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RESONANCE WIDTHS FOR GENERAL HELMHOLTZ RESONATORS WITH STRAIGHT NECK

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Abstract. We prove an optimal exponential lower bound on the widths of resonances for a general Helmholtz resonator with straight neck.

1. Introduction

A resonator consists of a bounded cavity (the chamber) connected to the exterior by a thin tube (the neck of the chamber). The frequencies of the sounds it produces are determined by the shape of the chamber, while their duration by the length and the width of the neck in a non-obvious way, and our goal is to understand these. Mathematically, this phenomenon is described by the resonances of the Dirichlet Laplacian $-\Delta_\Omega$ on the domain Ω consisting of the union of the chamber, the neck and the exterior.

In [MN1], an optimal bound has been optioned for very particular two-dimensional Helmholtz resonators, for which the exterior consists of an infinite straight half tube. Then, the result has been generalized in [MN2] for much more general two-dimensional Helmholtz resonators, under the condition that that the neck meets the boundary of the external region perpendicularly to it, and that the boundary is flat there. Moreover, an extension to larger dimensions (up to 12) was obtained, too, but only for necks with a square section.

Here, we plan to generalize the result to any n-dimensional Helmholtz resonator with straight neck, without particular assumption on the section of the neck or on the flatness of the boundary near the mouth of the neck. The only assumption we need is that the boundary of the exterior is analytic there.

We recall that resonances are the eigenvalues of a complex deformation of $-\Delta_\Omega$; their real and imaginary parts are the frequencies and inverses of the half-lives, respectively, of the corresponding vibrational modes. It is of obvious physical interest to estimate these two quantities as precisely as possible. One practical way to do this involves studying this problem

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in the asymptotic limit when the width ε of the neck tends to zero. Those resonances whith imaginary parts tending to zero converge to the eigenvalues of the Dirichlet Laplacian on the cavity, and there is an exponentially small upper bound for the absolute values of the imaginary parts (the widths) of the resonances [HM]. However, without very restrictive hypotheses, no lower bound is known. We mention in particular that lower bounds are known in the one-dimensional case [Ha, HaSi]. As for the higher dimensional case, we mention [Be, Fe, FL, Bu2, HS] which contain results concerning exponentially small widths of quantum resonances, but these do not apply to a Helmholtz resonator. We also mention that the semiclassical lower bound obtained in [HS] is optimal (see also [FLM] for a generalization).

Here, we obtain an optimal lower bound in a very general case (see Theorem 2.1) under a somehow natural condition of analyticity near the mouth of the neck. As in [MN2], the problem is first related to a lower bound on the resonant function in a large . Assuming, by contradiction, that this function is small there, the smallness can be propagated up to a small neighborhood of the end part of the neck, by means of general Carleman estimates. Then, we construct highly oscillating local solutions to the equation by solving a Cauchy problem (this is where the assumption of analyticity is used) that permit us to propagate the estimate up to the exterior boundary of the neck (in a way similar to that of propagation of analytic singularities). Finally, this estimate is propagated in the inside part of the neck by using an explicit version of Carleman estimates. At that point, the contradiction is obtained as in [MN2], by using a result of [BHM] on the size of the Dirichlet eigenfunctions of the cavity.

2. Assumtions and results

Let $\mathcal{C}$ and $\mathcal{B}$ be two bounded domains in $\mathbb{R}^n$ ($n \geq 2$), with $\mathcal{C}^\infty$ boundary, and denote by $\overline{\mathcal{C}}$, $\overline{\mathcal{B}}$ their closures, and by $\partial\mathcal{C}$, $\partial\mathcal{B}$ their boundaries. We assume that Euclidean coordinates $x = (x_1, \dots, x_n) =: (x_1, x')$ can be chosen in such a way that, for some $L > 0$, one has,

$$\overline{\mathcal{C}} \subset \mathcal{B} \quad ; \quad 0 \in \partial\mathcal{C} \quad ; \quad (L, 0_{\mathbb{R}^{n-1}}) \in \partial\mathcal{B} \quad ; \quad [0, L] \times \{0_{\mathbb{R}^{n-1}}\} \subset \overline{\mathcal{B}} \backslash \mathcal{C}. \tag{2.1}$$

We also assume,

$$\begin{aligned} &[0, L] \times \{0_{\mathbb{R}^{n-1}}\} \text{ is transversal to } \partial\mathcal{B} \text{ at } (L, 0_{\mathbb{R}^{n-1}}); \\ &\partial\mathcal{B} \text{ is analytic at } (L, 0_{\mathbb{R}^{n-1}}). \end{aligned} \tag{2.2}$$

Let $D_1 \subset \mathbb{R}^{n-1}$ be a bounded domain containing the origin, with smooth boundary ∂D_1. For $\varepsilon > 0$ small enough, we set $D_\varepsilon := \varepsilon D_1$ and,

$$\begin{aligned} &\mathbf{E} := \mathbb{R}^n \backslash \overline{\mathcal{B}}; \\ &\mathcal{T}(\varepsilon) := ([-\varepsilon_0, L + \varepsilon_0] \times D_\varepsilon) \cap (\mathbb{R}^n \backslash (\mathbf{E} \cup \mathcal{C})); \\ &\mathcal{C}(\varepsilon) = \mathcal{C} \cup \mathcal{T}(\varepsilon), \end{aligned}$$

where $\varepsilon_0 > 0$ is fixed sufficiently small in order that $[-\varepsilon_0, L + \varepsilon_0] \times \{0_{\mathbb{R}^{n-1}}\}$ crosses $\partial\mathcal{C}$ and $\partial\mathcal{B}$ at one point only. Then, the resonator is defined as,

$$\Omega(\varepsilon) := \mathcal{C}(\varepsilon) \cup \mathbf{E}.$$

As $\varepsilon \to 0_+$, the resonator $\Omega(\varepsilon)$ collapses to $\Omega_0 := \mathcal{C} \cup [0, M_0] \cup \mathbf{E}$, where M_0 is the point $(L, 0_{\mathbb{R}^{n-1}}) \in \mathbb{R}^n$.

For any domain Q, let P_Q denote the Laplacian $-\Delta_Q$ with Dirichlet boundary conditions on ∂Q, and set $P_\varepsilon := P_{\Omega_\varepsilon}$.

The resonances of P_ε are defined as the eigenvalues of the operator obtained by performing a complex dilation with respect to x, for $|x|$ large. We are interested in those resonances of P_ε that are close to the eigenvalues of $P_{\mathcal{C}}$. Thus, let $\lambda_0 > 0$ be an eigenvalue of $P_{\mathcal{C}}$ with u_0 the corresponding normalized eigenfunction. As in [MN2], we assume,

$$\begin{aligned} &\lambda_0 \text{ is simple;} \\ &u_0 \text{ does not vanish on } \mathcal{C} \text{ near } 0. \end{aligned} \tag{2.3}$$

Observe that these two properties are automatically satisfied when λ_0 is the lowest eigenvalue of $-\Delta_{\mathcal{C}}$, and when it is a higher eigenvalue, the last property just means that 0 does not lie on the closure of a nodal line of u_0.

By the arguments of [Be, HM], we know that there is a resonance $\rho(\varepsilon) \in \mathbb{C}$ of P_ε such that $\rho(\varepsilon) \to \lambda_0$ as $\varepsilon \to 0_+$. Furthermore, denoting by α_0 the square root of the first eigenvalue of $-\Delta_{D_1}$, there is an eigenvalue $\lambda(\varepsilon)$ of $P_{\mathcal{C}(\varepsilon)}$ such that, for any $\delta > 0$,

$$|\rho(\varepsilon) - \lambda(\varepsilon)| \leq C_\delta e^{-2\alpha_0(1-\delta)L/\varepsilon}, \tag{2.4}$$

for some $C_\delta > 0$ and all sufficiently small $\varepsilon > 0$. In particular, since $\lambda(\varepsilon) \in \mathbb{R}$, this gives

$$|\operatorname{Im} \rho(\varepsilon)| \leq C_\delta e^{-2\alpha_0(1-\delta)L/\varepsilon}. \tag{2.5}$$

We now state our main result.

Theorem 2.1. *Under Assumptions (2.1)-(2.3), for any $\delta > 0$ there exists $C_\delta > 0$ such that, for all $\varepsilon > 0$ small enough, one has,*

$$|\operatorname{Im} \rho(\varepsilon)| \geq \frac{1}{C_\delta} e^{-2\alpha_0(1+\delta)L/\varepsilon}.$$

Remark 2.2. *Gathering (2.5) and Theorem 2.1, we can reformulate the result as,*

$$\lim_{\varepsilon \to 0_+} \varepsilon \ln |\operatorname{Im} \rho(\varepsilon)| = -2\alpha_0 L. \tag{2.6}$$

3. Background properties

By definition, the resonance $\rho(\varepsilon)$ is an eigenvalue of the complex distorted operator,

$$P_\varepsilon(\mu) := U_\mu P_\varepsilon U_\mu^{-1},$$

where $\mu > 0$ is a small parameter, and U_μ is a complex distortion of the form,

$$U_\mu \varphi(x) := \varphi(x + i\mu f(x)),$$

with $f \in C^\infty(\mathbb{R}^n;\mathbb{R}^n)$, $f = 0$ near $\overline{\mathcal{B}}$, $f(x) = x$ for $|x|$ large enough. (Observe that by Weyl Perturbation Theorem, the essential spectrum of $P_\varepsilon(\mu)$ is $e^{-2i\theta}\mathbb{R}_+$, with $\theta = \arctan\mu$.)

It is well known that such eigenvalues do not depend on μ (see, e.g., [SZ, HeM]), and that the corresponding eigenfunctions are of the form $U_\mu u_\varepsilon$ with u_ε independent of μ, smooth on $\mathbb{R}^n$ and analytic in a complex sector around $\mathbf{E}$. In other words, u_ε is a non trivial analytic solution of the equation $-\Delta u_\varepsilon = \rho(\varepsilon)u_\varepsilon$ in $\Omega(\varepsilon)$, such that $u_\varepsilon\big|_{\partial\Omega(\varepsilon)} = 0$ and, for all $\mu > 0$ small enough, $U_\mu u_\varepsilon$ is well defined and is in $L^2(\Omega(\varepsilon))$ (in our context, this latter property will be taken as a definition of the fact that u_ε is *outgoing*). Moreover, u_ε can be normalized by setting, for some fixed $\mu > 0$,

$$\|U_\mu u_\varepsilon\|_{L^2(\Omega(\varepsilon))} = 1.$$

In that case, we learn from [HM] (in particular Proposition 3.1 and formula (5.13)), that, for any $\delta > 0$, and for any $R > 0$ large enough, one has,

$$\|u_\varepsilon\|_{L^2(\Omega(\varepsilon)\cap\{|x|<R\})} \geq 1 - \mathcal{O}(e^{(\delta-\alpha_0)/\varepsilon}), \tag{3.1}$$

and

$$\|u_\varepsilon\|_{H^1(\mathbf{E}\cap\{|x|<R\})} = \mathcal{O}(e^{(\delta-\alpha_0))/\varepsilon}). \tag{3.2}$$

Now, we take $R > 0$ such that $\overline{\mathcal{B}} \subset \{|x| < R\}$. Using the equation $-\Delta u_\varepsilon = \rho u_\varepsilon$ and Green's formula on the domain $\Omega(\varepsilon) \cap \{|x| < R\}$, and using polar coordinates (r,ω), we obtain,

$$\operatorname{Im}\rho\int_{\Omega(\varepsilon)\cap\{|x|<R\}} |u_\varepsilon|^2 dx = -\operatorname{Im}\int_{\mathbb{S}^{n-1}} \frac{\partial u_\varepsilon}{\partial r}(R,\omega)\overline{u}_\varepsilon(R,\omega)R^{n-1}d\sigma_{n-1}(\omega),$$

(where $d\sigma_{n-1}(\omega)$ stands for the surface measure on $\mathbb{S}^{n-1}$), and thus, by (3.1)-(3.2), and for some $\delta_0 > 0$,

$$\operatorname{Im}\rho = -(1 + \mathcal{O}(e^{(\delta-2\alpha_0)/\varepsilon}))\ \operatorname{Im}\int_{\mathbb{S}^{n-1}} \frac{\partial u_\varepsilon}{\partial r}(R,\omega)\overline{u}_\varepsilon(R,\omega)R^{n-1}d\sigma_{n-1}(\omega) \tag{3.3}$$

where the $\mathcal{O}$ is locally uniform with respect to R.

Therefore, to prove our result, it is sufficient to obtain a lower bound on $\operatorname{Im}\int_{\mathbb{S}^{n-1}} \frac{\partial u_\varepsilon}{\partial r}(R,\omega)\overline{u}_\varepsilon(R,\omega)R^{n-1}d\sigma_{n-1}(\omega)$. Note that, by using (3.2), we immediately obtain (2.5).

Starting from this formula, the following proposition has been proved in [MN2] (the proof is actually done in 2 dimensions only, but can be generalized easily to any dimension: see [MN2], Remark 4.6):

Proposition 3.1 (Martinez-Nédélec [MN2]). *Let $R_1 > R_0 > 0$ be fixed in such a way that $\overline{\mathcal{B}} \subset \{|x| < R_0\}$. Then, for any $C > 0$, there exists a constant $C' = C'(R_0, R_1, C) > 0$ such that, for all $\varepsilon > 0$ small enough, one has,*

$$|\operatorname{Im}\rho| \geq \frac{1}{C'}\|u_\varepsilon\|^2_{L^2(R_0<|x|<R_1)} - C'e^{-C/\varepsilon}.$$

Then, reasoning by contradiction as in [MN2], we assume the existence of $\delta_0 > 0$ such that, along a sequence $\varepsilon \to 0_+$, one has

$$|\operatorname{Im}\rho| = \mathcal{O}(e^{-(2\alpha_0+\delta_0)/\varepsilon}). \tag{3.4}$$

Proposition 3.1 (added to standard Sobolev estimates) tells us that for any $R_1 > R_0 > 0$ such that $\overline{\mathcal{B}} \subset \{|x| < R_0\}$, we have,

$$\|u_\varepsilon\|_{H^1(R_0<|x|<R_1)} = \mathcal{O}(e^{-(\alpha_0+\delta_0)/\varepsilon}). \tag{3.5}$$

Still following the procedure used in [MN2], we see that this estimate can be propagated up to the boundary of $\mathcal{B}$, away from an arbitrarily small neighborhood of $M_0 := (L, 0_{\mathbb{R}^{n-1}})$ (this is done by means of Carleman inequalities up to the boundary [LR, LL]), and one obtains (see [MN2], Proposition 6.1),

Proposition 3.2 (Martinez-Nédélec [MN2])**.** *Under the assumption (3.4), for any neighborhood $\mathcal{U}$ of M_0 and any compact set $K \subset \mathbb{R}^n$, there exists $\delta_K > 0$ such that,*

$$\|u_\varepsilon\|_{H^1(\mathbf{E}\cap K\setminus\mathcal{U})} = \mathcal{O}(e^{-(\alpha_0+\delta_K)/\varepsilon}),$$

uniformly as $\varepsilon \to 0_+$.

From this point, the proof starts to differ completely from that of [MN2]. As a first step, we will propagate this estimate on u up to $\partial\mathcal{B}$ at bounded frequencies, by solving a Cauchy problem and use the solution as a propagator (in the same spirit as for the propagation of analytic singularities: see, e.g., [Sj]). Next, for high frequencies, the propagation will be obtained by taking a partial Fourier transform of u and by using the properties of the differential equation in x_1 it is solution to. The final step will consist in propagating the estimate inside the thin tube, by using some “hand-made” Carleman estimate on such an ε-dependent domain. After that, the proof can be completed exactly as in [MN2].

4. Estimate near M_0 at bounded frequencies

We first make a translation and rotation of Euclidean coordinates, in such a way that M_0 becomes the origin and, in the new coordinates, $\mathcal{B}$ is given, near M_0, by an inequality $f(x) < 0$ with $f(x) = x_1 + \mathcal{O}(x^2)$. In particular,

$$T_{M_0}\partial\mathcal{B} = \{x_1 = 0\}.$$

Then, for $\eta \in \mathbb{R}^{n-1}\backslash\{0\}$ and $a > 0$ small enough, we set,

$$\psi_{\eta,a,\pm}(x) := e^{i\eta\cdot x'/\varepsilon \pm (\eta^2-\varepsilon^2\rho)^{\frac{1}{2}}(x_1-a)/\varepsilon}.$$

In particular, $\psi_{\eta,a,\pm}$ is solution to the Cauchy problem,

$$\begin{cases} -\Delta\psi_{\eta,a,\pm} = \rho\psi_{\eta,a,\pm}; \\ \psi_{\eta,a,\pm}\,|_{x_1=a} = e^{i\eta\cdot x'/\varepsilon}; \\ \dfrac{\partial\psi_{\eta,a,\pm}}{\partial x_1}\,|_{x_1=a} = \pm\varepsilon^{-1}(\eta^2-\varepsilon^2\rho)^{\frac{1}{2}}e^{i\eta\cdot x'/\varepsilon}. \end{cases} \tag{4.1}$$

Now, let $\chi \in C_0^\infty(\mathbb{R}^n)$ be supported in a small neighborhood of M_0, and such that $\chi = 1$ near M_0, and set,

$$v := \chi u.$$

We have,

$$-\Delta v = \rho v - [\Delta, \chi]u,$$

and thus, by the Green formula on $\Omega_a := \mathbf{E} \cap \{x > a\}$, and by using Proposition 3.2, (2.5) and (3.2),

$$\begin{aligned}\int_{\partial\Omega_a} \left(v \, \frac{\partial \overline{\psi_{\eta,a,\pm}}}{\partial \nu} - \frac{\partial v}{\partial \nu} \, \overline{\psi_{\eta,a,\pm}} \right) d\sigma_a &= \langle -2i(\operatorname{Im}\rho)v + [\Delta,\chi]u, \psi_{\eta,a,\pm}\rangle_{\Omega_a} \\ &= \mathcal{O}(e^{-(\alpha_0+\delta_1-r_0|\eta|)/\varepsilon}),\end{aligned}$$

where ν stands for the normal interior unit vector to Ω_a (defined almost everywhere on $\partial\Omega_a$), $d\sigma_a$ is the surface measure on Ω_a, $\delta_1 > 0$ is independent of η, and $r_0 := \sup\{x_1 - a\,;\, x \in \operatorname{Supp}\chi\}$.

5. Estimate near M_0 at high frequencies

6. Estimate inside the neck

7. Completion of the proof of Theorem 2.1

By Assumption (2.3), we see that the Dirichlet eigenfunction u_0 satisfies the hypothesis of [BHM] Lemma 3.1. Then, following the arguments of [BHM] leading to (13) in that paper, and using again [HM], Proposition 3.1 and Formula (5.13), we conclude that for any $\delta > 0$ and any $x \in (0, L)$, there exists C_1 such that the resonant state u_ε verifies (see [BHM], Formula (13)),

$$\|u_\varepsilon\|_{L^2([x,L]\times[-\varepsilon,\varepsilon])} \geq \frac{1}{C_0}\varepsilon^{??+\delta} e^{-\alpha_0 x/\varepsilon}. \tag{7.1}$$

But this contradicts the inequality **??**, and thus completes the proof the theorem 2.1.

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