Identification and geometric characterization of Lie triple screw systems and their exponential images

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Highlights of *Identification and Geometric Characterization of Lie Triple Screw Systems and Their Exponential Images*

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- a geometric characterization of the Lie product and the Lie triple product of a generic screw system.
- a systematic identification of all Lie triple screw systems of $\mathfrak{se}(3)$, by an approach based on both algebraic Lie group theory and descriptive screw theory.
- derivation of the exponential motion manifolds of the Lie triple screw systems in dual quaternion representation.

Identification and Geometric Characterization of Lie Triple Screw Systems and Their Exponential Images☆

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Abstract

The twist space of a plunging constant-velocity (CV) coupling with intersecting shafts consists, in all configurations, of a planar field of zero-pitch screws. Recently, we reported an important discovery about this screw system: it is closed under two consecutive Lie bracket operations, thus being referred to as a Lie triple system; taking the exponential of all its twists generates the motion manifold of the coupling. In this paper, we first give a geometric characterization of the Lie product and the Lie triple product of a generic screw system. Then, we present a systematic identification of *all* Lie triple screw systems of $\mathfrak{se}(3)$, by an approach based on both algebraic Lie group theory and descriptive screw theory. We also derive the exponential motion manifolds of the Lie triple screw systems in dual quaternion representation. Finally, several important applications of Lie triple screw systems in mechanism and machine theory are highlighted in the conclusions.

Keywords: screw system, Lie triple system, constant-velocity coupling, parallel mechanism, type synthesis

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Nomenclature

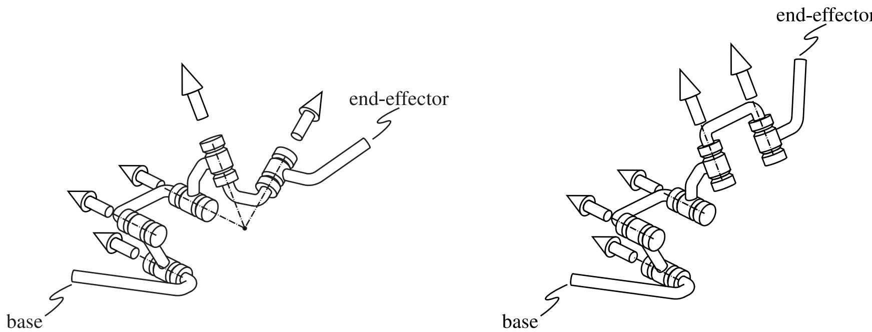
$\mathcal{R}$	revolute joint
$\mathcal{S}$	spherical-joint-equivalent subchain
$\mathcal{E}$	planar-joint-equivalent subchain
	zero-pitch screw
	infinite-pitch screw
	finite-non-zero-pitch screw
SE(3)	the special Euclidean group
$\mathfrak{se}(3)$	the Lie algebra of SE(3)
$\boldsymbol{\xi}, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots$	twists or screws
S	generic screw system with unspecified type
$S_{i,j}$	the jth special i-system (see [2, Ch. 12])
$S^0_{i,j}$	jth special i-system whose finite-pitch screws all have zero pitch
L,$L_{i,j}$,$L^0_{i,j}$	Lie product of S,$S_{i,j}$ and $S^0_{i,j}$ respectively: $L = [S, S]$
T,$T_{i,j}$,$T^0_{i,j}$	triple product of S,$S_{i,j}$ and $S^0_{i,j}$ respectively: $T = [[S, S], S]$

1. Introduction

Methods for inferring finite motions from infinitesimal ones are of crucial importance in kinematics of mechanisms and robotics [2, 3]. For example, a one degree-of-freedom (1-DoF) finite motion is a parameterized curve in the displacement group SE(3), which may be integrated from its axode surface, a parameterized curve in the Lie algebra $\mathfrak{se}(3)$ of SE(3) [3, 4]. From mechanism synthesis point of view, it is important to consider multi-DoF finite motions or *parameterized submanifolds* of SE(3) that may be generated by the end-effector of an open-chain or closed-chain mechanism [5, 6, 7, 8, 9]. Several researchers characterize the end-effector motion set of mechanisms by the more general notion of algebraic varieties [10, 11]. In practice, we may always assume that such varieties are submanifolds by restricting them to an open subset about the identity element of SE(3).

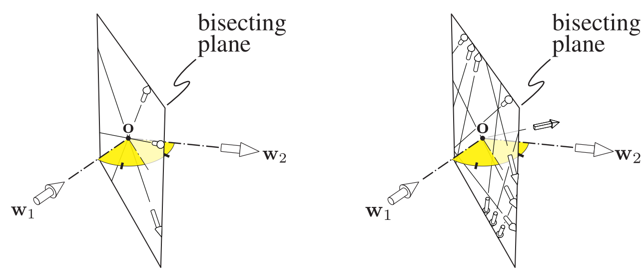
Inferring finite end-effector motions of mechanisms from their *twist space* [1] has been extensively studied in the literature, in particular in type synthesis of parallel mechanisms [13, 5, 6, 7, 14, 8, 15, 9, 16, 17, 12, 1]. The methods of inference involved in such works are mainly heuristic and developed on a case-by-case basis. Hunt pointed out that the full-cycle mobility of a kinematic chain may be inferred from its twist space if the latter remains invariant for all configurations [2]. The inferred finite motions are simply the connected subgroups of SE(3), with their twist spaces being the corresponding Lie subalgebras of $\mathfrak{se}(3)$[18, 5, 19]. For this reason, Lie subalgebras of $\mathfrak{se}(3)$ are sometimes referred to as *invariant screw system* (ISS) [12]. More generally, when the twist space of a mechanism remains invariant *up to a rigid displacement* under arbitrary

[1]The twist space of a mechanism is the vector space of all possible end-effector twists at a given configuration. In other words, it is a vector subspace of the Lie algebra $\mathfrak{se}(3)$ of the special Euclidean group SE(3).



(a) (b)

Figure 1: Examples of PSS serial generators [12]: (a) a $\mathcal{ES}$-type serial chain; (b) $\mathcal{EE}$-type serial chain.

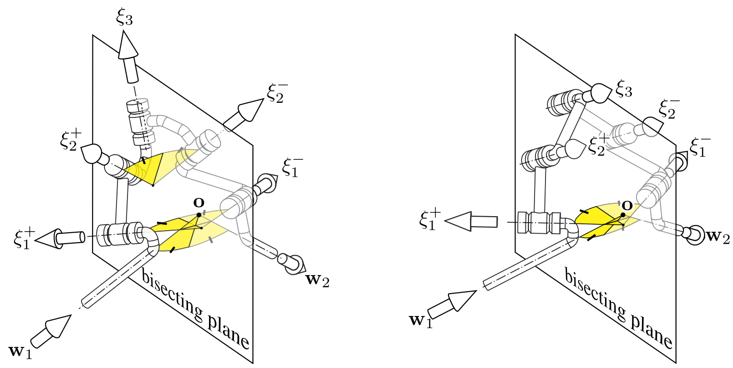


(a) $S^0_{2,1}$ (b) $S^0_{3,4}$

Figure 2: Screw systems of a CV coupling with intersecting shafts, where $\mathbf{w}_1$ and $\mathbf{w}_2$ are the input and the output twist, respectively: (a) non-plunging coupling; (b) plunging coupling. $S^0_{2,1}$ comprises a planar pencil of instantaneous rotations converging into a point. $S^0_{3,4}$ comprises a planar field of instantaneous rotations as well as an instantaneous translation perpendicular to the plane.

finite motions away from singularities, thus having a constant *class* [20], the mechanism is said to have a *persistent screw system* (PSS) of the end-effector (see Fig. 1 for examples) [12]. The notion of PSS is important as it allows us to expand the theoretical framework under which the inference of finite motion properties from instantaneous ones is guaranteed. The exhaustive derivation and classification of all *serial* chains with an m-dimensional (m-D) PSS of the end-effector is complete up to $m \leq 4$ [21, 22, 23].

However, not all persistent motions are generated by serial chains. In a seminal paper on the analysis and synthesis of CV couplings with parallel-kinematics structure [24], Hunt identified the twist space of a CV coupling with intersecting shafts to be either the fourth special three-system (with zero-pitch screws), denoted $S^0_{3,4}$, or the first special two-system (with zero-pitch screws), denoted $S^0_{2,1}$, depending on whether plunging is allowed or not (see Fig. 2 and also [2,



(a) (b)

Figure 3: Typical connecting chains of plunging CV coupling with intersecting shafts: (a) a $\mathcal{RSR}$ chain; (b) a $\mathcal{RER}$ chain. Each pair of revolute joint twists $(\boldsymbol{\xi}_i^+, \boldsymbol{\xi}_i^-)$ remains mirror symmetric about the bisecting plane.

Ch. 13.5]). Focusing, without loss of generality, on the plunging CV coupling, CV transmission is provided for any input and articulation angles if the twist space always remains a $S^0_{3,4}$ system, with all its screws lying on the bisecting plane [24]. The persistence property of the coupling is enforced by a parallel-kinematics architecture with prescribed types of connecting chains with five (in some cases four) DoF. All of the latter respect mirror symmetry about the bisecting plane (see Fig. 3 and also [24, 15]). The twist space of a CV coupling with intersecting shafts, although not fixed at a particular location, remains congruent to itself under arbitrary configuration changes, thus being persistent. It is briefly stated in [24] and later proved in [21] that there exists no serial chain whose twist space remains $S^0_{3,4}$ for all its configurations. The serial 5-DoF connecting chains in a plunging CV coupling are not persistent either. For example, the twist space of the $5\mathcal{R}$ chain shown in Fig. 3(b) has a reciprocal screw with zero pitch at a mirror symmetric configuration [24]; however, the chain may also have configurations where the reciprocal screw has infinite pitch [25]. This shows that the persistent behavior of $S^0_{3,4}$ in a CV coupling is not the result of the in-parallel connection of higher-dimensional-PSS serial generators.

Bonev et al. [26] studied CV-like generators of $S^0_{3,4}$ from a finite-motion perspective, by the so-called tilt-and-torsion orientation parameters. They pointed out that the end-effector motion of a 3-$\mathcal{RSR}$ reflected tripod, which is a parallel mechanism equivalent to a plunging CV coupling (see [2, 15, pp. 397]), has zero torsion for all its configurations. In other words, the reflected tripod has a zero-torsion motion type[2] [27]. It is further shown in [27] that the motion

[2]The motion type [16] or motion pattern [9] of a mechanism is characterized by a submanifold Q of SE(3), with the twist spaces of the mechanism being tangent spaces of Q at corresponding configurations. Once the motion type is given, the twist spaces can be derived

type of a plunging CV coupling is exactly the image $\exp(S^0_{3,4})$ of $S^0_{3,4}$ under the exponential map exp [3]. The fact that its tangent spaces are all copies of $S^0_{3,4}$ can be readily verified by the half-angle property [28, 29].

Through a series of recent studies, we discovered that the many geometric properties of $\exp(S^0_{3,4})$ are best understood from a symmetric space viewpoint [28, 29]. A symmetric space is a smooth manifold which has an inversion symmetry about every point [30]. The inversion symmetry condition for a screw system S is [28, 29]:

$$\forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in S: \quad e^{\boldsymbol{\xi}_1} e^{\boldsymbol{\xi}_2} e^{\boldsymbol{\xi}_1} \in \exp(S) \tag{1}$$

Both $S^0_{2,1}$ and $S^0_{3,4}$ satisfy condition (1), turning $\exp(S^0_{2,1})$ and $\exp(S^0_{3,4})$ into symmetric spaces. We showed in [29] that the inversion symmetry of S leads to the mirror symmetric arrangement of joint axes in a CV coupling. The necessary and sufficient condition for a screw system S to satisfy (1) is being closed under two consecutive *Lie bracket* or *commutator* [19] operations, namely [30]

$$\forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3 \in S: \quad [[\boldsymbol{\xi}_1, \boldsymbol{\xi}_2], \boldsymbol{\xi}_3] \in S \tag{2}$$

The double commutator operation $[[\cdot,\cdot],\cdot]$ is sometimes referred to as a *Lie triple product* [31]. A screw system S that satisfies (2) is called a *Lie triple system* (LTS, see [30, pp. 78]). LTSs can be considered a generalization of Lie subalgebras (or ISSs) of $\mathfrak{se}(3)$, since *taking exponentials of these screw systems generates the corresponding motion manifolds*. However, as shown by Hunt's general theory of CV couplings, LTSs admit distinct and more complex type synthesis rules compared to ISSs. Hence, it is important to identify all LTSs of $\mathfrak{se}(3)$ and investigate their type synthesis.

A full list of the LTSs of $\mathfrak{se}(3)$, which all share the many geometric properties of $S^0_{2,1}$ and $S^0_{3,4}$, is given in [28, 29] [4]. However, details of their identification is not reported therein. In this paper, accordingly, we provide the full derivation and classification. Our main contribution here is to offer a more complete view of the geometry of LTSs from both a Lie algebraic and screw system point of view. So far, as the authors are aware of, this is the first time that two derived screw systems of a screw system S, namely the *Lie product* $[S, S]$

$$[S, S] := \{\, [\boldsymbol{\xi}_1, \boldsymbol{\xi}_2] \mid \forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in S\}$$

---

by differentiation.

[3] In this paper, we use exp to denote the exponential map from $\mathfrak{se}(3)$ to SE(3) [19]. The exponential of a twist $\boldsymbol{\xi}$ is simply denoted by $e^{\boldsymbol{\xi}}$.

[4] It is worth pointing out that, after our early work in [28] and the presentation of the preliminary version of this paper in [1], Selig proposed a slightly different classification of LTSs for addressing optimal motion planning problems [32].

and the *Lie triple product* [5] $[[S, S], S]$

$$[[S, S], S] := \{\, [[\boldsymbol{\xi}_1, \boldsymbol{\xi}_2], \boldsymbol{\xi}_3] \mid \forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3 \in S\}$$

are *systematically* studied from a screw theoretic point of view. In addition, by using a dual-vector representation for the screws in S, we may study the exponential manifolds of the LTSs as sub-varieties of the Study quadric, thus offering more insight into the geometry of LTSs.

Without loss of generality, we base our identification process on the classification of screw systems provided in [2]. The classification of screw systems provided by Hunt in [2] is exhaustive, even though Gibson and Hunt [20, 18] later refined such classification by distinguishing several sub cases. While on the one hand these refinements increase the description complexity, on the other they are not relevant for the purpose of our work. Accordingly, Hunt's original (and simpler) classification is used here.

The paper outline is as follows. In Sec. 2, we review concepts of Lie group theory and screw geometry with the help of dual-vector representation. In Sec. 3, we give a geometric characterization of Lie products and Lie triple products, and introduce the concept of LTSs. In Sec. 4, we provide a systematic identification of the LTSs of $\mathfrak{se}(3)$, by analyzing Hunt's complete list of screw systems. Finally, in Sec. 5, we derive the dual quaternion representation of the motion manifolds corresponding to the exponentials of the LTSs.

2. Definitions and notations

Denote the scalar and vector product of two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ by $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{u} \times \mathbf{v}$, respectively. The dual-vector representation of the instantaneous *twist* of a rigid body motion is given by [34, Ch. 12]:

$$\begin{aligned} \boldsymbol{\xi} &= \mu(\mathbf{w} + \varepsilon \mathbf{v}) \\ &= \mu(\mathbf{w} + \varepsilon(\mathbf{q} \times \mathbf{w} + h\mathbf{w})) \\ &= \mu(1 + \varepsilon h)(\mathbf{w} + \varepsilon \mathbf{q} \times \mathbf{w}) \end{aligned} \qquad \begin{cases} \mu, h \in \mathbb{R} \\ \mathbf{w}, \mathbf{v} \in \mathbb{R}^3 \end{cases} \quad \|\mathbf{w}\| = 1$$

where the dual number ε satisfies $\varepsilon^2 = 0$. $\boldsymbol{\xi}$ corresponds to the instantaneous twist of a screw motion along the line with Plücker coordinate $\boldsymbol{\xi}^\circ \triangleq \mathbf{w} + \varepsilon \mathbf{q} \times \mathbf{w}$, where $\mathbf{q}$ is any point on the line, usually chosen to be $\mathbf{w} \times \mathbf{v}$. The *pitch*, *magnitude* and *dual magnitude* are given by $h = \mathbf{w} \cdot \mathbf{v}$, μ and $\mu(1 + \varepsilon h)$ respectively [34]. We shall refer to $\boldsymbol{\xi}^\circ$ as the *axis* of $\boldsymbol{\xi}$. For the special case $\|\mathbf{w}\| = 0$, $\boldsymbol{\xi}$ defines an infinite-pitch screw along direction $\mathbf{v}$, namely $\boldsymbol{\xi} = \varepsilon \mathbf{v}$ with $\|\mathbf{v}\| = 1$. The collection of all twists forms the 6-D vector space $\mathfrak{se}(3)$, i.e. the Lie algebra of SE(3) [19]. A *screw system* is a vector subspace of $\mathfrak{se}(3)$.

[5] The Lie product and Lie triple product have the same definition as the second and the third term in the lower central series of a Lie algebra [33]. However, we find the former locutions more easily intelligible for a non-specialized audience, since they may be related to the familiar concepts of vector product $\mathbf{u} \times \mathbf{v}$ and vector triple product $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ of 3-D vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$.

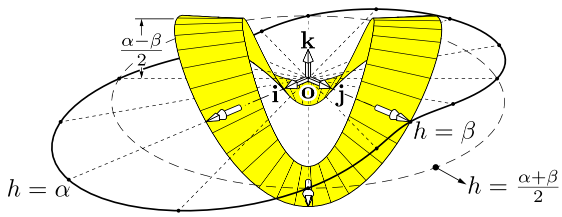


Figure 4: Cylindroid of the general two-system $S_{2,g}$ with basis pitches α and β, $\alpha > \beta$. The length of the cylindroid is given by $\alpha - \beta$, with the plane $\mathbf{ij}$ of the basis screws located in the middle. The variation of the pitch along the screws of the cylindroid is illustrated by the solid curve.

In particular, a two-system is spanned by two linearly independent screws, and it may either have a general form $S_{2,g}$ or one of five special forms $S_{2,j}, j = 1, \ldots, 5$ [2, pp. 344]. The screw axes of a general two-system $S_{2,g}$ envelope a cubic surface known as the cylindroid (see Fig. 4). $S_{2,g}$ admits two concurrent and perpendicular screws with maximal pitch α and minimal pitch β, which we may use as *basis screws* of the system. For the purpose of our exposition, the basis screws of a screw subspace are chosen, without loss of generality, by reason of computational convenience.

Similarly, a three-system is spanned by three linearly independent screws, and it may have either a general form $S_{3,g}$ or one of ten special forms $S_{3,j}, j = 1, \ldots, 10$ [2, pp. 356]. The screw axes of a general three-system $S_{3,g}$ envelope a pencil of ∞^1 concentric hyperboloids as shown in Fig. 5. $S_{3,g}$ admits three concurrent and perpendicular screws, which we use as basis screws (one has maximal pitch α, another has minimal pitch γ, and the last has an intermediate pitch β).

The commutator $[\boldsymbol{\xi}_1, \boldsymbol{\xi}_2] := \frac{1}{2}(\boldsymbol{\xi}_1\boldsymbol{\xi}_2 - \boldsymbol{\xi}_2\boldsymbol{\xi}_1)$ ([19, pp. 212]) of two dual vectors $\boldsymbol{\xi}_i = \mathbf{w}_i + \varepsilon\mathbf{v}_i, i = 1, 2$ is simply equal to the vector product [19, 5]:

$$\begin{aligned}
[\boldsymbol{\xi}_1, \boldsymbol{\xi}_2] &= \boldsymbol{\xi}_1 \times \boldsymbol{\xi}_2 = -\boldsymbol{\xi}_2 \times \boldsymbol{\xi}_1 \\
&= \mu_1\mu_2(\mathbf{w}_1 \times \mathbf{w}_2 + \varepsilon(\mathbf{w}_1 \times \mathbf{v}_2 - \mathbf{w}_2 \times \mathbf{v}_1)) \\
&= \mu_1\mu_2(1 + \varepsilon h_1)(1 + \varepsilon h_2)\boldsymbol{\xi}_1^\circ \times \boldsymbol{\xi}_2^\circ
\end{aligned}$$

It is clear that the commutator is a bilinear operator satisfying, respectively, skew-symmetry and the Jacobi identity [19, 5], i.e.

$$\boldsymbol{\xi}_1 \times \boldsymbol{\xi}_2 = -\boldsymbol{\xi}_2 \times \boldsymbol{\xi}_1$$

and

$$(\boldsymbol{\xi}_1 \times \boldsymbol{\xi}_2) \times \boldsymbol{\xi}_3 + (\boldsymbol{\xi}_2 \times \boldsymbol{\xi}_3) \times \boldsymbol{\xi}_1 + (\boldsymbol{\xi}_3 \times \boldsymbol{\xi}_1) \times \boldsymbol{\xi}_2 = 0 \tag{3}$$

In particular, $[\boldsymbol{\xi}, \boldsymbol{\xi}] = 0, \forall \boldsymbol{\xi} \in \mathfrak{se}(3)$. The scalar product of two dual vectors $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$ gives

$$\boldsymbol{\xi}_1 \cdot \boldsymbol{\xi}_2 = \mu_1\mu_2(\mathbf{w}_1 \cdot \mathbf{w}_2 + \varepsilon(\mathbf{w}_1 \cdot \mathbf{v}_2 + \mathbf{w}_2 \cdot \mathbf{v}_1))$$

Given a unit twist $\boldsymbol{\xi} = (1 + \varepsilon h)\boldsymbol{\xi}^\circ$, the screw motion about $\boldsymbol{\xi}$ with angle θ, in dual quaternion representation (see [34]-12.4.4), is given by the exponential

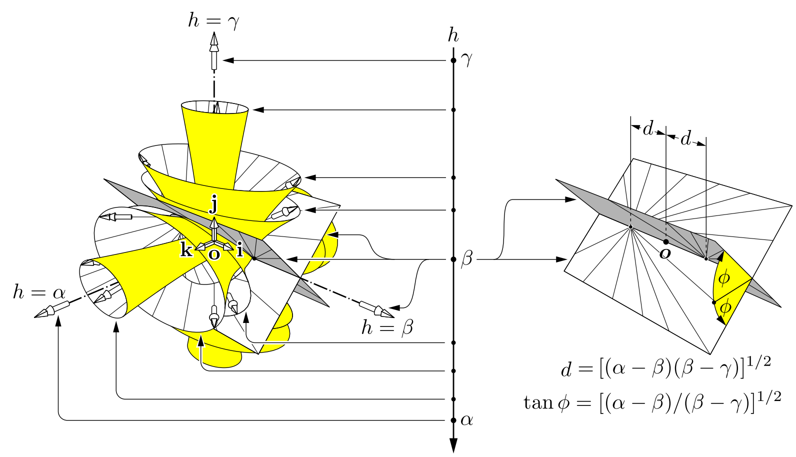


Figure 5: Left: a general three-system $S_{3,g}$ with basis pitches α, β and γ, $\alpha > \beta > \gamma$; equal-pitch screws with $h \neq \beta$ form a family of ∞^1 reguli lying on concentric hyperboloids; right: the screws of pitch β lie on a degenerate regulus consisting of two line pencils.

map

$$e^{\theta\boldsymbol{\xi}/2} = \cos(\hat{\theta}/2) + \sin(\hat{\theta}/2)\boldsymbol{\xi}^{\circ} \tag{4}$$

where the dual angle $\hat{\theta}/2$ is defined as $\theta(1+\varepsilon h)/2$ and

$$\begin{cases} \cos(\hat{\theta}/2) = \cos(\theta/2) - \varepsilon\dfrac{h\theta}{2}\sin(\theta/2) \\ \sin(\hat{\theta}/2) = \sin(\theta/2) + \varepsilon\dfrac{h\theta}{2}\cos(\theta/2) \end{cases}$$

or, for a unit infinite-pitch screw $\boldsymbol{\xi} = \varepsilon\boldsymbol{\xi}^{\circ}$,

$$e^{\theta\boldsymbol{\xi}/2} = 1 + \varepsilon\frac{\theta}{2}\boldsymbol{\xi}^{\circ} \tag{5}$$

Here, we adopt a half dual angle in order to be consistent with the usual matrix exponential $e^{\theta\boldsymbol{\xi}}$ [34, Ch. 15.2.2]. For simplicity, we shall hereafter denote $\sin(\theta)$ and $\cos(\theta)$ by s_θ and c_θ, respectively.

Given the canonical basis $\mathbf{i}, \mathbf{j}, \mathbf{k}$ of $\mathbb{R}^3$ and a reference point $\mathbf{o} \in \mathbb{R}^3$, we define the following basis for $\mathfrak{se}(3)$:

$$\begin{cases} \mathbf{I} = \mathbf{i} + \varepsilon\mathbf{o}\times\mathbf{i} \\ \mathbf{J} = \mathbf{j} + \varepsilon\mathbf{o}\times\mathbf{j} \\ \mathbf{K} = \mathbf{k} + \varepsilon\mathbf{o}\times\mathbf{k} \end{cases} \quad \begin{cases} \varepsilon\mathbf{I} = \varepsilon\mathbf{i} \\ \varepsilon\mathbf{J} = \varepsilon\mathbf{j} \\ \varepsilon\mathbf{K} = \varepsilon\mathbf{k} \end{cases} \tag{6}$$

with commutator relations given by:

$$\begin{cases} \mathbf{I}\times\mathbf{J} = \mathbf{K} \\ \mathbf{J}\times\mathbf{K} = \mathbf{I} \\ \mathbf{K}\times\mathbf{I} = \mathbf{J} \end{cases} \quad \begin{cases} \varepsilon\mathbf{I}\times\mathbf{J} = \mathbf{I}\times\varepsilon\mathbf{J} = \varepsilon\mathbf{K} \\ \varepsilon\mathbf{J}\times\mathbf{K} = \mathbf{J}\times\varepsilon\mathbf{K} = \varepsilon\mathbf{I} \\ \varepsilon\mathbf{K}\times\mathbf{I} = \mathbf{K}\times\varepsilon\mathbf{I} = \varepsilon\mathbf{J} \end{cases} \quad \begin{cases} \varepsilon\mathbf{I}\times\varepsilon\mathbf{J} = \mathbf{0} \\ \varepsilon\mathbf{J}\times\varepsilon\mathbf{K} = \mathbf{0} \\ \varepsilon\mathbf{K}\times\varepsilon\mathbf{I} = \mathbf{0} \end{cases}$$

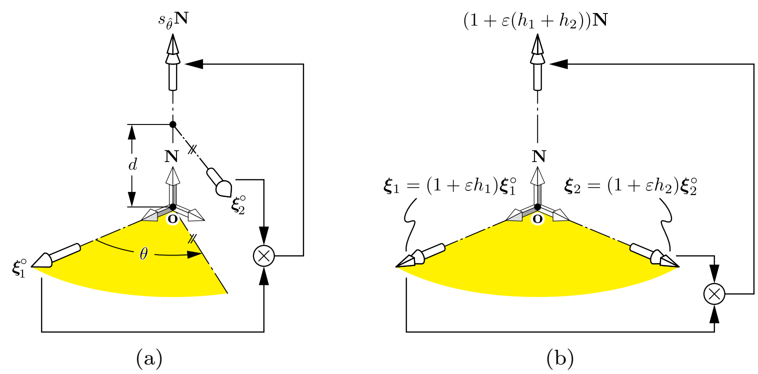


Figure 6: (a) Commutator of two zero-pitch screws $\boldsymbol{\xi}_1^\circ$ and $\boldsymbol{\xi}_2^\circ$. $\mathbf{N}$ is a unit line along the common perpendicular. (b): Commutator for two perpendicularly intersecting screws $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$ with pitches h_1, h_2. In the figures, the commutator operator is denoted by $\otimes$.

The commutator of two unit twists $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$ with pitch $h_1, h_2 \neq \infty$ is given by

$$\begin{aligned}\boldsymbol{\xi}_1 \times \boldsymbol{\xi}_2 &= (1+\varepsilon h_1)(1+\varepsilon h_2)\boldsymbol{\xi}_1^\circ \times \boldsymbol{\xi}_2^\circ \\ &= (1+\varepsilon(h_1+h_2))\boldsymbol{\xi}_1^\circ \times \boldsymbol{\xi}_2^\circ.\end{aligned}$$

Since the commutator for two lines (see Fig. 6(a)) is given by [34]

$$\boldsymbol{\xi}_1^\circ \times \boldsymbol{\xi}_2^\circ = s_{\hat{\theta}}\mathbf{N}$$

where $\hat{\theta} = \theta + \varepsilon d$, and $\mathbf{N}$ is the common perpendicular line between $\boldsymbol{\xi}_1^\circ$ and $\boldsymbol{\xi}_2^\circ$, we have

$$\begin{aligned}\boldsymbol{\xi}_1 \times \boldsymbol{\xi}_2 &= (1+\varepsilon(h_1+h_2))s_{\hat{\theta}}\mathbf{N} \\ &= s_\theta(1+\varepsilon(d\cot(\theta)+(h_1+h_2)))\mathbf{N}.\end{aligned} \tag{7}$$

Hence, $\boldsymbol{\xi}_1 \times \boldsymbol{\xi}_2$ is a screw along $\mathbf{N}$, with pitch $d\cot(\theta)+(h_1+h_2)$ and magnitude s_θ (see also [35]). In particular, when $d=0, \theta=\frac{\pi}{2}$, the above equation simplifies to $\boldsymbol{\xi}_1 \times \boldsymbol{\xi}_2 = (1+\varepsilon(h_1+h_2))\mathbf{N}$ (see Fig. 6(b)). When $h_2=\infty$ or $h_1=\infty$, one has

$$\boldsymbol{\xi}_1 \times \boldsymbol{\xi}_2 = s_\theta \varepsilon \mathbf{N}$$

The commutator vanishes if both h_1 and h_2 are infinite.

3. Lie Product, Lie Triple Product and Lie Triple System

Definition 1. *The **Lie product** L of a screw system S is the vector space spanned by the commutator of all twists in S:*

$$L \triangleq [S,S] = \{\, \boldsymbol{\xi}_1 \times \boldsymbol{\xi}_2 \mid \forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in S \}\,.$$

By bilinearity of the commutator, if $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_k$ is a basis of S, then the twists $\boldsymbol{\xi}_i \times \boldsymbol{\xi}_j, i, j \in \{1, \ldots, k\}$ (though not necessarily linearly independent) span L.

Definition 2. *The **Lie triple product** T of a screw system S is the vector space spanned by the triple products of all twists in S:*

$$\begin{aligned} T &\triangleq [[S,S],S] = [L,S] = [S,L] \\ &= \{ (\boldsymbol{\xi}_1 \times \boldsymbol{\xi}_2) \times \boldsymbol{\xi}_3 \mid \forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3 \in S \} . \end{aligned}$$

By bilinearity of the commutator, if $\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_k$ is a basis of S, then the twists $(\boldsymbol{\xi}_i \times \boldsymbol{\xi}_j) \times \boldsymbol{\xi}_l, i, j, l \in \{1, \dots, k\}$ (though not necessarily linearly independent) span T. We say that S is a **Lie triple system** (LTS) if:

$$T = [[S,S],S] \subseteq S. \tag{8}$$

In general, it is shown in [28, 29] that the exponential $\exp(S)$ of a screw system S can be generated as in (1) if and only if S is a LTS. We shall give in the next section an exhaustive identification of all such screw systems. Note that Lie subalgebras are automatically closed under the triple product, and therefore are *trivial* LTSs. There are ten classes of Lie subalgebras, which correspond to ten classes of connected Lie subgroups of SE(3) [19, 5]. The Lie subalgebras are well known and, therefore, they will not be considered under the LTS framework. The following theorem concerning LTSs holds [30, 36]. The proof is reported here for illustrative purpose.

Theorem 1. *If S is a LTS, then $L = [S,S]$ is a Lie subalgebra of $\mathfrak{se}(3)$, i.e.:*

$$\forall \boldsymbol{\xi}_1 \times \boldsymbol{\xi}_2, \boldsymbol{\xi}_3 \times \boldsymbol{\xi}_4 \in L \quad \Rightarrow \quad (\boldsymbol{\xi}_1 \times \boldsymbol{\xi}_2) \times (\boldsymbol{\xi}_3 \times \boldsymbol{\xi}_4) \in L$$

Proof. Since S is a LTS, we have $T = [[S,S],S] \subseteq S$. Given two general elements $\boldsymbol{\xi}_1 \times \boldsymbol{\xi}_2, \boldsymbol{\xi}_3 \times \boldsymbol{\xi}_4$ of L, by skew-symmetry (2) and Jacobi identity (3), we have:

$$\begin{aligned} (\boldsymbol{\xi}_1 \times \boldsymbol{\xi}_2) \times (\boldsymbol{\xi}_3 \times \boldsymbol{\xi}_4) &= (\underbrace{(\boldsymbol{\xi}_3 \times \boldsymbol{\xi}_4) \times \boldsymbol{\xi}_2}_{\in S}) \times \boldsymbol{\xi}_1 \\ &\quad - (\underbrace{(\boldsymbol{\xi}_3 \times \boldsymbol{\xi}_4) \times \boldsymbol{\xi}_1}_{\in S}) \times \boldsymbol{\xi}_2 \in L. \end{aligned}$$

Therefore, $L = [S,S]$ is closed under Lie product and is a Lie subalgebra. □

Note that Theorem 1 is relevant for screw systems with dimension three or higher, since the Lie product of a two-system is, in general, a one-system and, thus, is automatically a Lie subalgebra. We also emphasize that Theorem 1 provides a *necessary* condition for a screw system to be LTS. Hence, it may be used to simplify the identification process for 3 to 5-D LTSs, by allowing us to rule out screw systems that are not LTSs without computing their triple product.

4. Identification of Lie Triple Screw Systems

In this section, we shall verify the triple product closure condition (2) for all screw systems of $\mathfrak{se}(3)$ using Hunt's classification (up to conjugation) of general and special screw systems [2], thereby exhaustively identifying all Lie triple systems.

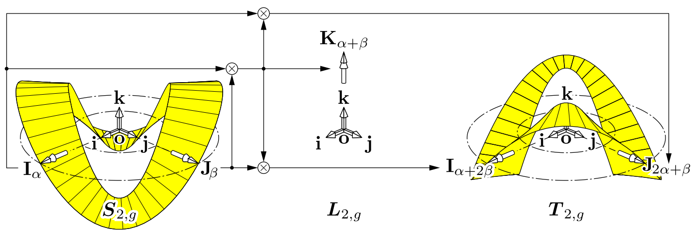


Figure 7: Lie product and triple product of a general two-system $S_{2,g}$.

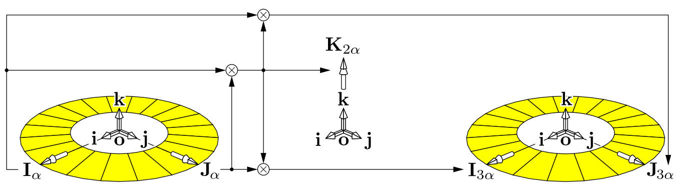


Figure 8: Lie product and triple product of the first special two-system $S_{2,1}$.

4.1. Two-systems

4.1.1. $S_{2,g}$ and $S_{2,1}$

In order to investigate whether a general two-system $S_{2,g}$ with basis pitches α and β, $\alpha \geq \beta$, is a LTS, we choose the reference point $\mathbf{o}$ to be the center of the system, and the basis (6) so that the basis screws lie on $\mathbf{I}$ and $\mathbf{J}$ (denoted $\mathbf{I}_\alpha$ and $\mathbf{J}_\beta$, respectively). It is clear from (7) that the Lie product $L_{2,g}$ is spanned by a single screw along $\mathbf{K}$ with pitch $\alpha + \beta$, namely

$$\mathbf{I}_\alpha \times \mathbf{J}_\beta = (1 + \varepsilon(\alpha + \beta))\mathbf{K} = \mathbf{K}_{\alpha+\beta}.$$

The triple product $T_{2,g}$ is the following two-system:

$$T_{2,g} = \mathrm{span}(\mathbf{I}_{\alpha+2\beta}, \mathbf{J}_{2\alpha+\beta})$$

where

$$\mathbf{I}_{\alpha+2\beta} = \mathbf{J}_\beta \times \mathbf{K}_{\alpha+\beta}, \quad \mathbf{J}_{2\alpha+\beta} = \mathbf{K}_{\alpha+\beta} \times \mathbf{I}_\alpha.$$

$T_{2,g}$ shares the same center and basis lines as $S_{2,g}$ (see Fig. 7). Accordingly, the LTS condition (8) , namely the former is contained in the latter, only when their basis pitches are identical, namely

$$\begin{cases} \alpha + 2\beta = \alpha \\ 2\alpha + \beta = \beta \end{cases} \quad \Leftrightarrow \quad \alpha = \beta = 0$$

which leads to the first special two-system $S^0_{2,1}$ with two zero basis pitches (see Fig. 8).

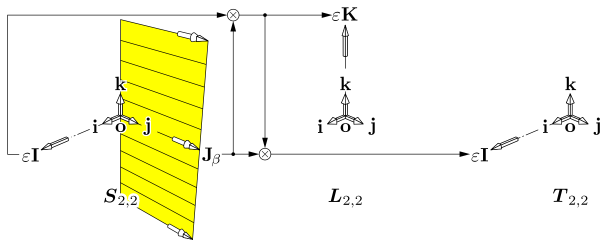


Figure 9: Lie product and triple product of the second special two-system $S_{2,2}$.

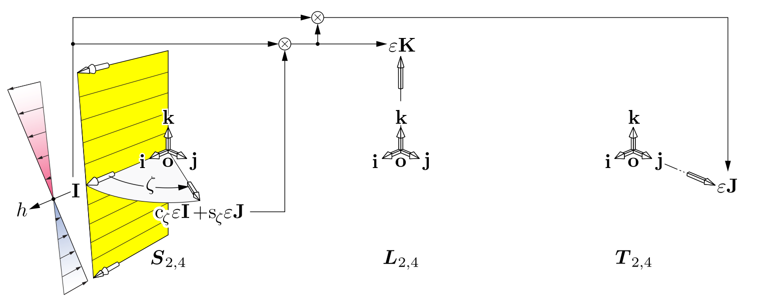


Figure 10: Lie product and triple product of the fourth special two-system $S_{2,4}$.

The above computation illustrates the efficacy of Hunt's classification in comparison to Gibson's subtypes, which may be collectively verified by the same equation. It is interesting to note that the cylindroid of $T_{2,g}$ has the same length as that of $S_{2,g}$, but it is rotated 90° counter-clockwise about the $\mathbf{K}$ axis. By this geometric reasoning alone, we may see that $S_{2,g}$ is not a LTS.

4.1.2. $S_{2,2}$ and $S_{2,3}$

When $\alpha = \infty, \beta \neq \infty$, then $S_{2,2} = \text{span}(\varepsilon\mathbf{I}, \mathbf{J}_\beta)$, and it is easy to verify that:

$$L_{2,2} = \text{span}(\varepsilon\mathbf{K}), \quad T_{2,2} = \text{span}(\varepsilon\mathbf{I}) \subset S_{2,2}.$$

Therefore, $S_{2,2}$ is a LTS (see Fig. 9)[6]. When $\alpha = \beta = \infty$, we end up with the third special two-system $S_{2,3} = \text{span}(\varepsilon\mathbf{I}, \varepsilon\mathbf{J})$, namely the planar translation algebra.

4.1.3. $S_{2,4}$ and $S_{2,5}$

Let $\beta = -k^2\alpha$ for some positive real number $k \neq 1$ as $\alpha \to \infty$. Then, shift the reference point $\mathbf{o}$ as shown in [2], and choose an appropriate basis $\mathbf{i}, \mathbf{j}, \mathbf{k}$ so that $S_{2,4}$ is given by

$$S_{2,4} = \text{span}(\mathbf{I}, c_\zeta\varepsilon\mathbf{I} + s_\zeta\varepsilon\mathbf{J})$$

[6]Note that, here, β can either be 0 or take a finite non-zero real value, thus giving rise to two classes of LTSs if the Gibson-Hunt classification [20, 18] is adopted.

	$S_{2,j}$		$L_{2,j}$		$T_{2,j}$		
	basis	condition	basis	type	basis	type	
$S_{2,g}$	$\mathbf{I}_\alpha, \mathbf{J}_\beta$	$\alpha \neq \beta$	$\mathbf{K}_{\alpha+\beta}$	$S_{1,g}$	$\mathbf{I}_{\alpha+2\beta}, \mathbf{J}_{2\alpha+\beta}$	$S_{2,g}$	
$S_{2,1}$	$\mathbf{I}_\alpha, \mathbf{J}_\alpha$	$\alpha \neq 0$	$\mathbf{K}_{2\alpha}$	$S_{1,g}$	$\mathbf{I}_{3\alpha}, \mathbf{J}_{3\alpha}$	$S_{2,1}$	
$S^0_{2,1}$	$\mathbf{I}, \mathbf{J}$		$\mathbf{K}$	$S_{1,g}$	$\mathbf{I}, \mathbf{J}$	$S^0_{2,1}$	LTS
$S_{2,2}$	$\varepsilon\mathbf{I}, \mathbf{J}_\beta$		$\varepsilon\mathbf{K}$	$S_{1,s}$	$\varepsilon\mathbf{I}$	$S_{1,s}$	LTS
$S_{2,3}$	$\varepsilon\mathbf{I}, \varepsilon\mathbf{J}$		N/A	N/A	N/A	N/A	subalg.
$S_{2,4}$	$\mathbf{I}, c_\zeta\varepsilon\mathbf{I} + s_\zeta\varepsilon\mathbf{J}$	$\zeta \notin \{0, \pi/2\}$	$\varepsilon\mathbf{I}$	$S_{1,s}$	$\varepsilon\mathbf{K}$	$S_{1,s}$	
$S_{2,5}$	$\mathbf{I}, \varepsilon\mathbf{I}$		N/A	N/A	N/A	N/A	subalg.

Table 1: Lie products and Lie triple products of 2-systems. All subalgebras are highlighted with background shading.

with $\cot\zeta = 2k/(k^2-1), \zeta \notin \left\{0, \frac{\pi}{2}\right\}$. It is straightforward to see that (Fig. 10):

$$L_{2,4} = \mathrm{span}(\varepsilon\mathbf{K}), \quad T_{2,4} = \mathrm{span}(\varepsilon\mathbf{J}).$$

$S_{2,4}$ is not a LTS since $c_\zeta\varepsilon\mathbf{I} + s_\zeta\varepsilon\mathbf{J}$ is the only infinite-pitch screw in $S_{2,4}$ but is not aligned with $\varepsilon\mathbf{J}$, and therefore $T_{2,4} \nsubseteq S_{2,4}$.

The fifth special two-system emerges by setting $k = 1$, namely $\zeta = 0$. In this case, an appropriate basis (6) may be chosen so that:

$$S_{2,5} = \mathrm{span}(\mathbf{I}, \varepsilon\mathbf{I}).$$

$S_{2,5}$ is the cylindrical algebra, and it comprises screws of all pitches along $\mathbf{I}$.

In summary, we have identified two two-systems as nontrivial LTSs, namely the first special two-system $S^0_{2,1} = \mathrm{span}(\mathbf{I}, \mathbf{J})$, comprising a pencil of converging zero-pitch screws, and the second special two-system $S_{2,2} = \mathrm{span}(\varepsilon\mathbf{I}, \mathbf{J}_\beta)$, comprising a planar pencil of parallel equal-pitch screws and an infinite-pitch screw perpendicular to the pencil plane. All two-systems, along with their Lie product and Lie triple product, are listed in Table 1 for further reference.

4.2. Three-systems

4.2.1. Lie subalgebras

It is well known that the following three-systems are Lie subalgebras of $\mathfrak{se}(3)$:

- $S^0_{3,2}$ with three zero-pitch basis screws, i.e. the spherical algebra, comprising all zero-pitch screws passing through a point (see Fig. 12 with $\alpha = 0$);
- $S_{3,5}$, i.e. the planar (or helicoidal planar) algebra of all screws of equal pitch parallel to a common direction, as well as infinite-pitch screws perpendicular to this direction (see Fig. 13(b));
- $S_{3,6}$, i.e. the spatial translation algebra of infinite-pitch screws along all directions in space.

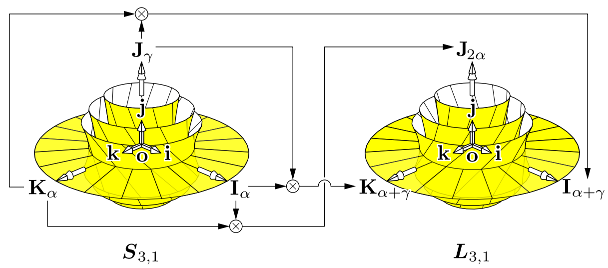


Figure 11: The first special three-system $S_{3,1}$ and its Lie product $L_{3,1}$.

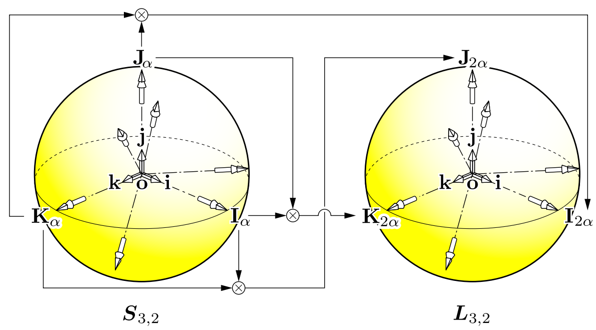


Figure 12: The second special three-system $S_{3,2}$ and its Lie product $L_{3,2}$.

4.2.2. $S_{3,g}$, $S_{3,1}$ and $S_{3,2}$

Given a general three-system $S_{3,g}$ (see Fig. 5) with finite basis pitches α, β and γ, $\alpha \geq \beta \geq \gamma$, we choose the reference point $\mathbf{o}$ to be the center of the system, and the basis (6) so that the basis screws lie on $\mathbf{K}, \mathbf{I}, \mathbf{J}$ respectively (denoted $\mathbf{K}_\alpha, \mathbf{I}_\beta, \mathbf{J}_\gamma$ respectively).

If $\alpha = \beta$ or $\beta = \gamma$, $S_{3,g}$ becomes a first special three-system $S_{3,1}$, as illustrated in Fig. 11. Equal-pitch screws with $h \neq \alpha$ form ∞^1 reguli lying on a pencil of coaxial hyperboloids of revolution. Screws with pitch α form a planar pencil located at the reference point $\mathbf{o}$ in the $\mathbf{ki}$ plane. The Lie product $L_{3,1}$ is also a $S_{3,1}$ system, with basis pitches 2α and $\alpha + \gamma$. In this case, equal-pitch screws with $h \neq \alpha + \gamma$ lie on the same hyperboloids as those of the original $S_{3,1}$ system, but on the complementary reguli.

If $\alpha = \beta = \gamma$, $S_{3,g}$ becomes a second special three-system $S_{3,2}$, as illustrated in Fig. 12. $S_{3,2}$ comprises a bundle of equal-pitch screws. Its Lie product $L_{3,2}$ is also a $S_{3,2}$ system with a bundle of equal-pitch screws with pitch twice as that of the original system.

The Lie product $L_{3,g}$ is also a 3-system sharing the same center and basis lines with $S_{3,g}$, but with finite basis pitches $\beta + \gamma, \alpha + \gamma, \alpha + \beta$:

$$L_{3,g} = \mathrm{span}(\mathbf{K}_{\beta+\gamma}, \mathbf{I}_{\alpha+\gamma}, \mathbf{J}_{\alpha+\beta}).$$

According to Theorem 1, $S_{3,g}$ is a LTS only if (meaning necessarily but not

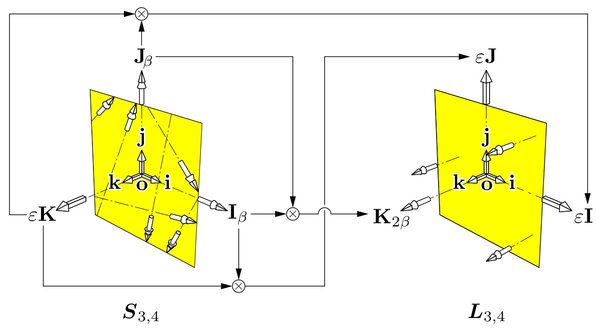


Figure 13: The fourth special three-system $S_{3,4}$ and its Lie product $L_{3,4}$.

sufficiently) $L_{3,g}$ is one of the Lie subalgebras listed in Sec. 4.2.1. Since both $S_{3,5}$ and $S_{3,6}$ have one or more infinite basis pitches, the only possible case is $L_{3,g} = S^0_{3,2}$ which is true if and only if

$$\beta + \gamma = \alpha + \gamma = \alpha + \beta = 0 \quad \Leftrightarrow \quad \alpha = \beta = \gamma = 0$$

namely, if all basis pitches of the original system $S_{3,g}$ are zero. Therefore, $S_{3,g}$ is a LTS only if it coincides with the spherical algebra $S^0_{3,2}$, thus being a trivial LTS. The same reasoning applies to $S_{3,1}$.

4.2.3. $S_{3,3}$ and $S_{3,4}$

When one basis pitch is infinite, say $\alpha = \infty$, i.e.

$$S_{3,3} = \mathrm{span}(\varepsilon\mathbf{K}, \mathbf{I}_\beta, \mathbf{J}_\gamma),$$

we have

$$L_{3,3} = \mathrm{span}(\mathbf{K}_{\beta+\gamma}, \varepsilon\mathbf{I}, \varepsilon\mathbf{J})$$

which is the Lie subalgebra $S_{3,5}$. The triple product is given by

$$T_{3,3} = \mathrm{span}(\varepsilon\mathbf{K}, \mathbf{I}_{\beta+2\gamma}, \mathbf{J}_{2\beta+\gamma}).$$

The three-system is a LTS if and only if:

$$\begin{cases} \beta + 2\gamma = \beta, \\ 2\beta + \gamma = \gamma. \end{cases} \quad \Leftrightarrow \quad \beta = \gamma = 0.$$

Therefore, only $S^0_{3,4}$ (with $\beta = \gamma = 0$) is a LTS. The fourth special three-system $S_{3,4}$ and its Lie product $L_{3,4}$ are shown in Fig. 13. In particular, $S^0_{3,4}$ consists of zero-pitch screws lying on a field of lines and an infinite-pitch screw perpendicular to the field plane. Its Lie product $L^0_{3,4}$ is the planar algebra $S^0_{3,5}$.

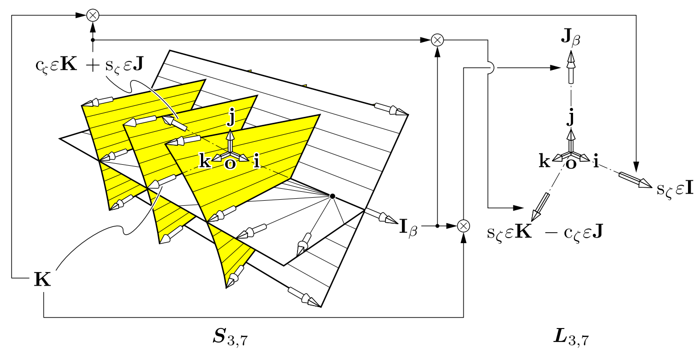


Figure 14: The seventh special three-system $S_{3,7}$ and its Lie product $L_{3,7}$.

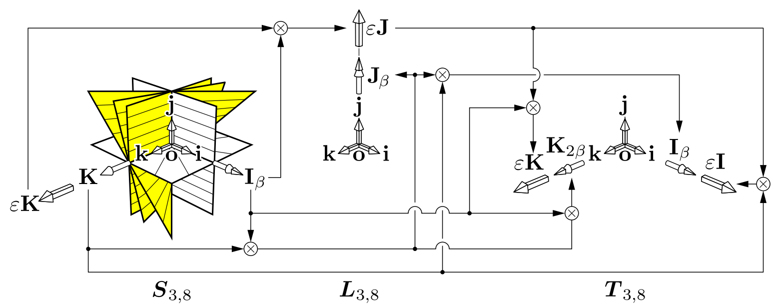


Figure 15: The eighth special three-system $S_{3,8}$, its Lie product $L_{3,8}$ and triple product $T_{3,8}$.

4.2.4. $S_{3,7}$ and $S_{3,8}$

Let $\gamma = -k^2\alpha$ for some positive real number $k \neq 1$ as $\alpha \to \infty$, while β remains finite. As with $S_{2,4}$ and $S_{2,5}$, conveniently shift the reference point $\mathbf{o}$ and choose a basis $\mathbf{i}, \mathbf{j}, \mathbf{k}$ so that $S_{3,7}$ is given by (see Fig. 14):

$$S_{3,7} = \mathrm{span}(\mathbf{K}, c_\zeta\varepsilon\mathbf{K} + s_\zeta\varepsilon\mathbf{J}, \mathbf{I}_\beta).$$

with $\cot\zeta = 2k/(k^2-1), \zeta \neq 0$. Equal-pitch screws with $h \neq \beta$ form ∞^1 reguli lying on a pencil of hyperbolic paraboloids. The screws of pitch β lie on a degenerate regulus consisting of two line pencils, one consisting of concurrent lines and the other consisting of coplanar parallel lines.

The fact that:

$$L_{3,7} = \mathrm{span}(s_\zeta\varepsilon\mathbf{K} - c_\zeta\varepsilon\mathbf{J}, \varepsilon\mathbf{I}, \mathbf{J}_\beta).$$

fails to be a Lie subalgebra (it is instead a ninth special three-system $S_{3,9}$) rules out the possibility for $S_{3,7}$ to be a LTS.

The eighth special three-system emerges by setting $k = 1$, i.e. $\zeta = 0$. We can choose an appropriate basis (6) so that $S_{3,8}$ is given by (see Fig. 15):

$$S_{3,8} = \mathrm{span}(\mathbf{K}, \varepsilon\mathbf{K}, \mathbf{I}_\beta).$$

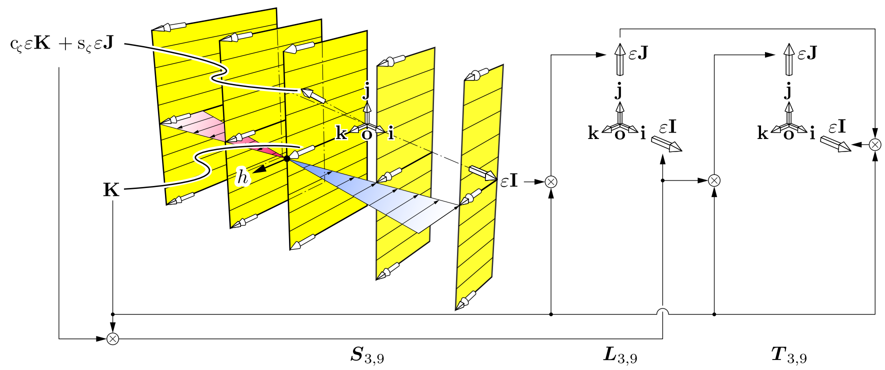


Figure 16: The ninth special three-system $S_{3,9}$, its Lie product $L_{3,9}$ and triple product $T_{3,9}$.

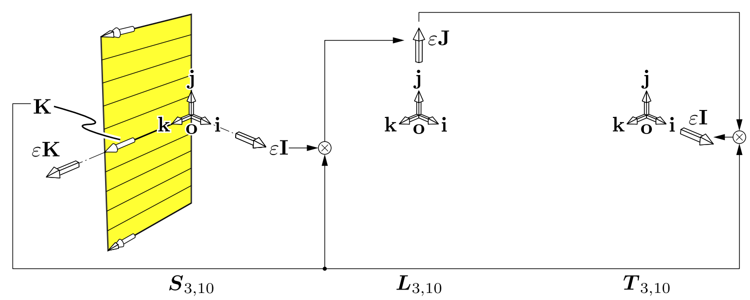


Figure 17: The tenth special three-system $S_{3,10}$, its Lie product $L_{3,10}$ and triple product $T_{3,10}$.

Equal-pitch screws with $h \neq \beta$ form ∞^1 reguli lying on a pencil of hyperbolic paraboloids, which have one line in common. The screws of pitch β lie on a degenerate regulus consisting of two perpendicular line pencils, one consisting of concurrent lines and the other consisting of coplanar parallel lines.

In this case, $L_{3,8}$ and $T_{3,8}$ are given by

$$\begin{cases} L_{3,8} = \mathrm{span}(\mathbf{J}_\beta, \varepsilon\mathbf{J}) = \mathrm{span}(\mathbf{J}, \varepsilon\mathbf{J}) \\ T_{3,8} = \mathrm{span}(\mathbf{K}_{2\beta}, \varepsilon\mathbf{K}, \mathbf{I}, \varepsilon\mathbf{I}) = \mathrm{span}(\mathbf{K}, \varepsilon\mathbf{K}, \mathbf{I}, \varepsilon\mathbf{I}) \end{cases}$$

$T_{3,8}$ is the fifth special four-system $S_{4,5}$, with dimension exceeding that of $S_{3,8}$. Thus $T_{3,8} \not\subseteq S_{3,8}$ and $S_{3,8}$ is not a LTS.

4.2.5. $S_{3,9}$ and $S_{3,10}$

Using the same bases adopted for $S_{3,7}$ and $S_{3,8}$, we let $\beta = \infty$. When $\zeta \neq 0$, we have $S_{3,9}$, which is given by (see Fig. 16):

$$S_{3,9} = \mathrm{span}(\mathbf{K}, c_\zeta \varepsilon\mathbf{K} + s_\zeta \varepsilon\mathbf{J}, \varepsilon\mathbf{I})$$

Aside from a planar pencil of infinite-pitch screws spanned by $\varepsilon\mathbf{I}$ and $c_\zeta\varepsilon\mathbf{K} + s_\zeta\varepsilon\mathbf{J}$, finite equal-pitch screws of $S_{3,9}$ form ∞^1 parallel planar pencils lying on

	$S_{3,j}$		$L_{3,j}$		$T_{3,j}$		
	basis	condition	basis	type	basis	type	
$S_{3,g}$	$\mathbf{K}_\alpha, \mathbf{I}_\beta, \mathbf{J}_\gamma$	$\alpha > \beta > \gamma$	$\mathbf{K}_{\beta+\gamma}, \mathbf{I}_{\alpha+\gamma}, \mathbf{J}_{\alpha+\beta}$	$S_{3,g}$	$\mathbf{K}, \mathbf{I}, \mathbf{J}, \varepsilon\mathbf{K}, \varepsilon\mathbf{I}, \varepsilon\mathbf{J}$	$\mathfrak{se}(3)$	
$S_{3,1}$	$\mathbf{K}_\alpha, \mathbf{I}_\alpha, \mathbf{J}_\gamma$	$\alpha \neq \gamma$	$\mathbf{K}_{\alpha+\gamma}, \mathbf{I}_{\alpha+\gamma}, \mathbf{J}_{2\alpha}$	$S_{3,1}$	$\mathbf{K}, \mathbf{I}, \mathbf{J}_{2\alpha+\gamma}, \varepsilon\mathbf{K}, \varepsilon\mathbf{I}$	$S_{5,g}$	
$S_{3,2}$	$\mathbf{K}_\alpha, \mathbf{I}_\alpha, \mathbf{J}_\alpha$	$\alpha \neq 0$	$\mathbf{K}_{2\alpha}, \mathbf{I}_{2\alpha}, \mathbf{J}_{2\alpha}$	$S_{3,1}$	$\mathbf{K}_{3\alpha}, \mathbf{I}_{3\alpha}, \mathbf{J}_{3\alpha}$	$S_{3,2}$	
$S^0_{3,2}$	$\mathbf{K}, \mathbf{I}, \mathbf{J}$		$\mathbf{K}, \mathbf{I}, \mathbf{J}$	$S^0_{3,2}$	$\mathbf{K}, \mathbf{I}, \mathbf{J}$	$S^0_{3,2}$	subalg.
$S_{3,3}$	$\varepsilon\mathbf{K}, \mathbf{I}_\beta, \mathbf{J}_\gamma$	$\beta \neq \gamma$	$\mathbf{K}_{\beta+\gamma}, \varepsilon\mathbf{I}, \varepsilon\mathbf{J}$	$S_{3,5}$	$\varepsilon\mathbf{K}, \mathbf{I}_{\beta+2\gamma}, \mathbf{J}_{2\beta+\gamma}$	$S_{3,3}$	
$S_{3,4}$	$\varepsilon\mathbf{K}, \mathbf{I}_\beta, \mathbf{J}_\beta$		$\mathbf{K}_{2\beta}, \varepsilon\mathbf{I}, \varepsilon\mathbf{J}$	$S_{3,5}$	$\varepsilon\mathbf{K}, \mathbf{I}_{3\beta}, \mathbf{J}_{3\beta}$	$S_{3,4}$	
$S^0_{3,4}$	$\varepsilon\mathbf{K}, \mathbf{I}, \mathbf{J}$		$\mathbf{K}, \varepsilon\mathbf{I}, \varepsilon\mathbf{J}$	$S_{3,5}$	$\varepsilon\mathbf{K}, \mathbf{I}, \mathbf{J}$	$S^0_{3,4}$	LTS
$S_{3,5}$	$\varepsilon\mathbf{K}, \varepsilon\mathbf{I}, \mathbf{J}_\gamma$		$\varepsilon\mathbf{K}, \varepsilon\mathbf{I}$	$S_{2,3}$	$\varepsilon\mathbf{K}, \varepsilon\mathbf{I}$	$S_{2,3}$	subalg.
$S_{3,6}$	$\varepsilon\mathbf{K}, \varepsilon\mathbf{I}, \varepsilon\mathbf{J}$		N/A	N/A	N/A	N/A	subalg.
$S_{3,7}$	$\mathbf{K}, c_\zeta\varepsilon\mathbf{K} + s_\zeta\varepsilon\mathbf{J}, \mathbf{I}_\beta$		$s_\zeta\varepsilon\mathbf{K} - c_\zeta\varepsilon\mathbf{J}, \varepsilon\mathbf{I}, \mathbf{J}_\beta$	$S_{3,9}$	$\mathbf{K}, \mathbf{I}, \varepsilon\mathbf{K}, \varepsilon\mathbf{I}, \varepsilon\mathbf{J}$	$S_{5,s}$	
$S_{3,8}$	$\mathbf{K}, \varepsilon\mathbf{K}, \mathbf{I}_\beta$		$\mathbf{J}, \varepsilon\mathbf{J}$	$S_{2,5}$	$\mathbf{K}$,$\mathbf{I}$,$\varepsilon\mathbf{K}$,$\varepsilon\mathbf{I}$	$S_{4,5}$	
$S_{3,9}$	$\mathbf{K}, c_\zeta\varepsilon\mathbf{K} + s_\zeta\varepsilon\mathbf{J}, \varepsilon\mathbf{I}$		$\varepsilon\mathbf{I}, \varepsilon\mathbf{J}$	$S_{2,3}$	$\varepsilon\mathbf{I}, \varepsilon\mathbf{J}$	$S_{2,3}$	
$S_{3,10}$	$\mathbf{K}, \varepsilon\mathbf{K}, \varepsilon\mathbf{I}$		$\varepsilon\mathbf{J}$	$S_{1,s}$	$\varepsilon\mathbf{I}$	$S_{1,s}$	LTS

Table 2: Lie products and Lie triple products of 3-systems. All subalgebras are highlighted with background shading.

∞^1 parallel planes perpendicular to $\mathbf{i}$. It is easy to verify that:

$$L_{3,9} = \mathrm{span}(\varepsilon\mathbf{I}, \varepsilon\mathbf{J}), \quad T_{3,9} = \mathrm{span}(\varepsilon\mathbf{I}, \varepsilon\mathbf{J}) \not\subseteq S_{3,9}$$

so that $S_{3,9}$ is not a LTS.

The tenth special three-system arises by setting $k = 1$ ($\zeta = 0$; see Fig. 17):

$$S_{3,10} = \mathrm{span}(\mathbf{K}, \varepsilon\mathbf{K}, \varepsilon\mathbf{I}).$$

Aside from a planar pencil of infinite-pitch screws spanned by $\varepsilon\mathbf{I}$ and $\varepsilon\mathbf{K}$, $S_{3,10}$ comprises screws of arbitrary finite pitch lying on parallel lines of a planar pencil perpendicular to $\mathbf{i}$. It is easy to verify that:

$$L_{3,10} = \{\varepsilon\mathbf{J}\}, \quad T_{3,10} = \{\varepsilon\mathbf{I}\} \subset S_{3,10}.$$

and therefore $S_{3,10}$ is a LTS.

In summary, we have identified two three-systems as nontrivial LTSs, namely the fourth special three-system $S^0_{3,4} = \mathrm{span}(\varepsilon\mathbf{K}, \mathbf{I}, \mathbf{J})$, comprising a field of zero-pitch screws, and the tenth special three-system $S_{3,10} = \mathrm{span}(\varepsilon\mathbf{K}, \mathbf{K}, \varepsilon\mathbf{I})$, comprising a planar pencil of parallel screws with arbitrary pitches. All the three-systems along with their Lie product and triple product are listed in Table 2.

4.3. Four-systems

4.3.1. Lie subalgebras

Only one four-system, namely $S_{4,3}$:

$$S_{4,3} = \mathrm{span}(\varepsilon\mathbf{I}, \varepsilon\mathbf{J}, \mathbf{K}, \varepsilon\mathbf{K})$$

is a Lie subalgebra of $\mathfrak{se}(3)$, known as the Schönflies algebra [19, 5]. It consists of all finite-pitch screws along lines parallel to a common direction and infinite-pitch screws along any direction in space.

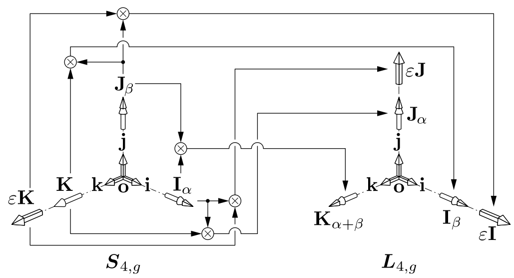


Figure 18: The general four-system $S_{4,g}$ and its Lie product $L_{4,g}$.

4.3.2. $S_{4,g}$ and $S_{4,1}$

Any four-system is characterized by a reciprocal two-system, whose reciprocal basis screws have pitches $-\alpha$ and $-\beta$. In the case of $S_{4,g}$ and $S_{4,1}$, we choose the reference point $\mathbf{o}$ to be the center point of the corresponding reciprocal two-system, and an appropriate basis (6) such that $S_{4,g}$ is given by (see Fig. 18):

$$S_{4,g} = \mathrm{span}(\mathbf{I}_\alpha, \mathbf{J}_\beta, \mathbf{K}, \varepsilon\mathbf{K}).$$

As long as α, β are finite, both in the case of $\alpha \neq \beta$ and in the case $\alpha = \beta$, the Lie product:

$$\begin{aligned} L_{4,g} &= \mathrm{span}(\mathbf{I}_\beta, \varepsilon\mathbf{I}, \mathbf{J}_\alpha, \varepsilon\mathbf{J}, \mathbf{K}_{\alpha+\beta}) \\ &= \mathrm{span}(\mathbf{I}, \varepsilon\mathbf{I}, \mathbf{J}, \varepsilon\mathbf{J}, \mathbf{K}_{\alpha+\beta}) \end{aligned}$$

is a 5-D vector subspace and cannot be a Lie subalgebra. Therefore, neither $S_{4,g}$ nor $S_{4,1}$ is a LTS.

4.3.3. $S_{4,2}$

When the basis pitch $\alpha = \infty$, we have the second special four-system $S_{4,2}$ (see Fig. 19):

$$S_{4,2} = \mathrm{span}(\varepsilon\mathbf{I}, \mathbf{J}_\beta, \mathbf{K}, \varepsilon\mathbf{K}).$$

The Lie product:

$$L_{4,2} = \mathrm{span}(\mathbf{I}_\beta, \varepsilon\mathbf{I}, \varepsilon\mathbf{J}, \varepsilon\mathbf{K}) = \mathrm{span}(\mathbf{I}, \varepsilon\mathbf{I}, \varepsilon\mathbf{J}, \varepsilon\mathbf{K})$$

is the Schönflies algebra. However, the triple product

$$T_{4,2} = \mathrm{span}(\varepsilon\mathbf{I}, \mathbf{J}, \varepsilon\mathbf{J}, \mathbf{K}_\beta, \varepsilon\mathbf{K}) = \mathrm{span}(\varepsilon\mathbf{I}, \mathbf{J}, \varepsilon\mathbf{J}, \mathbf{K}, \varepsilon\mathbf{K})$$

is a special five-system; then $T_{4,2} \not\subseteq S_{4,2}$. Therefore, $S_{4,2}$ is not a LTS.

4.3.4. $S_{4,4}$ and $S_{4,5}$

Let the basis pitches satisfy $\beta = -k^2\alpha$ for some positive number $k \neq 1$ as $\alpha \to \infty$. Shift the reference point $\mathbf{o}$ and choose an appropriate basis $\mathbf{i}, \mathbf{j}, \mathbf{k}$ such that $S_{4,4}$ is given by (see Fig. 20)

$$S_{4,4} = \mathrm{span}(\mathbf{I}, c_\zeta\varepsilon\mathbf{I} + s_\zeta\varepsilon\mathbf{J}, \mathbf{K}, \varepsilon\mathbf{K})$$

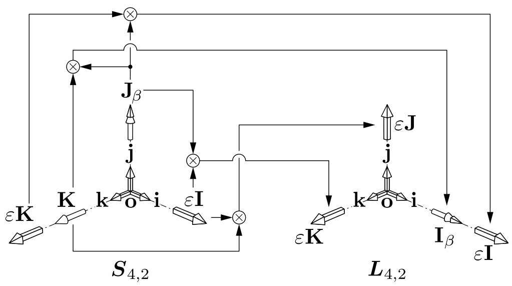


Figure 19: The second special four-system $S_{4,2}$ and its Lie product $L_{4,2}$.

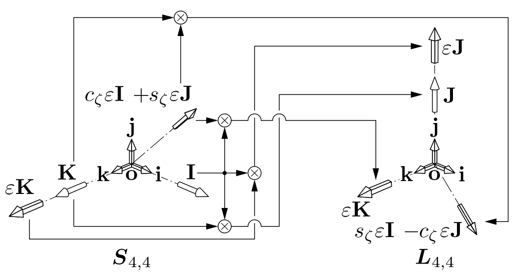


Figure 20: The fourth special four-system $S_{4,4}$ and its Lie product $L_{4,4}$.

with $\cot\zeta = 2k/(k^2-1), \zeta \neq 0$. We have:

$$\begin{aligned} L_{4,4} &= \operatorname{span}(s_\zeta\varepsilon\mathbf{I} - c_\zeta\varepsilon\mathbf{J}, \mathbf{J}, \varepsilon\mathbf{J}, \varepsilon\mathbf{K}) \\ &= \operatorname{span}(\varepsilon\mathbf{I}, \mathbf{J}, \varepsilon\mathbf{J}, \varepsilon\mathbf{K}) \end{aligned}$$

which is a Schönflies algebra, and

$$T_{4,4} = \operatorname{span}(\mathbf{I}, \varepsilon\mathbf{I}, \varepsilon\mathbf{J}, \mathbf{K}, \varepsilon\mathbf{K})$$

which is a special five-system. Since $T_{4,4} \not\subseteq S_{4,4}$, $S_{4,4}$ is not a LTS.

The fifth special four-system arises by letting $k = 1$ (and therefore $\zeta = 0$; see Fig. 21)

$$S_{4,5} = \operatorname{span}(\mathbf{I}, \varepsilon\mathbf{I}, \mathbf{J}, \varepsilon\mathbf{J})$$

$$L_{4,5} = \operatorname{span}(\mathbf{K}, \varepsilon\mathbf{K})$$

and

$$T_{4,5} = \operatorname{span}(\mathbf{I}, \varepsilon\mathbf{I}, \mathbf{J}, \varepsilon\mathbf{J}) = S_{4,5}.$$

Therefore $S_{4,5}$ is a LTS.

All four-systems along with their Lie product and triple product are listed in Table 3.

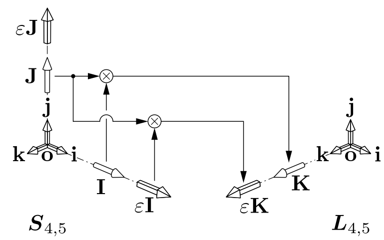


Figure 21: The fifth special four-system $S_{4,5}$ and its Lie product $L_{4,5}$.

	$S_{4,j}$		$L_{4,j}$		$T_{4,j}$		
	basis	condition	basis	type	basis	type	
$S_{4,g}$	$\mathbf{I}_\alpha, \mathbf{J}_\beta, \mathbf{K}, \varepsilon\mathbf{K}$	$\alpha \neq \beta$	$\mathbf{I}, \mathbf{J}, \mathbf{K}_{\alpha+\beta}, \varepsilon\mathbf{I}, \varepsilon\mathbf{J}$	$S_{5,g}$	$\mathbf{I}, \mathbf{J}, \mathbf{K}, \varepsilon\mathbf{I}, \varepsilon\mathbf{J}, \varepsilon\mathbf{K}$	$\mathfrak{se}(3)$	
$S_{4,1}$	$\mathbf{I}_\alpha, \mathbf{J}_\alpha, \mathbf{K}, \varepsilon\mathbf{K}$		$\mathbf{I}, \mathbf{J}, \mathbf{K}_{2\alpha}, \varepsilon\mathbf{I}, \varepsilon\mathbf{J}$	$S_{5,g}$	$\mathbf{I}, \mathbf{J}, \mathbf{K}, \varepsilon\mathbf{I}, \varepsilon\mathbf{J}, \varepsilon\mathbf{K}$	$\mathfrak{se}(3)$	
$S_{4,2}$	$\varepsilon\mathbf{I}, \mathbf{J}_\beta, \mathbf{K}, \varepsilon\mathbf{K}$		$\mathbf{I}, \varepsilon\mathbf{I}, \varepsilon\mathbf{J}, \varepsilon\mathbf{K}$	$S_{4,3}$	$\mathbf{J}, \mathbf{K}, \varepsilon\mathbf{I}, \varepsilon\mathbf{J}, \varepsilon\mathbf{K}$	$S_{5,s}$	
$S_{4,3}$	$\varepsilon\mathbf{I}, \varepsilon\mathbf{J}, \mathbf{K}, \varepsilon\mathbf{K}$		$\varepsilon\mathbf{I}, \varepsilon\mathbf{J}$	$S_{2,3}$	$\varepsilon\mathbf{I}, \varepsilon\mathbf{J}$	$S_{2,3}$	subalg.
$S_{4,4}$	$\mathbf{I}, c_\zeta\varepsilon\mathbf{I} + s_\zeta\varepsilon\mathbf{J}, \mathbf{K}, \varepsilon\mathbf{K}$	$\zeta \notin \{0, \pi/2\}$	$\mathbf{J}, \varepsilon\mathbf{I}, \varepsilon\mathbf{J}, \varepsilon\mathbf{K}$	$S_{4,3}$	$\mathbf{I}, \mathbf{K}, \varepsilon\mathbf{I}, \varepsilon\mathbf{J}, \varepsilon\mathbf{K}$	$S_{5,s}$	
$S_{4,5}$	$\mathbf{I}, \mathbf{J}, \varepsilon\mathbf{I}, \varepsilon\mathbf{J}$		$\mathbf{K}, \varepsilon\mathbf{K}$	$S_{2,5}$	$\mathbf{I}, \mathbf{J}, \varepsilon\mathbf{I}, \varepsilon\mathbf{J}$	$S_{4,5}$	LTS

Table 3: Lie products and Lie triple products of 4-systems. All subalgebras are highlighted with background shading.

4.4. Five-systems

4.4.1. $S_{5,g}$

A general five-system $S_{5,g}$ is reciprocal to a one-system with pitch $-\alpha$. Choose the reference point $\mathbf{o}$ to be a point on the axis of the system, and an appropriate basis $\mathbf{i}, \mathbf{j}, \mathbf{k}$ such that $S_{5,g}$ is given by:

$$S_{5,g} = \mathrm{span}(\mathbf{I}, \varepsilon\mathbf{I}, \mathbf{J}, \varepsilon\mathbf{J}, \mathbf{K}_\alpha).$$

It is straightforward to verify that:

$$L_{5,g} = \mathrm{span}(\mathbf{I}_\alpha, \varepsilon\mathbf{J}, \mathbf{J}_\alpha, \varepsilon\mathbf{I}, \mathbf{K}, \varepsilon\mathbf{K}) = \mathfrak{se}(3).$$

However, $T_{5,g} = [[S_{5,g}, S_{5,g}], S_{5,g}] = [\mathfrak{se}(3), S_{5,g}] = \mathfrak{se}(3) \nsubseteq S_{5,g}$. Accordingly, $S_{5,g}$ is not a LTS.

4.4.2. $S_{5,s}$

The special five-system $S_{5,s}$ is given by

$$S_{5,s} = \mathrm{span}(\mathbf{I}, \varepsilon\mathbf{I}, \mathbf{J}, \varepsilon\mathbf{J}, \varepsilon\mathbf{K}).$$

It is straightforward to verify that

$$L_{5,s} = \mathrm{span}(\varepsilon\mathbf{I}, \varepsilon\mathbf{J}, \mathbf{K}, \varepsilon\mathbf{K})$$

which is a Schönflies algebra, and:

$$T_{5,s} = S_{5,s}.$$

i	j	$S_{i,j}$	$L_{i,j}$	$T_{i,j}$
2	1	$\mathbf{I}, \mathbf{J}$	$\mathbf{K}$	$\mathbf{I}, \mathbf{J}$
	2	$\varepsilon\mathbf{I}, \mathbf{J}_\beta$	$\varepsilon\mathbf{K}$	$\varepsilon\mathbf{I}$
3	4	$\mathbf{I}, \mathbf{J}, \varepsilon\mathbf{K}$	$\varepsilon\mathbf{I}, \varepsilon\mathbf{J}, \mathbf{K}$	$\mathbf{I}, \mathbf{J}, \varepsilon\mathbf{K}$
	10	$\varepsilon\mathbf{I}, \mathbf{K}, \varepsilon\mathbf{K}$	$\varepsilon\mathbf{J}$	$\varepsilon\mathbf{I}$
4	5	$\mathbf{I}, \varepsilon\mathbf{I}, \mathbf{J}, \varepsilon\mathbf{J}$	$\mathbf{K}, \varepsilon\mathbf{K}$	$\mathbf{I}, \varepsilon\mathbf{I}, \mathbf{J}, \varepsilon\mathbf{J}$
5	1	$\mathbf{I}, \varepsilon\mathbf{I}, \mathbf{J}, \varepsilon\mathbf{J}, \varepsilon\mathbf{K}$	$\varepsilon\mathbf{I}, \varepsilon\mathbf{J}, \mathbf{K}, \varepsilon\mathbf{K}$	$\mathbf{I}, \varepsilon\mathbf{I}, \mathbf{J}, \varepsilon\mathbf{J}, \varepsilon\mathbf{K}$

Table 4: Lie triple screw systems of $\mathfrak{se}(3)$ which are not Lie subalgebras (represented by basis screws).

Therefore, $S_{5,s}$ is a LTS.

In this section, we have identified all Lie triple systems from Hunt's complete classification of screw systems [2]. Altogether, there are six classes of screw systems that fall in the category of Lie triple screw systems and, at the same time, are not Lie subalgebras. Our results are summarized in Table 4.

5. Dual Quaternion Characterization of the Exponential Images of LTSs

The exponential images of the LTSs identified in Sec. 4 correspond to symmetric subspaces of SE(3) [29]. A full analysis of their geometric properties and applications is developed in [29]. In this paper, we investigate symmetric subspaces from a different perspective, namely their characterization by using dual quaternions.

Elements of SE(3) can be represented as points in $\mathbb{R}\mathrm{P}^7$ under the dual quaternion representation [19, Ch. 9]:

$$\mathbf{g} = a_0 + a_1\mathbf{I} + a_2\mathbf{J} + a_3\mathbf{K} + b_0\varepsilon + b_1\varepsilon\mathbf{I} + b_2\varepsilon\mathbf{J} + b_3\varepsilon\mathbf{K} \in \mathrm{SE}(3)$$

where $\mathbf{I}, \mathbf{J}, \mathbf{K}, \varepsilon\mathbf{I}, \varepsilon\mathbf{J}, \varepsilon\mathbf{K}$ are as defined in (6). The homogeneous coordinate $(a_0 : a_1 : a_2 : a_3 : b_0 : b_1 : b_2 : b_3)$ satisfies the homogeneous quadratic constraint $a_0b_0 + a_1b_1 + a_2b_2 + a_3b_3 = 0$, which defines a 6-D quadric in $\mathbb{R}\mathrm{P}^7$ known as the *Study quadric* (denoted Q_s; see Fig. 22(a)). In particular, a pure rotation about an unit vector $\mathbf{w} = (\mathrm{w}_1, \mathrm{w}_2, \mathrm{w}_3)$ with magnitude θ is given by:

$$e^{\theta\mathbf{w}/2} = c_{\theta/2} + s_{\theta/2}\mathbf{w} = c_{\theta/2} + s_{\theta/2}\mathrm{w}_1\mathbf{I} + s_{\theta/2}\mathrm{w}_2\mathbf{J} + s_{\theta/2}\mathrm{w}_3\mathbf{K}$$

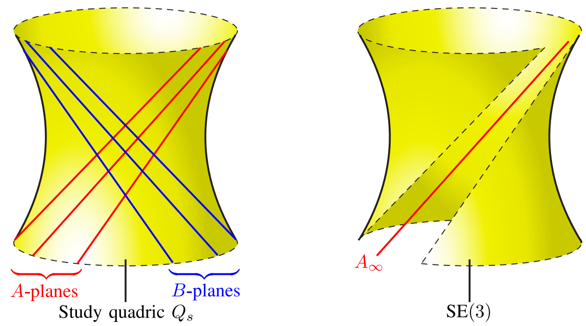
and a pure translation along a unit vector $\mathbf{v} = (\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3)$ with magnitude t is given by:

$$e^{\varepsilon t\mathbf{v}/2} = 1 + \frac{t}{2}\varepsilon\mathbf{v} = 1 + \frac{t}{2}\mathrm{v}_1\varepsilon\mathbf{I} + \frac{t}{2}\mathrm{v}_2\varepsilon\mathbf{J} + \frac{t}{2}\mathrm{v}_3\varepsilon\mathbf{K}$$

Note that the ideal three-plane defined by:

$$A_\infty := \{\, (a_0 : a_1 : a_2 : a_3 : b_0 : b_1 : b_2 : b_3) \mid a_0 = a_1 = a_2 = a_3 = 0\}$$

is completely contained in Q_s, and it comprises the only elements on Q_s that do not represent elements of SE(3) (see Fig. 22(b)). We shall refer to $Q_s - A_\infty \simeq$



(a) (b)

Figure 22: (a) A 2-D analogue of the 6-D Study quadric Q_s in $\mathbb{RP}^7$;(b) the subset of Q_s that corresponds to SE(3)

SE(3) as the set of proper dual quaternions. We shall also denote the ideal plane by $\varepsilon \wedge \varepsilon\mathbf{I} \wedge \varepsilon\mathbf{J} \wedge \varepsilon\mathbf{K}$, where $\wedge$ is the projective span operator and $\varepsilon, \varepsilon\mathbf{I}, \varepsilon\mathbf{J}, \varepsilon\mathbf{K}$ are the corresponding basis vectors.

Closely related to our work is a series of studies on sub-varieties of the Study quadric conducted by Selig [19, 37, 38]. Such sub-varieties arise either in motions satisfying natural geometric constraints (such as the point-plane constraint) [37], or in the dual quaternion representation of Lie subgroups of SE(3) [38].

5.1. LTSs containing only zero- or infinite-pitch screws

From (4) and (5), we see that the exponential image of a one-system spanned by a zero-pitch or infinite-pitch screw $\boldsymbol{\xi}$ is given by:

$$\left\{ e^{\theta\boldsymbol{\xi}^\circ/2} \;\middle|\; \forall\theta\in\mathbb{R}\right\} = \left\{ c_{\theta/2} + s_{\theta/2}\boldsymbol{\xi} \;\middle|\; \forall\theta\in\mathbb{R}\right\} = 1 \wedge \boldsymbol{\xi}^\circ$$

or

$$\left\{ e^{\varepsilon\theta\boldsymbol{\xi}^\circ/2} \;\middle|\forall\theta\in\mathbb{R}\right\} = \left\{ 1 + \varepsilon\frac{\theta}{2}\boldsymbol{\xi}^\circ \;\middle|\forall\theta\in\mathbb{R}\right\} = 1 \wedge \varepsilon\boldsymbol{\xi}^\circ$$

It is a one-parameter subgroup of SE(3), and corresponds to a line spanned by 1 and $\boldsymbol{\xi}^\circ$ (or $\varepsilon\boldsymbol{\xi}^\circ$) that lies completely on Q_s [19]. More generally, if a k-D screw system S comprises only zero- or infinite-pitch screws, its exponential image is given by:

$$\exp S = 1 \wedge S$$

which is a k-plane completely contained in Q_s. Based on this observation, we can immediately deduce that the exponential image of $S^0_{2,1}$, $S^0_{2,2}$ and $S^0_{3,4}$ are given by:

$$\begin{cases} \exp(S^0_{2,1}) = 1 \wedge S^0_{2,1} = 1 \wedge \mathbf{I} \wedge \mathbf{J} \\ \exp(S^0_{2,2}) = 1 \wedge S^0_{2,2} = 1 \wedge \mathbf{J} \wedge \varepsilon\mathbf{I} - \varepsilon\mathbf{I} \\ \exp(S^0_{3,4}) = 1 \wedge S^0_{3,4} = 1 \wedge \mathbf{I} \wedge \mathbf{J} \wedge \varepsilon\mathbf{K} - \varepsilon\mathbf{K} \end{cases}$$

	normal form	proper submanifolds	general form
A-plane	$A_0 = 1 \wedge \mathbf{I} \wedge \mathbf{J} \wedge \mathbf{K}$	spatial rotation group	cosets of conjugates
	$A_2 = 1 \wedge \mathbf{K} \wedge \varepsilon\mathbf{I} \wedge \varepsilon\mathbf{J}$	planar Euclidean group	cosets of conjugates
	$A_\infty = \varepsilon \wedge \varepsilon\mathbf{I} \wedge \varepsilon\mathbf{J} \wedge \varepsilon\mathbf{K}$	N/A	invariant
B-plane	$B_1 = 1 \wedge \mathbf{I} \wedge \mathbf{J} \wedge \varepsilon\mathbf{K}$	$\exp(S^0_{3,4})$ (symmetric subspace)	symmetric subspace
	$B_3 = 1 \wedge \varepsilon\mathbf{I} \wedge \varepsilon\mathbf{J} \wedge \varepsilon\mathbf{K}$	spatial translation group	cosets

Table 5: 3-planes of the Study quadric. Note that left/right transformation of a symmetric subspace is still a symmetric subspace.

where, in the latter two cases, the ideal point $\varepsilon\mathbf{I}$ or $\varepsilon\mathbf{K}$ (which does not correspond to proper rigid transformations) is excluded. In general, the exponential image of a one-system spanned by a finite non-zero pitch screw is not an algebraic sub-variety under dual quaternion coordinates. Consequently, $S_{2,2}$ with $\beta \notin \{0, \infty\}$ is not an algebraic sub-variety of Q_s either.

It is also interesting to note that according to Selig's earlier study [19, 39], $\exp(S^0_{3,4})$ (without removing the ideal point) is exactly the B-plane that meets A_∞ at a single point (called B_1-plane in [39]), while $\exp(S^0_{2,1})$ and $\exp(S^0_{2,2})$ are two 2-planes contained in $\exp(S^0_{3,4})$. For convenience, we recall Selig's complete classification of 3-planes of Q_s in Table 5 along with their proper submanifolds. It should be clear that this classification is up to multiplication on the left and/or right by proper dual quaternions [39]: such proper transformations are linear transformations sending planes in Q_s to planes of the same dimension, and they leave both A_∞ and $Q_s - A_\infty \simeq \mathrm{SE}(3)$ invariant. This should give rise to all 3-planes in Q_s from the normal forms listed in Table 5 (see also Fig. 22(a) for an analogue of the two families of 3-planes in Q_s).

5.2. LTSs comprising cylindrical algebras

Consider now the exponential image of the fifth special two-system $S_{2,5} = \mathrm{span}(\mathbf{I}, \varepsilon\mathbf{I})$ (the cylindrical algebra):

$$\exp(\frac{\theta}{2}\mathbf{I} + \frac{\lambda}{2}\varepsilon\mathbf{I}) = c_{\theta/2} + s_{\theta/2}\mathbf{I} + \frac{\lambda}{2}\varepsilon(-s_{\theta/2} + c_{\theta/2}\mathbf{I}),\ \forall\theta, \lambda \in \mathbb{R}$$

It is shown in [19] that this is exactly the intersection of Q_s with a three-space $1 \wedge \mathbf{I} \wedge \varepsilon \wedge \varepsilon\mathbf{I}$, a quadric prescribed by:

$$\begin{cases} a_0 b_0 + a_1 b_1 = 0 \\ a_2 = a_3 = b_2 = b_3 = 0 \end{cases} \tag{9}$$

This quadric contains an ideal line $\varepsilon \wedge \varepsilon\mathbf{I}$ which does not correspond to proper rigid transformations. By using this observation, we can deduce the exponential image of $S_{3,10}$, $S_{4,5}$ and $S_{5,\mathrm{s}}$.

First, it is clear from Fig. 17 that $S_{3,10}$ comprises a parallel pencil of cylindrical algebras:

$$\exp(S_{3,10}) = \bigcup_{l \in \mathbb{R}} \exp(\mathrm{span}(\mathbf{K} + \varepsilon l\mathbf{I}, \varepsilon\mathbf{K}))$$

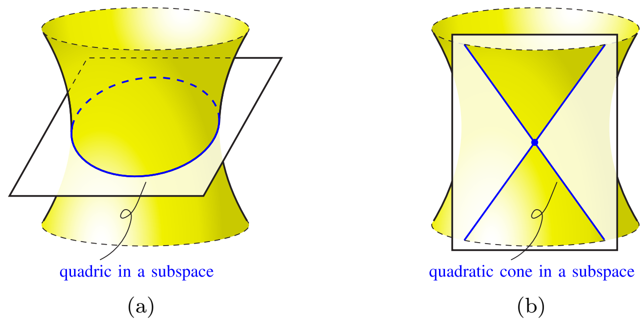


Figure 23: (a) A 1-D analogue of an intersection quadric Q_s with a subspace in $\mathbb{RP}^7$;(b) a 1-D analogue of a tangential intersection of Q_s with a subspace of $\mathbb{RP}^7$, resulting in a quadratic cone with the vertex being the point of tangency (denoted by the blue dot).

where $\mathrm{span}(\mathbf{K}+\varepsilon l\mathbf{I}, \varepsilon\mathbf{K})$ is the cylindrical algebra along the line $\mathbf{K}+\varepsilon l\mathbf{I} = \mathbf{k}+\varepsilon(l\mathbf{j})\times\mathbf{k}$. It is easy to verify that the exponential formula for this general cylindrical algebra is:

$$\exp(\mathrm{span}(\mathbf{K}+\varepsilon l\mathbf{I}, \varepsilon\mathbf{K})) = (1 \wedge (\mathbf{K}+\varepsilon l\mathbf{I}) \wedge \varepsilon \wedge \varepsilon\mathbf{K}) \cap Q_s - \varepsilon \wedge \varepsilon\mathbf{K}$$

Consequently, we have:

$$\begin{aligned}\exp(S_{3,10}) &= \bigcup_{l\in\mathbb{R}} \exp(\mathrm{span}(\mathbf{K}+\varepsilon l\mathbf{I}, \varepsilon\mathbf{K})) \\ &= \bigcup_{l\in\mathbb{R}} (1 \wedge (\mathbf{K}+\varepsilon l\mathbf{I}) \wedge \varepsilon \wedge \varepsilon\mathbf{K}) \cap Q_s - \varepsilon \wedge \varepsilon\mathbf{K} \\ &= (1 \wedge \mathbf{K} \wedge \varepsilon \wedge \varepsilon\mathbf{I} \wedge \varepsilon\mathbf{K} - \varepsilon\mathbf{I}) \cap Q_s - \varepsilon \wedge \varepsilon\mathbf{K} \\ &= (1 \wedge \mathbf{K} \wedge \varepsilon \wedge \varepsilon\mathbf{I} \wedge \varepsilon\mathbf{K}) \cap Q_s - \varepsilon \wedge \varepsilon\mathbf{I} \wedge \varepsilon\mathbf{K}\end{aligned}$$

We have shown that $\exp(S_{3,10})$ corresponds to a quadric prescribed by (see Fig. 23(a) for a 1-D analogue):

$$\begin{cases} a_0b_0 + a_3b_3 = 0 \\ a_1 = a_2 = b_2 = 0 \end{cases}$$

which is a quadratic cone [40] in a 3-plane, with base quadric

$$\begin{cases} a_0b_0 + a_3b_3 = 0 \\ a_1 = a_2 = b_1 = b_2 = 0 \end{cases}$$

which is exactly a conjugate of the cylindrical group prescribed by (9), and vertex $\varepsilon\mathbf{I}$ (see Fig. 23(b) for a 1-D analogue).

By similar arguments, the exponential image of $S_{4,5}$ is given by:

$$\exp(S_{4,5}) = (1 \wedge \mathbf{I} \wedge \mathbf{J} \wedge \varepsilon \wedge \varepsilon\mathbf{I} \wedge \varepsilon\mathbf{J}) \cap Q_s - \varepsilon \wedge \varepsilon\mathbf{I} \wedge \varepsilon\mathbf{J}$$

which corresponds to a quadric in a 5-plane prescribed by (see Fig. 23(a) for a 1-D analogue)

$$\begin{cases} a_0b_0 + a_1b_1 + a_2b_2 = 0 \\ a_3 = b_3 = 0 \end{cases}$$

The exponential image of $S_{5,\mathrm{s}}$ is given by:

$$\exp(S_{5,\mathrm{s}}) = (1 \wedge \mathbf{I} \wedge \mathbf{J} \wedge \varepsilon \wedge \varepsilon\mathbf{I} \wedge \varepsilon\mathbf{J} \wedge \varepsilon\mathbf{K}) \cap Q_s - \varepsilon \wedge \varepsilon\mathbf{I} \wedge \varepsilon\mathbf{J} \wedge \varepsilon\mathbf{K}$$

which corresponds to a quadratic cone in a 6-plane, with base quadric $\exp(S_{4,5})$ and vertex $\varepsilon\mathbf{K}$ (see Fig. 23(b) for a 1-D analogue). Referring to [39], $\exp(S_{4,5})$ is exactly the space of all line symmetric motions (with a fixed base line).

In summary, all symmetric subspaces except $\exp(S_{2,2})$ (with $\beta \neq 0$) are either projective subspaces (excluding a subspace of ideal points) that are completely contained in Q_s, or quadrics (including quadratic cones) of Q_s (i.e. intersection of Q_s with a projective subspace of $\mathbb{RP}^7$; see [40, Ch. 1] for the concept of quadric and quadratic cone). Some of these spaces turn out to be certain sub-varieties of Q_s that have been discovered earlier. However, the connection to LTSs and symmetric subspaces was never made. Hopefully, such connections may be useful for analysis and design of over-constrained linkages and special motion generators [37, 39, 41, 42, 11, 29].

6. Conclusions

In this paper, we identified all six non-trivial Lie triple systems from Hunt's complete classification of screw systems [2], as shown in Table 4. The presented derivation was based on both algebraic Lie-group theory and screw geometry. We paid special attention to the geometric features characterizing each Lie triple system, as well as its associated Lie product and Lie triple product. Lie triple screw systems have been playing, unknowingly, an important role in the type synthesis of constant-velocity couplings and torsion-free parallel manipulators for many years. We have expanded the known portfolio of screw systems that have the same special algebraic and geometric properties as the screw systems characterizing the latter mechanisms, thus possibly opening the way to new designs.

For example, we illustrated in [43] that the 2-D LTS $S^0_{2,1}$ can be used to provide an elegant kinematic analysis of linkage-based 2-D CV couplings such as the Thompson CV joint [44] and Kocabas's CV joint [45]. In [43], we implemented the same kinematic principles in the design of a novel 2-DoF parallel wrist with both an ample rotation range and a fixed center of rotation compared to Omni-wrist III [46]. In [29], we developed a systematic type synthesis method for kinematic chains persistently generating Lie triple screw systems. A systematic type synthesis for parallel mechanisms generating LTSs is currently being developed.

Aside from applications in mechanism analysis and synthesis, the symmetric spaces corresponding to the Lie triple screw systems that we identified may serve as important motion types for robotic applications. For example, $\exp(S^0_{2,1})$ serves as the 2-D orientation workspace for robotic wrists (excluding the final roll axis, see [47]), quadcopters (row and pitch), and the spindle of a five-axis machine [48]. The computational advantage of $\exp(S^0_{2,1})$ in comparison to the traditional unit sphere model [47] may allow one to conduct spline interpolation on the orientation workspace using the exponential map [49, 50].

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