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ON A FAMILY OF KP MULTI–LINE SOLITONS ASSOCIATED TO RATIONAL DEGENERATIONS OF REAL HYPERELLIPTIC CURVES AND TO THE FINITE NON–PERIODIC TODA HIERARCHY

SIMONETTA ABENDA

Abstract. We continue the program started in [1] of associating rational degenerations of M–curves to points in $Gr^{\mathrm{TNN}}(k,n)$ using KP theory for real finite gap solutions. More precisely, we focus on the inverse problem of characterizing the soliton data which produce Krichever divisors compatible with the KP reality condition when Γ is a certain rational degeneration of a hyperelliptic M–curve. Such choice is motivated by the fact that Γ is related to the curves associated to points in $Gr^{\mathrm{TP}}(1,n)$ and in $Gr^{\mathrm{TP}}(n-1,n)$ in [1]. We prove that the reality condition on the Krichever divisor on Γ singles out a special family of KP multi–line solitons (T–hyperelliptic solitons) in $Gr^{\mathrm{TP}}(k,n)$, $k \in [n-1]$, naturally connected to the finite non-periodic Toda hierarchy. We discuss the relations between the algebraic-geometric description of KP T–hyperelliptic solitons and of the open Toda system. Finally, we also explain the effect of the space–time transformation which conjugates soliton data in $Gr^{\mathrm{TP}}(k,n)$ to soliton data in $Gr^{\mathrm{TP}}(n-k,n)$ on the Krichever divisor for such KP solitons.

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Keywords. Total positivity, KP equation, real bounded solitons, M-curves, hyperelliptic curves, finite gap theory, duality of Grassmann cells, space–time inversion, finite non periodic Toda hierarchy

Contents

1. Introduction 2
2. $(n-k,k)$-line solitons via Darboux transformation, in the Sato Grassmannian and in finite–gap theory 6

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3. Vacuum KP–wavefunction on Γ 8
4. T-hyperelliptic solitons and k–compatible divisors 11
5. Comparison with the construction in [1] 18
6. The finite non–periodic Toda lattice hierarchy 23
6.1. IVP solutions to the Toda hierarchy 25
6.2. The spectral problem and the Baker–Akhiezer function for the finite Toda system 27
7. Toda data, Darboux transformations and Toda Baker–Akhiezer functions 29
8. T–hyperelliptic KP solitons and solutions to the Toda hierarchy 32
9. Reconstruction of soliton data and Toda solutions from k–compatible divisors 35
10. Duality of Grassmann cells, space–time inversion and divisors 37
10.1. Dual Toda flows 39
10.2. Duality and divisors of KP–soliton solutions 41
11. Summary and concluding remarks 44
References 48

1. Introduction

Regular bounded KP $(n-k,k)$-line solitons are associated to soliton data $(\mathcal{K},[A])$ with $\mathcal{K}=\{\kappa_1<\cdots<\kappa_n\}$ and $[A]\in Gr^{\mathrm{TNN}}(k,n)$, the totally non–negative part of the real Grassmannian, which is a reduction of the infinite dimensional Sato Grassmannian [36]. The asymptotic properties of such solitons have been successfully related to the combinatorial structure of $Gr^{\mathrm{TNN}}(k,n)$ in [6, 18, 19].

On the other side, in principle, such soliton solutions may be obtained assigning Krichever data, which satisfy the KP reality condition, on rational degenerations of regular `M`–curves[9, 10, 21, 22].

In [1], we have started the program of connecting such two areas of mathematics - the theory of totally positive Grassmannians and the rational degenerations of regular `M`-curves - using the real finite–gap theory for regular bounded KP $(n-k,k)$-line solitons: we have associated to any soliton data $(\mathcal{K},[A])$ with $[A]\in Gr^{\mathrm{TP}}(k,n)$, a curve Γ which is the rational degeneration of a regular `M`-curve of genus $g=k(n-k)$ and a Krichever divisor $\mathcal{D}$ compatible with the reality conditions settled in [10].

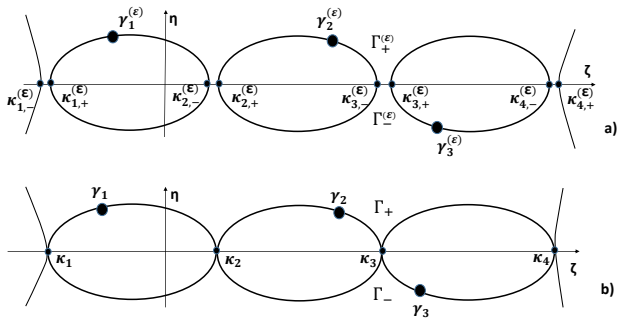

FIGURE 1. *Figure 1.a): KP regular real quasi–periodic solutions on the genus 3 hyperelliptic curve* $\Gamma^{(\epsilon)} = \Gamma_+^{(\epsilon)} \cup \Gamma_-^{(\epsilon)}$ *are parametrized by 3–point divisors such that there is exactly one divisor point* $\gamma_j^{(\epsilon)}$, $j \in [3]$ *in each finite oval. Figure 1.b): In the limit* $\epsilon \to 0$, $\Gamma^{(\epsilon)}$ *rationally degenerates to* $\Gamma = \Gamma_+ \cup \Gamma_-$ *and the limiting KP solution is a real bounded regular KP multi–soliton solution.*

In the present paper, we focus on the inverse problem: we choose Γ a given rational degeneration of an M–curve, we fix the point $P_+ \in \Gamma$ where the KP wavefunction has its essential singularity and a local coordinate ζ on Γ such that $\zeta^{-1}(P_+) = 0$, and we use the reality condition for the Krichever divisor to classify the regular bounded KP $(n-k,k)$–line solitons compatible with such algebraic–geometric setting, (Γ, P_+, ζ).

More precisely, we successfully investigate the case where Γ is obtained in the limit $\epsilon \to 0$ from regular real hyperelliptic curves with affine part $\Gamma^{(\epsilon)} = \{(\zeta,\eta) \ : \ \eta^2 = -\epsilon^2 + \prod_{j=1}^n (\zeta - \kappa_j)^2\}$, *i.e.*

$$\Gamma = \Gamma_+ \sqcup \Gamma_- = \{(\zeta,\eta) \ : \ \eta^2 = \prod_{m=1}^n (\zeta - \kappa_m)^2\}. \tag{1}$$

We choose Γ as in (1) because it is a desingularization of the curve associated in [1] to soliton data $(\mathcal{K}, [A])$, with $[A] \in Gr^{\mathrm{TP}}(n-1,n)$. Moreover, by a straightforward modification of the construction presented in [1], Γ is also associated to soliton data $(\mathcal{K}, [A])$, with $[A] \in Gr^{\mathrm{TP}}(1,n)$. We make the following ansatz:

(1) the number of phases n and the number k of divisor points belonging to the intersection of the finite ovals with Γ_+, the copy of $\mathbb{C}P^1$ containing the essential singularity of KP wavefunction, identifies the Sato finite dimensional reduction $Gr(k,n)$ corresponding to the KP solutions;

(2) the arithmetic genus of Γ, $n-1$, equals the dimension of the divisor and the dimension of the subvariety in $Gr(k,n)$ described by such KP soliton solutions;

(3) the real boundedness and regularity of the KP soliton solution due to the algebraic geometric data implies that the soliton data are realizable in $Gr^{\mathrm{TNN}}(k,n)$.

The above ansatz is compatible both with real KP finite–gap theory, with Sato dressing of the vacuum with finite dimensional operators and the characterization of real regular bounded $(n-k,k)$–line solitons.

In the first part of the paper, we review some known facts about $(n-k,k)$–line soliton KP solutions and the finite–gap setting (Section 2), we justify the above ansatz by explicitly characterizing the KP–soliton data producing such divisor structure and call such KP–solitons T–hyperelliptic (Sections 3 and 4) and explain the relation with the results in [1] in the case $Gr^{\mathrm{TP}}(n-1,n)$ (Section 5). The main results of this part are:

(1) We identify the points in $Gr^{\mathrm{TNN}}(k,n)$ which correspond to real regular bounded KP $(n-k,k)$–line solitons with algebraic geometric data on Γ;

(2) We prove that the divisor structure on Γ is compatible with the KP reality condition if and only if the soliton data $(\mathcal{K},[A])$ correspond to points $[A]\in Gr^{\mathrm{TP}}(k,n)$, with

$$A^i_j = a_j k_j^{i-1}, \quad i\in[k],\, j\in[n], \qquad [a]\in Gr^{\mathrm{TP}}(1,n).$$

We call such KP solitons T–hyperelliptic and explicitly construct the KP–wavefunction on Γ. k divisor points belong to the intersection of the real ovals with Γ_+ and the remaining $(n-k-1)$ to the intersection of the real ovals with Γ_-. For instance the divisor structure shown in picture 1.b) corresponds to soliton data $\mathcal{K}=\{\kappa_1<\cdots<\kappa_4\}$ and $[A]\in Gr^{\mathrm{TP}}(2,4)$ as above.

The special form of the KP τ–function associated to T–hyperelliptic solitons relates such class of solutions to the finite non–periodic Toda system[33]. Since Γ is related to the algebraic–geometric description of the finite non–periodic Toda hierarchy[28, 4, 25], it is then natural to investigate the relations between the algebraic–geometric approach for the two systems. The asymptotics of the KP wavefunction and of the Toda Baker–Akhiezer functions are different at the essential singularity P_+ since they are modeled respectively on regular finite gap KP theory and on the periodic Toda system.

In the second part of the paper we thouroughly investigate such relations and we prove that the divisor structure of the two systems are connected. In section 6 we review known results on the finite non–periodic Toda system to settle notations. In section 7, we introduce two sequences of Darboux transformations and express the Toda Baker–Akhiezer functions of [25]

using the the Toda resolvent. In section 8, we explain the relations between the divisor structure of KP T–hyperelliptic solitons and the spectral problem for the Toda system. In section 9 we solve the inverse problem of reconstructing the KP soliton data from a k–compatible divisor and express the Toda hierarchy solutions in function of the zero–divisor dynamics of the Toda Baker–Akhiezer functions. The main results for this part are:

(1) The KP vacuum divisor associated here to T–hyperelliptic solitons is the Toda divisor found in [25];
(2) The Darboux transformations generating T–hyperelliptic $(n-k,k)$–line solitons are associated to well–known recurrencies for the Toda system;
(3) The pole divisor of the normalized KP $(n-k,k)$–line T–hyperelliptic soliton wavefunction coincides with the zero divisor at times $\vec{t}\equiv\vec{0}$ of the k–th component of the Toda Baker–Akhiezer function defined in [25].

In the last part of the paper we discuss the duality correspondence between KP T–hyperelliptic soliton data in $Gr(k,n)$ and in $Gr(n-k,n)$ induced by space–time inversion from the algebraic–geometric point of view. For the Toda system, such transformation corresponds to the composition of space–time inversion with the reflection of the entries of the Toda Lax matrix $\mathfrak{A}$ with respect to the antidiagonal. In section 10, we discuss the relation between space time–inversion, duality in Grassmann cells and divisors of dual T–hyperelliptic solitons and its relation to Toda. In particular, we give the explicit formula to compute the divisor of the dual soliton in $Gr^{\mathrm{TP}}(n-k,n)$ from the soliton data in $Gr^{\mathrm{TP}}(k,n)$.

In section 11, we summarize the results of the paper. We are convinced that the results presented in this paper may be generalized in many directions and open the way to novel interpretations of the KP wavefunctions associated to generic points in $Gr^{\mathrm{TP}}(k,n)$[1] and its generalization to the whole $Gr^{\mathrm{TNN}}(k,n)$[2].

2. $(n-k,k)$-line solitons via Darboux transformation, in the Sato Grassmannian and in finite–gap theory

The KP-II equation[17]

$$(-4u_t+6uu_x+u_{xxx})_x+3u_{yy}=0, \tag{2}$$

is the first non–trivial flow of an integrable hierarchy [8, 9, 15, 32, 36] and in the following we denote $\vec{t}=(t_1=x,t_2=y,t_3=t,t_4,\dots)$. In this section, we characterize the real bounded

regular $(n-k,k)$-line soliton solutions in the general class of KP–soliton solutions via Darboux transformations, Sato's dressing transformations and finite gap–theory.

For any $k,n \in \mathbb{N}$ with $k<n$, we denote $[k,n]=\{k,k+1,\dots,n-1,n\}$ and $[n]=[1,n]$. Following Postnikov [35], a $k\times n$ real matrix $A\in Mat^{\rm TNN}_{k,n}$ if all the maximal $(k\times k)$ minors of A are non–negative and at least one of them is positive. The totally non–negative Grassmannian is $Gr^{\rm TNN}(k,n)=GL^+_k\backslash Mat^{\rm TNN}_{k,n}$, where GL^+_k are the $k\times k$ real matrices with positive determinant. The totally positive Grassmannian is $Gr^{\rm TP}(k,n)=\mathcal{S}\cap Gr^{\rm TNN}(k,n)$, where $\mathcal{S}$ is the top cell in the Gelfand–Serganova decomposition of $Gr(k,n)$, *i.e.* $[A]\in Gr^{\rm TP}(k,n)$ if and only if all maximal $(k\times k)$ minors of A are positive.

The simplest way to construct KP solitons is via the Wronskian method [13, 29]: suppose that $f^{(1)}(\vec{t}),\dots,f^{(k)}(\vec{t})$ satisfy the heat hierarchy $\partial_{t_l}f^{(r)}=\partial_x^l f^{(r)}$, with $l\geq 1$, $r\in[k]$, and let $\tau(\vec{t})={\rm Wr_x}(f^{(1)},\dots,f^{(k)})$. Then $u(\vec{t})=2\partial_x^2\log(\tau(\vec{t}))$, is a solution to KP-II. Let $\mathcal{K}=\{\kappa_1<\kappa_2<\dots<\kappa_n\}$ and $A=(A^r_j)$ be a real $k\times n$ matrix. The $(n-k,k)$–line soliton solutions are obtained choosing

$$f^{(r)}(\vec{t})=\sum_{j=1}^n A^r_j E_j(\vec{t}),\quad r=1,\dots,k,\qquad E_j(\vec{t})=e^{\theta(\kappa_j;\vec{t})},\ \text{ with }\ \theta(\zeta;\vec{t})=\sum_{n\geq 1}\zeta^n t_n. \tag{3}$$

In such a case $\tau(\vec{t})=\sum_{1\leq i_1<\dots<i_k\leq n}\Delta(i_1,\dots,i_k)E_{[i_1,\dots,i_k]}(\vec{t})$, where $\Delta(i_1,\dots,i_k)$ are the Plücker coordinates of the corresponding point in the real Grassmannian, $[A]\in Gr(k,n)$, and $E_{[i_1,\dots,i_k]}(\vec{t})={\rm Wr}_x(E_{i_1},\dots,E_{i_k})$. Then, following [18], the $(n-k,k)$–line soliton $u(\vec{t})$ is regular and bounded for all $\vec{t}=(x,y,t,0,\dots)$ if and only if $[A]\in Gr^{\rm TNN}(k,n)$, *i.e.* all $k\times k$ minors $\Delta(i_1,\dots,i_k)\geq 0$.

The KP solitons are also realized as special solutions in the Sato theory of the KP hierarchy [36, 32] using the Dressing transformation. Indeed let the vacuum hierarchy be

$$\partial_x\Psi^{(0)}=\lambda\Psi^{(0)},\qquad \partial_{t_n}\Psi^{(0)}=\partial_x^n\Psi^{(0)}=\lambda^n\Psi^{(0)},\quad n\geq 1,$$

and suppose that the dressing operator $W=1-w_1\partial_x^{-1}-w_2\partial_x^{-2}-\cdots$ satisfies the Sato equations $\partial_{t_n}W=(W\partial_x^nW^{-1})_+W-W\partial_x^n$, $n\geq 1$, where the symbol $(\cdot)_+$ denotes the differential part of the given operator. Then the KP hierarchy is generated by the inverse gauge (dressing) transformation $L=W\partial_xW^{-1}$

$$L\tilde{\Psi}^{(0)}=\lambda\tilde{\Psi}^{(0)},\qquad \partial_{t_n}\tilde{\Psi}^{(0)}=B_n\tilde{\Psi}^{(0)},\quad B_n=(W\partial_x^nW^{-1})_+,\quad n\geq 1,$$

with $\tilde{\Psi}^{(0)}=W\Psi^{(0)}$ and $x=t_1$, $y=t_2$, $t=t_3$. In such a case the Lax operator takes the form $L=\partial_x+u_2\partial_x^{-1}+u_3\partial_x^{-2}+\cdots$, and $u_2=\partial_x w_1$ satisfies the KP equation. $u(x,y,t)$ is the $(n-k,k)$–line soliton associated to soliton data $(\mathcal{K},[A])$, if and only if the dressing operator

takes the form $W = 1 - w_1\partial_x^{-1} - w_2\partial_x^{-2} - \cdots - w_k\partial_x^{-k}$, and $Df^{(r)} = 0$, $r = 1, \dots, k$, where D is the Darboux transformation[30]

$$D \equiv W\partial_x^k = \partial_x^k - w_1(\vec{t})\partial_x^{k-1} - \dots - w_k(\vec{t}). \tag{4}$$

Regular finite–gap solutions are the complex periodic or quasi–periodic meromorphic solutions to the KP equation (2). Krichever [21, 22] has classified this class of solutions: for any non-singular genus g complex algebraic curve Γ with a marked point P_0 and a local parameter λ such that $\lambda^{-1}(P_0) = 0$, there exists a family of regular complex finite–gap solutions $u(\vec{t})$ to (2) parametrized by non special divisors $\mathcal{D} = (P_1, \dots, P_g)$ on $\Gamma\backslash\{P_0\}$. More precisely, the Baker–Akhiezer function $\tilde{\Psi}(P; \vec{t})$ meromorphic on $\Gamma\backslash\{P_0\}$ with poles on $\mathcal{D}$ and an essential singularity at P_0 with the following asymptotics $\tilde{\Psi}(\zeta; \vec{t}) = \big(1 + \dfrac{\chi_1(\vec{t})}{\zeta} + O(\zeta^{-2})\big)e^{\zeta x + \zeta^2 y + \zeta^3 t + \cdots}$, as $(\zeta \to \infty)$, is a solution to $\dfrac{\partial\tilde{\Psi}}{\partial y} = B_2\tilde{\Psi}$, $\dfrac{\partial\tilde{\Psi}}{\partial t} = B_3\tilde{\Psi}$, where $B_2 \equiv (L^2)_+ = \partial_x^2 + u$, $B_3 = (L^3)_+ = \partial_x^3 + \frac{3}{4}(u\partial_x + \partial_x u) + u_3$ satisfy the compatibility conditions $[-\partial_y + B_2, -\partial_t + B_3] = 0$. If the divisor $\mathcal{D}$ is non–special, then $\tilde{\Psi}$ is uniquely identified by its normalization for $P \to P_0$. Finally, $\partial_x u_3 = \frac{3}{4}\partial_y u$, and the KP regular finite–gap solution is

$$u(x, y, t) = 2\partial_x\chi_1(x, y, t, 0, \dots) = 2\partial_x^2 \log(\Theta(Ux + Vy + Zt + z_0)) + c,$$

where $c \in \mathbb{C}$, $\Theta(z)$, $z \in \mathbb{C}^g$, is the Riemann theta–function associated to Γ, $z_0 \in \mathbb{C}^g$ is a constant vector which depends on the divisor $\mathcal{D}$, and U, V, $Z \in \mathbb{C}^g$ are the periods of certain normalized meromorphic differentials on Γ.

According to [10], a regular finite–gap KP–solution u is real (quasi)–periodic if and only if it corresponds to Krichever data on a regular M–curve Γ. More precisely Γ must possess an anti–holomorphic involution which fixes the maximum number of ovals, $\Omega_0, \dots, \Omega_g$ such that $P_0 \in \Omega_0$ and there is exactly one divisor point in each other oval, $P_j \in \Omega_j$, $j \in [g]$. We recall that the ovals are topologically circles and, by a theorem of Harnack [14], the maximal number of components (ovals) of a real algebraic curve in the projective plane is equal to $(d-1)(d-2)/2 + 1$, where d denotes the degree of the curve. According to finite–gap theory [8, 9], soliton solutions are obtained from finite–gap regular solutions in the limit in which some of the cycles of Γ become singular. In particular, the real smooth bounded $(n-k, k)$-line solitons may be obtained from regular real quasi–periodic solutions in the rational limit of M–curves where some cycles shrink to double points.

Example 2.1. *The hyperelliptic involution σ fixes the n real ovals of the genus $(n-1)$ real hyperelliptic curve $\Gamma^{(\epsilon)} = \{(\zeta,\eta) \ : \ \eta^2 = -\epsilon^2 + \prod_{j=1}^n (\zeta-\kappa_j)^2\}$ and the divisor $(\gamma_1^{(\epsilon)},\dots,\gamma_{n-1}^{(\epsilon)})$ such that $\zeta(\gamma_j^{(\epsilon)}) \in [\kappa_j+\epsilon, \kappa_{j+1}-\epsilon]$, for any $j \in [n-1]$ satisfies the reality condition in* [10] *for any $\epsilon^2 > 0$ sufficiently small.*

3. Vacuum KP–wavefunction on Γ

In this section, we define a family of vacuum wavefunctions $\Psi(P,\vec{t})$ on $\Gamma = \Gamma_+ \sqcup \Gamma_-$ as in (5), which coincide with Sato vacuum wavefunction on Γ_+ and are parametrized by non special divisors $\mathcal{B} = \{b_1 < \dots < b_{n-1}\} \subset \Gamma_-$, such that $b_r \in]\kappa_r, \kappa_{r+1}[$, $r \in [n-1]$. It is straightforward to verify that there is a bijection between such non special divisors and points $[a] \in Gr^{\mathrm{TP}}(1,n)$.

We model such vacuum wavefunctions on those constructed in [1] for KP soliton data in $Gr^{\mathrm{TP}}(1,n)$: in this case the effect of the Darboux transformation is to move the vacuum divisor points inside the finite ovals in such a way that exactly one pole belongs to Γ_+.

In the next sections, we investigate which Darboux transformations move the vacuum divisor points inside the finite ovals in such a way that exactly k poles belong to Γ_+, for $k \in [n-1]$.

In the following, the algebraic setting is (Γ, P_+, ζ), where Γ is the rational degeneration of a real hyperelliptic curve of genus $g = n-1$, with affine part

$$\Gamma \ : \ \{(\zeta;\eta) \in \mathbb{C}^2 \ : \ \eta^2 = \prod_{j=1}^n (\zeta-\kappa_j)^2\}. \tag{5}$$

$\Gamma = \Gamma_+ \sqcup \Gamma_-$, with $\Gamma_+ = \{(\zeta;\eta(\zeta))\,;\, \zeta \in \mathbb{C}\}$, $\Gamma_- = \sigma(\Gamma_+)$, where σ is the hyperelliptic involution, *i.e.* $\sigma(\zeta;\eta(\zeta)) = (\zeta;-\eta(\zeta))$. The marked point is $P_+ \in \Gamma_+$ such that $\zeta^{-1}(P_+) = 0$ and $P_- = \sigma(P_+) \in \Gamma_-$. To simplify the notations, ζ denotes the local coordinate in both copies $\Gamma_\pm$.

Γ possesses n ovals Ω_j, $j \in [0,n-1]$. Ω_0 is the oval containing the points $P_\pm$ and we call it infinite oval. We enumerate the other (finite) ovals according to the double points belonging to them, *i.e.* Ω_j is the oval containing κ_j and κ_{j+1}, $j \in [n-1]$ (see Figure 2 for the example $n=4$).

Definition 3.1. *Let $\Gamma = \Gamma_+ \sqcup \Gamma_-$ as in (5). On Γ_- we take $n-1$ real ordered points $\mathcal{B} = \{b_1,\dots,b_{n-1}\}$ such that $b_r \in]\kappa_r,\kappa_{r+1}[$, for any $r \in [n-1]$. To such data we associate a vacuum wave function*

$$\Psi(P,\vec{t}) = \begin{cases} \Psi^{(+)}(\zeta;\vec{t}) = e^{\theta(\zeta;\vec{t})}, & \zeta \in \Gamma_+, \\ \Psi^{(-)}(\zeta;\vec{t}) = \dfrac{\sum_{l=1}^n a_l E_l(\vec{t}) \prod_{j\neq l}^n (\zeta-\kappa_j)}{\prod_{r=1}^{n-1}(\zeta-b_r)}, & \zeta \in \Gamma_- \end{cases} \tag{6}$$

where

$$\theta \equiv \theta(\zeta;\vec{t}) = \sum_{i\geq 1} \zeta^i t_i, \qquad a_l = \frac{\prod_{r=1}^{n-1}(\kappa_l - b_r)}{\prod_{s\neq l}^{n}(\kappa_l - \kappa_s)}, \qquad l \in [n].$$

In Figure 2, we show the real part of the curve Γ in the case $n = 4$. The divisor points b_j are in the intersection of the finite ovals Ω_j with Γ_-, for any $j \in [3]$.

The wavefunction as in Definition 3.1 has the following properties

Lemma 3.1. *Let Γ, $\mathcal{B}$ and $\Psi(\zeta;\vec{t})$ as in Definition 3.1. Then $\Psi(\zeta,\vec{t})$ is a regular function of the variables $\vec{t} = \{t_1 = x, t_2 = y, t_3 = t, t_4, \ldots\}$ and, as a function of ζ, it is defined on $\Gamma\backslash\{P_+\}$. Moreover*

(1) $\Psi^{(\pm)}(\zeta;\vec{t})$ is real for real ζ and real $\vec{t}$;

(2) $\Psi(P,\vec{0}) \equiv 1$, for all $P \in \Gamma$;

(3) $\Psi^{(+)}(\zeta;\vec{t})$ is the Sato vacuum KP wave function;

(4) Ψ has an essential singularity at the marked infinity point $P_+ \in \Omega_0$;

(5) The coefficients a_j, $j \in [n]$, are positive and $\sum_{j=1}^{n} a_j = 1$;

*(6) **Divisor of poles of** $\Psi(P,\vec{t})$**:** for any $\vec{t}$, $\Psi^{(-)}(\zeta;\vec{t})$ is meromorphic in ζ on Γ_- with simple poles at the points b_r, $r \in [n-1]$, whose position is independent of $\vec{t}$;*

*(7) **Divisor of zeros of** $\Psi(P,\vec{t})$**:** In each finite oval Ω_r, ($r \in [n-1]$), $\Psi(\zeta,\vec{t})$ possesses exactly one simple pole b_r and exactly one simple zero $b_r(\vec{t})$, such that $b_r(\vec{0}) = b_r$ and $b_r(\vec{t}) \in]\kappa_r, \kappa_{r+1}[\subset \Gamma_- \cap \Omega_r$, for all $\vec{t}$;*

*(8) **Gluing rules between** $\Gamma_\pm$**:** For any $j \in [n]$, the values of $\Psi^{(\pm)}$ coincide at the marked points κ_j for all $\vec{t}$:*

$$\Psi^{(+)}(\kappa_j,\vec{t}) = \Psi^{(-)}(\kappa_j,\vec{t}) \equiv E_j(\vec{t}), \quad \forall \vec{t}, \tag{7}$$

with $E_j(\vec{t})$ as in (3).

We remark that the condition that each zero of $\Psi(\zeta,\vec{t})$ lies in a well-defined open interval $]\kappa_j, \kappa_{j+1}[$ for all $\vec{t}$, is natural since $\Psi(\zeta,\vec{t})$ represents a vacuum wave function: no collision is possible for the zero divisor.

To each vacuum divisor $\mathcal{B}$ as in Definition 3.1, we associate the point in $[a] \equiv [a_1, \ldots, a_n] \in Gr^{\mathrm{TP}}(1,n)$. Viceversa, to each point $[a] \in Gr^{\mathrm{TP}}(1,n)$, we uniquely associate a divisor $\mathcal{B}$ and vacuum wave–function $\Psi(\zeta;\vec{t})$, where $\mathcal{B} = \{\zeta : Q(\zeta) = 0\}$, with

$$Q(\zeta) = \prod_{j=1}^{n}(\zeta - \kappa_j)\left(\sum_{l=1}^{n} \frac{a_l}{\zeta - \kappa_l}\right), \tag{8}$$

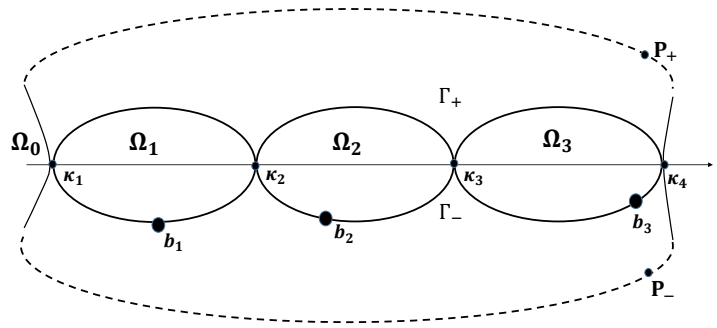

FIGURE 2. *A vacuum divisor* $\{b_1 < b_2 < b_3\}$ *on* Γ *of arithmetic genus 3. The essential singularity of the Sato vacuum wavefunction is at* P_+*. The hyperelliptic involution is reflection w.r.t. the horizontal axis and it leaves invariant the ovals* $\Omega_0, \dots, \Omega_3$*. The values of* $\Psi^{(+)}(\zeta, \vec{t})$*,* $\Psi^{(-)}(\zeta, \vec{t})$ *coincide at the double points* κ_j*,* $j \in [4]$*, for all* $\vec{t}$*.*

Indeed we have the following

Lemma 3.2. *Let* $\mathcal{K} = \{\kappa_1 < \kappa_2 < \cdots < \kappa_n\}$ *and* $[a] \in Gr^{TP}(1,n)$ *be fixed, with normalization* $\sum_{j=1}^{n} a_j = 1$*. Let* b_r*,* $r \in [n-1]$*, be the solutions to* $Q(\zeta) = 0$ *as in (8). Then* $b_r \in]\kappa_r, \kappa_{r+1}[$*,* $\forall r \in [n-1]$ *and*

$$\Psi(\zeta;\vec{t}) = \begin{cases} \Psi^{(+)}(\zeta;\vec{t}) \equiv e^{\theta(\zeta;\vec{t})}, & \text{if } \zeta \in \Gamma_+, \\ \Psi^{(-)}(\zeta;\vec{t}) \equiv \sum_{l=1}^{n} a_j E_j(\vec{t}) \ \dfrac{\prod_{s\neq l}(\zeta-\kappa_s)}{\prod_{r=1}^{n-1}(\zeta - b_r)}, & \text{if } \zeta \in \Gamma_-, \end{cases} \tag{9}$$

satisfies all the properties in Lemma 3.1.

Remark 3.1. *Lemmata 3.1 and 3.2 imply that the condition* $b_r \in]\kappa_r, \kappa_{r+1}[$*,* $r \in [n-1]$*, is equivalent to* $a_j > 0$*, for all* $j \in [n]$*. It follows that assigning the algebraic geometric data* $(\Gamma, P_+, \mathcal{B})$ *as in Definition 3.1 is equivalent to assigning the vacuum soliton data* $\mathcal{K}, [a]$*, with* $\mathcal{K} = \{\kappa_1 < \cdots \kappa_n\}$*,* $[a] \in Gr^{TP}(1,n)$*.*

Remark 3.2. *In* [1]*, we have introduced a parameter* ξ *which governs the position of the marked points at which we glue different copies of* $\mathbb{C}P^1$ *to construct a rational degeneration of an* M*–curve of arithmetic genus* $k(n-k)$*. At such marked points we control the asymptotics of the vacuum wave–function. In the case* $k = 1$*, in* [1] *we glue exactly two copies* Γ_0 *and* Γ_1 *of* $\mathbb{C}P^1$ *and the introduction of the scaling parameter* ξ *is unnecessary since any set of* n *ordered points on* Γ_1 *would work. In particular, for the choice* $\lambda_j^{(1)} = \kappa_j$*,* $j \in [n]$*, the vacuum wavefunction constructed in* [1] *coincides with (6) for soliton data* $(\mathcal{K}, [a])$*,* $[a] \in Gr^{TP}(1,n)$*.*

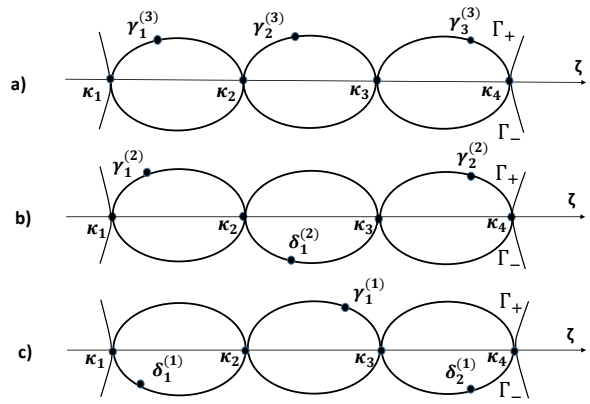

FIGURE 3. *According to Corollary 4.1, k–compatible divisors are realized in $(n-1)$–dimensional varieties of KP soliton data in $Gr^{TP}(k,n)$. Here Γ has arithmetic genus 3, vacuum divisor as in Figure 2 and $k=3$ (Figure 3.a), $k=2$ (Figure 3.b) and $k=1$(Figure 3.c). The case $k=0$ (3 divisor points on Γ_-) corresponds to the vacuum KP wavefunction in Figure 2 and the trivial KP solution $u(\vec{t})\equiv 0$.*

4. T-HYPERELLIPTIC SOLITONS AND k–COMPATIBLE DIVISORS

Any $(n-k,k)$–line soliton regular bounded KP-solution is uniquely identified by the soliton data $(\mathcal{K},[A])$, where $\mathcal{K}=\{\kappa_1<\cdots<\kappa_n\}$ and $[A]\in Gr^{\mathrm{TNN}}(k,n)$ define the heat hierarchy solutions $f^{(1)}(\vec{t}),\dots f^{(k)}(\vec{t})$, as in (3) and the Darboux transformation $D^{(k)}\equiv W\partial_x^k=\partial_x^k-w_1(\vec{t})\partial_x^{k-1}-\dots-w_k(\vec{t})$, as in (4) where W is the Dressing operator of the vacuum and $D^{(k)}f^{(i)}\equiv 0$, for any $i\in[k]$.

Let (Γ,P_+,ζ), with Γ as in (5). We make the ansatz that such real boundend soliton solution may be obtained from a real regular finite gap KP–solution on $\{(\zeta;\eta)\ :\ \eta^2=-\epsilon^2+\prod_{j=}^{n}(\zeta-\kappa_j)^2\}$, in the limit $\epsilon\to 0$. Such ansatz means that the Darboux transformation $D^{(k)}$, associated to the soliton data $(\mathcal{K},[A])$, acts on the vacuum divisor moving the poles inside the finite ovals in such a way that exactly k of them belong to Γ_+ and $n-k-1$ to Γ_- and that $D^{(k)}$ creates exactly k fixed zeros at P_-. Then the normalized KP–wavefunction $\frac{D^{(k)}\Psi(\zeta,\vec{t})}{D^{(k)}\Psi(\zeta,\vec{0})}$ has a Krichever divisor $\mathcal{D}$ which satisfies the reality condition [10] and is realized from the vacuum dressing via $W=D^{(k)}\partial_x^{-k}$, that is:

(1) the Krichever divisor is $\mathcal{D}=\{\gamma_1,\dots,\gamma_k,\delta_1,\dots,\delta_{n-k-1}\}\subset\Gamma\backslash\{P_+\}$;
(2) there is exactly one divisor point in each finite oval Ω_j, $j\in[n-1]$ according to the counting rule below;
(3) $\mathcal{D}\cap\Gamma_+=\{\gamma_1,\dots,\gamma_k\}$, $\mathcal{D}\cap\Gamma_-=\{\delta_1,\dots,\delta_{n-k-1}\}$.

We call such a divisor k–compatible and T–hyperelliptic the corresponding KP soliton data $(\mathcal{K}, [A])$. In this section we prove that the soliton data $(\mathcal{K}, [A])$ are T–hyperelliptic if and only if there is $[a] \in Gr^{\mathrm{TP}}(1,n)$ and

$$A^i_j = a_j \kappa_j^{i-1}, \qquad i \in [k],\ j \in [n].$$

In such a case the vacuum divisor $\mathcal{B}$ is the one associated to $[a] \in Gr^{\mathrm{TP}}(1,n)$ in the previous section. In particular for $\mathcal{K}$ and $k \in [2, n-2]$ fixed, such soliton data parametrize an $(n-1)$–dimensional variety in $Gr^{\mathrm{TP}}(k,n)$.

Remark 4.1. ***(The counting rule)*** *We call $\mathcal{D}$ generic, if no points of $\mathcal{D}$ lie at the double points of Γ, otherwise we call it non generic. In the non generic case, both $X = \kappa_m$ and $\sigma(X) = \kappa_m$ belong to $\mathcal{D}$, for some $m \in [n]$, and the wavefunction has simple zeroes (resp. simple poles) at κ_m at both the components Γ_- and Γ_+, i.e. we have a collision of 2 divisor points $\gamma_s \in \Gamma_+$ and $\delta_l \in \Gamma_-$. Then we use the following counting rule: if we have a pair of divisor points at a double point, one of them is assigned to the left oval and the other is assigned to the right oval.*

To characterize on (Γ, P_+, ζ) the admissible soliton data $(\mathcal{K}, [A])$, where $[A] \in Gr^{\mathrm{TP}}(k,n)$, we introduce the following definitions of k–compatible divisor and of T-hyperelliptic soliton.

Definition 4.1. ***(k–compatible divisor)*** *Let (Γ, P_+, ζ) be as above. Let $\Omega_j^{\pm} = \Gamma_{\pm} \cap [\kappa_j, \kappa_{j+1}]$, $j \in [n-1]$ so that the finite ovals are $\Omega_j = \Omega_j^+ \cup \Omega_j^-$. We call a divisor $\mathcal{D} = \{\gamma_1, \dots, \gamma_k, \delta_1, \dots, \delta_{n-k-1}\} \subset \Gamma \backslash \{P_+\}$ k–compatible if:*

(1) $\gamma_j \in \Gamma_+$, $j \in [k]$ are pairwise distinct and $\delta_l \in \Gamma_-$, $l \in [n-k-1]$ are pairwise distinct points;

(2) $\mathcal{D} \cap \Omega_j \neq \emptyset$, $j = 1, \dots, n-1$, and each finite oval contains exacly one divisor point according to the counting rule;

(3) $\mathcal{D}_{\pm} \cap \{\kappa_1, \kappa_n\} = \emptyset$, so that in particular no divisor point is in the infinite oval Ω_0;

(4) if $\kappa_m \in \mathcal{D}_+$ for some $m \in [2, n-1]$, then $\kappa_m \in \mathcal{D}_-$ and $\mathcal{D}_{\pm} \cap ([\kappa_{m-1}, \kappa_{m+1}] \backslash \{\kappa_m\}) = \emptyset$.

We define the k–compatible divisor generic if $\mathcal{D}_{\pm} \cap \{\kappa_1, \dots, \kappa_n\} = \emptyset$.

In figure Figure 3, we show k–compatible divisors for the case $n = 4$ and $k \in [3]$.

Definition 4.2. ***(T-hyperelliptic soliton)*** *Let $([A], \mathcal{K})$, $[A] \in Gr^{TNN}(k,n)$ be the soliton data of a regular bounded $(n-k,k)$–soliton solution to the KP equation and let $D^{(k)} = \partial_x^k - w_1(\vec{t})\partial_x^{k-1} -$*

$\cdots - w_k(\vec{t})$ *be the associated Darboux transformation. Let* (Γ, P_+, ζ)*, with* Γ *as in (5),* $P_+ \in \Gamma_+$ *such that* $\zeta^{-1}(P_+) = 0$*.*

We call the soliton $([A], \mathcal{K})$ *T-hyperelliptic, if there exists a* 0*–compatible vacuum divisor* $\mathcal{B} = \{b_1 < \cdots < b_{n-1}\} \subset \Gamma_-$ *i.e.* $\kappa_1 < b_1 < \kappa_2 < \cdots < b_{n-1} < \kappa_n$*, and the corresponding vacuum wavefunction* $\Psi(\zeta; \vec{t})$ *as in Definition 3.1 has the following property: after the Darboux transformation* $D^{(k)}$ *associated to the soliton data* $([A], \mathcal{K})$*, the zero–divisor of* $\Psi^{(k)}(\zeta; \vec{t}) \equiv D^{(k)}\Psi(\zeta; \vec{t})$ *is* $\mathcal{Z}^{(k)}_{\mathcal{B}}(\vec{t}) \equiv \mathcal{D}_{\mathcal{B}}(\vec{t}) \cup \{k\, P_-\} \subset \Gamma \backslash \{P_+\}$*, with* $\mathcal{D}_{\mathcal{B}}(\vec{t})$ *k–compatible for any* $\vec{t}$*.*

Remark 4.2. *We remark that the T-hyperelliptic solitons are a class of KP–soliton solutions whose algebraic geometric data are associated to rational degenerations of hyperelliptic curves, but they do not exhaust the whole class of KP–soliton solutions associated to algebraic geometric data on rational degenerations of real hyperelliptic curves.*

The vacuum wave–function defined in the previous section on Γ possesses a compatible 0–divisor according to the above definition and the corresponding T-hyperelliptic soliton is the trivial solution $u(\vec{t}) \equiv 0$.

Let $\mathcal{K} = \{\kappa_1 < \cdots < \kappa_n\}$, Γ as in (5), $\mathcal{B} = \{b_1 < \cdots < b_{n-1}\} \subset \Gamma_-$, and the vacuum wave–function as in (6)

$$\Psi_{\mathcal{B}}(\zeta; \vec{t}) = \begin{cases} e^{\theta(\zeta;\vec{t})}, & \text{if } \zeta \in \Gamma_+, \\ \Psi^{(-)}_{\mathcal{B}}(\zeta; \vec{t}) \equiv \sum_{l=1}^{n} a_j(\mathcal{B}) E_j(\vec{t}) \, \dfrac{\prod_{s \neq l}(\zeta - \kappa_s)}{\prod_{r=1}^{n-1}(\zeta - b_r)}, & \text{if } \zeta \in \Gamma_-, \end{cases} \tag{10}$$

where $a_j(\mathcal{B}) = \mathrm{Res}_{\zeta=\kappa_j} \frac{\prod_{r=1}^{n-1}(\zeta - b_r)}{\prod_{s=1}^{n}(\zeta - \kappa_s)} > 0$, $j \in [n]$, $\sum_{j=1}^{n} a_j(\mathcal{B}) = 1$.

Let $[A] \in Gr^{\mathrm{TNN}}(k, n)$ and choose a basis of heat hierarchy solutions $f^{(i)}(\vec{t}) = \sum_{j=1}^{n} A^i_j E_j(\vec{t})$, $i \in [k]$, representing $(\mathcal{K}, [A])$. The Darboux transformation $D^{(k)} = \partial_x^k - w_1(\vec{t})\partial_x^{k-1} - \cdots - w_k(\vec{t})$ is obtained solving the linear system $D^{(k)} f^{(i)}(\vec{t}) \equiv 0$, $i \in [k]$. The soliton data $(\mathcal{K}, [A])$ are T–hyperelliptic if and only if the Darboux transformed wave–function takes the form

$$\Psi^{(k)}_{\mathcal{B}}(\zeta; \vec{t}) \equiv D^{(k)}\Psi_{\mathcal{B}}(\zeta; \vec{t}) = \begin{cases} \left(\zeta^k - w_1(\vec{t})\zeta^{k-1} - \cdots - w_k(\vec{t})\right) e^{\theta(\zeta;\vec{t})}, & \zeta \in \Gamma_+, \\ \tilde{A}^{(k)}(\vec{t}) \dfrac{\prod_{l=1}^{n-k-1}(\zeta - \delta^{(k)}_l(\vec{t}))}{\prod_{r=1}^{n-1}(\zeta - b_r)}, & \zeta \in \Gamma_-, \end{cases} \tag{11}$$

with $\tilde{A}^{(k)}(\vec{t}) \neq 0$ for almost all $\vec{t}$ (we anticipate that, from our construction, necessarily $\tilde{A}^{(k)}(\vec{t}) > 0$ for all $\vec{t}$).

For any fixed 0–compatible (vacuum) divisor $\mathcal{B}$, the restriction of $D^{(k)}\Psi_{\mathcal{B}}(\zeta;\vec{t})$ to Γ_+ satisfies condition (1) in Definition 4.1, since $\mathcal{Z}^{(k)}_{\mathcal{B}}(\vec{t})\cap\Gamma_+=\{\gamma^{(k)}_1(\vec{t}),\dots,\gamma^{(k)}_k(\vec{t})\}$, with

$$\zeta^k - w_1(\vec{t})\zeta^{k-1}-\dots-w_k(\vec{t})=\prod_{n=1}^{k}(\zeta-\gamma^{(k)}(\vec{t})). \tag{12}$$

Moreover, by a Theorem in Malanyuk [27], for any real $\vec{t}$, $\kappa_1\le\gamma^{(k)}_1(\vec{t})<\dots<\gamma^{(k)}_k(\vec{t})\le\kappa_n$. On Γ_- for generic choice of $\mathcal{B}$,

(13)
$$D^{(k)}\Psi^{(-)}_{\mathcal{B}}(\zeta;\vec{t})=\sum_{j=1}^{n}a_j(\mathcal{B})\,E_j(\vec{t})\,\prod_{l=1}^{k}(\kappa_j-\gamma^{(k)}_l(\vec{t}))\,\frac{\prod_{s\neq j}^{n}(\zeta-\kappa_s)}{\prod_{r=1}^{n-1}(\zeta-b_r)}=\tilde{A}^{(k)}(\vec{t})\frac{\prod_{l=1}^{n-1}(\zeta-\delta^{(k)}_l(\vec{t}))}{\prod_{r=1}^{n-1}(\zeta-b_r)},$$

has non–special complex zero divisor $\mathcal{Z}^{(k)}_{\mathcal{B},-}(\vec{t})\equiv\mathcal{Z}^{(k)}_{\mathcal{B}}(\vec{t})\cap\Gamma_-=\left\{\delta^{(k)}_1(\vec{t}),\dots,\delta^{(k)}_{n-1}(\vec{t})\right\}$, such that, for all $\vec{t}$, $\#\left(\mathcal{Z}^{(k)}_{\mathcal{B},-}(\vec{t})\cap[k_1,k_n]\right)\ge n-k-1$.

Remark 4.3. *(12) and (13) imply that generic soliton data $(\mathcal{K},[A])$, with $[A]\in Gr^{TNN}(k,n)$, may be realized assigning on Γ, a non special divisor $\mathcal{D}\equiv\mathcal{D}(\vec{0})=\{\gamma^{(k)}_1,\dots,\gamma^{(k)}_k,\delta^{(k)}_1,\dots,\delta^{(k)}_{n-1}\}$ such that:*

(1) for all $j\in[k]$, $\gamma^{(k)}_j\in\Gamma_+\cap[\kappa_1,\kappa_n]$ is a root of (12) for $\vec{t}=\vec{0}$;
(2) $\delta^{(k)}_s\in\Gamma_-$, for all $s\in[n-1]$ are such that $\#\left(\{\delta^{(k)}_s\}\cap[k_1,k_n]\right)\ge n-k-1$;
(3) $\mathcal{D}$ satisfies the counting rule;
(4) in each finite oval there is an odd number of divisor points.

In the following Lemma we establish the necessary and sufficient conditions so that $\{k\,P_-\}\subset\mathcal{Z}^{(k)}_{\mathcal{B},-}(\vec{t})$.

Lemma 4.1. *Let $(\mathcal{K},[A])$, with $[A]\in Gr^{TNN}(k,n)$ and $\mathcal{K}=\{\kappa_1<\dots<\kappa_n\}$ be given. Let $D^{(k)}$ be the Darboux transformation for $(\mathcal{K},[A])$. Let $\mathcal{B}=\{b_1<\dots<b_{n-1}\}$ be a 0–compatible divisor on Γ as in (5) and $\Psi_{\mathcal{B}}(\zeta;\vec{t})$ as in (10). Let $s\in[k]$ be fixed. Then the following assertions are equivalent*

(i) $\{s\,P_-\}\subset\mathcal{Z}^{(k)}_{\mathcal{B},-}(\vec{t})$ and $\left(\mathcal{Z}^{(k)}_{\mathcal{B},-}(\vec{t})\backslash\{s\,P_-\}\right)\subset\Gamma_-\backslash\{P_-\}$;
(ii) For all $\vec{t}$, $\sum_{j=1}^{n}\kappa_j^{i-1}a_j(\mathcal{B})\prod_{l=1}^{k}(\kappa_j-\gamma^{(k)}_l(\vec{t}))E_j(\vec{t})\equiv 0$, $\forall i\in[s]$ and $\sum_{j=1}^{n}\kappa_j^{s}a_j(\mathcal{B})\prod_{l=1}^{k}(\kappa_j-\gamma^{(k)}_l(\vec{t}))E_j(\vec{t})\neq 0$;
(iii) The heat hierarchy solutions $\mu_i(\vec{t})=\sum_{j=1}^{n}k^i a_j(\mathcal{B})E_j(\vec{t})$, $i\ge 0$, satisfy $D^{(k)}\mu_i(\vec{t})=0$, for all $i\in[0,s-1]$, and $D^{(k)}\mu_s(\vec{t})\neq 0$.

The proof is trivial and it is omitted.

Remark 4.4. *Let $k \in [n-1]$ be fixed and let $(\mathcal{K}, [A])$, $[A] \in Gr^{TNN}(k,n)$ be the soliton data. For $s = 1$, the condition $D^{(k)}\mu_0(\vec{t}) = 0$ in Lemma 4.1 is realized taking $\mu_0(\vec{t}) = \sum_{j=1}^{n} A^i_j E_j(\vec{t})$, for some fixed $i \in [k]$ and choosing $\mathcal{B}$ as in Lemma 3.2, for $a_j \equiv \frac{A^i_j}{\sum_{l=1}^{n} A^i_l}$, for all $j \in [n]$.*

For $1 < s < k$, generically, there does not exist a 0–divisor $\mathcal{B} \subset \Gamma_-$ such that the heat hierarchy solutions $\mu_0(\vec{t}), \dots, \mu_{s-1}(\vec{t})$ satisfy Lemma 4.1.

For $s = k$, if such a divisor $\mathcal{B} \subset \Gamma_-$ exists, it is unique. Moreover in such case $\mu_0(\vec{t}), \dots, \mu_{k-1}(\vec{t})$ generate the Darboux transformation $D^{(k)}$.

In the following, we restrict ourselves to the case $s = k$.

Corollary 4.1. *Let the soliton data $(\mathcal{K}, [A])$ be given, with $[A] \in Gr^{TNN}(k,n)$, and suppose that there exists a vacuum divisor $\mathcal{B}$ such that Lemma 4.1 holds for $s = k$. Then $[A] \in Gr^{TP}(k,n)$ and $[A] = [B]$, where*

$$B^i_j = \kappa_j^{i-1} a_j(\mathcal{B}), \qquad j \in [n], \quad i \in [k]. \tag{14}$$

Let $\mathcal{K} = \{\kappa_1, \dots \kappa_4\}$. Then the divisors in Figure 3 correspond to KP–soliton data $(\mathcal{K}, [A])$, respectively with $[A] \in Gr^{\mathrm{TP}}(3,4)$ (Figure 3.a)), $[A] \in Gr^{\mathrm{TP}}(2,4)$ (Figure 3.b)) and $[A] \in Gr^{\mathrm{TP}}(1,4)$ (Figure 3.c)) where $[A]$ satisfies Corollary 4.1.

Corollary 4.2. *Let $\mathcal{K}$ be given. Then for any $[A] \in Gr^{TP}(n-1,n)$, there exists $[a] \in Gr^{TP}(1,n)$ such that $[A] = [B]$, with B as in (14).*

Proof. Indeed, let $x_i > 0$, $i = 1 \dots, n-1$ and

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 & (-1)^{n-2}x_1 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & -x_{n-2} \\ 0 & \cdots & 0 & 1 & x_{n-1} \end{pmatrix} \tag{15}$$

be the representative matrix in the reduced row echelon form of the given point $[A] \in Gr^{\mathrm{TP}}(n-1,n)$. For $[a] \in Gr^{\mathrm{TP}}(1,n)$, let B the matrix as in (14) and denote

$$y_i \equiv \Delta_{[1,\dots,\hat{i},\dots,n]}(B) = \left(\prod_{s \neq i} a_s\right) \prod_{\substack{1 \le j < l \le n \\ j,l \neq i}} (\kappa_l - \kappa_j), \qquad i \in [n],$$

its $(n-1)$ minors with the i–th column omitted. Then $[A]=[B]$ if and only if

$$x_i = \frac{y_i}{y_n} = (-1)^{n-i-1}\frac{a_n}{a_i}\prod_{\substack{s=1\\ s\neq i}}^{n-1}\frac{(\kappa_n-\kappa_s)}{(\kappa_i-\kappa_s)}, \qquad i\in[n-1].$$

□

For fixed k, $1<k<n$, (14) identify a $(n-1)$–dimensional variety in $Gr^{\mathrm{TP}}(k,n)$. Indeed, let the matrix in reduced row echelon form

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 & (-1)^{k-1}x_1^k & \cdots & (-1)^{k-1}x_{n-k}^k \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & x_1^1 & \cdots & x_{n-k}^1 \end{pmatrix}, \tag{16}$$

represent a point $[A]\in Gr^{\mathrm{TP}}(k,n)$ for which Lemma 4.1 holds for $n=k$. The $k\times(n-k)$ matrix

$$X = \begin{pmatrix} x_1^1 & \cdots & x_{n-k}^1 \\ \vdots & \vdots & \vdots \\ x_1^k & \cdots & x_{n-k}^k \end{pmatrix},$$

is totally positive in classical sense and the explicit relations between $[a]=[a_1,\dots,a_n]\in Gr^{\mathrm{TP}}(1,n)$ and the coefficients of X are as follows:

Lemma 4.2. *Let B the $k\times n$ matrix defined in (14) and associated to $[a]=[a_1,\dots,a_n]\in Gr^{TP}(1,n)$, such that $\sum_{j=1}^{n} a_n = 1$. Then the coefficients of the associated reduced row echelon form matrix A as in (16) take the form*

$$x_j^i \;=\; (-1)^{k-i}\,\frac{a_j}{a_i}\prod_{\substack{l=1\\ l\neq i}}^{k}\frac{(\kappa_j-\kappa_l)}{(\kappa_i-\kappa_l)}, \qquad i\in[k], \quad j\in[n-k+1,k].$$

We end this section summarizing the divisor properties of T–hyperelliptic $(n-k,k)$–solitons.

Theorem 4.1. *Let $k\in[n-1]$, $\mathcal{K}=\{\kappa_1<\cdots<\kappa_n\}$, $[a]=[a_1,\dots,a_n]\in Gr^{TP}(1,n)$ and $[A]\in Gr^{TNN}(k,n)$ satisfy Corollary 4.1. Let $D^{(k)}$ and $\tilde{\Psi}^{(k)}(\zeta;\vec{t})=\frac{D^{(k)}\Psi(\zeta;\vec{t})}{D^{(k)}\Psi(\zeta;\vec{0})}$ respectively be the Darboux transformation and the KP–wavefunction associated to the soliton data $(\mathcal{K},[A])$, with $\Psi(\zeta;\vec{t})$ as in (9). Then, for all $\vec{t}$, the zero divisor $\mathcal{D}^{(k)}(\vec{t})$ of $\tilde{\Psi}^{(k)}$ has the following properties:*

(1) Γ_+ contains exactly k points of $\mathcal{D}^{(k)}(\vec{t})$: $\mathcal{D}_+^{(k)}(\vec{t})=\{\gamma_1^{(k)}(\vec{t}),\dots,\gamma_k^{(k)}(\vec{t})\}$;

(2) Γ_- contains exactly $n-k-1$ points of $\mathcal{D}^{(k)}(\vec{t})$: $\mathcal{D}_-^{(k)}(\vec{t})=\{\delta_1^{(k)}(\vec{t}),\dots,\delta_{n-k-1}^{(k)}(\vec{t})\}$;

(3) All points $\gamma_l^{(k)}(\vec{t})$, $l\in[k]$, lying in Γ_+ are pairwise different;

(4) All points $\delta_s^{(k)}(\vec{t})$, $s\in[n-k-1]$, lying in Γ_- are pairwise different;

(5) $\mathcal{D}(\vec{t}) \cap \Omega_0 = \emptyset$*;*

(6) $\mathcal{D} \subset \bigcup_n \Omega_n$*, that is each divisor point is real and lies in some finite oval;*

(7) Each finite oval Ω_n *contains exactly one point of* $\mathcal{D}(\vec{t})$ *both for the generic and the non generic case, according to the counting rule.*

Remark 4.5. *For any fixed* $\vec{t}$*, no zero or pole of* $\tilde{\Psi}^{(k)}(\zeta, \vec{t})$ *lies at the double points* κ_1 *or* κ_n*, and, thanks to the counting rule,* $\#\left(\mathcal{D}_+^{(k)}(\vec{t}) \cap \{\kappa_2, \dots, \kappa_{n-1}\}\right) = \#\left(\mathcal{D}_-^{(k)}(\vec{t}) \cap \{\kappa_2, \dots, \kappa_{n-1}\}\right) \leq min\{k, n-k-1\}$*.*

The proof of Theorem 4.1 follows the same lines as for Theorem 7 in [1] and is omitted. Notice that the pole divisor of $\tilde{\Psi}^{(k)}$ is just $\mathcal{D}^{(k)}(\vec{0})$.

5. Comparison with the construction in [1]

In this section we show that Γ as in (5) is a desingularization of the curve Γ_ξ constructed in [1] for points in $Gr^{\mathrm{TP}}(n-1, n)$ and that the respective KP wavefunctions coincide.

In [1], for any fixed $\xi >> 1$ we have associated to any soliton data $(\mathcal{K}, [A])$, with $\mathcal{K} = \{\kappa_1 < \kappa_2 < \cdots < \kappa_n\}$ and $[A] \in Gr^{\mathrm{TP}}(k, n)$,

(1) a connected curve $\Gamma_\xi = \Gamma_0 \sqcup \Gamma_{\xi,1} \sqcup \cdots \sqcup \Gamma_{\xi,k}$, which is the rational degeneration of a regular M–curve of genus $(n-k)k$ with $1 + (n-k)k$ ovals, Ω_0, $\Omega_{i,j}$, $i \in [k]$, $j \in [n-k]$;

(2) a vacuum wavefunction $\Psi_\xi(\lambda; \vec{t})$ on $\Gamma(\xi)$ with the following properties:

  (a) $\Psi_\xi(\lambda; \vec{t})$ is real for $\lambda \in \Omega_0 \cup_{i,j} \Omega_{i,j}$ and real $\vec{t}$;

  (b) on Γ_0, $\Psi_\xi(\lambda; \vec{t})$ coincides with Sato vacuum wavefunction and has essential singularity at $P_0 \in \Gamma_0 \cap \Omega_0$;

  (c) on each $\Gamma_{\xi,i}$, $i \in [k]$, $\Psi_\xi(\lambda; \vec{t})$ is meromorphic and possesses $n-k$ divisor points $b_{\xi,1}^{(i)}, \dots, b_{\xi,n-k}^{(i)}$, whose position is independent of $\vec{t}$;

  (d) in each finite oval $\Omega_{i,j}$, $i \in [k]$, $j \in [n-k]$, there is exactly one such divisor point according to the counting rule;

and we have proven that, after the Darboux transformation $D^{(k)}$ associated with the given soliton data $(\mathcal{K}, [A])$, the normalized wavefunction $\tilde{\Psi}_\xi(\lambda; \vec{t}) = \frac{D^{(k)}\Psi_\xi(\lambda; \vec{t})}{D^{(k)}\Psi_\xi(\lambda; \vec{0})}$ satisfies Dubrovin–Natanzon conditions, *i.e.*

(1) $\tilde{\Psi}_\xi(\lambda; \vec{t})$ is real for $\lambda \in \Omega_0 \cup_{i,j} \Omega_{i,j}$ and real $\vec{t}$;

(2) on Γ_0 it has an essential singularity at P_0 and it possesses k divisor points $\gamma_{\xi,1}^{(0)}, \dots, \gamma_{\xi,k}^{(0)}$ whose position is independent of time;

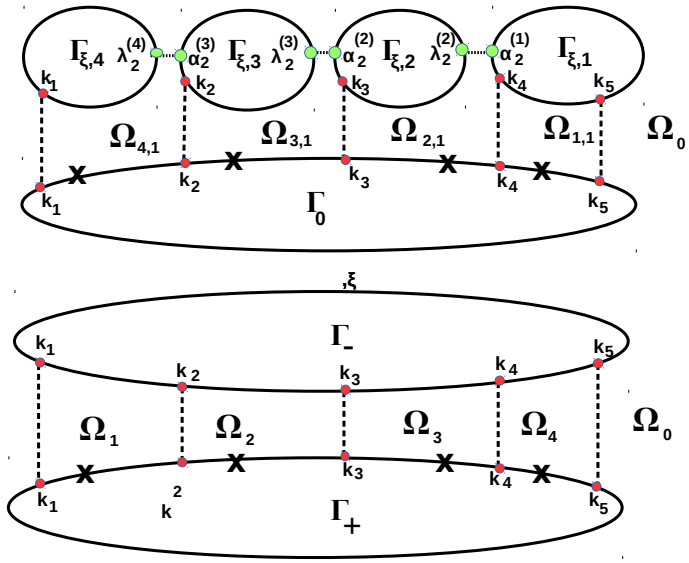

FIGURE 4. *The desingularization of* Γ_ξ *to* Γ *in the case* $n = 5$. $\Gamma = \Gamma_+ \sqcup \Gamma_-$ *(below) is a desingularization of* $\Gamma_\xi = \Gamma_0 \sqcup \hat{\Gamma}_{\xi,-}$ *(above), with* $\Gamma_- = \Gamma_0$ *and* $\hat{\Gamma}_{\xi,-} = \Gamma_{\xi,1} \sqcup \cdots \sqcup \Gamma_{\xi,4}$. *The divisor points on* Γ *and* Γ_ξ *are represented by crosses and are left unchanged by the desingularization.*

(3) in each $\Gamma_{\xi,i}$, $i \in [k]$, $\tilde{\Psi}_\xi(\lambda; \vec{t})$ is meromorphic and possesses $n-k-1$ divisor points $\delta^{(i)}_{\xi,1}, \dots, \delta^{(i)}_{\xi,n-k-1}$ whose position is independent of $\vec{t}$;

(4) in each finite oval $\Omega_{i,j}$, $i \in [k]$, $j \in [n-k]$, there is exactly one such divisor point according to the counting rule.

Let $\mathcal{K} = \{\kappa_1 < \kappa_2 < \cdots < \kappa_n\}$ be fixed and $k = n-1$. According to Corollary 4.2, any $[A] \in Gr^{\mathrm{TP}}(n-1, n)$ contains a representative matrix B as in (14). Let $\xi >> 1$ be fixed and $\Gamma_\xi = \Gamma_0 \sqcup \Gamma_{\xi,1} \sqcup \cdots \sqcup \Gamma_{\xi,n-1}$ as in Theorem 6 in [1]. Then $\Gamma_0 = \Gamma_+$ and the double points on $\Gamma_{\xi,r}$, $r \in [n-1]$, in the local coordinate λ are

$$\lambda_1^{(r)} = 0, \qquad \lambda_2^{(r)} = -1, \qquad \alpha_2^{(r)} = \xi^{-1}. \tag{17}$$

On each $\Gamma_{\xi,r}$, $r = 1, \dots, n-1$, let us perform the linear substitution

$$\zeta = M_\xi^{(r)}(\lambda) \equiv c^{(r)}_{\xi,0}\lambda + c^{(r)}_{\xi,1}, \tag{18}$$

where $c^{(r)}_{\xi,1} = \kappa_{n-r}$, and $c^{(r)}_{\xi,0}$ are recursively defined

$$c^{(1)}_{\xi,0} = \kappa_{n-1} - \kappa_n, \qquad c^{(r)}_{\xi,0} = \kappa_{n-r} - M_\xi^{(r-1)}(\frac{1}{\xi}) = \sum_{j=0}^{r-1} (-1)^r \frac{\kappa_{n-r+j} - \kappa_{n-r+j+1}}{\xi^j}, \ r \in [2, n-1].$$

In the local coordinate ζ the marked points on $\Gamma_{\xi,r}$ are $\lambda_1^{(r)} = \kappa_{n-r}$, $r \in [n-1]$,

$$\lambda_2^{(r)} = \begin{cases} \kappa_n, & \text{if } r = 1, \\ \frac{c_{\xi,0}^{(r-1)}}{\xi} + \kappa_{n-r+1}, & \text{if } r \in [2, n-1], \end{cases}$$

and $\alpha_2^{(r)} = \frac{c_{\xi,0}^{(r)}}{\xi} + \kappa_{n-r}$, $r \in [n-1]$. If ξ is sufficiently big, $\lambda_2^{(r)} = \alpha_2^{(r-1)} \in]\kappa_{n-r}, \kappa_{n-r+1}[$, since $c_{\xi,0}^{(r-1)} < \kappa_{n-r+1} - \kappa_{n-r+2} < 0$; moreover $\lim_{\xi\to\infty} \alpha_2^{(r)} = \kappa_{n-r}$, for any $r \in [n-1]$.

Remark 5.1. *For any fixed $\xi >> 1$, Γ_ξ itself is a desingularization of $\Gamma_\infty = \Gamma_0 \sqcup \Gamma_{\infty,1} \sqcup \cdots \sqcup \Gamma_{\infty,n-1}$. On Γ_∞, Γ_0 is glued at κ_n with $\Gamma_{\infty,1}$, at κ_1 with $\Gamma_{\infty,n-1}$, and, for $r \in [2, n-1]$, at κ_{n-r+1} with $\Gamma_{\infty,r-1}$ and $\Gamma_{\infty,r}$.*

Let us denote $\Psi_{\xi,r}$, $\tilde{\Psi}_r$ respectively the vacuum and the normalized Darboux transformed wavefunctions on $\Gamma_{\xi,r}$, $r \in [0, n-1]$. Following [1], on $\Gamma_0 = \Gamma_+$, the vacuum wave–function is $\Psi_{\xi,0}(\zeta;\vec{t}) = e^{\theta(\zeta;\vec{t})}$ and $D^{(n-1)}\Psi_{\xi,0}(\zeta;\vec{t}) = \prod_{j=1}^{n-1}(\zeta - \gamma_j^{(n-1)}(\vec{t}))e^{\theta(\zeta;\vec{t})} = \Psi_+^{(n-1)}(\zeta;\vec{t})$ (see (47)).

On each $\Gamma_{\xi,r}$, $r \in [n-1]$, applying the inverse of (18), $\lambda = M_\xi^{-1}(\zeta)$, the vacuum wave–function is $\Psi_{\xi,r}(\zeta;\vec{t}) = C_r(\xi,\vec{t})\frac{\zeta - \chi_1^{(r)}(\vec{t})}{\zeta - b_1^{(r)}}$, with $C_r(\xi,\vec{t}) > 0$ for all $\vec{t}$, $\chi_1^{(r)}(\vec{t}), b_1^{(r)} \in]\kappa_{n-r}, \lambda_2^{(r)}[$, for all $\vec{t}$, and $\chi_1^{(r)}(\vec{0}) = b_1^{(r)}$. After the Darboux transformation, $D^{(n-1)}\Psi_{\xi,r}(\zeta;\vec{t}) = \frac{\tilde{C}_r(\xi,\vec{t})}{\zeta - b_1^{(r)}}$ and the normalized wavefunction restricted to $\Gamma_{\xi,r}$ is

$$\tilde{\Psi}_r(\zeta;\vec{t}) = \frac{\tilde{C}_r(\xi,\vec{t})}{\tilde{C}_r(\xi,\vec{0})} = \phi(\vec{t}). \tag{19}$$

ϕ is constant in ζ and also in ξ, since the glueing condition between Γ_0 and $\Gamma_{\xi,r}$ implies $\tilde{\Psi}_r(\kappa_{n-r};\vec{t}) = \tilde{\Psi}_+^{(n-1)}(\kappa_{n-r};\vec{t})$, where the right hand side is independent of ξ. Finally $\phi(\vec{t})$ does not depend on $r \in [n-1]$ as well, because of the glueing condition between $\Gamma_{\xi,r+1}$ and $\Gamma_{\xi,r}$, $\tilde{\Psi}_{r+1}(\lambda_2^{(r+1)};\vec{t}) = \tilde{\Psi}_r(\alpha_2^{(r)};\vec{t})$, $r \in [n-2]$. In conclusion, the double points $\lambda_2^{(r)} \in \Gamma_{\xi,r}$, $\alpha_2^{(r-1)} \in \Gamma_{\xi,r-1}$, $r = 2, \ldots, n-1$, are due to the technical conditions posed in [1], but they play no role since the normalized KP wave–function $\tilde{\Psi}(\zeta;\vec{t})$ takes the same constant value on $\hat{\Gamma}_{\xi,-} = \Gamma_{\xi,1} \sqcup \cdots \sqcup \Gamma_{\xi,n-1}$ for any $\xi > 1$. So we may desingularize $\hat{\Gamma}_{\xi,-}$ to Γ_- without modifying $\tilde{\Psi}$ for any $\xi > 1$.

Finally the normalized wavefunction as in (47) also takes the constant value $\phi(\vec{t})$ for any $\zeta \in \Gamma_-$, $\tilde{\Psi}_-^{(n-1)}(\zeta,\vec{t}) \equiv \phi(\vec{t})$, since it is constant w.r.t. the spectral parameter ζ and $\tilde{\Psi}_-^{(n-1)}(\kappa_j,\vec{t}) = \tilde{\Psi}_+^{(n-1)}(\kappa_j,\vec{t}) = \tilde{\Psi}(\kappa_j,\vec{t})$, for all $j \in [n]$ and $\vec{t}$. We thus have proven

Theorem 5.1. *Let $\xi >> 1$ be fixed. Let the soliton data $(\mathcal{K}, [A])$ be fixed with $\mathcal{K} = \{\kappa_1 < \cdots < \kappa_n\}$ and $[A] \in Gr^{TP}(n-1, n)$. Let $D^{(n-1)} = \partial_x^{n-1} - w_1^{(n-1)}(\vec{t})\partial_x^{n-2} - \cdots - w_{n-1}^{(n-1)}(\vec{t})$ be*

the Darboux transformation associated to $(\mathcal{K},[A])$, $\tilde{\Psi}(\zeta;\vec{t})$ *be the normalized wave-function on* $\Gamma_\xi=\Gamma_0\sqcup\Gamma_{\xi,1}\sqcup\cdots\sqcup\Gamma_{\xi,n-1}$ *constructed in* [1], *and* $\tilde{\Psi}^{(n-1)}(\zeta;\vec{t})$ *be the normalized wave-function on* $\Gamma=\Gamma_+\sqcup\Gamma_-$ *as in Theorem 8.1 for* $k=n-1$. *Then*

(1) *The curve* $\Gamma_\xi=\Gamma_0\sqcup\hat{\Gamma}_-$, *with* $\hat{\Gamma}_{\xi,-}\equiv\Gamma_{\xi,1}\sqcup\ldots\sqcup\Gamma_{\xi,n-1}$, *is the rational degeneration of a regular hyperelliptic curve of genus* $g=n-1$. Γ_ξ *may be desingularized to* Γ, *where* $\Gamma_0=\Gamma_+$ *and* $\hat{\Gamma}_-$ *is* Γ_- *with the extra double points* $\lambda_2^{(r)}\in\Gamma_{\xi,r}$ *and* $\alpha_2^{(r-1)}\in\Gamma_{\xi,r-1}$, *at which we connect* $\Gamma_{\xi,r}$ *to* $\Gamma_{\xi,r-1}$, *for* $r=2,\ldots,n-1$;

(2) *The wavefunctions associated to* Γ_ξ *and to its desingularization* Γ *are the same. More precisely, for any* $\zeta\in\Gamma_0\equiv\Gamma_+$ *and for any* $\vec{t}$,

$$\tilde{\Psi}_{\xi,0}(\zeta;\vec{t})=\tilde{\Psi}_+^{(n-1)}(\zeta;\vec{t})=\left(\frac{\zeta^{n-1}-w_1^{(n-1)}(\vec{t})\zeta^{n-2}-\cdots-w_{n-1}^{(n-1)}(\vec{t})}{\zeta^{n-1}-w_1^{(n-1)}(\vec{0})\zeta^{n-2}-\cdots-w_{n-1}^{(n-1)}(\vec{0})}\right)e^{\theta(\zeta;\vec{t})},$$

and there exists a regular function $\phi(\vec{t})$ *which is the common value respectively of* $\tilde{\Psi}(\zeta;\vec{t})$ *on* $\hat{\Gamma}_{\xi,-}$ *and of* $\tilde{\Psi}_-^{(n-1)}(\zeta;\vec{t})$ *on* Γ_-:

$$\phi(\vec{t})=\tilde{\Psi}_+^{(n-1)}(\kappa_j;\vec{t}),\qquad\forall j\in[n],\ \forall\vec{t}.$$

In Figure 4, we show the desingularization of Γ_ξ to Γ when $n=5$.

Let us fix the soliton data $(\mathcal{K},[A])$, with $[A]\in Gr^{\mathrm{TP}}(n-1,n)$. The representative matrix

$$A=\begin{pmatrix}1&\frac{x_1}{x_2}&0&\cdots&\cdots&0\\0&1&\frac{x_2}{x_3}&0&\cdots&0\\\vdots&\ddots&\ddots&\ddots&\ddots&\vdots\\0&\cdots&0&1&\frac{x_{n-2}}{x_{n-1}}&0\\0&\cdots&\cdots&0&1&x_{n-1}\end{pmatrix},\tag{20}$$

is equivalent to A in (15), and the Darboux transformation for $([A],\mathcal{K})$, $D^{(n-1)}=\partial_x^{n-1}-w_1^{(n-1)}(\vec{t})\partial_x^{n-2}-\cdots-w_{n-1}^{(n-1)}(\vec{t})$, has the following kernel $f^{(r)}(\vec{t})=x_{n-r+1}E_{n-r}(\vec{t})+x_{n-r}E_{n-r+1}(\vec{t})$, $r\in[n-1]$.

Corollary 5.1. *The pole divisor* $\mathcal{D}^{(n-1)}=\{\gamma_1^{(n-1)},\ldots,\gamma_{n-1}^{(n-1)}\}\subset\Gamma_+$ *associated to the soliton data* $([A],\mathcal{K})$ *satisfies*

$$x_{i+1}\prod_{l=1}^{n}(\kappa_i-\gamma_l^{(n-1)})+x_i\prod_{l=1}^{n}(\kappa_{i+1}-\gamma_l^{(n-1)})=0,\qquad i=1,\ldots,n-1,\tag{21}$$

with $\gamma_l^{(n-1)}\in]\kappa_l,\kappa_{l+1}[\cap\Gamma_+$, *for any* $l\in[n-1]$.

Identities (21) are easily deduced from the equations $D^{(n-1)}f^{(r)}=0$, $r=1,\dots,n$.

Soliton data in $Gr^{\mathrm{TP}}(n-1,n)$, are rather special since the divisor satisfies $\mathcal{D}^{(n-1)}(\vec{t})\subset\Gamma_+$.

Proposition 5.1. *Let us fix the soliton data $(\mathcal{K},[A])$, with $[A]\in Gr^{TP}(n-1,n)$. Let $D^{(n-1)}$ be the Darboux transformation associated to $(\mathcal{K},[A])$ and Γ as in (5). For any given $[c]\in Gr^{TP}(1,n)$, let the vacuum wavefunction be*

$$\Psi_{[c]}(\zeta;\vec{t})=\begin{cases} e^{\theta(\zeta;\vec{t})} & \zeta\in\Gamma_+,\\ \displaystyle\sum_{j=1}^{n}\frac{c_jE_j(\vec{t})}{\sum_{l=1}^{n}c_l}\frac{\prod_{s\neq j}^{n}(\zeta-\kappa_s)}{\prod_{r=1}^{n-1}(\zeta-b_r(c))}. & \zeta\in\Gamma_-.\end{cases}$$

Then, $\tilde{\Psi}^{(n-1)}_{[c]}(\zeta;\vec{t})\equiv\frac{D^{(n-1)}\Psi_{[c]}(\zeta;\vec{t})}{D^{(n-1)}\Psi_{[c]}(\zeta;\vec{0})}=\phi(\vec{t})$, for all $\zeta\in\Gamma_-$ and for all $\vec{t}$, with $\phi(\vec{t})$ as in Theorem 5.1.

The proof is trivial and is omitted. The above proposition means that, for any $[c]\in Gr^{\mathrm{TP}}(1,n)$, the zero divisor of the *un–normalized* wave–function, $D^{(n-1)}\Psi_{[c]}(\zeta;\vec{t})$ is $\mathcal{D}(\vec{t})\cup\{P_{[c],1},\dots,P_{[c],n-1}\}$, where the points $P_{[c],j}\in\Gamma_-$ are independent of $\vec{t}$, for all $j\in[n-1]$. However, there is a unique point $[a]\in Gr^{\mathrm{TP}}(1,n)$ such that the zero divisor of the *un–normalized* wavefunction $D^{(n-1)}\Psi_{[a]}(\zeta,\vec{t})$ is $\mathcal{D}^{(n-1)}\cup\{(n-1)P_-\}$. This is one of the reasons why we have defined k–compatibility for the un-normalized wave–function $D^{(k)}\Psi$ instead that for the normalized wavefunction $\tilde{\Psi}^{(k)}$.

6. The finite non–periodic Toda lattice hierarchy

The special form of the KP τ–functions associated to T–hyperelliptic solitons relates such class of solutions to those of the finite non–periodic Toda system[37, 33]. Also Γ is related to the algebraic–geometric description of the finite non–periodic Toda hierarchy[28, 4, 25]. For these reasons, it is then natural to investigate the relations between the algebraic–geometric approach for the two systems. In this section we review known results on the solutions to the Toda hierarchy and on the Toda Baker–Akhiezer function and then, in the following sections, we discuss the relations between the two systems.

Toda [37] proposed a model of a chain of n mass points moving on the real axis, with position $\mathfrak{q}_l$, $l\in[n]$, Hamiltonian $H=\frac{1}{2}\sum_{l=1}^{n}\mathfrak{p}_l^2+\sum_{l=1}^{n-1}e^{(\mathfrak{q}_l-\mathfrak{q}_{l+1})}$, which is integrable both in the periodic and non–periodic case [12, 33]. The Toda system is the first flow of an integrable hierarchy and it has been generalized in many ways [7, 11, 16, 20, 26, 34, 38].

The finite non–periodic Toda lattice system corresponds to formal boundary conditions $\mathfrak{q}_0 = -\infty$, $\mathfrak{q}_{n+1} = +\infty$, and, under the transformation $\mathfrak{a}_k = e^{\mathfrak{q}_k - \mathfrak{q}_{k+1}}$, $k \in [n-1]$, $\mathfrak{b}_k = -\mathfrak{p}_k$, $k \in [n]$, it is equivalent to

$$\frac{d\mathfrak{a}_k}{dt_1} = \mathfrak{a}_k \left(\mathfrak{b}_{k+1} - \mathfrak{b}_k\right), \quad k \in [n-1], \qquad \frac{d\mathfrak{b}_k}{dt_1} = \mathfrak{a}_k - \mathfrak{a}_{k-1}, \quad k \in [n], \tag{22}$$

with boundary conditions $\mathfrak{a}_0 = \mathfrak{a}_n = 0$. The space of configurations in the new variables is

$$D = \left\{(\mathfrak{a}, \mathfrak{b}) \in \mathbb{R}^{n-1} \times \mathbb{R}^n \ : \ \mathfrak{a}_k > 0,\, k \in [n-1]\right\}. \tag{23}$$

The system (22) may be put in Lax form $\dfrac{d\mathfrak{A}}{dt_1} = [\mathfrak{B}_1, \mathfrak{A}]$, with

$$\mathfrak{A} = \begin{pmatrix} \mathfrak{b}_1 & \mathfrak{a}_1 & 0 & \cdots & 0 \\ 1 & \mathfrak{b}_2 & \mathfrak{a}_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & \mathfrak{b}_{n-1} & \mathfrak{a}_{n-1} \\ 0 & \cdots & 0 & 1 & \mathfrak{b}_n \end{pmatrix}, \tag{24}$$

$\mathfrak{B}_1 = (\mathfrak{A})_+$, where $(P)_+$ denotes the strict upper triangular part of the matrix P. (22) is the first flow of an integrable hierarchy

$$\frac{d\mathfrak{A}}{dt_j} = [\mathfrak{B}_j, \mathfrak{A}], \qquad j \geq 1, \qquad \mathfrak{B}_j = \left(\mathfrak{A}^j\right)_+. \tag{25}$$

(25) are the equations associated to the symmetries of the Toda lattice generated by $H_j = \frac{1}{j+1}\,\mathrm{Tr}\,\mathfrak{A}^{j+1}$. Since the 0-th flow is trivial, $\frac{d\mathfrak{A}}{dt_0} \equiv 0$, in the following we take $t_0 = 0$ and denote $\vec{t} = (t_1, t_2, t_3, \dots)$.

In the configuration space D, the eigenvalues of $\mathfrak{A}$, $\kappa_j^{(T)}$ are real, distinct and independent of all t_j, *i.e.* they are constants of the motion. We order them in increasing order, $\kappa_1^{(T)} < \kappa_2^{(T)} < \cdots < \kappa_n^{(T)}$, and denote the characteristic polynomial and the resolvent of $\mathfrak{A}(\vec{t})$, respectively,

$$\Delta_n(\zeta) = \ \det\ (\zeta I - \mathfrak{A}) = \prod_{j=1}^{n} (\zeta - \kappa_j^{(T)}). \qquad \mathfrak{R}(\zeta; \vec{t}) = \left(\zeta \mathfrak{I}_n - \mathfrak{A}(\vec{t})\right)^{-1}.$$

Let $\Delta_j(\zeta; \vec{t})$, $\hat{\Delta}_j(\zeta; \vec{t})$, $j \in [n]$, respectively be the minors formed by the last j rows and columns and by the first j rows and columns of $\zeta \mathfrak{I}_n - \mathfrak{A}$, $j \in [n-1]$:

$$\Delta_j(\zeta; \vec{t}) = \det\ \left(\zeta \mathfrak{I}_n - \mathfrak{A}(\vec{t})\right)_{[n-j+1, \dots, n]}, \qquad \hat{\Delta}_j(\zeta; \vec{t}) = \det\ \left(\zeta \mathfrak{I}_n - \mathfrak{A}(\vec{t})\right)_{[1, \dots, j]}, \tag{26}$$

where $\Delta_0(\zeta;\vec{t}) \equiv \hat{\Delta}_0(\zeta;\vec{t}) \equiv 1$. The vectors $(\Delta_{n-1},\dots,\Delta_0)$ and $(\hat{\Delta}_0,\dots,\hat{\Delta}_{n-1})$ are respectively the first column and the last row of $\Delta_n(\zeta)\,\mathfrak{R}(\zeta;\vec{t})$ and the following identities hold for $j\in[n-1]$,

$$\begin{aligned}
&\Delta_{j+1}(\zeta;\vec{t}) = (z-\mathfrak{b}_{n-j}(\vec{t}))\Delta_j(\zeta;\vec{t}) - \mathfrak{a}_{n-j}(\vec{t})\Delta_{j-1}(\zeta;\vec{t}),\\
&\hat{\Delta}_{j+1}(\zeta;\vec{t}) = (z-\mathfrak{b}_{j+1}(\vec{t}))\hat{\Delta}_j(\zeta;\vec{t}) - \mathfrak{a}_j(\vec{t})\hat{\Delta}_{j-1}(\zeta;\vec{t}),\\
&\Delta_n(\zeta) = \hat{\Delta}_n(\zeta) = \Delta_{n-j}(\zeta;\vec{t})\hat{\Delta}_j(\zeta;\vec{t}) - \mathfrak{a}_j(\vec{t})\Delta_{n-j-1}(\zeta;\vec{t})\hat{\Delta}_{j-1}(\zeta;\vec{t}).
\end{aligned} \tag{27}$$

Remark 6.1. *We remark that reflection w.r.t. to the anti–diagonal transforms $\mathfrak{A}(\vec{t})$ into $\mathfrak{A}^*(\vec{t})$ where*

$$\mathfrak{A}^*(\vec{t}) = \begin{pmatrix} \mathfrak{b}_n(\vec{t}) & \mathfrak{a}_{n-1}(\vec{t}) & 0 & \cdots & 0\\ 1 & \mathfrak{b}_{n-1}(\vec{t}) & \mathfrak{a}_{n-2}(\vec{t}) & \ddots & \vdots\\ 0 & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & 1 & \mathfrak{b}_2(\vec{t}) & \mathfrak{a}_1(\vec{t})\\ 0 & \cdots & 0 & 1 & \mathfrak{b}_1(\vec{t}) \end{pmatrix}. \tag{28}$$

Such transformation preserves the spectrum and inverts the role of the minors since

$$\begin{aligned}
&\Delta^*_j(\zeta;\vec{t}) \equiv \det\ \left(\zeta\mathfrak{I}_n - \mathfrak{A}^*(\vec{t})\right)_{[n-j+1,\dots,n]} = \hat{\Delta}_j(\zeta;\vec{t}),\\
&\hat{\Delta}^*_j(\zeta;\vec{t}) \equiv \det\ \left(\zeta\mathfrak{I}_n - \mathfrak{A}^*(\vec{t})\right)_{[1,\dots,j]} = \Delta_j(\zeta;\vec{t}),
\end{aligned} \qquad j\in[0,n]. \tag{29}$$

As a consequence we may equivalently use Δ_j or $\hat{\Delta}_j$ to represent IVP solutions of the finite Toda system (25). It is well known that assigning the initial datum $\mathfrak{A}(\vec{0})$ in the configuration space D as in (23) is equivalent to the Toda data $(\mathcal{K},[a])$, where $\mathcal{K}=\{\kappa_1^{(T)}<\cdots<\kappa_n^{(T)}\}$ is the spectrum of $\mathfrak{A}_0$ and $[a]\in Gr^{TP}(1,n)$, with $a_l = \mathrm{Res}_{\zeta=\kappa_l^{(T)}}\,\mathfrak{R}_{11}(\zeta;\vec{0})$, $l\in[n]$. The reflection transformation induces a transformation of $Gr^{TP}(1,n)$ into itself, and $\mathfrak{A}(\vec{0})$ is also associated to the reflected set $(\mathcal{K},[\hat{a}])$, with $\mathcal{K}$ as before and $[\hat{a}]\in Gr^{TP}(1,n)$ uniquely identified by the residues of $\mathfrak{R}_{nn}(\zeta;\vec{0})$.

6.1. **IVP solutions to the Toda hierarchy.** We recall below different characterizations of the solutions to the IVP $\mathfrak{A}(\vec{0})=\mathfrak{A}_0$ for the Toda hierarchy flows (25). Let us define the generating functions

$$\begin{aligned}
&\mathfrak{f}(\zeta;\vec{t}) \equiv \prec e_1,\mathfrak{R}(\zeta;\vec{t})e_1\succ \equiv \frac{\Delta_{n-1}(\zeta;\vec{t})}{\Delta_n(\zeta)} = \sum_{j\geq 0}\frac{h_j(\vec{t})}{\zeta^{j+1}} = \sum_{j=1}^{n}\frac{\mathfrak{M}_j(\vec{t})}{\zeta-\kappa_j^{(T)}},\\
&\hat{\mathfrak{f}}(\zeta;\vec{t}) \equiv \prec e_n,\mathfrak{R}(\zeta;\vec{t})e_n\succ \equiv \frac{\hat{\Delta}_{n-1}(\zeta;\vec{t})}{\Delta_n(\zeta)} = \sum_{j\geq 0}\frac{\hat{h}_j(\vec{t})}{\zeta^{j+1}} = \sum_{j=1}^{n}\frac{\hat{\mathfrak{M}}_j(\vec{t})}{\zeta-\kappa_j^{(T)}}.
\end{aligned} \tag{30}$$

For any initial datum $\mathfrak{A}_0$ in the configuration space D, the exponential matrix $\psi(\vec{t})$, $\vec{t}=(t_1,\dots,t_s)$, $s\geq n-1$, admits a Bruhat decomposition

$$\psi(\vec{t}) = \exp\left(\mathfrak{A}_0 t_1+\cdots+\mathfrak{A}_0^s t_s\right) = \mathfrak{L}(\vec{t})\mathfrak{U}(\vec{t}), \tag{31}$$

where $\mathfrak{L}(\vec{t})$ is lower triangular with positive entries on the diagonal and $\mathfrak{U}(\vec{t})$ is unit upper triangular, $\mathfrak{L}(\vec{0}) = \mathfrak{U}(\vec{0}) = \mathfrak{I}_n$. Such decomposition gives the following explicit characterization of the IVP solution to (25).

Proposition 6.1. *Let $\mathfrak{A}_0$ be a Jacobi matrix of the form (24) with $a_j(\vec{0}) > 0$, $j \in [n-1]$ and eigenvalues $\kappa_1^{(T)} < \kappa_2^{(T)} < \cdots < \kappa_n^{(T)}$. Define $\psi(\vec{t})$, $\mathfrak{f}(\zeta;\vec{t})$ and $\hat{\mathfrak{f}}(\zeta;\vec{t})$ as in (31) and (30). Then*

$$\mathfrak{A}(\vec{t}) = \mathfrak{L}(\vec{t})^{-1}\mathfrak{A}_0\mathfrak{L}(\vec{t}) = \mathfrak{U}(\vec{t})\mathfrak{A}_0\mathfrak{U}(\vec{t})^{-1},$$

Finally, in this representation of the solution the generating functions and the Hankel coefficients take the form

$$\mathfrak{f}(\zeta;\vec{t}) = \frac{\prec e_1, \psi(\vec{t})(\zeta\mathfrak{I}_n - \mathfrak{A}_0)^{-1}e_1 \succ}{\prec e_1, \psi(\vec{t})e_1 \succ}, \qquad \hat{\mathfrak{f}}(\zeta;\vec{t}) = \frac{\prec e_n, (\zeta\mathfrak{I}_n - \mathfrak{A}_0)^{-1}\psi^{-1}(\vec{t})e_n \succ}{\prec e_n, \psi^{-1}(\vec{t})e_n \succ},$$
$$h_j(\vec{t}) = \frac{\prec e_1, \psi(\vec{t})\mathfrak{A}_0^j e_1 \succ}{\prec e_1, \psi(\vec{t})e_1 \succ}, \qquad \hat{h}_j(\vec{t}) = \frac{\prec e_n, \psi^{-1}(\vec{t})\mathfrak{A}_0^j e_n \succ}{\prec e_n, \psi^{-1}(\vec{t})e_n \succ}, \qquad j \geq 0.$$

Proposition 6.2. *Under the hypotheses of Proposition 6.1, let us define, for $j \geq 0$,*

$$\mu_j(\vec{t}) \equiv \prec e_1, \psi(\vec{t})\mathfrak{A}_0^j e_1 \succ = \partial_{t_1}^j \mu_0(\vec{t}), \qquad \hat{\mu}_j(\vec{t}) \equiv \prec e_n, \psi^{-1}(\vec{t})\mathfrak{A}_0^j e_n \succ = (-1)^j \partial_{t_1}^j \hat{\mu}_0(\vec{t}). \tag{32}$$

Then $\mu_{l+j}(\vec{t}) = \partial_{t_l}\mu_j(\vec{t}) = \partial_{t_1}^l \mu_j(\vec{t}) = \partial_{t_j}\mu_l(\vec{t}) = \partial_{t_1}^j \mu_l(\vec{t})$, $\forall l, j \geq 0$, and

$$\hat{\mu}_{j+l}(\vec{t}) = -\partial_{t_l}\hat{\mu}_j(\vec{t}) = -\partial_{t_j}\hat{\mu}_l(\vec{t}), \quad \partial_{t_l}\hat{\mu}_j(\vec{t}) = (-1)^{l+1}\partial_{t_1}^l \hat{\mu}_j(\vec{t}), \qquad \forall l, j \geq 0. \tag{33}$$

The converse to Proposition 6.2 also holds true.

Proposition 6.3. *Let $\mu_0(\vec{t})$ be a solution to the heat hierarchy, $\mu_j(\vec{t}) \equiv \partial_{t_1}^j \mu_0(\vec{t})$, $\forall j \geq 1$, and define the Hankel matrix $H_\mu(\vec{t}) = \left[\mu_{i+j-2}(\vec{t})\right]_{i,j\geq 1}$. Then $\mathfrak{f}_\mu(\zeta;\vec{t}) = \sum_{j\geq 0} \frac{\mu_j(\vec{t})}{\mu_0(\vec{t})}\zeta^{-(j+1)}$ generates a solution to the Toda hierarchy (25) in the configuration space D as in (23), if and only if $H_\mu(\vec{t})$ has finite rank n with principal minors $\det H_{\mu,j} > 0$, for all $j \in [n]$, that is if and only if there exists $(\mathcal{K}, [a])$ with $\mathcal{K} = \{\kappa_1^{(T)} < \cdots < \kappa_n^{(T)}\}$ and $[a] \in Gr^{TP}(1,n)$, such that $\mu_0(\vec{t}) = \sum_{j=1}^n a_j E_j(\vec{t})$.*

Similarly, let $\hat{\mu}_0(\vec{t})$ be a solution to (33), $\hat{\mu}_j(\vec{t}) = (-1)^j \partial_{t_1}^j \hat{\mu}_0(\vec{t})$, $\forall j \geq 1$, and define the Hankel matrix $\hat{H}_{\hat{\mu}}(\vec{t}) = \left[\hat{\mu}_{i+j-2}(\vec{t})\right]_{i,j\geq 1}$. Then $\hat{\mathfrak{f}}_\mu(\zeta;\vec{t}) = \sum_{j\geq 0} \frac{\hat{\mu}_j(\vec{t})}{\hat{\mu}_0(\vec{t})}\zeta^{-(j+1)}$ generates a solution to the Toda flows system (25) in the configuration space D if and only if $\hat{H}_{\hat{\mu}}(\vec{t})$ has finite rank n with principal minors $\det \hat{H}_{\hat{\mu},j} > 0$, for all $j \in [n]$, that is if and only if there exists $(\mathcal{K}, [\hat{a}])$ with $\mathcal{K} = \{\kappa_1^{(T)} < \cdots < \kappa_n^{(T)}\}$ and $[\hat{a}] \in Gr^{TP}(1,n)$, such that $\hat{\mu}_0(\vec{t}) = \sum_{j=1}^n \hat{a}_j E_j(-\vec{t})$.

The explicit form of the solution is then given in the following Proposition

Proposition 6.4. *Let $\mathfrak{A}_0$ be a Jacobi matrix as in (24) in the configuration space D (23) and let simple spectrum be $\kappa_1^{(T)} < \kappa_2^{(T)} < \cdots < \kappa_n^{(T)}$. Let $\hat{\mathfrak{f}}(\zeta;\vec{0}) =\prec e_n, (\zeta\mathfrak{I}_n - \mathfrak{A}_0)^{-1}e_n \succ$, $\mathfrak{f}(\zeta;\vec{0}) =\prec e_1, (\zeta\mathfrak{I}_n - \mathfrak{A}_0)^{-1}e_1 \succ$. Let*

$$a_l \equiv \mathfrak{M}_l(\vec{0}) = \operatorname*{Res}_{\zeta=\kappa_l^{(T)}} \mathfrak{f}(\zeta;\vec{0}) = \frac{\Delta_{n-1}(\kappa_l^{(T)},\vec{0})}{\prod\limits_{s\neq l}(\kappa_l^{(T)} - \kappa_s^{(T)})},$$

$$\hat{a}_l \equiv \hat{\mathfrak{M}}_l(\vec{0}) = \operatorname*{Res}_{\zeta=\kappa_l^{(T)}} \hat{\mathfrak{f}}(\zeta;\vec{0}) = \frac{\hat{\Delta}_{n-1}(\kappa_l^{(T)},\vec{0})}{\prod\limits_{s\neq l}(\kappa_l^{(T)} - \kappa_s^{(T)})}, \quad l \in [n],$$

$\theta_l(\vec{t}) = \sum\limits_{j\geq 1}\left(\kappa_l^{(T)}\right)^j t_l$ and $E_l(\vec{t}) = \exp(\theta_l(\vec{t}))$. Let $\tau_0(\vec{t}) \equiv 1$, $\tau_1(\vec{t}) = \mu_0(\vec{t}) = \sum\limits_{j=1}^n a_j E_j(\vec{t})$, $\tau_j(\vec{t}) = \mathrm{Wr}_{t_1}(\mu_0(\vec{t}), \partial_{t_1}\mu_0(\vec{t}), \cdots, \partial_{t_1}^{j-1}\mu_0(\vec{t}))$, $j \geq 2$.

Then $\mu_0(\vec{t})$ generates the solution to (25) with initial condition $\mathfrak{A}(\vec{0}) = \mathfrak{A}_0$ and $\mathfrak{M}_l(\vec{t}) = \dfrac{a_l E_l(\vec{t})}{\sum\limits_{j=1}^n a_j E_j(\vec{t})}$, $l \in [n]$,

$$\mathfrak{a}_k(\vec{t}) = \frac{\tau_{k-1}(\vec{t})\tau_{k+1}(\vec{t})}{\tau_k^2(\vec{t})}, \quad k \in [n-1], \qquad \mathfrak{b}_k(\vec{t}) = \frac{\partial_{t_1}\tau_k(\vec{t})}{\tau_k(\vec{t})} - \frac{\partial_{t_1}\tau_{k-1}(\vec{t})}{\tau_{k-1}(\vec{t})}, \quad k \in [n]. \tag{34}$$

Similarly $\hat{\mu}_0(\vec{t}) = \sum\limits_{j=1}^n \hat{a}_j \exp(-\theta_j(\vec{t}))$ generates the same solution and, up to a multiplicative constant $c > 0$

$$\hat{a}_j a_j = c \prod_{\substack{1\leq i<l\leq n \\ i,l\neq j}} (\kappa_l^{(T)} - \kappa_i^{(T)})^{-2}, \qquad j \in [n]. \tag{35}$$

Remark 6.2. *The meaning of the above Proposition is the following: the solution of the Toda hierarchy (25) with initial condition $\mathfrak{A}(\vec{0}) = \mathfrak{A}_0$, is thus uniquely identified by the data $(\mathcal{K}, [a])$, where $\mathcal{K} = \{\kappa_1^{(T)} < \kappa_2^{(T)} < \cdots < \kappa_n^{(T)}\}$ is the spectrum of $\mathfrak{A}_0$ and $[a] \in Gr^{TP}(1,n)$, via the heat hierarchy solution $\mu_0(\vec{t}) = \sum_{j=1}^n a_j E_j(\vec{t})$ and $f(\zeta;\vec{t})$. The same solution may be also associated to the reflected set of data $(\mathcal{K}, [\hat{a}])$, with $[\hat{a}]$ related to $[a]$ via (35), using $\hat{\mu}_0(\vec{t}) = \sum_{j=1}^n \hat{a}_j E_j(-\vec{t})$ and $\hat{f}(\zeta;\vec{t})$.*

6.2. **The spectral problem and the Baker–Akhiezer function for the finite Toda system.** The idea of singularizing the smooth spectral curve of the periodic Toda to obtain a spectral curve for the finite Toda system goes back to Mc Kean [31]. More recently new interest in the problem [28, 4] has come from the connections of the Toda lattice with Seiberg–Witten

theory of supersymmetric $SU(n)$ gauge theory. The spectral curve proposed for the open Toda lattice in [28, 4] is determined by the equation

$$\hat{\eta} = \prod_{j=1}^{n} (\zeta - \kappa_j^{(T)}), \tag{36}$$

considered as the limit $\epsilon \to 0$ of the hyperelliptic spectral curve

$$\hat{\eta} + \frac{\epsilon^2}{4\hat{\eta}} = \prod_{j=1}^{n} (\zeta - \kappa_j^{(T)}),$$

of the periodic Toda system. In [25], the Baker-Akhiezer function approach is used to provide a solution to the inverse spectral problem for the singular curve (36) and action–angle variables are constructed following the approach in [23, 24]. In [25] they use the self-adjoint representation of the finite non–periodic Toda system and introduce the following finite–dimensional operators

$$\begin{aligned} \mathfrak{A}_w^{(sa)} &= \begin{pmatrix} \mathfrak{b}_1 & \sqrt{\mathfrak{a}_1} & 0 & \cdots & 0 \\ \sqrt{\mathfrak{a}_1} & \mathfrak{b}_2 & \sqrt{\mathfrak{a}_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & 1 & \mathfrak{b}_{n-1} & \sqrt{\mathfrak{a}_{n-1}} \\ \frac{w}{\sqrt{\prod_{s=1}^{n-1} \mathfrak{a}_s}} & \cdots & 0 & \sqrt{\mathfrak{a}_{n-1}} & \mathfrak{b}_n \end{pmatrix}, \\ \mathfrak{B}_w^{(sa)} &= \frac{1}{2} \begin{pmatrix} 0 & \sqrt{\mathfrak{a}_1} & 0 & \cdots & 0 \\ -\sqrt{\mathfrak{a}_1} & 0 & \sqrt{\mathfrak{a}_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & 1 & 0 & \sqrt{\mathfrak{a}_{n-1}} \\ \frac{w}{\sqrt{\prod_{s=1}^{n-1} \mathfrak{a}_s}} & \cdots & 0 & -\sqrt{\mathfrak{a}_{n-1}} & 0 \end{pmatrix}, \end{aligned} \tag{37}$$

The time–dependent Baker–Akhiezer functions $\Psi_j^{(T)}(\zeta, \vec{t})$, $\Psi_j^{(T)\sigma}(\zeta, \vec{t})$ are, respectively, eigenvectors of $\mathfrak{A}_w^{(sa)} \Psi^{(T)} = \zeta \Psi^{(T)}$ and of $\mathfrak{A}_w^{(sa),\sigma} \Psi^{(T),\sigma} = \zeta \Psi^{(T),\sigma}$, where $\mathfrak{A}_w^{(sa),\sigma}$ is the adjoint of $\mathfrak{A}_w^{(sa)}$. Their components take the form

$$\begin{aligned} \Psi_j^{(T)}(\zeta, \vec{t}) &= e^{\frac{\theta(\zeta,\vec{t})}{2}} \left(\sum_{l=0}^{j} c_j(\vec{t}, n) \zeta^l \right), \\ \Psi_j^{(T),\sigma}(\zeta, \vec{t}) &= e^{-\frac{\theta(\zeta,\vec{t})}{2}} \left(\frac{\sum_{i=0}^{n-j-1} c_j^{\sigma}(\vec{t}, n) \zeta^i}{\prod_{s=1}^{n-1} (\zeta - b_s^{(T)})} \right), \qquad j \in [0, n-1], \end{aligned} \tag{38}$$

where $\theta(\zeta,\vec{t})=\sum_{j\geq 1}t_j\zeta^j$, and the coefficients in (38) are uniquely defined by the gluing conditions

$$\Psi_j^{(T)}(\kappa_l^{(T)},\vec{t})=\Psi_j^{(T),\sigma}(\kappa_l^{(T)},\vec{t}),\qquad l\in[n],$$

and the normalization $c_j(j)c_j^\sigma(n-i-j)=1$. In particular, the Toda divisor $\mathcal{D}^{(T)}=\{b_1^{(T)}<\cdots<b_{n-1}^{(T)}\}$ is the spectrum of the matrix obtained from $\mathfrak{A}_0^{(sa)}(\vec{0})$ deleting the first row and the first column and

$$\kappa_1^{(T)}<b_1^{(T)}<\kappa_2^{(T)}<\cdots<\kappa_{n-1}^{(T)}<b_{n-1}^{(T)}<\kappa_n^{(T)}.$$

Finally, the Toda Baker–Akhiezer components are uniquely recovered from the recurrences

$$\begin{aligned}\zeta\Psi_0^{(T)}&=\sqrt{\mathfrak{a}_1}\Psi_1^{(T)}+\mathfrak{b}_1\Psi_0^{(T)},\\ \zeta\Psi_j^{(T)}&=\sqrt{\mathfrak{a}_j}\Psi_{j+1}^{(T)}+\mathfrak{b}_{j+1}\Psi_j^{(T)}+\sqrt{\mathfrak{a}_j}\Psi_{j-1}^{(T)},\qquad j\in[N-2],\end{aligned}\tag{39}$$

$$\begin{aligned}\Psi_j^{(T),\sigma}&=\tfrac{\Phi_j^{(T)}}{\Phi_0^{(T)}},\qquad j\in[0,N-1],\\ \zeta\Phi_{n-1}^{(T)}&=\sqrt{\mathfrak{a}_{n-1}}\Phi_{n-2}^{(T)}+\mathfrak{b}_n\Phi_{n-1}^{(T)},\\ \zeta\Psi_j^{(T)}&=\sqrt{\mathfrak{a}_{j+1}}\Phi_{j+1}^{(T)}+\mathfrak{b}_{j+1}\Phi_j^{(T)}+\sqrt{\mathfrak{a}_j}\Psi_{j-1}^{(T)},\qquad j\in[N-2],\end{aligned}\tag{40}$$

where the last equation which determines w implies that the zeroes of

$$w(\zeta)=\prod_{j=1}^n(\zeta-\kappa_j^{(T)}),$$

are the eigenvalues of the matrix $\mathfrak{A}^{(sa)}$. In [25], the explicit form of the coefficients $c_j(\vec{t},n)$, $c_j^\sigma(\vec{t},n)$ is given solving the gluing conditions under the assumption $\sum_{l=1}^n\kappa_l^{(T)}=0$.

7. Toda data, Darboux transformations and Toda Baker–Akhiezer functions

In this section, we associate explicitly the Toda Baker–Akhiezer functions (38) to the Toda data $(\mathcal{K},[a])$, with $[a]\in Gr^{TP}(1,n)$ and $\mathcal{K}=\{\kappa_1^{(T)}<\kappa_2^{(T)}<\cdots<\kappa_n^{(T)}\}$ and we introduce two finite sequences of Darboux transformations, $k\in[n-1]$,

$$\begin{aligned}D^{(1)}&=\partial_{t_1}-\mathfrak{b}_1(\vec{t}),\qquad & D^{(k)}&=\left(\partial_{t_1}-\mathfrak{b}_k(\vec{t})\right)D^{(k-1)},\\ \hat{D}^{(1)}&=\partial_{t_1}-\mathfrak{b}_n(-\vec{t}),\qquad & \hat{D}^{(k)}&=\left(\partial_{t_1}-\mathfrak{b}_k(-\vec{t})\right)\hat{D}^{(k-1)}.\end{aligned}\tag{41}$$

In the next section, we show that for any given datum $(\mathcal{K},[a])$, the KP vacuum divisor coincides with the Toda divisor, the Darboux transformation $D^{(k)}$ in (41) generates a T–hyperelliptic soliton with k–compatible divisor, for $k\in[n-1]$, and that such k–compatible divisor may be reconstructed from the zeroes at $\vec{t}=\vec{0}$ of the k–th entry of the Toda Baker–Akhiezer function.

Finally, we shall use the second set of Darboux transformations in (41), $\hat{D}^{(k)}$, when discussing the duality of Grassmann cells under space–time inversion in Section 10.

The following proposition contains the necessary information which will be used in this section to re–express the Toda Baker–Akhiezer function in our representation and will be used in the next section in connection with KP T–hyperelliptic solitons.

Proposition 7.1. *Let $(\mathcal{K},[a])$, with $[a]\in Gr^{TP}(1,n)$ and $\mathcal{K}=\{\kappa_1^{(T)}<\kappa_2^{(T)}<\cdots<\kappa_n^{(T)}\}$, be the initial Toda data, $\mathfrak{A}(\vec{t})$ be the corresponding Toda hierarchy solution and $\mathfrak{R}(\zeta;\vec{t})=(\zeta\mathfrak{I}_n-\mathfrak{A}(\vec{t}))^{-1}$. Let $\Delta_k(\zeta;\vec{t})=\zeta^k-\hat{w}_1^{(k)}(\vec{t})\zeta^{k-1}-\cdots\hat{w}_k^{(k)}(\vec{t})$, $\hat{\Delta}_k(\zeta;\vec{t})=\zeta^k-w_1^{(k)}(\vec{t})\zeta^{k-1}-\cdots w_k^{(k)}(\vec{t})$, $k\in[n-1]$, as in (26). Define*

$$\Phi(\zeta;\vec{t})=\mu_0(\vec{t})\Delta_{n-1}(\zeta;\vec{t}),\qquad \hat{\Phi}(\zeta;\vec{t})=\hat{\mu}_0(-\vec{t})\hat{\Delta}_{n-1}(\zeta;-\vec{t}). \tag{42}$$

with $\mu_0(\vec{t})=\sum_{j=1}^n a_jE_j(\vec{t})$, $\hat{\mu}_0(\vec{t})=\sum_{j=1}^n\hat{a}_jE_j(-\vec{t})$ and a_j related to $\hat{a}_j$ as in Proposition 6.4. Then, for any $k\in[n-1]$,

(43)

$$\begin{aligned}
&D^{(k)}=\partial_{t_1}^k-w_1^{(k)}(\vec{t})\partial_{t_1}^{k-1}-\cdots-w_k^{(k)}(\vec{t}),\\
&D^{(k)}\Phi(\zeta;\vec{t})=\mu_0(\vec{t})\prod_{j=1}^k\mathfrak{a}_j(\vec{t})\Delta_{n-1-k}(\zeta;\vec{t})=\frac{\tau_{k+1}(\vec{t})}{\tau_k(\vec{t})}\Delta_{n-k-1}(\zeta;\vec{t})=\mu_0(\vec{t})\mathfrak{R}^1_{k+1}(\zeta;\vec{t})\Delta_n(\zeta),\\
&D^{(k)}\mu_0(\vec{t})=D^{(k)}\mu_1(\vec{t})=\cdots D^{(k)}\mu_{k-1}(\vec{t})=0,
\end{aligned}$$

$$\begin{aligned}
&\hat{D}^{(k)}=\partial_{t_1}^k-\hat{w}_1^{(k)}(-\vec{t})\partial_{t_1}^{k-1}-\cdots-\hat{w}_k^{(k)}(-\vec{t}),\\
&\hat{D}^{(k)}\hat{\Phi}(\zeta;\vec{t})=\hat{\mu}_0(-\vec{t})\prod_{j=1}^k\mathfrak{a}_{n-j}(-\vec{t})\hat{\Delta}_{n-1-k}(\zeta;-\vec{t})=\hat{\mu}_0(-\vec{t})\mathfrak{R}^{n-k}_n(\zeta;-\vec{t})\Delta_n(\zeta),\\
&\hat{D}^{(k)}\hat{\mu}_0(-\vec{t})=\hat{D}^{(k)}\hat{\mu}_1(-\vec{t})=\cdots\hat{D}^{(k)}\hat{\mu}_{k-1}(-\vec{t})=0,
\end{aligned}\tag{44}$$

where $\mu_j(\vec{t})=\partial_{t_1}^j\mu_0(\vec{t})$, $\hat{\mu}_j(\vec{t})=(-1)^j\partial_{t_1}^j\hat{\mu}_0(\vec{t})$, $\tau_0(\vec{t})\equiv1$, $\tau_j(\vec{t})=\mathrm{Wr}_{t_1}(\mu_0,\partial_{t_1}\mu_0,\cdots,\partial_{t_1}^{j-1}\mu_0)$, $j\geq1$.

Proof. The proof is by induction in k. By definition $\mathfrak{b}_1(\vec{t})=h_1(\vec{t})=\dfrac{\partial_{t_1}\mu_0(\vec{t})}{\mu_0(\vec{t})}$ and $\mathfrak{b}_n(\vec{t})=\hat{h}_1(\vec{t})=-\dfrac{\partial_{t_1}\hat{\mu}_0(\vec{t})}{\hat{\mu}_0(\vec{t})}$, so that $D^{(1)}\mu_0(\vec{t})=0$, $\hat{D}^{(1)}\hat{\mu}_0(-\vec{t})=0$ and we directly verify that

$$D^{(1)}\Phi(\zeta;\vec{t})=\mu_0(\vec{t})\mathfrak{a}_1(\vec{t})\Delta_{n-2}(\zeta;\vec{t}),\qquad \hat{D}^{(1)}\hat{\Phi}(\zeta;\vec{t})=\hat{\mu}_0(-\vec{t})\mathfrak{a}_{n-1}(-\vec{t})\hat{\Delta}_{n-2}(\zeta;-\vec{t}).$$

We easily prove the second identity in (43) and (44) by induction using (22) and

$$\partial_{t_1}\Delta_{n-j}(\zeta;\vec{t})=\mathfrak{a}_j(\vec{t})\Delta_{n-j-1}(\zeta;\vec{t}),\qquad \partial_{t_1}\hat{\Delta}_j(\zeta;\vec{t})=-\mathfrak{a}_j(\vec{t})\hat{\Delta}_{j-1}(\zeta;\vec{t})\qquad j\in[n-1].$$

Since

$$D^{(k)}\Phi(\zeta;\vec{t}) = O(\zeta^{n-k-1}) = \Delta_n(\zeta)\left[\frac{D^{(k)}\mu_0(\vec{t})}{\zeta} + \cdots + \frac{D^{(k)}\mu_{k-1}(\vec{t})}{\zeta^k}\right] + O(\zeta^{n-k-1})$$

$\mu_0, \ldots, \partial_{t_1}^{k-1}\mu_0$ are a basis of solutions for the linear differential operator $D^{(k)}$. Finally,

$$\mu_{s+k}(\vec{t}) = w_1^{(k)}(\vec{t})\mu_{s+k-1}(\vec{t}) + \cdots + w_k^{(k)}(\vec{t})\mu_s(\vec{t}), \qquad s \in [k-1],$$

are the explicit relations between the Hankel coefficients of $H_\mu(\vec{t})$ defined in Proposition 6.3 and the minors $\Delta_k(\vec{t})$, so that then the coefficients of $D^{(k)}$ satisfy (43). The proof of the remaining identities in (44) is similar. □

Let $\mathfrak{C} = \operatorname{diag}\left(1, \sqrt{\mathfrak{a}_1}, \sqrt{\mathfrak{a}_1\mathfrak{a}_2}, \ldots, \sqrt{\prod_{s=1}^{n-1}\mathfrak{a}_s}\right)$. Then $\mathfrak{A}(\vec{t}) \equiv \mathfrak{C}^{-1}\mathfrak{A}_0^{(sa)}(\vec{t})\mathfrak{C}$ and $\mathfrak{C}^{-1}\Psi^{(T)}$ and $\mathfrak{C}\Psi^{(T),\sigma}$ are the Toda Baker–Akhiezer functions in our representation. It is straightforward to check that (39) and (40) are equivalent to the first two recurrences in (27). Then we may use the Toda data $(\mathcal{K}, [a])$ to obtain the following equivalent representation of Toda Baker–Akhiezer functions.

Corollary 7.1. *Let $\Psi^{(T)}(\zeta;\vec{t})$, $\Psi^{(T),\sigma}(\zeta;\vec{t})$ the Toda Baker–Akhiezer functions associated to the Toda datum $(\mathcal{K}, [a])$, $[a] \in Gr^{TP}(1,n)$ and let $\Phi(\zeta;\vec{t})$ as in Proposition 7.1. Then*

$$\Psi^{(T)}(\zeta;\vec{t}) = e^{\frac{\theta(\zeta,\vec{t})}{2}}\mathfrak{C}\begin{pmatrix}\hat{\Delta}_0(\zeta;\vec{t})\\ \hat{\Delta}_1(\zeta;\vec{t})\\ \vdots\\ \hat{\Delta}_{n-1}(\zeta;\vec{t})\end{pmatrix}, \qquad \Psi^{(T),\sigma}(\zeta;\vec{t}) = e^{-\frac{\theta(\zeta,\vec{t})}{2}}\mathfrak{C}^{-1}\begin{pmatrix}\frac{\Phi(\zeta;\vec{t})}{\Phi(\zeta,\vec{0})}\\ \frac{D^{(1)}\Phi(\zeta;\vec{t})}{\Phi(\zeta,\vec{0})}\\ \vdots\\ \frac{D^{(n-1)}\Phi(\zeta;\vec{t})}{\Phi(\zeta,\vec{0})}\end{pmatrix}. \tag{45}$$

Remark 7.1. *In section 8, we prove that $\Phi(\zeta;\vec{t})$ is the vacuum KP-wavefunction on Γ as in (5) for the soliton data $(\mathcal{K}, [a])$, and show that $D^{(k)}\Phi(\zeta;\vec{t})$ is the un–normalized KP–wavefunction associated to the $(n-k,k)$-line T–hyperelliptic soliton generated by the Darboux transformation $D^{(k)}$, $k \in [n-1]$. It then follows that the Toda divisor $\mathcal{D}^{(T)}$ and the KP vacuum divisor $D^{(0)} = \{b_1 < \cdots < b_{n-1}\}$ coincide and (45) settle a natural correspondence between the pole divisor of the $(n-k,k)$-line T–hyperelliptic soliton and the zero divisor at $\vec{t} \equiv 0$ of the k–th component of the Toda Baker–Akhiezer function.*

In section 10, we associate (44) to the dual Toda hierarchy solution and the dual KP line-soliton solutions generated by the space–time inversion and associated to heat hierarchy solution $\sum_{j=1}^n \hat{a}_j E_j(\vec{t})$ and use the third recurrence in (27) for the Toda system to compute the dual KP divisor.

8. T–hyperelliptic KP solitons and solutions to the Toda hierarchy

Propositions 6.3, 7.1 and Corollary 4.1 imply a strict connection between T–hyperelliptic solitons and solutions to the finite non–periodic Toda hierarchy. Indeed let $\mathcal{K} = \{\kappa_1 < \cdots < \kappa_n\}$, $[a] \in Gr^{\mathrm{TP}}(1,n)$ be given, so that Toda spectrum and the KP phases coincide, that is $\kappa_j^{(T)} \equiv \kappa_j$, $j \in [n]$. Then $\mu_0(\vec{t}) = \sum_{l=1}^n a_j E_j(\vec{t})$[1] generates the τ–functions with $\tau_k(\vec{t}) = Wr(\mu_0(\vec{t}), \ldots, \mu_{k-1}(\vec{t}))$, $k \in [n]$, which are the building blocks of

(1) the Toda hierarchy solution $\mathfrak{a}_k(\vec{t}) = \frac{\tau_{k-1}(\vec{t})\tau_{k+1}(\vec{t})}{\tau_k^2(\vec{t})}$, $k \in [n-1]$, $\mathfrak{b}_k(\vec{t}) = \frac{\partial_{t_1}\tau_k(\vec{t})}{\tau_k(\vec{t})} - \frac{\partial_{t_1}\tau_{k-1}(\vec{t})}{\tau_{k-1}(\vec{t})}$, $k \in [n]$;

(2) the set of KP T–hyperelliptic solitons, $u_k(\vec{t}) = 2\partial_x^2 \log \tau_k(\vec{t})$, $k \in [n-1]$.

The identities above suggest a relation between the spectral problems for the finite non periodic Toda system and for KP T–hyperelliptic solitons. Indeed, for any data $(\mathcal{K}, [a])$, with $\mathcal{K} = \{\kappa_1 < \cdots < \kappa_n\}$ and $[a] \in Gr^{\mathrm{TP}}(1,n)$, we prove:

(1) upon identifying Γ_- with the copy of $\mathbb{C}P^1$ containing the Toda divisor, the KP vacuum divisor $\{b_1, \ldots, b_{n-1}\}$ and the Toda divisor $\{b_1^{(T)}, \ldots, b_{n-1}^{(T)}\}$ coincide;

(2) The Darboux transformations which generate T–hyperelliptic solitons with k–compatible divisors, coincide with the Darboux transformations recursively defined in (41) for the Toda system;

(3) the divisor $\mathcal{D}^{(k)} = \{\gamma_1^{(k)}, \ldots, \gamma_k^{(k)}, \delta_1^{(k)}, \ldots, \delta_{n-k-1}^{(k)}\}$ of the k–th T hyperelliptic soliton is the zero divisor of the k–th component of the Toda Baker–Akhiezer function at times $\vec{t} = \vec{0}$.

We then use the third identity in (27) to recursively compute the divisor of T–hyperelliptic solitons as k varies from 1 to $n-1$.

Theorem 8.1. *Let $\mathcal{K} = \{\kappa_1 < \cdots < \kappa_n\}$, $[a] \in Gr^{TP}(1,n)$. Let $\mu_0(\vec{t}) = \sum\limits_{j=1}^n a_j E_j(\vec{t})$, with the normalization $\mu_0(\vec{0}) = \sum_{j=1}^n a_j = 1$, and $\mu_s(\vec{t}) = \partial_x^{s-1}\mu_0(\vec{t})$, $s \geq 1$. Let $\Delta_j(\zeta;\vec{t})$, $\hat{\Delta}_j(\zeta;\vec{t})$, $j \in [n-1]$, $\mathfrak{f}_\mu(\vec{t})$ and $\Phi(\zeta;\vec{t})$ be as in Propositions 6.3 and 7.1. For any $j \in [n-1]$, let $\tau^{(j)}(\vec{t}) = Wr\left(\mu_0(\vec{t}), \ldots, \mu_{j-1}(\vec{t})\right)$ and $D^{(j)}$ be the Darboux transformation such that $D^{(j)}\mu_s(\vec{t}) \equiv 0$, $s \in [j-1]$. Then*

[1]We remark that in our setting Toda and KP times coincide, that is $(t_1, t_2, t_3, t_4, \cdots) = (x, y, z, t_4, \cdots)$. If one uses Flasckha original change of variables there is a scaling factor 2^j between each j-th Toda and j-th KP time.

(1) The vacuum KP–wavefunction associated to the soliton data $(\mathcal{K},[a])$ as in (9) satisfies

(46)
$$\Psi^{(-)}(\zeta;\vec{t})=\frac{\Phi(\zeta;\vec{t})}{\Phi(\zeta;\vec{0})}=\mu_0(\vec{t})\,\frac{\Delta_{n-1}(\zeta;\vec{t})}{\Delta_{n-1}(\zeta;\vec{0})}=\mu_0(\vec{t})\,\frac{\mathfrak{f}(\zeta;\vec{t})}{\mathfrak{f}(\zeta;\vec{0})}=\mu_0(\vec{t})\prod_{j=1}^{n-1}\frac{(\zeta-b_j(\vec{t}))}{(\zeta-b_j)},\qquad \forall\vec{t},\ \zeta\in\Gamma_-;$$

(2) For each given $k\in[n-1]$, the normalized KP-wavefunction of the $(n-k,k)$–soliton solution associated to the Darboux transformation $D^{(k)}$ is $\tilde{\Psi}^{(k)}(\zeta;\vec{t})=\dfrac{\Psi^{(k)}(\zeta;\vec{t})}{\Psi^{(k)}(\zeta;\vec{0})}$, where, $\forall\vec{t}$,

$$\Psi^{(k)}(\zeta;\vec{t})\equiv D^{(k)}\Psi(\zeta;\vec{t})=\begin{cases}\hat{\Delta}_k(\zeta;\vec{t})e^{\theta(\zeta;\vec{t})}, & \zeta\in\Gamma_+,\\ \dfrac{D^{(k)}\Phi(\zeta;\vec{t})}{\Phi(\zeta;\vec{0})}=\dfrac{\tau_{k+1}(\vec{t})}{\tau_k(\vec{t})}\dfrac{\Delta_{n-k-1}(\zeta;\vec{t})}{\Delta_{n-1}(\zeta;\vec{0})}, & \zeta\in\Gamma_-,\end{cases}\tag{47}$$

(3) For any $k\in[n-1]$, the divisor $\mathcal{D}^{(k)}(\vec{t})=\{\gamma_1^{(k)}(\vec{t}),\dots,\gamma_k^{(k)}(\vec{t}),\delta_1^{(k)},\dots,\delta_{n-k-1}^{(k)}(\vec{t})\}$ of $\tilde{\Psi}^{(k)}(\zeta;\vec{t})$ satisfies

$$\prod_{j=1}^{k}(\zeta-\gamma_j^{(k)}(\vec{t}))=\hat{\Delta}_k(\zeta;\vec{t}),\qquad \prod_{j=1}^{n-k-1}(\zeta-\delta_j^{(k)}(\vec{t}))=\Delta_{n-k-1}(\zeta;\vec{t}).\tag{48}$$

Proof. Inserting $h_j(\vec{t})=\frac{\mu_j(\vec{t})}{\mu_j(\vec{0})}$ in (30), and $\mathfrak{M}_l(\vec{t})=\frac{a_lE_l(\vec{t})}{\sum_{j=1}^{n}a_jE_j(\vec{t})}$ in (42), we get

$$\Phi(\zeta;\vec{t})=\mu_0(\vec{t})\left(\sum_{j=1}^{n}\mathfrak{M}_j(\vec{t})\prod_{s\neq j}(\zeta-\kappa_s)\right)=\sum_{j=1}^{n}a_jE_j(\vec{t})\prod_{s\neq j}(\zeta-\kappa_s).$$

In particular, with our normalization of a and by the definition of the vacuum divisor,

$$\Phi(\zeta;\vec{0})=\Delta_{n-1}(\zeta;\vec{0})=\prod_{j=1}^{n}(\zeta-\kappa_j)\left(\sum_{i=1}^{n}\frac{a_i}{\zeta-\kappa_i}\right)=\prod_{r=1}^{n-1}(\zeta-b_r).$$

Then, using Proposition 7.1, the other assertions follow. □

Notice that in (11), $\tilde{A}^{(k)}(\vec{t})=\dfrac{\tau^{(k+1)}(\vec{t})}{\tau^{(k)}(\vec{t})}>0$, $\forall\vec{t}$. Moreover, for any fixed $k\in[n-1]$ and for all $\vec{t}$, the divisor $\mathcal{D}^{(k)}$ is k–compatible according to the counting rule.

Corollary 8.1. *Let the Toda and KP soliton datum be $\mathcal{K}=\{\kappa_1<\dots<\kappa_n\}$, $[a]\in Gr^{TP}(1,n)$. Let $\tilde{\Psi}^{(k)}(\zeta;\vec{t})$ be the KP wavefunction for the T–hyperelliptic $(n-k,k)$–soliton as in Theorem 8.1 and let $\Psi^{(T)}(\zeta;\vec{t})$, $\Psi^{(T),\sigma}(\zeta;\vec{t})$ be the Toda Baker–Akhiezer functions associated to such datum. Let $\{b_1<\dots<b_{n-1}\}$, $\{b_1^{(T)}<\dots<b_{n-1}^{(T)}\}$ respectively be the KP vacuum divisor and the Toda*

divisor for such data, $\mathcal{D}^{(k)} = \{\gamma_1^{(k)}, \dots, \gamma_k^{(k)}, \delta_1^{(k)}, \dots, \delta_{n-k-1}^{(k)}\}$ *be the pole divisor of* $\tilde{\Psi}^{(k)}(\zeta; \vec{t})$, $k \in [n-1]$. *Then*

$$\begin{aligned} &b_j = b_j^{(T)}, \quad j \in [n-1]; \\ &\Psi_k^{(T)}(\gamma_l^{(k)}, \vec{0}) = 0, \quad \forall l \in [k], \qquad \Psi_k^{(T),\sigma}(\delta_s^{(k)}, \vec{0}) = 0, \quad \forall s \in [n-k-1]. \end{aligned} \tag{49}$$

The corollary easily follows comparing (45) and (47) and the definition of the vacuum KP and of the Toda divisors in Sections 3 and 7.

In view of (48), we interpret the third identity in (27) as a recursive relation to compute the divisor of T–hyperelliptic solitons as k varies from 1 to $n-1$.

Corollary 8.2. *Under the hypotheses of Theorem 8.1, for any fixed* $k \in [n-1]$ *the divisor of* $\tilde{\Psi}^{(k)}(\zeta; \vec{t})$ *may be computed from the divisor of* $\tilde{\Psi}^{(k-1)}(\zeta; \vec{t})$, *for all* $\vec{t}$, *using (27)*

$$\prod_{j=1}^{n}(\zeta - \kappa_j) = \prod_{l=1}^{k}(\zeta - \gamma_l^{(k)}(\vec{t})) \prod_{s=1}^{n-k}(\zeta - \delta_s^{(k-1)}(\vec{t})) - \mathfrak{a}_k(\vec{t}) \prod_{i=1}^{k-1}(\zeta - \gamma_i^{(k-1)}(\vec{t})) \prod_{r=1}^{n-k-1}(\zeta - \delta_r^{(k)}(\vec{t})), \tag{50}$$

where $\mathfrak{a}_k(\vec{t})$ *are as in (34). Moreover, for any given* $\vec{t}$ *and* $k \in [n-1]$, *the divisor* $\mathcal{D}^{(k)}(\vec{t})$ *may be computed from the vacuum divisor* $(b_1(\vec{t}), \dots, b_{n-1}(\vec{t}))$ *solving the system of equation*

$$\prod_{r=1}^{n-1}(\kappa_j - b_r(\vec{t})) \prod_{i=1}^{k}(\kappa_j - \gamma_i^{(k)}(\vec{t})) - \left(\prod_{s=1}^{k} \mathfrak{a}_s(\vec{t})\right) \prod_{r=1}^{n-k-1}(\kappa_j - \delta_r^{(k)}(\vec{t})) = 0, \qquad j \in [n]. \tag{51}$$

Proof. If the divisors $\mathcal{D}^{(k)}(\vec{t})$ are all generic for a given $\vec{t}$, *i.e.* $\Delta_k(\kappa_j, \vec{t}), \hat{\Delta}_k(\kappa_j, \vec{t}) \neq 0$, for all $k \in [n-1]$ and $j \in [n]$, the proof of (51) is by induction in k using the third identity in (27).

Suppose now, that, for a given $\vec{t}$ the divisors are generic for $k \in [l-1]$ and $\mathcal{D}^{(l)}(\vec{t})$ is not generic and contains the point $\kappa_{\hat{\jmath}}$. Then (51) hold for $k \in [l]$. By the intertwining properties of the zeros of the polynomials Δ_k and $\hat{\Delta}_k$, $\hat{\Delta}_l(\kappa_{\hat{\jmath}}, \vec{t}) = \Delta_{n-l-1}(\kappa_{\hat{\jmath}}, \vec{t}) = 0$ implies that $\hat{\Delta}_{l+1}(\kappa_{\hat{\jmath}}, \vec{t}), \hat{\Delta}_{l-1}(\kappa_{\hat{\jmath}}, \vec{t}), \Delta_{n-l}(\kappa_{\hat{\jmath}}, \vec{t}), \Delta_{n-l-2}(\kappa_{\hat{\jmath}}, \vec{t}) \neq 0$. Let $\hat{\Delta}_l(\zeta; \vec{t}) = (\zeta - \kappa_{\hat{\jmath}})\hat{\Delta}_l'(\zeta; \vec{t})$ and $\Delta_{n-l-1}(\zeta; \vec{t}) = (\zeta - \kappa_{\hat{\jmath}})\Delta_{n-l-1}'(\zeta; \vec{t})$. For $k = l+1$ and $j \neq \hat{\jmath}$, identities (51) still hold, while if $j = \hat{\jmath}$, using (27), we get

$$\begin{aligned} &\prod_{s=1}^{l-1} \mathfrak{a}_s(\vec{t}) \prod_{r \neq \hat{\jmath}} (\kappa_{\hat{\jmath}} - \kappa_r) \left(\hat{\Delta}_{l+1}(\kappa_{\hat{\jmath}}, \vec{t}) + \mathfrak{a}_l(\vec{t})\hat{\Delta}_{l-1}(\kappa_{\hat{\jmath}}, \vec{t})\right) \\ &\quad = \hat{\Delta}_l'(\kappa_{\hat{\jmath}}, \vec{t})\hat{\Delta}_{l-1}(\kappa_{\hat{\jmath}}, \vec{t}) \left(\Delta_{n-1}(\kappa_{\hat{\jmath}}, \vec{t})\hat{\Delta}_{l+1}(\kappa_{\hat{\jmath}}, \vec{t}) - \big(\prod_{s=1}^{l+1} \mathfrak{a}_l(\vec{t})\big)\Delta_{n-l-2}(\kappa_{\hat{\jmath}}, \vec{t})\right). \end{aligned}$$

Since $\hat{\Delta}_{l+1}(\kappa_{\hat{\jmath}}, \vec{t}) + \mathfrak{a}_l(\vec{t})\hat{\Delta}_{l-1}(\kappa_{\hat{\jmath}}, \vec{t}) = 0$, and $\hat{\Delta}_l'(\kappa_{\hat{\jmath}}, \vec{t}), \hat{\Delta}_{l-1}(\kappa_{\hat{\jmath}}, \vec{t}) \neq 0$, we conclude that (51) holds also for $j = \hat{\jmath}$. □

Remark 8.1. *For any fixed $k \in [n-1]$, the conditions $D^{(k)}\Psi_+(\kappa_j, \vec{t}) = D^{(k)}\Psi_-(\kappa_j, \vec{t})$, for all $j \in [n]$, are equivalent to (51), which may be rewritten as*

$$\Delta_{n-1}(\kappa_j, \vec{t})\hat{\Delta}_k(\kappa_j, \vec{t}) - \left(\prod_{s=1}^{k} \mathfrak{a}_s(\vec{t})\right) \Delta_{n-k+1}(\kappa_j, \vec{t}) = 0, \quad \forall j \in [n].$$

Moreover, for any fixed $l \in [n]$,

$$\tau_1(\vec{t}) = E_l(\vec{t}) \prod_{r=1}^{n-1} \frac{\kappa_l - b_r(\vec{0})}{\kappa_l - b_r(\vec{t})}, \qquad \forall \vec{t}. \tag{52}$$

9. Reconstruction of soliton data and Toda solutions from k–compatible divisors

Let $\mathcal{K}$ be fixed. The relations found in the previous section, allow to reconstruct the soliton data associated to a k–compatible divisor and to express the solution of the Toda hierarchy in function of the Toda/KP zero–divisor dynamics. Indeed, for any given $k \in [n-1]$, equations (51) allow to solve both the direct and the inverse problem. If we assign the soliton datum $[a] \in Gr^{\mathrm{TP}}(1, n)$, we first compute the vacuum divisor $(b_1, \dots, b_{n-1})$ using the identity

$$\prod_{r=1}^{n-1}(\zeta - b_r) = \prod_{j=1}^{n}(\zeta - \kappa_j) \left(\sum_{s=1}^{n} \frac{a_s}{\sum_{l=1}^{n} a_l}(\zeta - \kappa_s)^{-1}\right)$$

and then the k–compatible divisor $\mathcal{D}^{(k)} = (\gamma_1^{(k)}, \dots, \gamma_k^{(k)}, \delta_1^{(k)}, \dots, \delta_{n-k-1}^{(k)})$ from (51).

Viceversa, if we assign a k–compatible divisor $\mathcal{D}^{(k)}$ on Γ, we may reconstruct the soliton datum $[a] \in Gr^{\mathrm{TP}}(1, n)$, by first computing the vacuum divisor from (51) and then taking $a_j = \frac{\prod_{r=1}^{n-1}(\kappa_j - b_r)}{\prod_{s \neq j}^{n}(\kappa_j - \kappa_s)}$. Indeed we have the following

Theorem 9.1. *Let $\mathcal{K} = \{\kappa_1 < \dots < \kappa_n\}$, (Γ, P_+, ζ) as in (5) and let $\mathcal{D}^{(k)} = (\gamma_1^{(k)}, \dots, \gamma_k^{(k)}, \delta_1^{(k)}, \dots, \delta_{n-k-1}^{(k)})$ be a k–compatible divisor on $\Gamma \backslash \{P_+\}$. If $\mathcal{D}^{(k)}$ is generic, then the un-normalized soliton datum is*

$$a_j = \frac{\prod_{s=1}^{n-1}(\kappa_j - \delta_s^{(k)})}{\prod_{r=1}^{k}(\kappa_j - \gamma_r^{(k)}) \prod_{l \neq j}^{n}(\kappa_j - \kappa_l)}, \qquad j \in [n]. \tag{53}$$

In the non–generic case, (53) holds for j if $\kappa_j \notin \mathcal{D}^{(k)}$. For any $\hat{\jmath}$ such that $\kappa_{\hat{\jmath}} \in \mathcal{D}^{(k)}$, let $\kappa_{\hat{\jmath}} = \gamma_{\hat{r}}^{(k)} = \delta_{\hat{s}}^{(k)}$. Then (53) is substituted by

$$a_{\hat{\jmath}} = -\frac{\prod_{s \neq \hat{s}}^{n-1}(\kappa_{\hat{\jmath}} - \delta_s^{(k)})}{\prod_{r \neq \hat{r}}^{k}(\kappa_{\hat{\jmath}} - \gamma_r^{(k)}) \prod_{l \neq \hat{\jmath}}^{n}(\kappa_{\hat{\jmath}} - \kappa_l)}. \tag{54}$$

Proof. If the divisor is generic plugging (51) into $a_j = \frac{\prod_{r=1}^{n-1}(\kappa_j - b_r)}{\prod_{s\neq j}^{n}(\kappa_j-\kappa_s)}$, we get (53) up to the constant normalization factor $\frac{\tau_{k+1}(\vec{0})}{\tau_k(\vec{0})}$. The non generic divisor containing $\kappa_{\hat{\jmath}} = \gamma_{\hat{r}}^{(k)} = \delta_{\hat{s}}^{(k)}$ is the limit of the generic divisor $\mathcal{D}_\epsilon^{(k)} = \Big(\mathcal{D}^{(k)}\backslash\{\gamma_{\hat{r}}^{(k)}, \delta_{\hat{s}}^{(k)}\})\Big) \cup \{\gamma_{\hat{r}}^{(k)} + \epsilon, \delta_{\hat{s}}^{(k)} - \epsilon\}$, when $\epsilon \to 0$. So $a_{\hat{\jmath}}$ satisfies (54). □

Let $\mathcal{K} = \{\kappa_1 < \cdots < \kappa_n\}$ be fixed and let $\mathcal{D}^{(k)}$ be a k–compatible divisor on Γ. Then, using Theorem 9.1 we reconstruct the initial data of a solution to the Toda hierarchy (25) and, using Corollary 8.2, we may express the solution to the Toda hierarchy in function of the system of compatible divisors associated to such soliton data.

Proposition 9.1. *Let $(\mathcal{K}, [a])$ be soliton data with $[a] \in Gr^{TP}(1,n)$, $\sum_{j=1}^n a_j = 1$. Let $\mathcal{D}^{(k)}(\vec{t}) = \{\gamma_1^{(k)}(\vec{t}), \dots, \gamma_k^{(k)}(\vec{t}), \delta_1^{(k)}(\vec{t}), \dots, \delta_{n-k-1}^{(k)}(\vec{t})\}$, $k \in [n-1]$, be the set of k–compatible divisors associated to such soliton data, with $k \in [n-1]$, and let $\mathcal{B}(\vec{t}) = \{b_1(\vec{t}) < \cdots < b_{n-1}(\vec{t})\}$ be the zero divisor of $\Psi(\zeta;\vec{t})$ in (46). Let $j \in [n]$ be fixed. Then the solution to (25) with initial datum $(\mathcal{K}, [a])$ is, for any $\vec{t}$,*

(55)

$$\mathfrak{a}_1(\vec{t}) = \frac{(\kappa_j - \gamma_1^{(1)}(\vec{t}))\ \prod_{r=1}^{n-1}(\kappa_j - b_r(\vec{t}))}{\prod_{s=1}^{n-1}(\kappa_j - \delta_s^{(1)}(\vec{t}))}, \quad \mathfrak{b}_1(\vec{t}) = \kappa_j + \sum_{r=1}^{n-1} \frac{\partial_x b_r(\vec{t})}{\kappa_j - b_r(\vec{t})},$$

$$\mathfrak{a}_k(\vec{t}) = \frac{\prod_{i=1}^{k}(\kappa_j - \gamma_i^{(k)}(\vec{t}))\ \prod_{l=1}^{n-k}(\kappa_j - \delta_l^{(k-1)}(\vec{t}))}{\prod_{r=1}^{k-1}(\kappa_j - \gamma_r^{(k-1)}(\vec{t}))\ \prod_{s=1}^{n-k-1}(\kappa_j - \delta_s^{(k)}(\vec{t}))}, \qquad k = 2, \dots, n-1;$$

$$\mathfrak{b}_k(\vec{t}) = \kappa_j + \sum_{l=1}^{n-k} \frac{\partial_x \delta_l^{(k-1)}(\vec{t})}{\kappa_j - \delta_l^{(k-1)}(\vec{t})} - \sum_{i=1}^{k-1} \frac{\partial_x \gamma_i^{(k-1)}(\vec{t})}{\kappa_j - \gamma_i^{(k-1)}(\vec{t})}, \qquad k = 2, \dots, n,$$

where, if for some $\vec{t}$ and $\hat{k} \in [n-1]$, $\gamma_{\hat{\imath}}^{(\hat{k})}(\vec{t}) = \delta_{\hat{s}}^{(\hat{k})}(\vec{t}) = \kappa_{\hat{\jmath}}$, we substitute $\mathfrak{a}_{\hat{k}}(\vec{t}), \mathfrak{a}_{\hat{k}+1}(\vec{t}), \mathfrak{b}_{\hat{k}+1}(\vec{t})$ in (55) with

$$\mathfrak{a}_{\hat{k}}(\vec{t}) = -\frac{\prod_{i\neq\hat{\imath}}^{\hat{k}}(\kappa_{\hat{\jmath}} - \gamma_i^{(\hat{k})}(\vec{t}))\ \prod_{l=1}^{n-\hat{k}}(\kappa_{\hat{\jmath}} - \delta_l^{(\hat{k}-1)}(\vec{t}))}{\prod_{r=1}^{\hat{k}-1}(\kappa_{\hat{\jmath}} - \gamma_r^{(\hat{k}-1)}(\vec{t}))\ \prod_{s\neq\hat{s}}^{n-\hat{k}-1}(\kappa_{\hat{\jmath}} - \delta_s^{(\hat{k})}(\vec{t}))},$$

$$\mathfrak{a}_{\hat{k}+1}(\vec{t}) = -\frac{\prod_{r=1}^{\hat{k}+1}(\kappa_{\hat{\jmath}} - \gamma_r^{(\hat{k}+1)}(\vec{t}))\ \prod_{s\neq\hat{s}}^{n-\hat{k}-1}(\kappa_{\hat{\jmath}} - \delta_s^{(\hat{k})}(\vec{t}))}{\prod_{i\neq\hat{\imath}}^{\hat{k}}(\kappa_{\hat{\jmath}} - \gamma_i^{(\hat{k})}(\vec{t}))\ \prod_{l=1}^{n-\hat{k}-2}(\kappa_{\hat{\jmath}} - \delta_l^{(\hat{k}+1)}(\vec{t}))},$$

$$\mathfrak{b}_{\hat{k}+1}(\vec{t}) = \kappa_{\hat{\jmath}} + \sum_{s\neq\hat{s}}^{n-\hat{k}-1} \frac{\partial_x \delta_s^{(\hat{k})}(\vec{t})}{\kappa_{\hat{\jmath}} - \delta_s^{(\hat{k})}(\vec{t})} - \sum_{i\neq\hat{\imath}}^{\hat{k}} \frac{\partial_x \gamma_i^{(\hat{k})}(\vec{t})}{\kappa_{\hat{\jmath}} - \gamma_i^{(\hat{k})}(\vec{t})}.$$

Proof. (55) easily follow using (34), (51) and (52), since

$$\prod_{s=1}^{k} \mathfrak{a}_s(\vec{t}) = \frac{\tau_{k+1}(\vec{t})}{\tau_k(\vec{t})\tau_1(\vec{t})} = \frac{\prod_{i=1}^{k}(\kappa_j - \gamma_i^{(k)}(\vec{t}))\ \prod_{r=1}^{n-1}(\kappa_j - b_r(\vec{t}))}{\prod_{s=1}^{n-k-1}(\kappa_j - \delta_s^{(k)}(\vec{t}))}.$$

The case of the non–generic divisor as usual follows from the limit of the generic case. □

10. Duality of Grassmann cells, space–time inversion and divisors

Space–time inversion in KP soliton solutions induces a duality transformation of $Gr(k,n)$ to $Gr(n-k,n)$. In this section, we investigate the effect of such transformation on the algebraic geometric description of T–hyperelliptic soliton solutions and we show that (50) lead to a natural characterization of the dual divisor in $Gr(n-k,n)$ through hyperelliptic involution. For the Toda system such duality corresponds to the composition of the reflection w.r.t the antidiagonal defined in (28) with Toda times inversion, *i.e.* to pass from Toda solutions for $\mathfrak{A}(\vec{t})$ to Toda solutions for $\mathfrak{A}^*(-\vec{t})$.

The KP equation is invariant under the space–time inversion $\vec{t}\to-\vec{t}$. In particular, if $u_{[A]}(\vec{t})$[2] is the KP solution for the soliton data $(\mathcal{K},[A])$ with in $[A]\in Gr^{\mathrm{TNN}}(k,n)$ then there exists $[\hat{A}]\in Gr^{\mathrm{TNN}}(n-k,n)$ such that $u_{[\hat{A}]}(\vec{t})\equiv u_{[A]}(-\vec{t})$ is the solution associated to the dual soliton data $(\mathcal{K},[\hat{A}])$. The combinatorial interpretation of this transformation has been given in [5] (see also [39]).

Since the phases are invariant with respect to the space–time inversion, the curve Γ is preserved. If $(\mathcal{K},[A])$ are the data of a T–hyperelliptic soliton, also the dual data $(\mathcal{K},[\hat{A}])$ are associated to a T–hyperelliptic soliton solution. In this section we investigate the relations between the divisors of T–hyperelliptic dual solitons with A and $\hat{A}$ as in (14),

$$A^i_j=a_j\kappa_j^{i-1},\qquad \hat{A}^l_j=\hat{a}_j\kappa_j^{l-1},\qquad \forall i\in[k],\ l\in[n-k],\ j\in[n],$$

$[a_1,\dots,a_n],[\hat{a}_1,\dots,\hat{a}_n]\in Gr^{\mathrm{TP}}(1,n)$, for some $k\in[n-1]$. Let us denote, respectively, the heat hierarchy solutions

$$\mu_{a,i}(\vec{t})=\sum_{j=1}^n a_j\kappa_j^iE_j(\vec{t}),\qquad\qquad \mu_{\hat{a},i}(\vec{t})=\sum_{j=1}^n \hat{a}_j\kappa_j^iE_j(\vec{t}),\qquad i\geq 0,$$

the τ–functions

(56)

$$\tau_a^{(k)}(\vec{t})=\mathrm{Wr}(\mu_{a,0},\dots,\mu_{a,k-1})=\sum_{1\leq i_1<\dots<i_k\leq n}\left(\prod_{s=1}^k a_{i_s}E_{i_s}(\vec{t})\right)\prod_{1<r<s<k}(\kappa_{i_s}-\kappa_{i_r})^2,$$

$$\tau_{\hat{a}}^{(n-k)}(\vec{t})=\mathrm{Wr}(\mu_{\hat{a},0},\dots,\mu_{\hat{a},n-k-1})=\sum_{1\leq l_1<\dots<l_{n-k}\leq n}\left(\prod_{s=1}^{n-k}\hat{a}_{l_s}E_{l_s}(\vec{t})\right)\prod_{1<r<s<n-k}(\kappa_{l_s}-\kappa_{l_r})^2,$$

[2]In the following the subscripts $[x]$, respectively x, mean that the value of the expression depends on the point in the Grassmannian, respectively on the representative matrix of the point in the Grassmannian.

and the Darboux transformations

$$D^{(k)}_{[a]} = \partial_x^k - w^{(k)}_{[a],1}(\vec{t})\partial_x^{k-1} - \cdots w^{(k)}_{[a],k}(\vec{t}), \qquad D^{(n-k)}_{[\hat{a}]} = \partial_x^{n-k} - w^{(n-k)}_{[\hat{a}],1}(\vec{t})\partial_x^{k-1} - \cdots w^{(n-k)}_{[\hat{a}],n-k}(\vec{t}), \tag{57}$$

where $D^{(k)}_{[a]}\mu_{a,i}(\vec{t}) \equiv 0, \quad D^{(n-k)}_{[\hat{a}]}\mu_{\hat{a},l}(\vec{t}) \equiv 0$, for all $i \in [0, k-1]$, $l \in [0, n-k-1]$, $\vec{t}$. The KP solutions

$$u_{[a],k}(\vec{t}) = 2\partial_x^2 \log \tau_a^{(k)}(\vec{t}), \qquad u_{[\hat{a}],n-k}(\vec{t}) = 2\partial_x^2 \log \tau_{\hat{a}}^{(n-k)}(\vec{t}) \tag{58}$$

are related by the space–time transformation $u_{[\hat{a}]}(\vec{t}) = u_{[a]}(-\vec{t})$ if and only if there exists a constant $C_k(a, \hat{a}) > 0$ such that

$$\tau_{\hat{a}}^{(n-k)}(\vec{t}) = C_k(a, \hat{a})\, \tau_a^{(k)}(-\vec{t}) \prod_{j=1}^{n} E_j(\vec{t}), \qquad \forall \vec{t}, \quad k \in [0, n].$$

To characterize the duality condition, it is convenient to use a different set of coordinates. Let $[\alpha], [\hat{\alpha}] \in Gr^{\mathrm{TP}}(1, n)$ be related to $[a], [\hat{a}] \in Gr^{\mathrm{TP}}(1, n)$ by

$$a_j = \frac{(-1)^{n-j}\alpha_j}{\prod\limits_{m \neq j}^{n} (\kappa_j - \kappa_m)}, \qquad \hat{a}_j = \frac{(-1)^{n-j}\hat{\alpha}_j}{\prod\limits_{m \neq j}^{n} (\kappa_j - \kappa_m)}, \qquad j \in [n]. \tag{59}$$

Then the duality condition is equivalent to the following relations between $[\alpha]$ and $[\hat{\alpha}]$.

Lemma 10.1. *Let $k \in [n-1]$ be fixed and $\mathcal{K} = \{\kappa_1 < \cdots < \kappa_n\}$. Let $[a], [\hat{a}], [\alpha], [\hat{\alpha}] \in Gr^{TP}(1, n)$, $\tau_a^{(k)}(\vec{t})$, $\tau_{\hat{a}}^{(n-k)}(\vec{t})$, $u_{[a],k}(\vec{t})$, $u_{[\hat{a}],n-k}(\vec{t})$ as in (59),(56) and (58). Then the following statements are equivalent*

(1) $u_{[\hat{a}],n-k}(\vec{t}) = u_{[a],k}(-\vec{t})$ for all $\vec{t}$;

(2) it is possible to normalize the representative vector of $[\hat{a}]$ so that $\tau_{\hat{a}}^{(n-k)}(\vec{t}) = \tau_a^{(k)}(-\vec{t}) \prod\limits_{j=1}^{n} \frac{E_j(\vec{t})}{\alpha_j}$, $\forall \vec{t}$;

(3) $[\hat{\alpha}_1, \ldots, \hat{\alpha}_n] = \left[\alpha_1^{-1}, \ldots, \alpha_n^{-1}\right]$.

Proof. (1) and (2) are equivalent since soliton solutions depend just on the point in $Gr^{\mathrm{TP}}(1, n)$ and τ–functions are defined up to multiplicative constants. Condition (2) on the τ–functions is equivalent to $\prod_{i \in I} a_i \prod\limits_{r,s \in I, r<s} (\kappa_s - \kappa_r)^2 = \prod_{j \in J} \hat{a}_j \prod_{l,k \in J, k<l} (\kappa_l - \kappa_k)^2$, for all $I \in \binom{[n]}{k}$, $J = [n] - I$. Inserting (59), in the above identities it is straightforward to show that $\prod\limits_{i \in I} \alpha_i = \prod_{j \in [n] \setminus I} \hat{\alpha}_j$, for all $I \in \binom{[n]}{k}$, which is is equivalent to (3). □

Corollary 10.1. *Let $\mathcal{K}$ be given. Let $[a], [\hat{a}] \in Gr^{TP}(1, n)$, $\tau_a^{(k)}(\vec{t})$, $\tau_{\hat{a}}^{(k)}(\vec{t})$, $u_{[a],k}(\vec{t})$, $u_{[\hat{a}],n-k}(\vec{t})$, $\forall k \in [0, n]$, $\forall \vec{t}$, as in (59), (56) and (58). Then the following statements are equivalent*

(1) There exists $\bar{k} \in [n-1]$ *such that, for any* $\vec{t}$, $u_{[\hat{a}],n-\bar{k}}(\vec{t}) = u_{[a],\bar{k}}(-\vec{t})$;

(2) For any $k \in [1, n-1]$ *and for any* $\vec{t}$, $u_{[\hat{a}],n-k}(\vec{t}) = u_{[a],k}(-\vec{t})$;

(3) $[\hat{\alpha}_1, \dots, \hat{\alpha}_n] = [\alpha_1^{-1}, \dots, \alpha_n^{-1}]$.

Condition (3) in Corollary 10.1 is equivalent to (35), that is the duality of Grassmann cells induces dual Toda hierarchy solutions and dual KP soliton solutions which are naturally linked.

10.1. **Dual Toda flows.** The space–time inversion settles a duality condition in the space of KP line soliton solutions which is also a duality condition between Toda flows. In this subsection, we list the relevant relations between such dual Toda hierarchies using Proposition 7.1 and then in the next subsection we use them to determine the relations among the divisors associated to dual soliton data.

To the initial data $(\mathcal{K}, [a])$ and $(\mathcal{K}, [\hat{a}])$, with $[a]$, $[\hat{a}]$, $[\alpha]$, $[\hat{\alpha}] \in Gr^{\mathrm{TP}}(1, n)$ satisfying condition (3) in Lemma 10.1 and (59), we associate dual Toda hierarchies, $j \geq 1$,

$$\frac{d\mathfrak{A}_{[a]}}{dt_j}(\vec{t}) = [\mathfrak{B}_{[a],j}(\vec{t}), \mathfrak{A}_{[a]}(\vec{t})], \qquad \frac{d\mathfrak{A}_{[\hat{a}]}}{dt_j}(\vec{t}) = [\mathfrak{B}_{[\hat{a}],j}(\vec{t}), \mathfrak{A}_{[\hat{a}]}(\vec{t})], \tag{60}$$

with

(61)

$$\mathfrak{A}_{[a]}(\vec{t}) = \begin{pmatrix} \mathfrak{b}_{[a],1}(\vec{t}) & \mathfrak{a}_{[a],1}(\vec{t}) & 0 & \cdots \\ 1 & \mathfrak{b}_{[a],2}(\vec{t}) & \mathfrak{a}_{[a],2}(\vec{t}) & \ddots \\ 0 & \ddots & \ddots & \ddots \\ 0 & \cdots & 1 & \mathfrak{b}_{[a],n}(\vec{t}) \end{pmatrix}, \quad \mathfrak{A}_{[\hat{a}]}(\vec{t}) = \begin{pmatrix} \mathfrak{b}_{[\hat{a}],1}(\vec{t}) & \mathfrak{a}_{[\hat{a}],1}(\vec{t}) & 0 & \cdots \\ 1 & \mathfrak{b}_{[\hat{a}],2}(\vec{t}) & \mathfrak{a}_{[\hat{a}],2}(\vec{t}) & \ddots \\ 0 & \ddots & \ddots & \ddots \\ 0 & \cdots & 1 & \mathfrak{b}_{[\hat{a}],n}(\vec{t}) \end{pmatrix},$$

$\mathfrak{B}_{[a],j}(\vec{t}) = \left(\mathfrak{A}^j_{[a]}(\vec{t})\right)_+$, $\mathfrak{B}_{[\hat{a}],j}(\vec{t}) = \left(\mathfrak{A}^j_{[\hat{a}]}(\vec{t})\right)_+$, where $(\cdot)_+$ denotes the strictly upper triangular part of the matrix, via the generating functions

$$\begin{aligned} &\mathfrak{f}_{[a]}(\zeta; \vec{t}) \equiv \prec e_1, (\zeta \mathfrak{I}_n - \mathfrak{A}_{[a]}(\vec{t}))^{-1} e_1 \succ = \frac{\Delta_{[a],n-1}(\zeta; \vec{t})}{\Delta_n(\zeta)} = \mu^{-1}_{a,0}(\vec{t}) \sum_{j \geq 0} \frac{\mu_{a,j}(\vec{t})}{\zeta^{j+1}}, \\ &\mathfrak{f}_{[\hat{a}]}(\zeta; \vec{t}) \equiv \prec e_1, (\zeta \mathfrak{I}_n - \mathfrak{A}_{[\hat{a}]}(\vec{t}))^{-1} e_1 \succ = \frac{\Delta_{[\hat{a}],n-1,}(\zeta; \vec{t})}{\Delta_n(\zeta)} = \mu^{-1}_{\hat{a},0}(\vec{t}) \sum_{j \geq 0} \frac{\mu_{\hat{a},j}(\vec{t})}{\zeta^{j+1}}, \\ &\Delta_n(z) = \det \left(zI - \mathfrak{A}_{[a]}(\vec{t})\right) = \det \left(zI - \mathfrak{A}_{[\hat{a}]}(\vec{t})\right) = \prod_{j=1}^{n} (z - \kappa_j), \end{aligned} \tag{62}$$

where $\mu_{a,0}(\vec{t}) = \sum_{j=1}^n a_j E_j(\vec{t})$, $\mu_{\hat{a},0}(\vec{t}) = \sum_{j=1}^n \hat{a}_j E_j(\vec{t})$. Moreover, let

$$
\begin{aligned}
\hat{\mathfrak{f}}_{[a]}(\zeta;\vec{t}) &\equiv \prec e_n, (\zeta\mathfrak{I}_n - \mathfrak{A}_{[a]}(\vec{t}))^{-1} e_n \succ = \frac{\hat{\Delta}_{[a],n-1}(\zeta;\vec{t})}{\Delta_n(\zeta)} = \hat{\mu}_{a,0}^{-1}(\vec{t}) \sum_{j\geq 0} \frac{\hat{\mu}_{a,j}(\vec{t})}{\zeta^{j+1}}, \\
\hat{\mathfrak{f}}_{[\hat{a}]}(\zeta;\vec{t}) &\equiv \prec e_n, (\zeta\mathfrak{I}_n - \mathfrak{A}_{[\hat{a}]}(\vec{t}))^{-1} e_n \succ = \frac{\hat{\Delta}_{[\hat{a}],n-1}(\zeta;\vec{t})}{\Delta_n(\zeta)} = \hat{\mu}_{\hat{a},0}^{-1}(\vec{t}) \sum_{j\geq 0} \frac{\hat{\mu}_{\hat{a},j}(\vec{t})}{\zeta^{j+1}}.
\end{aligned} \tag{63}
$$

Then the following relations hold between such Toda hierarchies

Proposition 10.1. *Let $\mathcal{K} = \{\kappa_1 < \cdots < \kappa_n\}$, $[a], [\hat{a}], \alpha], [\hat{\alpha}] \in Gr^{TP}(1,n)$ such that (59) and Lemma 10.1 holds. Let $\mathfrak{A}_{[a]}(\vec{t})$, $\mathfrak{A}_{[\hat{a}]}(\vec{t})$, be as in (61), with associated Toda flows and generating functions as in (60), (62) and (63) . Then the following relations hold true for all $\vec{t}$,*

$$
\begin{aligned}
&\hat{\mu}_{\hat{a},0}(\vec{t}) = \mu_{a,0}(-\vec{t}) = \textstyle\sum_{j=1}^n a_j E_j(-\vec{t}), && \hat{\mu}_{a,0}(\vec{t}) = \mu_{\hat{a},0}(-\vec{t}) = \textstyle\sum_{j=1}^n \hat{a}_j E_j(-\vec{t}), \\
&\hat{\mu}_{\hat{a},j}(\vec{t}) = \mu_{a,j}(-\vec{t}), && \hat{\mu}_{a,j}(\vec{t}) = \mu_{\hat{a},j}(-\vec{t}), \qquad \forall j \geq 0, \\
&\hat{\mathfrak{f}}_{[a]}(\zeta;\vec{t}) = \mathfrak{f}_{[\hat{a}]}(\zeta;-\vec{t}), && \hat{\mathfrak{f}}_{[\hat{a}]}(\zeta;\vec{t}) = \mathfrak{f}_{[a]}(\zeta;-\vec{t}), \\
&\hat{\Delta}_{[a],k}(\zeta;\vec{t}) = \Delta_{[\hat{a}],k}(\zeta;-\vec{t}) && \Delta_{[a],k}(\zeta;\vec{t}) = \hat{\Delta}_{[\hat{a}],k}(\zeta;-\vec{t}), \quad \forall k \in [1,n], \\
&\mathfrak{a}_{[\hat{a}],n-k}(\vec{t}) = \mathfrak{a}_{[a],k}(-\vec{t}), \quad k \in [n-1], && \mathfrak{b}_{[\hat{a}],n-k}(\vec{t}) = \mathfrak{b}_{[a],k+1}(-\vec{t}), \quad k \in [0,n-1].
\end{aligned} \tag{64}
$$

The proof trivially follows from Propositions 6.4, 7.1 and Corollary 10.1. In conclusion the dual initial data $(\mathcal{K}, [a])$ and $(\mathcal{K}, [\hat{a}])$, with $[a]$ related to $[\hat{a}]$ by (35) generate dual Toda hierarchy solutions which satisfy

$$
\begin{pmatrix}
\mathfrak{b}_{[\hat{a}],1}(\vec{t}) & \mathfrak{a}_{[\hat{a}],1}(\vec{t}) & 0 & \cdots \\
1 & \mathfrak{b}_{[\hat{a}],2}(\vec{t}) & \mathfrak{a}_{[\hat{a}],2}(\vec{t}) & \ddots \\
0 & \ddots & \ddots & \ddots \\
0 & \cdots & 1 & \mathfrak{b}_{[\hat{a}],n}(\vec{t})
\end{pmatrix}
=
\begin{pmatrix}
\mathfrak{b}_{[a],n}(-\vec{t}) & \mathfrak{a}_{[a],n-1}(-\vec{t}) & 0 & \cdots \\
1 & \mathfrak{b}_{[a],n-1}(-\vec{t}) & \mathfrak{a}_{[a],n-2}(-\vec{t}) & \ddots \\
0 & \ddots & \ddots & \ddots \\
0 & \cdots & 1 & \mathfrak{b}_{[a],1}(-\vec{t})
\end{pmatrix}.
$$

10.2. **Duality and divisors of KP–soliton solutions.** Lemma 10.1 implies the following: for any given set of phases $\mathcal{K} = \{\kappa_1 < \kappa_2 < \cdots < \kappa_n\}$ and any given point $[a] \in Gr^{\mathrm{TP}}(1,n)$, there exists $[\hat{a}] \in Gr^{\mathrm{TP}}(1,n)$, satisfying (59) with $\hat{\alpha}_j = 1/\alpha_j$, $j \in [n]$, such that the vacuum

wavefunctions

$$\Psi_{[a]}(\zeta;\vec{t}) = \begin{cases} e^{\theta(\zeta;\vec{t})}, & \text{if } \zeta \in \Gamma_+, \\ \Psi^{(-)}_{[a]}(\zeta;\vec{t}) \equiv \sum\limits_{l=1}^{n} \dfrac{a_j \prod\limits_{s\neq l}(\zeta-\kappa_s)}{\left(\sum\limits_{m=1}^{n} a_m\right)\prod\limits_{r=1}^{n-1}(\zeta-b_{[a],r})} E_j(\vec{t}), & \text{if } \zeta \in \Gamma_-, \end{cases}$$

(65)

$$\Psi_{[\hat{a}]}(\zeta;\vec{t}) = \begin{cases} e^{\theta(\zeta;\vec{t})}, & \text{if } \zeta \in \Gamma_+, \\ \Psi^{(-)}_{[\hat{a}]}(\zeta;\vec{t}) \equiv \sum\limits_{l=1}^{n} \dfrac{\hat{a}_j \prod\limits_{s\neq l}(\zeta-\kappa_s)}{\left(\sum\limits_{m=1}^{n} \hat{a}_m\right)\prod\limits_{r=1}^{n-1}(\zeta-b_{[\hat{a}],r})} E_j(\vec{t}), & \text{if } \zeta \in \Gamma_-, \end{cases}$$

generate dual $(k, n-k)$ and $(n-k, k)$–solitons respectively via the Darboux transformations $D^{(k)}_{[a]}$ and $D^{(n-k)}_{[\hat{a}]}$ for any $k \in [n-1]$. From (47) and (64), the vacuum divisors satisfy respectively $\prod_{r=1}^{n-1}(\zeta-b_{[a],r}) \equiv \Delta_{n-1,[a]}(\zeta;\vec{0}) = \hat{\Delta}_{n-1,[\hat{a}]}(\zeta;\vec{0})$, $\prod_{r=1}^{n-1}(\zeta-b_{[\hat{a}],r}) \equiv \Delta_{n-1,[\hat{a}]}(\zeta;\vec{0}) = \hat{\Delta}_{n-1,[a]}(\zeta;\vec{0})$.

Choosing the representative elements of the dual soliton solutions as in (59), with $\hat{\alpha}_j = \alpha_j^{-1}$, $j \in [n]$, the dual vacuum divisor $\left\{b_{[\hat{a}],1} < \cdots < b_{[\hat{a}],n-1}\right\}$ may be explicitly computed solving the following system of equations

$$\prod_{r=1}^{n-1}(\kappa_j - b_{[\hat{a}],r}) = \frac{(-1)^{n-j}\hat{\alpha}_j}{\sum\limits_{m=1}^{n}\hat{a}_m} = \frac{(-1)^{n-j}\prod\limits_{m=1}^{n}\alpha_m}{\alpha_j \tau_a^{(n-1)}(\vec{0})}, \qquad \forall j \in [n]. \tag{66}$$

In particular, the space–time inversion leaves the vacuum divisor invariant if and only if $b_{[\hat{a}],r} = b_{[a],r}$, $r \in [n-1]$, which is equivalent to $[\hat{a}] = [a] = [1/\hat{a}]$, that is

$$[\alpha_1, \ldots, \alpha_n] = [\hat{\alpha}_1, \ldots, \hat{\alpha}_n] = [1, \ldots, 1].$$

We have thus proven

Corollary 10.2. *Let $\mathcal{K} = \{\kappa_1 < \cdots < \kappa_n\}$, $[a], [\hat{a}] \in Gr^{TP}(1,n)$, $\Psi_{[a]}(\zeta;\vec{t})$, $\Psi_{[\hat{a}]}(\zeta;\vec{t})$, as in (59) and (65), where $[\alpha], [\hat{\alpha}] \in Gr^{TP}(1,n)$ satisfy Lemma 10.1. Then the space–time inversion leaves the 0–divisor invariant, $b_{[a],r} = b_{[\hat{a}],r}$, $r \in [n]$, if and only if $[\alpha] = [\hat{\alpha}] = [1, \ldots, 1]$.*

The self–dual (k,k)–soliton solutions $u(\vec{t}) = u(-\vec{t})$ are thus associated to $[\alpha] = [1, \ldots, 1] \in Gr^{\mathrm{TP}}(1, 2k)$.

Let us now return to the general case of dual T–hyperelliptic soliton data. The following theorem explains the relations between k–compatible divisors under the space–time inversion. Indeed, using (48) and (64) we have the following.

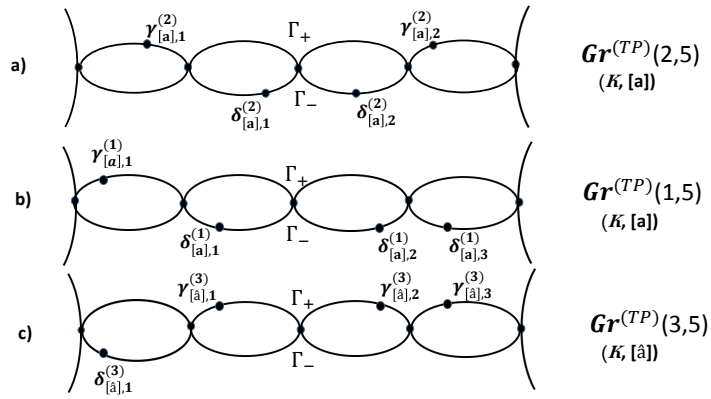

FIGURE 5. *We illustrate Theorem 10.1 on dual T–hyperelliptic divisors in $Gr^{TP}(2,5)$ and in $Gr^{TP}(3,5)$. a): $\mathcal{D}^{(2)}_{[a]} = \{\gamma^{(2)}_{[a],1}, \gamma^{(2)}_{[a],2}, \delta^{(2)}_{[a],1}, \delta^{(2)}_{[a],2}\}$ is the divisor of the (3,2)–line soliton in $Gr^{TP}(2,5)$ associated to the soliton data $(\mathcal{K}, [a])$. b): $\mathcal{D}^{(1)}_{[a]} = \{\gamma^{(1)}_{[a],1}, \delta^{(1)}_{[a],1}, \delta^{(1)}_{[a],2}, \delta^{(1)}_{[a],3}\}$ is the divisor of the (4,1)–line soliton in $Gr^{TP}(1,5)$ associated to the soliton data $(\mathcal{K}, [a])$. c): $\mathcal{D}^{(3)}_{[\hat{a}]} = \{\gamma^{(3)}_{[\hat{a}],1}, \gamma^{(3)}_{[\hat{a}],2}, \gamma^{(3)}_{[\hat{a}],3}, \delta^{(3)}_{[\hat{a}],1}\}$ is the divisor for the (2,3)–line soliton solution in $Gr^{TP}(3,5)$ associated to the dual soliton data $(\mathcal{K}, [\hat{a}])$ via space–time inversion and it is obtained applying the hyperelliptic involution σ to $\mathcal{D}^{(1)}_{[a]}$: $\mathcal{D}^{(3)}_{[\hat{a}]} = \sigma(\mathcal{D}^{(1)}_{[a]})$.*

Theorem 10.1. *Let $n > 1$, $\mathcal{K} = \{\kappa_1 < \cdots < \kappa_n\}$ and $[\alpha_1, \ldots, \alpha_n] \in Gr^{TP}(1,n)$ be given. Let $[\hat{\alpha}_1, \ldots, \hat{\alpha}_n] = [1/\alpha_1, \ldots, 1/\alpha_n]$ and $[a], [\hat{a}] \in Gr^{TP}(1,n)$ as in (59). Let $\Psi_{[a]}(\zeta; \vec{t})$ and $\Psi_{[\hat{a}]}(\zeta; \vec{t})$ be the dual vacuum wavefunctions as in (65).*

For any given $k \in [n-1]$, let $D^{(k)}_{[a]}$, $D^{(n-k)}_{[\hat{a}]}$, respectively be the dual Darboux transformations as in (57). Let σ be the hyperelliptic involution on Γ, i.e. $\sigma(\Gamma_\pm) = \Gamma_\mp$. Let, for any fixed $k \in [n-1]$

$$\mathcal{D}^{(k)}_{[a]} = \mathcal{D}^{(k)}_{[a],+} \cup \mathcal{D}^{(k)}_{[a],-}, \qquad \mathcal{D}^{(k)}_{[\hat{a}]} = \mathcal{D}^{(k)}_{[\hat{a}],+} \cup \mathcal{D}^{(n-k)}_{[\hat{a}],-}, \tag{67}$$

be the pole divisors respectively of $\tilde{\Psi}^{(k)}_{[a]}(\zeta; \vec{t}) \equiv \frac{D^{(k)}_{[a]} \Psi_{[a]}(\zeta; \vec{t})}{D^{(k)}_{[a]} \Psi_{[a]}(\zeta; \vec{0})}$, $\tilde{\Psi}^{(k)}_{[\hat{a}]}(\zeta; \vec{t}) \equiv \frac{D^{(k)}_{[\hat{a}]} \Psi_{[\hat{a}]}(\zeta; \vec{t})}{D^{(k)}_{[\hat{a}]} \Psi_{[\hat{a}]}(\zeta; \vec{0})}$, where

$$\begin{aligned}
\mathcal{D}^{(k)}_{[a],+} &\equiv \left\{\gamma^{(k)}_{[a],1}, \ldots, \gamma^{(k)}_{[a],k}\right\} = \{(\zeta; \mu) \in \Gamma_+ \,:\, \hat{\Delta}_{k,[a]}(\zeta; \vec{0}) = 0\}, \\
\mathcal{D}^{(k)}_{[a],-} &\equiv \left\{\delta^{(k)}_{[a],1}, \ldots, \delta^{(k)}_{[a],n-k+1}\right\} = \{(\zeta; \mu) \in \Gamma_- \,:\, \Delta_{n-k-1,[a]}(\zeta; \vec{0}) = 0\}, \\
\mathcal{D}^{(k)}_{[\hat{a}],+} &\equiv \left\{\gamma^{(k)}_{[\hat{a}],1}, \ldots, \gamma^{(k)}_{[\hat{a}],k}\right\} = \{(\zeta; \mu) \in \Gamma_+ \,:\, \hat{\Delta}_{k,[\hat{a}]}(\zeta; \vec{0}) = 0\}, \\
\mathcal{D}^{(k)}_{[\hat{a}],-} &\equiv \left\{\delta^{(k)}_{[\hat{a}],1}, \ldots, \delta^{(k)}_{[\hat{a}],n-k+1}\right\} = \{(\zeta; \mu) \in \Gamma_- \,:\, \Delta_{n-k-1,[\hat{a}]}(\zeta; \vec{0}) = 0\}.
\end{aligned}$$

Then for any fixed $k \in [n-1]$,

$$\mathcal{D}^{(n-k)}_{[\hat{a}],+} = \sigma\left(\mathcal{D}^{(k-1)}_{[a],-}\right), \qquad \mathcal{D}^{(n-k)}_{[\hat{a}],-} = \sigma\left(\mathcal{D}^{(k-1)}_{[a],+}\right). \tag{68}$$

In particular, if $k=1$, $\mathcal{D}^{(n-1)}_{[\hat{a}]} = \mathcal{D}^{(n-1)}_{[\hat{a}],+} \equiv \{\sigma(b_{[a],1}), \dots, \sigma(b_{[a],n-1})\}$.

Corollary 10.3. *Under the hypotheses of the above theorem, for any fixed $k \in [n-1]$ the pole divisor of $\tilde{\Psi}^{(n-k)}_{[\hat{a}]}(\zeta;\vec{t})$ may be computed from the pole divisor of $\tilde{\Psi}^{(k)}_{[a]}(\zeta;\vec{t})$,*

$$\prod_{j=1}^{n}(\zeta-\kappa_j) = \prod_{l=1}^{k}(\zeta-\gamma^{(k)}_{[a],l})\prod_{s=1}^{n-k}(\zeta-\gamma^{(n-k)}_{[\hat{a}],s}) - \mathfrak{a}_{[a],k}(\vec{0})\prod_{i=1}^{k-1}(\zeta-\delta^{(n-k)}_{[\hat{a}],i})\prod_{r=1}^{n-k-1}(\zeta-\delta^{(k)}_{[a],r}).$$

In particular, the dual compatible divisors $\mathcal{D}^{(n-k)}_{[\hat{a}]}$ and $\mathcal{D}^{(k)}_{[a]}$ satisfy

$$\frac{\prod_{l=1}^{k}\left(\kappa_j-\gamma^{(k)}_{[a],l}\right)\cdot\prod_{i=1}^{n-k}\left(\kappa_j-\gamma^{(n-k)}_{[\hat{a}],i}\right)}{\prod_{s=1}^{n-k-1}\left(\kappa_j-\delta^{(k)}_{[a],s}\right)\cdot\prod_{r=1}^{k-1}\left(\kappa_j-\delta^{(n-k)}_{[\hat{a}],r}\right)} = \mathfrak{a}_{[a],k}(\vec{0}), \qquad \forall j\in[n] \tag{69}$$

Moreover, if $[\hat{a}]=[a]=[1,\dots,1]\in Gr^{TP}(1,n)$, then

$$\mathcal{D}^{(n-k)}_{[a],+} = \sigma\left(\mathcal{D}^{(k-1)}_{[a],-}\right), \qquad \mathcal{D}^{(n-k)}_{[a],-} = \sigma\left(\mathcal{D}^{(k-1)}_{[a],+}\right), \qquad \forall k\in[n].$$

Proof. It is sufficient to insert (68) into the identities in Corollary 8.2. for all $j\in[n]$, such that $\kappa_j \notin \mathcal{D}^{(k)}_{\alpha}\cup\mathcal{D}^{(n-k)}_{\beta}$. If $\kappa_j\in\mathcal{D}^{(k)}_{\alpha}$, that is $\kappa_j=\gamma^{(k)}_{\alpha,\bar{l}}=\delta^{(k)}_{\alpha,\bar{s}}$, in (69) the factors corresponding to $s=\bar{s}$ and $l=\bar{l}$ are omitted and substituted by (-1). Similarly if $\kappa_j\in\mathcal{D}^{(n-k)}_{\beta}$. □

In Figure 5, we show the first non trivial example of such duality relation between T–hyperelliptic solitons in $Gr^{\mathrm{TP}}(2,5)$ and $Gr^{\mathrm{TP}}(3,5)$.

11. Summary and concluding remarks

In this paper we have characterized the KP–soliton data $(\mathcal{K},[A])$, $\mathcal{K}=\{\kappa_1<\dots<\kappa_n\}$, $[A]\in Gr^{\mathrm{TNN}}(k,n)$, which are compatible with a real divisor structure on a given rational degeneration of a real hyperelliptic curve. Indeed we start from the ansatz that

$$\Gamma=\Gamma_+\sqcup\Gamma_- = \{\eta^2=\prod_{j=1}^{N}(\zeta-\kappa_j)^2\},$$

is obtained from $\Gamma^{(\epsilon)}=\{\eta^2=\prod_{j=1}^{N}(\zeta-\kappa_j)^2-\epsilon^2\}$ when $\epsilon^2\to 0$ and that the divisor $\mathcal{D}=\{P_1,\dots,P_{n-1}\}\subset\Gamma\backslash\{P_+\}$ is the limit of a $(n-1)$–point non special divisor $(P^{(\epsilon)}_1,\dots,P^{(\epsilon)}_{n-1})$ on $\Gamma^{(\epsilon)}\backslash\{P_+\}$ satisfying Dubrovin–Natanzon reality conditions. For that reason we impose the

finite oval condition $\zeta(P_j) \in [\kappa_j, \kappa_{j+1}]$, $\forall j \in [n-1]$ (see Figure 1). By construction such a KP soliton solution is regular and bounded for all times.

Here $P_+ \in \Gamma_+$ is the marked point where the KP Baker–Akhiezer function has its essential singularity, k divisor points, say $P_1, \dots, P_k$ belong to Γ_+, and the remaining $n-k-1$ are in Γ_-, for some $k \in [0, n-1]$. In agreement with vacuum dressing, there exist a finite inverse gauge operator $W = 1 - w_1(\vec{t})\partial_x^{-1} - \cdots w_k(\vec{t})\partial_x^{-k}$ and k linearly independent solutions to the heat hierarchy, $f^{(1)}(\vec{t}), \dots, f^{(k)}(\vec{t})$, which form a basis of the linear operator (Darboux transformation) $D^{(k)} = W\partial_x^k$.

Such algebraic geometric structure is also compatible with the ansatz that the soliton solution corresponds to a point in $Gr(k,n)$ (the finite dimensional reduction of the Sato Grassmannian) and that the double points $\mathcal{K} = \{\kappa_1 < \cdots < \kappa_n\}$ are related to the phases of the KP soliton solutions, via the heat hierarchy solutions appearing in its τ–function.

Finally, according to [19], KP solitons corresponding to data in $Gr(k,n)$ (the so–called $(n-k,k)$–line solitons) are real regular bounded for all times (x,y,t) if and only if the soliton data belong to the totally non–negative part of the Grassmannian $Gr^{\mathrm{TNN}}(k,n)$.

For all of the above reasons we have investigated which soliton data $(\mathcal{K}, [A])$, with $[A] \in Gr^{\mathrm{TNN}}(k,n)$ are compatible with the divisor structure on Γ (Sections 3, 4). In particular we have proven that Γ is a desingularization of Γ_ξ as in [1] for points in $Gr^{\mathrm{TP}}(n-1,n)$ (Section 5).

More precisely, for any fixed $\mathcal{K} = \{\kappa_1 < \cdots < \kappa_n\}$ and $k \in [0, n-1]$, we have called $\mathcal{D} = \{P_1, \dots, P_{n-1}\} \subset \Gamma \backslash \{P_+\}$, with Γ as in (1), a *k–compatible divisor* if

(1) there is exactly one pole in each finite oval: $\zeta(P_j) \in [k_j, k_{j+1}]$, $\forall j \in [n-1]$;
(2) exactly k divisor points points belong to Γ_+.

In figure 3, we show possible k–compatible divisor configurations for $n = 4$ and $k = 3, 2, 1$.

We have called *T–hyperelliptic* the soliton data $(\mathcal{K}, [A])$, $[A] \in Gr^{\mathrm{TNN}}(k,n)$, which produce k–compatible divisors on Γ. To identify T–hyperelliptic soliton data in $Gr^{\mathrm{TNN}}(k,n)$, we have proceeded in two steps:

(1) On Γ we have defined a vacuum wavefunction $\Psi(\zeta; \vec{t})$, which coincides with Sato vacuum wavefunction on Γ_+. Vacuum divisors are 0–compatible by construction and are in bi-jection with points $[a] \in Gr^{\mathrm{TP}}(1,n)$ (see Lemmata 3.1 and 3.2);

(2) Then we have applied the Darboux transformation $D^{(k)}$ generated by the soliton data $(\mathcal{K},[A])$ and we have required that there exists a vacuum divisor $[a]\in Gr^{\mathrm{TP}}(1,n)$ such that the divisor of the normalized KP–wavefunction $\dfrac{D^{(k)}\Psi(\zeta;\vec{t})}{D^{(k)}\Psi(\zeta;\vec{0})}$ is k–compatible. Then
   (a) If $k=1,n-1$, we have obtained a bi–jection between vacuum divisors $[a]\in Gr^{\mathrm{TP}}(1,n)$ and soliton data $[A]\in Gr^{\mathrm{TP}}(1,n), Gr^{\mathrm{TP}}(n-1,n)$;
   (b) In Lemma 4.1 and Theorem 4.1, for any fixed $k\in]1,n-1[$, we have proven that the soliton data $(\mathcal{K},[A])$ are T–hyperelliptic if and only if $[A]\in Gr^{\mathrm{TP}}(n-1,n)$ with representative matrix

$$A^i_j=\kappa_j^{i-1}a_j,\qquad i\in[k],\ \ j\in[n]. \tag{70}$$

For any fixed $\mathcal{K}$ and $k\in]1,n-1[$, such soliton data parametrize an $(n-1)$–dimensional real connected variety in the $k(n-k)$ dimensional $Gr^{\mathrm{TP}}(k,n)$. For instance, varying the position of the 2–compatible divisor in Figure 1.b), we parametrize a 3–dimensional variety of KP soliton data in $Gr^{\mathrm{TP}}(2,4)$.

Soliton data satisfying (70) are naturally connected with the solutions to the finite non–periodic Toda lattice hierarchy (25) since, for any fixed $k\in[n-1]$, as observed in [3], the τ–function generating such KP $(n-k,k)$–soliton solution $u_{KP}(\vec{t})=2\partial_x^2\log\tau^{(k)}(\vec{t})$

$$\tau^{(k)}(\vec{t})=\mathrm{Wr}_x\left(\mu_0(\vec{t}),\partial_x\mu_0(\vec{t}),\dots,\partial_x^{k-1}\mu_0(\vec{t})\right),\qquad \mu_0(\vec{t})=\sum_{j=1}^n a_j\exp(\kappa_jx+\kappa_j^2y+\kappa_j^3t+\cdots),$$

is also the τ–function associated to the solutions to the finite non–periodic Toda lattice hierarchy (25) for the Toda datum $(\mathcal{K},[a])$, where $\mathcal{K}$ is the Toda spectrum and $[a]\in Gr^{\mathrm{TP}}(1,n)$ uniquely identifies the initial value Toda problem.

We have then discussed the relation between the spectral problem associated to KP T-hyperelliptic $(n-k,k)$-line KP solitons and the open Toda lattice. The spectral curve proposed for the open Toda lattice in [31, 28, 4] is determined by the equation $\hat{\eta}=\prod_{j=1}^n(\zeta-\kappa_j)$, considered as the limit $\epsilon\to0$ of the hyperelliptic spectral curve $\hat{\eta}+\frac{\epsilon^2}{4\hat{\eta}}=\prod_{j=1}^n(\zeta-\kappa_j)$, of the periodic Toda system. In [25], the Baker-Akhiezer function approach is used to provide a solution to the Toda inverse spectral problem for the singular curve (36).

The rational curve[3] is the same for both finite Toda and KP T–hyperelliptic solitons and the same τ–functions govern the solutions in both cases. Then, it is natural to expect a relation between the divisor structure of the two problems. In the rational setting, the two systems may be distinguished from one another from the different asymptotics at the essential singularities, asymptotics which is modeled on the algebraic geometric regular periodic setting for the two systems. In sections 8 and 9 we have investigated the relations between the two problems.

Here, to any KP soliton data $(\mathcal{K}, [a])$, $\mathcal{K} = \{\kappa_1 < \cdots \kappa_n\}$ and $[a] \in Gr^{\mathrm{TP}}(1, n)$:

(1) we associate the vacuum KP spectral data $(\mathcal{K}, \mathcal{B})$, where $\mathcal{B} = \{b_1 < \cdots < b_{n-1}\}$ is the vacuum divisor and satisfies $b_j \in]\kappa_j, \kappa_{j+1}[$, for all $j \in [n-1]$ and the vacuum KP–wavefunction $\Psi(\zeta; \vec{t})$ as in (9);

(2) for any fixed $k \in [n-1]$, we associate a Darboux transformation $D^{(k)}$, the T-hyperelliptic $(n-k, k)$-line KP soliton solution $(\mathcal{K}, [A])$, with $[A] \in Gr^{\mathrm{TP}}(k, n)$ as in (70), the k–compatible spectral data $(\mathcal{K}, \mathcal{D}^{(k)})$ and the normalized KP–wavefunction $\frac{\Psi_{\mathcal{B}}^{(k)}(\zeta;\vec{t})}{\Psi_{\mathcal{B}}^{(k)}(\zeta;\vec{0})}$, with $\Psi_{\mathcal{B}}^{(k)}$ as in (11).

The same data $(\mathcal{K}, [a])$ parametrize the solutions to the IVP of the finite non–periodic Toda hierarchy in the real configuration space (23). The results in [25] may be restated as follows:

(1) $(\mathcal{K}, [a])$ are in bi–jection with Toda spectral data $(\mathcal{K}, \mathcal{B}^{(T)})$, with $\mathcal{B}^{(T)} = \{b_1^{(T)} < \cdots < b_{n-1}^{(T)}\}$, $b_j^{(T)} \in]\kappa_j, \kappa_{j+1}[$.

(2) The time–dependent Baker–Akhiezer Toda vectors $\Psi_k^{(T)}(\zeta; \vec{t}) = e^{\zeta/2}\left(\sum_{i=0}^{k} c_i(\vec{t}, k)\zeta^i\right)$, $\Psi_k^{(T),\sigma}(\zeta; \vec{t}) = e^{-\zeta/2}\left(\frac{\sum_{l=0}^{n-1-k} c_l^{\sigma}(\vec{t},k)\zeta^l}{\prod_{s=1}^{n-1}(\zeta - b_s^{(T)})}\right)$, $k \in [0, n-1]$, satisfy the gluing conditions $\Psi_j^{(T)}(\kappa_l^{(T)}, \vec{t}) = \Psi_j^{(T),\sigma}(\kappa_l^{(T)}, \vec{t})$, and necessary normalization relations[25].

In Theorem 8.1), we have expressed the normalized KP wavefunction associated to the T–hyperelliptic $(n-k, k)$–solitons in function of the entries of the Toda resolvent $\mathfrak{R} = (\zeta \mathfrak{I}_n - \mathfrak{A})^{-1}$ for the data $(\mathcal{K}, [a])$. It then follows (Corollary 8.1):

(1) the KP vacuum divisor and the Toda divisor coincide for the datum $(\mathcal{K}, [a])$: $\mathcal{B} = \mathcal{B}^{(T)}$;

(2) the Darboux transformation associated to $(\mathcal{K}, [A])$ is the differential operator associated to Toda k–minors $\hat{\Delta}^{(k)}(\zeta, \vec{t})$;

[3]Notice that also the regular hyeperelliptic curves are the same, so it is natural to expect relations between Toda periodic hierarchy and regular real–periodic KP solutions on $\Gamma^{(\epsilon)}$.

(3) the KP divisor of the T–hyperelliptic $(n-k,k)$–soliton, $\mathcal{D}^{(k)} = \{\gamma_1^{(k)}, \dots, \gamma_k^{(k)}, \delta_1^{(k)}, \dots, \delta_{n-k-1}^{(k)}\}$ is the zero divisor at times $\vec{t} \equiv \vec{0}$, of the Toda Baker–Akhiezer components $\Psi_k^{(T)}$ and $\Psi_k^{(T),\sigma}$ (see (49)).

Then we have used the identities among the entries of the Toda resolvent $\mathfrak{R}$ as recursive sets of equations which allow to compute both the k–compatible divisors of the KP wavefunction and the KP Darboux transformations $D^{(k)}$, as k varies from 1 to $n-1$ (see Proposition 7.1 and Corollary 8.2). We have also solved the inverse problem: in Theorem 9.1, given a k–compatible divisor on Γ, we explicitly reconstruct the soliton data $[a]$.

The space–time inversion $\vec{t} \mapsto -\vec{t}$ allows to map $(n-k,k)$-soliton KP solutions to $(k,n-k)$-soliton KP solutions and induces a duality relation between the Grassmann cells in $Gr(k,n)$ and $Gr(n-k,n)$ (see [5] and references therein).

The space–time inversion leaves Γ invariant since it preserves the spectrum of $\mathfrak{A}$. For the Toda hierarchy, it corresponds to the composition of the space–time inversion with the reflection of the entries of $\mathfrak{A}$ with respect to the anti–diagonal. This correspondence for Toda is natural in view of the asymptotic behaviour of $\mathfrak{A}(\vec{t})$ when all times go either to $+\infty$ or to $-\infty$, which generalizes a result in [33] for the first flow to the Toda hierarchy.

As a consequence, the entries $\mathfrak{R}_{1,1}(\vec{t})$ and $\mathfrak{R}_{n,n}(-\vec{t})$ of the resolvent of $\mathfrak{A}(\vec{t})$ are the generating functions for such dual Toda hierarchies (Proposition 10.1), and are also associated to dual families of T–hyperelliptic soliton solutions.

We have used such correspondence to obtain explicit relations among the dual divisors. In particular if the initial k–compatible divisor in $(Gr^{\mathrm{TP}}(k,n)$ is associated to $[\alpha] \in Gr^{\mathrm{TP}}(1,n)$, with $[\alpha]$ as in (59), then the dual divisor is $(n-k)$–compatible in $(Gr^{\mathrm{TP}}(n-k,n)$ and associated to $[1/\alpha]$ (see Corollary 10.1). We have shown that such $(n-k)$–compatible dual divisor may be explicitly computed by applying the hyperelliptic involution to the $(k-1)$–compatible divisor associated to $[\alpha]$ (see Theorem 10.1 and Figure 5).

T–hyperelliptic solitons are not the unique family of KP solitons which may be associated to rational degenerations of hyperelliptic curves. In [2], we associate the rational degeneration of a regular $\mathtt{M}$–curve to generic soliton data $(\mathcal{K},[A])$, with $[A] \in Gr^{\mathrm{TNN}}(k,n)$ and we plan to classify other families of KP solitons associated to rational degenerations of hyperelliptic or trigonal curves. We also plan to investigate other connection between KP theory and Toda type systems in future.

References

[1] S. Abenda, P.G. Grinevich *Rational degenerations of M-curves, totally positive Grassmannians and KP–solitons*, arXiv:1506.00563 submitted.

[2] S. Abenda, P.G. Grinevich *Le–diagrams, KP–theory and rational degenerations of* M*–curves*, in preparation (2016).

[3] G. Biondini, Y. Kodama *On a family of solutions of the Kadomtsev-Petviashvili equation which also satisfy the Toda lattice hierarchy.* J. Phys. A **36** (2003), no. 42, pp. 10519-10536.

[4] H. W. Braden, A. Marshakov *Singular phases of Seiberg-Witten integrable systems: weak and strong coupling.* Nuclear Phys. B **595** (2001), no. 1-2, 417–466.

[5] S. Chakravarty, Y. Kodama. *Classification of the line-soliton solutions of KPII.* J. Phys. A **41** (2008), no. 27, 275209, 33 pp.

[6] S. Chakravarty, Y. Kodama. *Soliton solutions of the KP equation and application to shallow water waves.* Stud. Appl. Math. **123** (2009) 83-151.

[7] P. Deift, L.C. Li, T. Nanda, C. Tomei. *The Toda flow on a generic orbit is integrable.* Commun. Pure Appl. Math. **39** (2) (1986), 183-232.

[8] L.A, Dickey. Soliton equations and Hamiltonian systems. Second edition. Advanced Series in Mathematical Physics, 26. World Scientific Publishing Co., Inc., River Edge, NJ, 2003. xii+408 pp.

[9] B.A. Dubrovin, I.M. Krichever, S.P. Novikov. Integrable systems. Dynamical systems, IV, 177-332, Encyclopaedia Math. Sci., 4, Springer, Berlin, 2001.

[10] B. A.Dubrovin, S.M. Natanzon. *Real theta-function solutions of the Kadomtsev-Petviashvili equation.* Izv. Akad. Nauk SSSR Ser. Mat. **52** (1988) 267-286.

[11] N.M. Ercolani, H. Flaschka, S. Singer. *The geometry of the full Kostant–Toda lattice.* in Progress in Mathematics, **115** (1993), pp. 181-225, Birkhäuser, Boston.

[12] H. Flaschka. *The Toda lattice I. Existence of integrals.* Phys. Rev. B (3) **9** (1974), 1924-1925.

[13] N.C. Freeman, J.J.C. Nimmo. *Soliton-solutions of the Korteweg-deVries and Kadomtsev-Petviashvili equations: the Wronskian technique.* Phys. Lett. 95A (1983), 1-3

[14] A. Harnack. *Über die Vieltheiligkeit der ebenen algebraischen Curven.* Math. Ann. **10** (1876) 189-199.

[15] R. Hirota. The direct method in soliton theory. Cambridge Tracts in Mathematics, 155. Cambridge University Press, Cambridge, 2004. xii+200 pp.

[16] M. Jimbo, T. Miwa. *Solitons and infinite dimensional Lie algebras* Publ. Res. Inst. Math. Sci. **19** (1983), 943-1001.

[17] B.B. Kadomtsev, V.I. Petviashvili. *On the stability of solitary waves in weakly dispersive media.*, Sov. Phys. Dokl. **15** (1970) 539-541.

[18] Y. Kodama,L.K. Williams. *The Deodhar decomposition of the Grassmannian and the regularity of KP solitons.* Adv. Math. **244** (2013) 979-1032.

[19] Y. Kodama,L.K. Williams. *KP solitons and total positivity for the Grassmannian.* Invent. Math. **198** (2014) 637-699.

[20] B. Kostant. *The solution to a generalized Toda lattice and representation theory.* Adv. in Math. **34** (1979), no. 3, 195-338.

[21] I. M. Krichever. *An algebraic-geometric construction of the Zakharov-Shabat equations and their periodic solutions.* (Russian) Dokl. Akad. Nauk SSSR **227** (1976) 291-294.

[22] I. M. Krichever. *Integration of nonlinear equations by the methods of algebraic geometry.* (Russian) Funkcional. Anal. i Priložen. **11** (1977) 15-31, 96.

[23] I.M. Krichever, D.H. Phong, *On the integrable geometry of soliton equations and N=2 supersymmetric gauge theories.* J. Differential Geom. **45** (1997), no. 2, 349-389

[24] I.M. Krichever, D.H. Phong, *Symplectic forms in the theory of solitons.* Surveys in differential geometry: integral systems [integrable systems], Surv. Differ. Geom., IV, Int. Press, Boston, MA, (1998), 239-313.

[25] I.M. Krichever, K.L. Vaninsky, *The periodic and open Toda lattice.* AMS/IP Stud. Adv. Math., **33**, Amer. Math. Soc., Providence, RI, (2002), 139-158.

[26] A.N. Leznov, M.V. Saveliev. *Theory of group representations and integration of nonlinear systems* $x_{a,z\bar{z}} = \exp(kx)_a$ Physica D **3** (1981) 62-72.

[27] T.M. Malanyuk. *A class of exact solutions of the Kadomtsev-Petviashvili equation.* Russian Math. Surveys **46** (1991) 225-227.

[28] A. Marshakov, "Seiberg-Witten theory and integrable systems." World Scientific Publishing Co., Inc., River Edge, NJ, (1999) ii+253 pp.

[29] V.B. Matveev. *Darboux transformation and explicit solutions of the Kadomtcev-Petviaschvily equation, depending on functional parameters.* Lett. Math. Phys. **3** (1979), no. 3, 213-216.

[30] Matveev, V. B.; Salle, M.A., "Darboux transformations and solitons.", Springer Series in Nonlinear Dynamics. Springer-Verlag, Berlin, (1991)

[31] H.McKean *Singular curves and the Toda Lattice.* unpublished manuscript (1982).

[32] T. Miwa, M. Jimbo, E. Date. Solitons. Differential equations, symmetries and infinite-dimensional algebras. Cambridge Tracts in Mathematics, 135. Cambridge University Press, Cambridge, (2000) x+108 pp.

[33] J. Moser. *Finitely many mass points on the line under the influence of an exponential potential - an integrable system.* in Dynamical systems, theory and applications, Lecture Notes in Phys. **38**, Springer, Berlin (1975), 467-497.

[34] Y. Nakamura. *Moduli space of SU(2) monopoles and the complex cyclic-Toda hierarchy.* Comm. Math. Phys. **128** (1990), 509-520.

[35] A. Postnikov. *Total positivity, Grassmannians, and networks.*, arXiv:math/0609764 [math.CO]

[36] M. Sato. *Soliton equations as dynamical systems on infinite-dimensional Grassmann manifold.* in: Nonlinear PDEs in Applied Sciences (US-Japan Seminar, Tokyo), P. Lax and H. Fujita eds., North-Holland, Amsterdam (1982) 259-271.

[37] M. Toda. *Vibration of a Chain with Nonlinear Interaction* J. Phys. Soc. Japan **22** (1967), 431-6.

[38] K. Ueno, K.Takasaki. *Toda lattice hierarchy Group Representations and Systems of Differential Equations*, in Advanced Studies in Pure Mathematics **4** (1984), 1-95, Amsterdam: North-Holland.

[39] Y. Zarmi. *Vertex dynamics in multi-soliton solutions of Kadomtsev-Petviashvili II equation.* Nonlinearity **27** (2014), 1499-1523.

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