal{E}(1)$	$\to$	$\odot^2 T^*M(1)$	$\mathcal{D}^*$:	$\odot^2 T^*M(-1)$	$\to$	$\mathcal{E}(-5)$
	$\cup$		$\cup$		$\cup$		$\cup$
	σ	$\mapsto$	$\big(\nabla_a\nabla_b + Q_{ab}\big)\sigma$		γ_{ab}	$\mapsto$	$\big(\nabla^a\nabla^b + Q^{ab}\big)\gamma_{ab}$
$\mathcal{L}_0$:	$\mathcal{E}(1)$	$\to$	$\mathcal{T}_A$	$\mathcal{L}^0$:	$\mathcal{T}_A(-4)$	$\to$	$\mathcal{E}(-5)$
	$\cup$		$\cup$		$\cup$		$\cup$
	σ	$\mapsto$	$(\sigma, \nabla\sigma)$		(ρ, μ_a)	$\mapsto$	$\frac{\Lambda}{3}\rho - \nabla^a\mu_a$
$\mathcal{L}_1$:	$\odot^2 T^*M(1)$	$\to$	$\wedge^1(\mathcal{T}_A)$	$\mathcal{L}^1$:	$\wedge^1(\mathcal{T}_A(-2))$	$\to$	$\odot^2 T^*M(-1)$
	$\cup$		$\cup$		$\cup$		$\cup$
	φ_{ab}	$\mapsto$	$(0, \varphi_{ab})$		(ξ_a, ν_{ab})	$\mapsto$	$\nu_{(ab)}$

A straightforward computation shows that

$$\boldsymbol{M}^{\nabla}\mathcal{L}_1(\varphi_{ab}) = \begin{pmatrix} -\nabla.\boldsymbol{\varphi} + \boldsymbol{\nabla}\bar{\varphi} \\ \Delta\,\boldsymbol{\varphi}_a - \nabla^b\boldsymbol{\nabla}\varphi_{ab} - \boldsymbol{W}^{cd}{}_a\varphi_{cd} \end{pmatrix}. \tag{6.5}$$

Here, as earlier, $(\boldsymbol{\varphi}_a, \boldsymbol{W}^{cd}{}_a)$ denote the soldered one-forms made from φ_{ba} and $\mathsf{W}_b{}^{cd}{}_a$, while ∇ is the Levi-Civita coupled exterior derivative and $\Delta := \nabla^a\nabla_a$ is the Bochner Laplacian. By construction (see Theorem 5.1), the quantities in the above enjoy the gauge invariance (6.3). The top slot is exactly the constraint 6.2. The bottom slot has both a symmetric and antisymmetric piece. The latter vanishes on Einstein backgrounds using the constraint. The symmetric piece (which is the image of $\mathcal{L}^1$), again modulo the constraint, is precisely the PM equation of motion 6.1. Finally, note that $\boldsymbol{d}^{\nabla}\mathcal{L}_1(\varphi_{ab})$ produces a tractor for which only the bottom slot, $\boldsymbol{\nabla}\boldsymbol{\varphi}_b$, is nonvanishing, and hence both projectively and gauge invariant. This quantity is the PM curvature found in [33].

6.3. **Higher spin Einstein obstructions.** Spins greater than two do not enjoy the special status of metric fluctuations which arise as the linearization of a consistent interaction theory. Their propagation in constant curvature spaces has been studied in detail [27, 28, 29, 30, 31, 32, 34]. In this section we employ the BGG machine to characterize the obstruction to putative maximal depth PM spin three couplings to Einstein backgrounds.

Again our starting point is the projectively invariant operator $\mathcal{D}$ given in (5.8). However, now reconsider the diagram:

$$\begin{array}{ccc} \mathcal{T}_{(AB)} & \xrightarrow{\boldsymbol{d}^{\nabla}} & \wedge^1(\mathcal{T}_{(AB)}) \\ \uparrow{\scriptstyle \mathcal{L}_0} & & \uparrow{\scriptstyle \mathcal{L}_1} \\ \mathcal{E}(2) & \xrightarrow{\mathcal{D}} & \odot^3 T^*M(2) \end{array}$$

When the structure $\boldsymbol{p}$ is projectively flat the above diagram is commutative, while to construct the corresponding detour complex we must require the structure to be projective Einstein; these two conditions together yield constant curvature theories.

Relaxing the projectively flat condition, we may still construct a BGG detour, at the cost of replacing the space $\odot^3 T^*M(2)$ by the reducible bundle $(T^*M \otimes \odot^2 T^*M)(2)$. We chronicle this new detour complex in Proposition 6.6 at the end of this section. There is however, an alternate—well-known in the theory of prolongations for overdetermined systems [39, 42])—method to maintain commutativity of the above diagram. This is achieved by replacing ∇ by the *prolongation connection* $\widetilde{\nabla}$, defined such that the diagram above commutes. Of course, this new connection may not, in general, solve the Yang–Mills equations required for the detour sequence to be a complex. This gives a tight characterization of the obstruction to higher spin, Einstein, PM models.

We are now tasked with finding the prolongation connection $\widetilde{\nabla}$ on the vector bundle $\mathcal{T}_{(AB)}$ obeying

$$\boldsymbol{d}^{\widetilde{\nabla}} \circ \mathcal{L}_0 = \mathcal{L}_1 \circ \mathcal{D}\,.$$

For that computation we focus, for simplicity on Einstein projective structures. In the Einstein scale we find

$$\Big(\boldsymbol{d}^{\nabla} \circ \mathcal{L}_0 - \mathcal{L}_1 \circ \mathcal{D}\Big)(\sigma) = \frac{1}{3}\, Z^a Z^b\, \boldsymbol{W}_{(ab)}{}^d \nabla_d \sigma =: q\Big(\frac{1}{3} W_{c(ab)}{}^d \nabla_d \sigma\Big)\,. \tag{6.6}$$

Here we have used the map $Z : T^*M(1) \longrightarrow \mathcal{T}^*$ in the composition series (3.3) to define the insertion operator q into the bottom slot of $\wedge^1\big(\mathcal{T}_{(AB)}\big)$:

$$\begin{array}{ccccc} q & : & \big(T^*M \otimes \odot^2 T^*M\big)(2) & \longrightarrow & \wedge^1\big(\mathcal{T}_{(AB)}\big) \\ & & \cup & & \cup \\ & & \alpha_{cab} & \longmapsto & \begin{pmatrix} 0 \\ 0 \\ \boldsymbol{\alpha}_{ab} \end{pmatrix} \end{array} \tag{6.7}$$

The canonical projection $\ker(\iota(X)) \cap \wedge^1\big(\mathcal{T}_{(AB)}\big) \to \big(T^*M \otimes \odot^2 T^*M\big)(2)$ is denoted by $q^\star$. Note that for Einstein projective structures, $q^\star$ is the formal adjoint of q.

On sections of $\mathcal{T}_{(AB)}$ the prolongation connection is given by

$$\widetilde{\nabla}_a \begin{pmatrix} \tau \\ v_b \\ w_{bc} \end{pmatrix} := \begin{pmatrix} \nabla_a \tau - 2v_a \\ \nabla_a v_b + Q_{ab}\tau - w_{ab} \\ \nabla_a w_{bc} + 2Q_{a(b} v_{c)} - \frac{2}{3} W_{a(bc)}{}^d v_d \end{pmatrix}\,.$$

Hence we have by now established the following result:

Lemma 6.1. *The following diagram commutes for projective Einstein structures:*

$$\begin{array}{ccc} \mathcal{T}_{(AB)} & \xrightarrow{\boldsymbol{d}^{\widetilde{\nabla}}} & \wedge^1(\mathcal{T}_{(AB)}) \\ \uparrow{\scriptstyle \mathcal{L}_0} & \circlearrowleft & \uparrow{\scriptstyle \mathcal{L}_1} \\ \mathcal{E}(2) & \xrightarrow{\mathcal{D}} & \odot^3 T^*M(2) \end{array}$$

Remark 6.2. The above result can be trivially extended to general projective structures $\boldsymbol{p}$.

The curvature $\widetilde{\Omega} \in \wedge^2\big(\mathrm{End}(\mathcal{T}_{(AB)})\big)$ of the prolongation connection $\widetilde{\nabla}$ acting on a section of $\mathcal{T}_{(AB)}$ is given by:

$$\widetilde{\Omega} \circ \begin{pmatrix} \tau \\ v_a \\ w_{ab} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\frac{2}{3}(\boldsymbol{\nabla}\boldsymbol{W}_{(ab)}{}^c)v_c + 2\,\boldsymbol{W}_{(a}{}^c w_{b)c} + \frac{2}{3}\boldsymbol{W}_{(ab)}{}^c\,\boldsymbol{w}_c \end{pmatrix}.$$

As usual, $\boldsymbol{w}_b$ here denotes the one-form obtained by soldering w_{ab}. This curvature does not obviously obey the Yang–Mills equation (4.5) (replacing, of course $\nabla \to \widetilde{\nabla}$) unless the structure is projectively flat.

Remark 6.3. The detour operator arises as the second variation of a Yang–Mills action principle [54]. Hence, one might consider other gauge invariant action principles whose first and second variations would modify, respectively, the Yang–Mills equation and detour operator. Whether there exists such an action principle for which the connection $\widetilde{\nabla}$ is "Yang–Mills", is an open question.

Remark 6.4. It is interesting to note that some results do exist for massive spin three theories coupled to curved backgrounds. In particular, Zinoviev has given an action principle purported to describe massive spin three excitations on general Ricci flat spacetimes [72]. This action depends on a quartet of totally symmetric, rank $(0,1,2,3)$, tensor fields and enjoys accompanying gauge invariances. For Minkowski backgrounds, the rank $(0,1,2)$ "Stückelberg" fields of that model can be algebraically gauged away leaving a minimal field content that can be compared with that found within our BGG framework. This might be taken as evidence for the existence of curved versions of the spin three PM BGG detour complex. However we note that for general Ricci flat backgrounds, the auxiliaries in the approach of [72] can no longer be algebraically gauged away.

Finally, as promised, we describe a new detour complex whose long operator is defined for reducible bundles. First we recompute (6.6) for general projective structures:

$$\Big[\boldsymbol{d}^{\nabla} \circ \mathcal{L}_0 - \mathcal{L}_1 \circ \mathcal{D}\Big](\sigma) = \frac{1}{3}\big(\boldsymbol{W}_{(ab)}{}^c\nabla_c - 2\,\boldsymbol{C}_{(ab)}\big)\sigma\,.$$

This gives a projectively invariant operator

$$\begin{array}{rccc} \mathcal{D}_{\ydiagram{2,1}} & : & \mathcal{E}(2) & \longrightarrow & T^*_{\ydiagram{2,1}}M(2) \\ & & \cup & & \cup \\ & & \sigma & \longmapsto & \big(W_{a(bc)}{}^d\nabla_d - 2C_{a(bc)}\big)\sigma\,. \end{array}$$

The above is trivial for projectively flat structures. It is the mixed symmetry part of the map

$$\begin{array}{rccc} \widetilde{\mathcal{D}} & : & \mathcal{E}(2) & \longrightarrow & \big(T^*M \otimes \odot^2 T^*M\big)(2) \\ & & \cup & & \cup \\ & & \sigma & \longmapsto & \frac{1}{2}\nabla_c\nabla_a\nabla_b\sigma + Q_{c(a}\nabla_{b)}\sigma + \nabla_c(Q_{ab}\sigma) \end{array}$$

We use this map to establish the following result.

Lemma 6.5. *For any projective structure* $\boldsymbol{p}$*, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{T}_{(AB)} & \xrightarrow{\boldsymbol{d}^{\nabla}} & \wedge^1(\mathcal{T}_{(AB)}) \\ \uparrow{\scriptstyle \mathcal{L}_0} & \circlearrowleft & \uparrow{\scriptstyle q} \\ \mathcal{E}(2) & \xrightarrow{\widetilde{\mathcal{D}}} & \big(T^*M \otimes \odot^2 T^*M\big)(2) \end{array}$$

We have by now gathered together the main parts of the BGG detour machine and assemble them in the next proposition:

Proposition 6.6. *Let* $\boldsymbol{p}$ *be an Einstein projective structure and denote the formal adjoint of* $\widetilde{\mathcal{D}}$ *by* $\widetilde{\mathcal{D}}^\star$*. Then, calling*

$$M := q^\star \circ \boldsymbol{M}^{\nabla} \circ q\,,$$

we have the following complex

$$0 \longrightarrow C^\infty(M) \xrightarrow{\widetilde{\mathcal{D}}} \big(T^*M \otimes \odot^2 T^*M\big) \xrightarrow{M} T^*M \otimes \odot^2 T^*M \xrightarrow{\widetilde{\mathcal{D}}^\star} C^\infty(M) \longrightarrow 0.$$

Proof. We have constructed $\widetilde{\mathcal{D}}$ so that the analog of the first square in the diagram (4.7) commutes. Moreover the connection $\nabla^{\mathcal{T}}$ is Yang–Mills. Thus we are in the situation of Theorem 4.3. □

Remark 6.7. The above proposition defines a novel gauge theory on Einstein backgrounds. However, since the image of the gauge operator $\widetilde{\mathcal{D}}$ is no longer reducible, the irreducible gauge field content is a totally symmetric *and* a mixed symmetry tensor, both of rank three. These are, by construction, subject to a single, scalar, gauge invariance as well as a system of gauge invariant constraints and equations of motion determined by M^{∇}. The physical consequences of the mixed symmetry field content is currently unclear.

7. Action principles

Action principles for PM models were first constructed, for lower spin examples, in [28]. These were extended to arbitrary spins in [71] by integrating in additional auxiliary fields and then requiring there exist extra gauge invariances removing these leaving the PM systems. Shortly afterwards it was realized that action principles could be obtained geometrically by log-radially reducing [4] massless systems in one higher dimension [56]. As mentioned in the introduction and explicated in Appendix A, log-radial reduction is intimately related to projective geometry. The actions obtained from this reduction were "metric–like", meaning that the field content was arranged in sections of totally symmetric projective tractor bundles. The BGG machine gives what is often called a "frame-like formulation" because one deals with tractor-valued differential forms. Frame-like PM action principles have been given in [67, 73]. These methods introduce progressively more field content in order to make actions simple, and possibly amenable to interacting theories [69]. Conversely, first order, Hamiltonian action principles written in terms of only the physical DoF were introduced in [32]. Here, we give action principles germane to the detour set-up because we are interested in the connection of these systems to projective geometry.

Our key principle is to construct gauge invariant functionals; these are far from unique. Ensuring that gauge invariant equations also imply a gauge invariant system of constraints as their integrability conditions singles out the correct action principle.

The operator $\boldsymbol{M}^\nabla$ annihilates the gauge operator $\boldsymbol{d}^\nabla$. Therefore the functional, defined on projective Einstein structures,

$$S[\boldsymbol{V}] := \langle\, \boldsymbol{V}, \boldsymbol{M}^\nabla \boldsymbol{V} \,\rangle\,, \tag{7.1}$$

where $\boldsymbol{V} \in \bigwedge^1(\odot^k \mathcal{T}^*)$, is gauge invariant for any spin $s = k+1$. Here the pairing $\langle \boldsymbol{U}, \boldsymbol{V}\rangle$ denotes

$$\langle \boldsymbol{U}, \boldsymbol{V}\rangle := \int_M g^{ab}\, U_{aA_1\dots A_k} V_{bB_1\dots B_k}\, H^{A_1B_1}\cdots H^{A_kB_k}\,.$$

We now focus (partly for simplicity, but also because this case is special) on spin two. In this case, calling

$$V_{aA} := \begin{pmatrix} v_a \\ \psi_{ab} \end{pmatrix}\,,$$

the gauge invariance of (7.1) reads

$$\left\{ \begin{array}{rcl} \delta v_a &=& \nabla_a \varepsilon - \xi_a \\ \delta \psi_{ab} &=& \nabla_a \xi_b + Q_{ab}\varepsilon\,, \end{array}\right. \tag{7.2}$$

where $\begin{pmatrix} \xi_a & \varepsilon \end{pmatrix} \in \mathcal{T}_A$. Thus, since the one-form v_a enjoys an algebraic gauge invariance, it can be gauged away. However, there is still the freedom to further gauge transform ψ along the locus $\xi_a = \nabla_a \varepsilon$, yielding

$$\delta \psi_{ab} = \big(\nabla_a \nabla_b + Q_{ab}\big)\varepsilon\,.$$

Here only the symmetric part of ψ_{ab} transforms (while its antisymmetric piece $\psi_{[ab]}$ is gauge inert). Therefore, in the action (7.1), we may set

$$\psi_{ab} = \varphi_{ab}\,, \quad v_a = 0\,,$$

where φ_{ab} is symmetric, and so obtain a functional $S[\mathcal{L}_1(\varphi)]$ invariant under the gauge symmetry $\delta\varphi = \mathcal{D}\varepsilon$, which is, of course, the PM gauge symmetry 6.4.

We must still account for the gauge invariant constraints. These are encoded by the Bianchi identity corresponding to the gauge symmetry (7.2) of the original action (7.1),

$$\boldsymbol{\delta}^\nabla \boldsymbol{M}^\nabla \boldsymbol{V} = 0\,.$$

Denoting the two equations of motion of the action (7.1)

$$\boldsymbol{M}^\nabla V_a := \begin{pmatrix} \mathcal{G}^v_a \\ \mathcal{G}^\psi_{ab} \end{pmatrix}\,,$$

the above Bianchi identity implies

$$\nabla^a \mathcal{G}^\psi_{ab} + \mathcal{G}^v_b = 0\,. \tag{7.3}$$

Now observe that the equation of motion of the PM gauge invariant action $S[\mathcal{L}_1(\varphi)]$ is related to that of $S[V]$ by

$$\mathcal{G}^\varphi_{ab} = \mathcal{G}^\psi_{(ab)}\Big|_{\psi=\varphi, v=0}\,.$$

Applying the Bianchi identity (7.3) to the field configuration $V = \mathcal{L}_1(\varphi)$, we thus learn

$$\nabla^a \mathcal{G}^\varphi_{ab} + \Big(\nabla^a \mathcal{G}^\psi_{[ab]} + \mathcal{G}^v_b\Big)\Big|_{\psi=\varphi, v=0} = 0\,.$$

The second and third terms above are an integrability condition of the PM equation of motion and thus ought yield the divergence constraint. Indeed, the last of these is

$$\mathcal{G}^v_a\Big|_{\psi=\varphi, v=0} = X^A\big[\boldsymbol{M}^\nabla \mathcal{L}_1(\varphi)\big]_{aA} = -\nabla.\varphi_a + \nabla_a \bar\varphi\,,$$

which is precisely the divergence constraint. However, we still need to remove the term $\nabla^a \mathcal{G}^{\psi}_{[ab]}$. It is proportional to the divergence of the variation of the square of the divergence constraint so can be removed by adding such a term to the action. This yields the PM spin two action principle

$$S[\varphi] = \langle \mathcal{L}_1(\varphi), \boldsymbol{M}^{\nabla}\mathcal{L}_1(\varphi)\rangle - \frac{1}{2}\langle \iota(X)\boldsymbol{M}^{\nabla}\mathcal{L}_1(\varphi), \iota(X)\boldsymbol{M}^{\nabla}\mathcal{L}_1(\varphi)\rangle\,.$$

Note that $F_{abc} := \nabla_a\varphi_{bc} - \nabla_b\varphi_{ac}$ is a gauge invariant curvature for the PM field φ (the PM action in terms of this curvature first appeared in [33]). The top slot of the image of $\boldsymbol{M}^{\nabla}\mathcal{L}_1$, denoted above by $\iota(X)\boldsymbol{M}^{\nabla}\mathcal{L}_1(\varphi)$, is the both the trace of this curvature and the divergence constraint. Finally, we note, that by construction, the variation of the above action yields the PM equation of motion (6.1).

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Appendix A. Log-radial reduction

Actions and gauge operators for PM models with arbitrary spin were originally computed using the log-radial reduction method [4] in [56]. This technology is closely related to projective geometry. Here we sketch this link and extend the log-radial reduction technique to Ricci flat metric cone spaces, and therefore Einstein backgrounds. Note that, in a general projective setting, this ambient space is known as the Thomas cone space, in view of [68], and the link between this, (the metric cone [44]) and the projective tractor connection and calculus treated in [15, 51].

We wish to consider a Ricci flat, $(n+1)$-dimensional "ambient" manifold $(\tilde{M}, ds^2)$. For that we make the metric ansatz

$$ds^2 = e^{2u}\big(du^2 + d\Omega^2\big)\,.$$

Here we focus on Riemannian signature (but other signatures can be treated similarly). If the n-dimensional metric $d\Omega^2$ is u-independent, this metric ansatz implies that the vector field $\xi := \frac{\partial}{\partial u}$ is a homothety, namely

$$\pounds_\xi ds^2 = 2ds^2\,.$$

Furthermore, the diffeomorphism $r = e^u$ shows that ds^2 is a cone over the n-dimensional, constant u, manifold M with metric $d\Omega^2$. (Note that for the ambient metric $\tilde{g} = ds^2$, we have $\nabla^{\tilde{g}}_A \xi^B = \delta^B_A$; see [46] for the proof that this condition implies $\tilde{g}$ is a cone metric.) Thus u is a logarithm of the "radius" r, hence the terminology "log-radial reduction". Further, requiring that the metric $d\Omega^2$ is Einstein with scalar curvature $\mathsf{Sc} = n(n-1)$ makes ds^2 Ricci flat. For the metric $d\Omega^2$, we have $\Lambda = \frac{(n-1)(n-2)}{2}$. Choosing a frame

$$E^A = e^u \begin{pmatrix} \boldsymbol{e}^a \\ du \end{pmatrix}\,,$$

where $d\Omega^2 = \boldsymbol{e}^a \odot \boldsymbol{e}_a$, the Levi-Civita connection of ds^2, pulled back to M acts as

$$\boldsymbol{\nabla}^* \begin{pmatrix} v^a \\ \rho \end{pmatrix} = \begin{pmatrix} \boldsymbol{\nabla}\rho \ - \boldsymbol{v} \\ \boldsymbol{\nabla} v^a + \boldsymbol{e}^a \rho \end{pmatrix} = \boldsymbol{\nabla}^{\mathcal{T}} \begin{pmatrix} v^a \\ \rho \end{pmatrix} .$$

Here we employ $\boldsymbol{e}^a$ as the soldering form and recognize the above as exactly the tractor connection (3.4) for Einstein projective structures whose Schouten tensor obeys $Q_{ab} = g_{ab}$. Specializing to homogeneous sections $V^A \in T\tilde{M}$, with homogeneity condition

$$\frac{\partial}{\partial u} V^A = w V^A ,$$

we further see that $\nabla_A V^B$ evaluated along a constant u hypersurface M, yields precisely the Thomas D-operator acting on V^B, so $D_A V^B$, as given in (3.6).

Massless higher spins in flat space were first written down in [43, 23]. They can be described by a totally symmetric tensor $\Phi_{A_1 \dots A_s}$ subject to the gauge symmetry

$$\Phi_{A_1 \dots A_s} \sim \Phi_{A_1 \dots A_s} + \nabla_{(A_1} \Xi_{A_2 \cdots A_s)} , \tag{A.1}$$

where the fields and gauge parameters obey trace conditions

$$\Phi_A{}^A{}_B{}^B{}_{A_5 \dots A_s} = 0 = \Xi_A{}^A{}_{A_3 \dots A_{s-1}} .$$

Specializing to distinguished homogeneities, the gauge transformation (A.1) is known to encode the PM gauge operators [56]. Let us explicate this for the case of spin $s = 2$. We now identify Φ_{AB} with a section of $\mathcal{T}_{(AB)}(w)$ of homogeneity w and ∇_A with the Thomas D-operator. The gauge parameter is thus a section of $\mathcal{T}_A(w+1)$ and the gauge transformation becomes (denoting $\Xi_A := (\xi_a \quad \varepsilon)$)

$$D_{(A}\Xi_{B)} = \begin{pmatrix} \nabla_a \xi_b + g_{ab}\varepsilon & \nabla_a \varepsilon + w \xi_a \\ \nabla_b \varepsilon + w \xi_b & (w+1)\varepsilon \end{pmatrix} .$$

The PM model appears at homogeneity $w = -1$. Denoting

$$\mathcal{T}_{(AB)}(-1) \ni \Phi_{AB} =: \begin{pmatrix} \tau \\ v_a \\ \varphi_{ab} \end{pmatrix} ,$$

we have the PM gauge transformations

$$\begin{cases} \delta\tau = 0 \\ \delta v_a = \nabla_a \varepsilon - \xi_a \\ \delta \varphi_{ab} = \nabla_a \xi_b + g_{ab} \varepsilon . \end{cases} \tag{A.2}$$

Because the field τ is gauge inert, it can be consistently set to zero. Moreover, in our Einstein projective setting $Q_{ab} = g_{ab}$, so we see that the above transformation exactly matches those produced by the BGG detour machine in (7.2).

References

[1] S. Armstrong: Projective holonomy. I. Principles and properties. Ann. Global Anal. Geom., **33** (2008), 47–69, arXiv:math/0602620. Cited on page(s): 3, 9

[2] T.N. Bailey, M.G. Eastwood, and A.R. Gover: Thomas's structure bundle for conformal, projective and related structures. Rocky Mountain J. Math. **24** (1994), 1191–1217. Cited on page(s): 3, 5, 6, 7, 9

[3] Baston R.J.: Almost Hermitian symmetric manifolds. II. Differential invariants. Duke Math. J. **63** (1991), 113–138. Cited on page(s): 3, 10

[4] T. Biswas and W. Siegel: Radial dimensional reduction: Anti-de Sitter theories from flat. JHEP **0207**, 005–034 (2002) [hep-th/0203115]. Cited on page(s): 4, 23, 25

[5] Bokan, Neda; Gilkey, Peter; Živaljević, Rade An inhomogeneous elliptic complex. J. Anal. Math. 61 (1993), 367–393. Cited on page(s): 10

[6] I.N. Bernstein, I.M. Gelfand, S.I. Gelfand: Differential operators on the base affine space and a study of $\mathfrak{g}$-modules. In "Lie Groups and their Representations" (ed. I.M. Gelfand) Adam Hilger 1975, 21–64. Cited on page(s): 3, 10

[7] T. Branson, A. R. Gover: Conformally invariant operators, differential forms, cohomology and a generalisation of Q-curvature. Communications in Partial Differential Equations. **30** (2005), 1611–1669. Cited on page(s): 3, 10, 13

[8] T. Branson, A. R. Gover: The conformal deformation detour complex for the obstruction tensor. Proc. American Math. Soc., **135**, 2961–2965. Cited on page(s): 3, 10, 11

[9] E. Calabi: On compact Riemannian manifolds with constant curvature I. In Differential Geometry, Proc. Sympos. Pure Math., Vol. III, Amer. Math. Soc., Providence, R.I., 1961, 155–180. Cited on page(s): 3

[10] D.M.J. Calderbank and T. Diemer: Differential invariants and curved Bernstein-Gelfand-Gelfand sequences. J. Reine Angew. Math. **537** (2001) 67–103. Cited on page(s): 3, 10, 11

[11] D. Calderbank, T. Diemer and V. Souček: Ricci-corrected derivatives and invariant differential operators. Differential Geom. Appl. **23** (2005), 149?175, arXiv:math/0310311. Cited on page(s): 15

[12] A. Čap: Overdetermined systems, conformal differential geometry, and the BGG complex. Symmetries and overdetermined systems of partial differential equations, 1–24, IMA Vol. Math. Appl., **144**, Springer, New York, 2008, arXiv:math.dg/0610225 Cited on page(s): 11

[13] A. Čap, A.R. Gover: Tractor calculi for parabolic geometries. Trans. Amer. Math. Soc. **354** (2002), no. 4, 1511–1548. Cited on page(s): 7, 8

[14] A. Čap, A.R. Gover: Projective compactifications and Einstein metrics, to appear in J. reine angew. Math., DOI 10.1515/crelle-2014-0036, arXiv:1304.1869 Cited on page(s): 3, 10

[15] A. Čap, A. R. Gover, and M. Hammerl: Projective BGG equations, algebraic sets, and compactifications of Einstein geometries. J. London Math. Soc., **86** (2012), 433–454. Cited on page(s): 3, 25

[16] A. Čap, A. R. Gover, and M. Hammerl: Holonomy reductions of Cartan geometries and curved orbit decompositions. Duke Math. J. 163 (2014), no. 5, 1035–1070. Cited on page(s): 3

[17] A. Čap, A.R. Gover, and H. Macbeth: Einstein metrics in projective geometry. Geom. Dedicata **168** (2014) 235–244. Cited on page(s): 3, 5, 6, 7, 9, 10, 19

[18] A. Čap, J. Slovák, and V. Souček: Bernstein-Gelfand-Gelfand sequences. Ann. of Math. **154** (2001), 97–113. Cited on page(s): 3, 11

[19] Cartan: Sur les variétés à connexion projective. Bull. Soc. Math. France **52** (1924), 205–241. Cited on page(s): 3, 7

[20] D. Cherney, E. Latini and A. Waldron: BRST Detour Quantization. J. Math. Phys. **51**, 062302 (2010), 51–95 arXiv:0906.4814 [hep-th]. Cited on page(s): 4

[21] D. Cherney, E. Latini and A. Waldron: Generalized Einstein Operator Generating Functions Phys. Lett. B **682** (2010) 472–475, arXiv:0909.4578 [hep-th]. Cited on page(s): 4

[22] D. Cherney, E. Latini and A. Waldron: Quaternionic Kähler Detour Complexes and N=2 Supersymmetric Black Holes. Commun. Math. Phys. **302** (2011), 843–873 , arXiv:1003.2234 [hep-th]. Cited on page(s): 4

[23] T. Curtright: Massless Field Supermultiplets With Arbitrary Spin. Phys. Lett. B **85**, (1979) 219–242. Cited on page(s): 26

[24] S. Deser and R. I. Nepomechie: Anomalous Propagation of Gauge Fields in Conformally Flat Spaces. Phys. Lett. B **132** (1983) 321–331 (1983); Gauge Invariance Versus Masslessness in De Sitter Space. Annals Phys. **154** (1984) 396–434. Cited on page(s): 4, 19

[25] S. Deser, E. Joung and A. Waldron: Gravitational- and Self- Coupling of Partially Massless Spin 2. Phys. Rev. D **86**, (2012) 104004–104014 , arXiv:1301.4181. Cited on page(s): 19

[26] S. Deser, E. Joung and A. Waldron: Partial Masslessness and Conformal Gravity. J. Phys. A **46**, (2013) 214019–214038, arXiv:1208.1307. Cited on page(s): 19

[27] S. Deser and A. Waldron: Gauge invariances and phases of massive higher spins in (A)dS. Phys. Rev. Lett. **87**, (2001) 031601–031605, hep-th/0102166. Cited on page(s): 4, 17, 19, 20

[28] S. Deser and A. Waldron: Partial masslessness of higher spins in (A)dS. Nucl. Phys. B **607**, 577–604 (2001), hep-th/0103198. Cited on page(s): 4, 17, 20, 23

[29] S. Deser and A. Waldron: Stability of massive cosmological gravitons. Phys. Lett. B **508**, 347–353 (2001), hep-th/0103255. Cited on page(s): 4, 20
[30] S. Deser and A. Waldron: Null propagation of partially massless higher spins in (A)dS and cosmological constant speculations, Phys. Lett. B **513**, (2001) 137–147. [hep-th/0105181]. Cited on page(s): 4, 19, 20
[31] S. Deser and A. Waldron: Arbitrary spin representations in de Sitter from dS/CFT with applications to dS supergravity. Nucl. Phys. B **662**, (2003) 379–392, hep-th/0301068. Cited on page(s): 4, 20
[32] S. Deser and A. Waldron: Conformal invariance of partially massless higher spins. Phys. Lett. B **603**, (2004), 30–40 (2004), [hep-th/0408155]. Cited on page(s): 20, 23
[33] S. Deser and A. Waldron, Partially Massless Spin 2 Electrodynamics: Phys. Rev. D **74**, 084036 (2006), hep-th/0609113. Cited on page(s): 19, 20, 25
[34] S. Deser and A. Waldron: PM = EM: Partially Massless Duality Invariance," Phys. Rev. D **87**, (2013) 087702–087705, arXiv:1301.2238 [hep-th. 20
[35] T. Diemer: Conformal Geometry, Representation Theory and Linear Fields. Ph. D. Thesis, Universität Bonn, 1999. Cited on page(s): 10
[36] L. Dolan, C. R. Nappi and E. Witten: Conformal operators for partially massless states. JHEP **0110**, (2001) 016–029, hep-th/0109096. Cited on page(s): 19
[37] M. Eastwood, A complex from linear elasticity. The Proceedings of the 19th Winter School "Geometry and Physics" (SrnÃŋ, 1999). Rend. Circ. Mat. Palermo (2) Suppl. No. 63 (2000), 23–29. Cited on page(s): 3
[38] M. Eastwood, Prolongations of linear overdetermined systems on affine and Riemannian manifolds. Proceedings of the 24th Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo (2005), 89–108 11
[39] M. Eastwood, Symmetries and Overdetermined Systems of Partial Differential Equations, The IMA Volumes in Mathematics and its Applications Volume 144, 2008, pp 41-60. Cited on page(s): 5, 21
[40] M. Eastwood and A.R. Gover: The BGG Complex on Projective Space. SIGMA 7 (2011), 060–068, arXiv:1106.4623. Cited on page(s): 15
[41] M. Eastwood and J. Rice: Conformally invariant differential operators on Minkowski space and their curved analogues. Comm. Math. Phys. **109** (1987), 207–228; Erratum: Comm. Math. Phys. **144** (1992), 213. Cited on page(s): 10
[42] M.G. Eastwood, and V. Matveev: Metric connections in projective differential geometry. in "Symmetries and overdetermined systems of partial differential equations", 339–350, IMA Vol. Math. Appl., 144, Springer, New York, 2008. Cited on page(s): 3, 5, 7, 21
[43] C. Fronsdal: Massless Fields with Integer Spin. Phys. Rev. D **18**, (1978) 3624–3643. Cited on page(s): 26
[44] S. Gallot: Équations différentielles charactéristiques de la sphère. Ann. Sci. École Norm. Sup. (4), **12**(2) (1979) 235–267. Cited on page(s): 25
[45] J. Gasqui: Sur la résolubilitélocale des équations d'Einstein. Compositio Math. **47**, (1982) 43–69. Cited on page(s): 10
[46] G. W. Gibbons and P. Rychenkova, Cones, triSasakian structures and superconformal invariance. Phys. Lett. B **443**, (1998) 138, hep-th/9809158. Cited on page(s): 25
[47] A.R. Gover: Conformally invariant operators of standard type. Quart. J. Math. Oxford **40** (1989) 197–207. Cited on page(s): 15
[48] A. R. Gover, K. Hallowell and A. Waldron: Higher spin gravitational couplings and the Yang-Mills detour complex, Phys. Rev. D **75**, 024032 (2007), hep-th/0606160. Cited on page(s): 11
[49] A.R. Gover and H. Macbeth: Detecting Einstein geodesics: Einstein metrics in projective and conformal geometry. Differential Geometry and Its Applications, **33** (2014), 44. arXiv:1212.6286 Cited on page(s): 3, 4, 5, 6, 7, 9
[50] A. R. Gover and P.N. Nurowski: Calculus and invariants on almost complex manifolds, including projective and conformal geometry. Illinois Journal of Mathematics, **57** (2013), 383–427. arXiv:1208.0648 Cited on page(s): 5, 6
[51] A. R. Gover, R. Panai, T. Willse: Nearly KÃd'hler geometry and (2,3,5)-distributions via projective holonomy, arXiv:1403.1959. 25
[52] A. R. Gover, A. Shaukat and A. Waldron: Tractors, Mass and Weyl Invariance. Nucl. Phys. B **812**, (2009) 424–455, arXiv:0810.2867. Cited on page(s): 3, 4

[53] A. R. Gover, A. Shaukat and A. Waldron: Weyl Invariance and the Origins of Mass. Phys. Lett. B **675**, (2009) 93-97 arXiv:0812.3364. Cited on page(s): 3, 4
[54] A.R. Gover, P. Somberg, V. Souček: Yang-Mills detour complexes and conformal geometry. Communications in Mathematical Physics, **278** (2008) 307–327. arXiv:math/0606401. Cited on page(s): 3, 10, 11, 13, 14, 22
[55] M. Grigoriev and A. Waldron: Massive Higher Spins from BRST and Tractors. Nucl. Phys. B **853**, (2011) 291–343, [arXiv:1104.4994 [hep-th]]. Cited on page(s): 4
[56] K. Hallowell and A. Waldron: Constant curvature algebras and higher spin action generating functions. Nucl. Phys. B **724**, (2005) 453–497, [hep-th/0505255]. Cited on page(s): 4, 23, 25, 26
[57] A. Higuchi: Forbidden Mass Range for Spin-2 Field Theory in De Sitter Space-time. Nucl. Phys. B 282, (1987) 397–438. Cited on page(s): 4
[58] Kiosak, Volodymyr; Matveev, Vladimir S.(D-FSU-MI) Proof of the projective Lichnerowicz conjecture for pseudo-Riemannian metrics with degree of mobility greater than two. (English summary) Comm. Math. Phys. 297 (2010), no. 2, 401–426. Cited on page(s): 3
[59] B. Kostant: Lie algebra cohomology and the generalized Borel Weil theorem, Ann. Math. **74**, (1961) 329–387. Cited on page(s): 3, 4
[60] Lepowsky J.: A generalization of the Bernstein–Gelfand–Gelfand resolution. J. Algebra **49** (1977), 496–511. Cited on page(s): 10
[61] J. Maldacena: Einstein Gravity from Conformal Gravity. arXiv:1105.5632. Cited on page(s): 19
[62] , Vladimir S. Hyperbolic manifolds are geodesically rigid. Invent. Math. 151 (2003), no. 3, 579–609. Cited on page(s): 3
[63] J. Mikes: Geodesic mappings of affine-connected and Riemannian spaces. Jour. Math. Sci. **78** (1996) 311–333. Cited on page(s): 3, 6
[64] O. V. Shaynkman, I. Y. Tipunin and M. A. Vasiliev: Unfolded form of conformal equations in M dimensions and $o(M+2)$ modules. Rev. Math. Phys. **18**, (2006) 823–886, hep-th/0401086. Cited on page(s): 4
[65] N.S. Sinjukov: Geodesic mappings of Riemannian spaces. (Russian), "Nauka," Moscow, 1979, 256pp. Cited on page(s): 3, 6
[66] E. D. Skvortsov: Gauge fields in (A)dS(d) within the unfolded approach: algebraic aspects. JHEP **1001**, 106 (2010), arXiv:0910.3334. Cited on page(s): 4
[67] E. D. Skvortsov and M. A. Vasiliev: Geometric formulation for partially massless fields. Nucl. Phys. B **756**, 117 (2006), hep-th/0601095. Cited on page(s): 23
[68] T.Y. Thomas: Announcement of a projective theory of affinely connected manifolds. Proc. Nat. Acad. Sci., **11** (1925), 588–589. Cited on page(s): 3, 7, 25
[69] M. A. Vasiliev: Consistent equations for interacting massless fields of all spins in the first order in curvatures. Annals Phys. **190** (1989) 59–106. Cited on page(s): 23
[70] M. A. Vasiliev: Bosonic conformal higher-spin fields of any symmetry. Nucl. Phys. B **829**, (2010) 176–224, arXiv:0909.5226. Cited on page(s): 4
[71] Y. Zinoviev: On Massive High Spin Particles in (A)dS. arXiv:hep-th/0108192 Cited on page(s): 23
[72] Y. M. Zinoviev: On spin 3 interacting with gravity," Class. Quant. Grav. **26**, (2009) 035022–035022, arXiv:0805.2226. Cited on page(s): 22
[73] Y. M. Zinoviev: Massive spin-2 in the Fradkin-Vasiliev formalism. I. Partially massless case. Nucl. Phys. B **886**, (2014) 712–732, arXiv:1405.4065. Cited on page(s): 23
[74] G. Zuckerman: Tensor products of finite and infinite dimensional representations of semisimple Lie Groups. Ann. of Math. **106**, (1977) 295–308. Cited on page(s): 3

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