CAUCHY PROBLEM FOR EFFECTIVELY HYPERBOLIC OPERATORS WITH TRIPLE CHARACTERISTICS OF VARIABLE MULTIPLICITY

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Abstract. We study a class of third order hyperbolic operators P in $G = \{(t, x) : 0 \le t \le T,\ x \in U \Subset \mathbb{R}^n\}$ with triple characteristics at $\rho = (0, x_0, \xi)$, $\xi \in \mathbb{R}^n \setminus \{0\}$. We consider the case when the fundamental matrix of the principal symbol of P at ρ has a couple of non-vanishing real eigenvalues. Such operators are called *effectively hyperbolic*. V. Ivrii introduced the conjecture that every effectively hyperbolic operator is *strongly hyperbolic*, that is the Cauchy problem for $P + Q$ is locally well posed for any lower order terms Q. This conjecture has been solved for operators having at most double characteristics and for operators with triple characteristics in the case when the principal symbol admits a factorization. A strongly hyperbolic operator in G could have triple characteristics in G only for $t = 0$ or for $t = T$. We prove that the operators in our class are strongly hyperbolic if T is small enough. Our proof is based on energy estimates with a loss of regularity.

Contents

1. Introduction

1.1. Notations and main result. Consider a differential operator

$$P(t, x, D_t, D_x) = \sum_{\alpha+|\beta|\le m} c_{\alpha,\beta}(t, x) D_t^{\alpha} D_x^{\beta},\ D_t = -\mathbf{i}\partial_t, D_{x_j} = -\mathbf{i}\partial_{x_j} \tag{1.1}$$

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of order m with C^∞ coefficients $c_{\alpha,\beta}(t,x),\ t \in \mathbb{R},\ x \in \mathbb{R}^n$. Denote by

$$p_m(t,x,\tau,\xi) = \sum_{\alpha+|\beta|=m} c_{\alpha,\beta}(t,x)\tau^\alpha \xi^\beta$$

the principal symbol of P. We assume that $c_{m,0}(t,x) \neq 0$ for all (t,x). Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set and let $\Omega_\eta^- = \Omega \cap \{t \leq \eta\}, \Omega_\eta^+ = \Omega \cap \{t \geq \eta\},\ G = \Omega \cap \{0 \leq t \leq T\}$.

Set $P_m(t,x,D_t,D_x) = p_m(t,x,D_t,D_x)$.

Definition 1.1. *We say that the Cauchy problem*

$$Pu = f \text{ in } \Omega \cap \{t < T\},\ \operatorname{supp} u \subset \overline{G} \tag{1.2}$$

is well posed in G if
(i) (existence) for every $f \in C_0^\infty(\Omega)$, $\operatorname{supp} f \subset \overline{\Omega_0^+}$ there exists a solution $u \in \mathcal{D}'(\Omega)$ satisfying (1.2).
(ii) (uniqueness) if $u \in \mathcal{D}'(\Omega)$ satisfies (1.2), *then for every $s, 0 < s \leq T$, if $Pu = 0$ in Ω_s^-, then $u = 0$ in Ω_s^-.*

A necessary condition for the well posedeness of the Cauchy problem (WPC) is the hyperbolicity of the operator P in G (see [6] and the references cited there). This means that for every $(t_0, x_0, \xi) \in G \times \mathbb{R}^n \setminus \{0\}$ the equation

$$p_m(t_0, x_0, \tau, \xi) = 0 \tag{1.3}$$

with respect to τ has only real roots $\tau = \lambda_j(t_0, x_0, \xi)$.

Definition 1.2. *We say that the operator P with principal symbol p_m is strongly hyperbolic in G if for every point $z_0 = (t_0, x_0) \in G$ there exists a neighbourhood U of z_0, $T(U) > 0$ and $T_0 \geq 0$ ($T_0 < T$ if $t_0 = T$ and $T_0 = 0$ if $t_0 = 0$) such that the Cauchy problem* (1.2) *for the operator $L = P_m(t,x,D_t,D_x) + Q_{m-1}(t,x,D_t,D_x)$ is well posed in U_s^+ for every $T_0 \leq s < T(U)$ and for any operator $Q_{m-1}(t,x,D_t,D_x)$ of order less or equal to $m-1$.*

When P is strictly hyperbolic, that is when the equation (1.3) has simple roots $\lambda_j(t,x,\xi)$ with respect to the variable τ for all $(t,x,\xi) \in G \times \mathbb{R}^N \setminus \{0\}$, it is a classical result that P is strongly hyperbolic. If the equation (1.3) has real roots with constant multiplicity for $(t,x,\xi) \in G \times \mathbb{R}^n \setminus \{0\}$, the operator P is strongly hyperbolic **if and only if** it is strictly hyperbolic. Thus in the case of roots with constant multiplicity—greater than 1—we must impose conditions on the lower order terms Q_{m-1}, called Levi conditions, in order that the Cauchy problem be well posed. The analysis of the Cauchy problem for such operators is complete and we know the necessary [3] and sufficient [2] conditions for (WPC).

Passing to the case when the roots of (1.3) have variable multiplicity, notice that the roots $\lambda_j(t,x,\xi)$ in general are not smooth but only continuous. The case of operators with constant coefficients is also completely examined and P is strongly hyperbolic **if and only if** P is strictly hyperbolic. The necessary and sufficient condition of Gårding for (WPC) says that there exists a constant $c > 0$ such that for the full symbol $p(\tau,\xi)$ of P we have

$$p(\tau,\xi) \neq 0, \text{ for } |\operatorname{Im} \tau| > c,\ \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

In the following, for the sake of simplicity, we switch to a different notation and denote $t = x_0$, $x = (x_0, x_1, ..., x_n) \in \mathbb{R}^{n+1}$. The dual variables are denoted by $\xi = (\xi_0, \xi_1, ..., \xi_n) = (\xi_0, \xi')$.

Given a symbol $p(x,\xi)$, let

$$\Sigma(p) = \{z \in T^*G \setminus \{0\} :\ p(z) = 0\},\ \Sigma_1(p) = \{z \in T^*(G) \setminus \{0\} : p(z) = 0,\ dp(z) = 0\}.$$

In the case $\Sigma_1(p_m) = \emptyset$, the operator is of principal type and a hyperbolic operator P in G is strongly hyperbolic (see [7] and Section 23.4 in [5]).

Turning to the case $\Sigma_1(p_m) \neq \emptyset$, notice that if we have a critical point $(\hat{x}, \hat{\xi}) \in \Sigma_1(p)$, then the Hamiltonian system

$$\frac{dx}{ds} = \partial_\xi p, \qquad \frac{d\xi}{ds} = -\partial_x p$$

has a stationary point and it is natural to consider the differential of the Hamilton vector field. Thus we are led to define the fundamental matrix

$$F_p(\hat{x}, \hat{\xi}) = \begin{pmatrix} p_{\xi,x}(\hat{x}, \hat{\xi}) & p_{\xi,\xi}(\hat{x}, \hat{\xi}) \\ -p_{x,x}(\hat{x}, \hat{\xi}) & -p_{x,\xi}(\hat{x}, \hat{\xi}) \end{pmatrix}.$$

We recall two important properties of F_p (see [6], [4]):

1. For every point $z \in \Sigma_1(p)$ the Hessian $Q_p(X, Y)$, $X, Y \in T_z(T^*(G))$ at z of $\frac{p}{2}$ is well defined. Then $Q_p(X, Y) = \sigma(X, F_p(z)Y)$, σ being the symplectic form on $T^*(G)$. Thus after a canonical transformation the fundamental matrix is transformed into a similar one and its eigenvalues are invariant under canonical transformations. Hörmander [4] called $F_p(z)$ the Hamilton map of Q_p.
2. If P is hyperbolic in G and $(\hat{x}, \hat{\xi})$ is a critical point of $p_m(x, \xi)$, then $F_{p_m}(\hat{x}, \hat{\xi})$ has at most two non-vanishing real simple eigenvalues μ and $-\mu$ and all other eigenvalues μ_j are purely imaginary, that is $\operatorname{Re} \mu_j = 0$.

The existence of non-vanishing real eigenvalues of $F_{p_m}(\hat{x}, \hat{\xi})$ is a *necessary condition* for strong hyperbolicity. More precisely, let $p_{m-1}(x, \xi) = \sum_{|\alpha|=m-1} c_\alpha(x)\xi^\alpha$ and let

$$p'_{m-1}(x, \xi) = p_{m-1}(x, \xi) + \frac{i}{2} \sum_{j=0}^{n} \frac{\partial^2 p_m}{\partial x_j \partial \xi_j}(x, \xi)$$

be the subprincipal symbol of P which is invariantly defined for $(x, \xi) \in \Sigma_1(p_m)$. Then we have the following

Theorem 1.1 (Theorem 3 and Corollary 3 in [6])**.** *If P is strongly hyperbolic in G, then at every point $(\hat{x}, \hat{\xi}) \in \Sigma_1(p_m)$ the fundamental matrix $F_{p_m}(\hat{x}, \hat{\xi})$ has two non-vanishing real eigenvalues. Moreover, for $(x, \xi') \in \mathring{G} \times (\mathbb{R}^n \setminus \{0\})$ the multiplicities of the roots of* (1) *are not greater than two, and for $(x, \xi') \in \{x_0 = 0\} \times \mathbb{R}^n \setminus \{0\}$ or for $(x, \xi') \in \{x_0 = T\} \times \mathbb{R}^n \setminus \{0\}$ these multiplicities are not greater than three. If $F_{p_m}(\hat{x}, \hat{\xi})$ has only purely imaginary eigenvalues, the condition $\operatorname{Im} p'_{m-1}(\hat{x}, \hat{\xi}) = 0$ is necessary for (WPC).*

If $F_{p_m}(\hat{x}, \hat{\xi})$ has only purely imaginary eigenvalues, for (WPC) we have a second necessary condition

$$|\operatorname{Re} p'_{m-1}(\hat{x}, \hat{\xi})| \leq \frac{1}{4} \sum_{j=0}^{2n+2} |\mu_j|,$$

μ_j being the eigenvalues of $F_{p_m}(\hat{x}, \hat{\xi})$ repeated according to their multiplicities. This condition has been proved in [6] in some special cases concerning the structure of $F_{p_m}(\hat{x}, \hat{\xi})$ and without any restriction by Hörmander [4].

Definition 1.3. *A hyperbolic operator with principal symbol $p_m(x, \xi)$ is called effectively hyperbolic if at every point $(\hat{x}, \hat{\xi}) \in \Sigma_1(p_m)$, the fundamental matrix $F_{p_m}(\hat{x}, \hat{\xi})$ has two non-vanishing real eigenvalues.*

V. Ivrii introduced the following

Conjecture. *A hyperbolic operator is strongly hyperbolic if and only if it is effectively hyperbolic.*

For operators with at most double characteristics the sufficient part of the above conjecture has been established for some special class of effectively operators by Oleinik [14], Hörmander [4], Ivrii [7], Melrose [10] and in the general case by N. Iwasaki [8], [9] and T. Nishitani [11], [12], [13]. In particular, in the works of Nishitani many properties of effectively hyperbolic operators with double characteristics have been established. An important phenomenon for this class of operators is that we have always a loss of regularity which depends on the ratio of the subprincipal symbol and the non-vanishing real eigenvalue of F_{p_m} at double characteristic points (see Theorem 3 in [6] for a necessary condition, and [7], [12] for sufficient conditions).

However according to Theorem 1.1, there are also classes of effectively hyperbolic operators with characteristics of multiplicity 3 which should be strongly hyperbolic. The analysis of operators with double characteristics is fairly complete and over the last 30 years a lot of papers treating both the effectively hyperbolic and the non-effectively hyperbolic operators has appeared. The case when we have triple characteristics is more complicated for several reasons. So far the only result concerning effectively hyperbolic operators with triple characteristics appears to be that of Ivrii [7] for operators such that in a conic neighborhood of every point $(0, x_0, \xi_0)$ with triple characteristics p_3 admits a factorization

$$p_3(t, x, \tau, \xi) = ((\tau - \beta(x, \xi))^2 - D(t, x, \xi))(\tau - \gamma(t, x, \xi)) \tag{1.4}$$

as a product of two principal type symbols with smooth real-valued symbols β, γ, D and $D \geq 0$ for $t \geq 0$. For such a class of operators Ivrii established in [7] a strong hyperbolicity result. It is clear that, for a factorization of the form (1.4) to exist, it is necessary that the equation $p_3 = 0$ has a C^∞ real root $\tau = \gamma(t, x, \xi)$ defined in a small conic neighborhood of the point $(0, x_0, \xi_0)$. This is possible only if some terms of p_3 have a very special behavior. For example, consider the symbol

$$p_3 = \tau^3 - (ta_2(t, x, \xi) + \alpha(x, \xi))\tau + t^2 b_3(t, x, \xi) \tag{1.5}$$

with $a_2(t, x, \xi) \geq c|\xi|^2$, $c > 0$, $\alpha(x, \xi) \geq 0$ and $\alpha(0, \xi_0) = 0, 4(ta_2 + \alpha)^3 \geq 27t^4 b_2^3$ for $t \geq 0$, where $b_3(t, x, \xi)$ is a symbol of third order. Then if $b_3(0, 0, \xi_0) \neq 0$, the factorization (1.4) for fixed $\xi = \xi_0$ is possible only if $\alpha(x, \xi_0) \equiv 0$ for all x in a neighborhood of 0. We prove this result in the Appendix.

The purpose of the present paper is to show that for a class of third order weakly hyperbolic operators whose principal symbol p_3, after a reduction, has the form (1.5) we have strong hyperbolicity if the Hamilton map F_{p_3} has two real non-vanishing eigenvalues in its spectrum on the triple characteristic points. The last condition is satisfied if the symbol $a_2(t, x, \xi)$ is elliptic. According to Theorem 1.1, a strongly hyperbolic operator may have triple characteristics only for $t = 0$ or $t = T$ and in this paper we deal with the case when this may happen for some points on $t = 0$.

More precisely, we study operators having the form

$$\begin{aligned} P \quad = \quad & D_t^3 + q_1(t, x, D_x)D_t^2 + q_2(t, x, D_x)D_t + q_3(t, x, D_x) \\ & + r_2(t, x, D_x) + r_1(t, x, D_x)D_t + r_0(t, x)D_t^2 + m_1(t, x, D_x) + m_0(t, x)D_t + c_0(t, x). \end{aligned} \tag{1.6}$$

Here $q_j(t,x,D_x)$, $j=1,2,3$, are differential operators with C^∞ coefficients and real-valued symbols $q_j(t,x,\xi)$ which are homogeneous polynomials of order j in ξ, $r_j(t,x,D_x)$, $j=1,2$, are differential operators with C^∞ coefficients and symbols $r_j(t,x,\xi)$ homogeneous of order j with respect to ξ, $r_0(t,x), m_0(t,x), c_0(t,x)$ are C^∞ functions and $m_1(t,x,D_x)$ is a first order differential operator with C^∞ coefficients. Let $p_3(t,x,\tau,\xi)$ be the principal symbol of P and let $G=\{(t,x):\, 0\le t\le T,\ x\in U\}$, where $U\Subset\mathbb{R}^n$ is an open set in $\mathbb{R}^n$. Consider the symbols

$$\Delta_1=27q_3-9q_1q_2+2q_1^3,\ \Delta_0=q_1^2-3q_2,\ \Delta=-\frac{1}{27}(\Delta_1^2-4\Delta_0^3).$$

The symbol Δ is the *discriminant* of the equation $p_3=0$ with respect to τ and we have three real roots for $t\ge 0$ if and only if $\Delta\ge 0$. The symbol Δ_0 is the *discriminant* of the equation $\partial_\tau p_3=3\tau^2+2q_1\tau+q_2=0$ with respect to τ and if we have a triple root at $\rho=(0,x_0,\xi)$, we get $\Delta_0(\rho)=0$. Thus if the equation $p_3=\tau^3+q_1\tau^2+q_2\tau+q_3=0$ has a triple real root at $\rho=(0,x_0,\xi)$, we must have $\Delta_0(\rho)=0,\ \Delta(\rho)=0$. This implies $\Delta_1(\rho)=0$ and the triple root is $\tau=-\frac{q_1(\rho)}{3}$.

Since the polynomial p_3 is hyperbolic with respect to τ for $t\ge 0$, we deduce that at a point ρ with triple characteristics we have

$$(d_{t,x}p_3)\Big(0,x_0,-\frac{q_1(\rho)}{3},\xi\Big)=0$$

(see Lemma 8.1 in [6]). The last condition can be written as follows

$$(d_{t,x}q_1)(\rho)\frac{q_1^2(\rho)}{9}-(d_{t,x}q_2)(\rho)\frac{q_1(\rho)}{3}+(d_{t,x}q_3)(\rho)=0.$$

Taking the differential of Δ_1 at ρ, and using $\Delta_0(\rho)=0$, we deduce easily that $(d_{t,x}\Delta_1)(\rho)=0$.

In this paper we make the following assumptions:

(H_0) The roots of the equation $p_3(t,x,\tau,\xi)=0$ with respect to τ are real for all $(t,x)\in\bar{G},\xi\in\mathbb{R}^n$.

(H_1) If the equation $p_3(0,x,\tau,\xi)=0$ with respect to τ has a triple root $\tau=\lambda(0,x_0,\xi_0)$ for $t=0$, $x_0\in U$, $\xi_0\in\mathbb{R}^n\setminus\{0\}$, then we have a triple root $\tau=\lambda(0,x_0,\xi)$ for $(0,x_0,\xi),\forall\xi\in\mathbb{R}^n\setminus\{0\}$ and the Hamiltonian map $F_{p_3}(0,x_0,\lambda(0,x_0,\xi),\xi)$ of p_3 has non-zero real eigenvalues $\pm\mu(x_0,\xi)$ for $\xi\in\mathbb{R}^n\setminus\{0\}$.

(H_2) If (H_1) holds for $(0,x_0,\xi)$, then there exists an open neighborhood U_{x_0} of x_0 such that $\Delta_1(0,x,\xi)=(\partial_t\Delta_1)(0,x,\xi)=0$ for $x\in U_{x_0}$ and $\xi\in\mathbb{R}^n\setminus\{0\}$.

Our main result is the following

Theorem 1.2. *Let $x_0\in U$ be a point for which the hypothesis $(H_0)-(H_2)$ are satisfied. Then there exists a neighbourhood $V_{x_0}\subset U_{x_0}$ of x_0 such that for $T>0$ sufficiently small, the operator P is strongly hyperbolic in $\{(t,x);\ 0\le t\le T,\ x\in V_{x_0}\}$.*

The technique of the energy estimates can be modified to cover the case when (H_1) holds only in a conic neighbourhood of a point $\rho_0=(0,x_0,\xi_0)$. However, to obtain a (WPC) in a neighbourhood of x_0, it is necessary to have more sophisticated tools to deduce (WPC) from the microlocal a priori estimates with loss of regularity. As we mentioned above, the assumption (H_2) is valid at all triple characteristic points. Notice that if T is not small and if for $0<\delta\le t<T$ with sufficiently small δ the operator P is effectively hyperbolic with double characteristics at some points, we

can combine our result with those of [8], [9], [11], [12] to obtain a strongly hyperbolic operator in $\{0 \le t < T,\ x \in U\}$. Our arguments work also if we assume that q_1, q_2, q_3 are pseudodifferential operators with real-valued symbols.

To prove Theorem 1.2, we establish a theorem of existence and uniqueness for solutions of the Cauchy problem (see Theorem 8.3). To this purpose we obtain an a priori estimate with a loss of regularity of order $2N/3 - 2$ for the operator P near a triple characteristic point. We choose $N = \frac{13}{2}\Pi + N_0$, with N_0 an integer and

$$\Pi = \frac{2}{3} + \sup_{x \in \bar{U}_{x_0}, |\xi|=1} |p_2'(0, x, \lambda(0, x, \xi), \xi)(\mu(x, \xi))^{-1}|, \tag{1.7}$$

where $p_2'(t, x, \tau, \xi)$ is the subprincipal symbol of P, $\lambda(0, x, \xi)$ denotes the triple root of $p_3 = 0$ and $\mu(x, \xi)$ is the non-vanishing eigenvalue of $F_{p_3}(0, x, \lambda(0, x, \xi), \xi)$. Moreover, the integer N_0 depends only on $p_3(0, x, \lambda(0, x, \xi), \xi)$, $x \in U_{x_0}$, but we are not going to precise the optimal value of N_0. It seems that with a more complicated analysis of the contribution of the subprincipal symbol p_2' it should be possible to obtain a loss of regularity $2N/3 - 2$ with $N = \frac{3}{2}\Pi + N_0$ and this is an interesting open problem.

We may compare the number (1.7) with the loss of regularity for second order strongly hyperbolic operators L with principal symbol $l_2(t, x, \tau, \xi)$ given by

$$2 + \Big[\frac{1}{2} + \sup_{\rho \in \Sigma_1} |l_1'(\rho)(\mu(\rho)^{-1}|\Big],$$

where $[z]$ is the integer part of z (see [14], [7], [10] for a special class of operators with double characteristics and [12] for the general case). Here

$$\Sigma_1 = \{\rho = (t, x, \tau, \xi) \in G \times \mathbb{R}^{n+1} \setminus \{0\} :\ l_2(\rho) = 0, dl_2(\rho) = 0\}$$

is the double characteristic set of L, $l_1'(\rho)$ is the subprincipal symbol of L and $\mu(\rho)$ is the non-vanishing eigenvalue of the Hamiltonian map $F_{l_2}(\rho)$ at $\rho \in \Sigma_1$. It is important to note that the loss of regularity M for the solutions of the Cauchy problem for P is **bounded** from below by

$$\sup_{x \in \bar{U},\, |\xi|=1} |\operatorname{Im}(p_2'(0, x, \lambda(0, x, \xi), \xi))(\mu(x, \xi))^{-1}| \le 2n(M + 3). \tag{1.8}$$

This follows from the necessary condition (36) in Theorem 3 in [6]. Thus our result with $M = 2N/3 - 2$ is compatible with this lower bound.

1.2. **Comments on the proof of the main result.** The proof of Theorem 1.2 is long and technical. It is based on the energy estimates obtained in Theorems 8.1 and 8.2 and, as we mentioned above, we cannot avoid the loss of regularity which is related to the ratio of the subprincipal symbol and the non-vanishing eigenvalue of the Hamiltonian map. This is one of the main differences compared to the hyperbolic operators of principal type (see for example the analysis in Section 23.4 in [5].)

First by a change of variables (t, x) we reduce the analysis to the case when the principal symbol p_3 in the new variables, which we denote again by (t, x), has the form

$$p_3(t, x, \tau, \xi) = \tau^3 - (ta_2(t, x, \xi) + \alpha(x, \xi))\tau + t^2 b_3(t, x, \xi),$$

where a_2, α, are real-valued symbol homogeneous of order 2 with respect to ξ and $a_2(t,x,\xi) \geq c|\xi|^2$, $c > 0$, $\alpha(x,\xi) \geq 0$, while $b_3(t,x,\xi)$ is a real-valued symbol of ordre 3 in ξ (see Section 2). Here we use essentially the condition (H_2) to present the term with third order derivaties in x as $t^2 b_3(t,x,D_x)$. If (H_2) is not satisfied, we will have a term

$$a_3(t,x,\xi) = t^2 b_3(t,x,\xi) + tc_3(x,\xi) + \beta(x,\xi)$$

with third order symbols b_3, c_3, β. Of course, the hyperbolicity of p_3 implies that

$$|a_3|^2 \leq \frac{4}{27}(ta_2 + \alpha)^3,\ t \geq 0,$$

but it is quite difficult to exploit this condition working with the energy $\mathscr{E}_N(u)$ and the time function $f = \frac{t}{3} + \langle\xi\rangle^{-2/3}$ defined below. A different choice of f adapted to the structure of p_3 and some more complicated energy techniques seem to be necessary to cover the general case when $c_3 \neq 0$ and $\beta \neq 0$. The reader may consult the work of Nishitani [12], where the choice of f is related to a microlocal model of the principal symbol with double characteristics.

Next, we introduce, in Section 4, the scaling $t = \varepsilon^{2/3}s, x = \varepsilon y$, $\varepsilon > 0$, and we transform our operator into the operator $\mathcal{P}$ with respect to (s,y) (see Section 4 for the notations). Since we are interested in showing that the Cauchy problem is well posed for sufficiently small t and since P is strictly hyperbolic for t positive and small enough, we can investigate the operator $\mathcal{P}$. We denote below again by (t,x) the new variables. The symbols $a_2(t,x,\xi)$, $\alpha(x,\xi)$ are transformed to symbols

$$a_2^{\varepsilon} = a_2(\varepsilon^{2/3}t, \varepsilon x, \xi),\ \alpha^{\varepsilon} = \alpha(\varepsilon x, \xi)$$

and this is important for the pseudodifferential calculus developed in Sections 3-4. Eventually we choose $\varepsilon = \mathcal{O}(N^{-1})$, where $N = \frac{13}{2}\Pi + N_0$ is a large integer, so that $0 < \varepsilon_0 \leq \varepsilon N \ll 1$.

The so called *time function* $f(t,\xi) = \frac{t}{3} + \langle\xi\rangle^{-2/3}$ plays an important role in the calculus of pseudodifferential operators with order function $m_N^t = f^{-N}(t,\xi)$ and metric

$$g^{\varepsilon}_{(x,\xi)} = \varepsilon^2|dx|^2 + \langle\xi\rangle^{-2}|d\xi|^2.$$

The reason of this choice is that the commutator $[a_2^{\varepsilon}(t,x,D_x), f^{-N}(t,D_x)]$ has symbol in the space $S(f^{-N}\varepsilon N\langle\xi\rangle, g^{\varepsilon})$ so $[a_2^{\varepsilon}(t,x,D_x), f^{-N}(t,D_x)]f^N(t,D_x)$ becomes a first order operator whose norm is not depending on N and, in particular, on Π. This is proved in Proposition 5.2.

The main idea is to multiply $f^{-2N}(t,D_x)\mathcal{P}u$ by the *multiplier*

$$Mu = \psi(t)\Big[D_t^2 - \theta(ta_2^{\varepsilon}(t,x,D_x)u + \alpha^{\varepsilon}(x,D_x)u)\Big],$$

where we choose $\theta = 1/3$ and $\psi(t) = \frac{e^{-2\lambda t}}{t}$, $\lambda > 0$. For the analysis of the problem with data $D_t^j u(T,x) = 0,\ j = 0,1,2$, we use the function $\varphi(t) = te^{-2\lambda t}$. We study the expression

$$-2\,\mathrm{Im}\langle f^{-2N}(t,D_x)\mathcal{P}u, Mu\rangle, \tag{1.9}$$

$\langle,\rangle$ denoting the scalar product in $L^2(\mathbb{R}^n)$ and $u \in C_0^\infty([0,T]\times U)$ has traces $D_t^j u(t_0,x) = 0$, $j = 0,1,2$, $0 \leq t_0 \leq t \leq T$. The last condition guarantees that Mu is well defined for $t_0 = 0$. The above expression modulo lower order terms is a sum of 15 terms and we make a quite detailed analysis of all these terms in Section 5. The purpose is to find, by integration by parts, “positive” terms with a big coefficient of order $\mathcal{O}(N)$ which will absorb in the energy estimate the contributions with ”indefinite” (possibly negative) sign.

In fact, we have many indefinite terms, while the positive ones come from the expression

$$\partial_t \mathscr{E}_N(u) + \frac{2N}{3}\mathscr{E}_{N+1/2}(u) + 2\lambda \mathscr{E}_N(u),$$

where

$$\begin{aligned}\mathscr{E}_N(u) = \psi\Big[\|f^{-N}u''\|_0^2 + \frac{2}{3}\operatorname{Re}\langle f^{-2N}(ta_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)u', u'\rangle \\ + \frac{1}{3}\|f^{-N}(ta_2^\varepsilon u + \varepsilon^{-2/3}\alpha^\varepsilon)u\|_0^2 + \frac{2}{3}\operatorname{Re}\langle f^{-2N}u'', (ta_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)u\rangle\Big] \\ + \varepsilon^{1/3}te^{-2\lambda t}2\operatorname{Im}\langle f^{-2N}b_3^\varepsilon u, u'\rangle.\end{aligned}$$

The quantity in the third line above, involving powers of ε and t, is a lower order perturbation. We introduce the **energy** of order $k \in N$ by the expression

$$\begin{aligned}E_k(u) = \psi\Big(\frac{1}{3}\|f^{-k}u''\|_0^2 + \frac{2}{3}\operatorname{Re}\langle f^{-k}(t\mathbf{a}_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)u', f^{-k}u'\rangle \\ + \frac{1}{6}\|f^{-k}(t\mathbf{a}_2^\varepsilon u + \varepsilon^{-2/3}\alpha^\varepsilon)u\|_0^2 + \frac{2}{3}\|f^{-k}(u'' + \frac{1}{2}(t\mathbf{a}_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)u)\|_0^2\Big),\end{aligned} \tag{1.10}$$

where $\mathbf{a}_2^\varepsilon$ is the operator with symbol $a_2^\varepsilon(0,x,\xi)$ and $\|.\|_k$ denotes the $H^k(\mathbb{R}^n)$ norms. Next we prove that modulo lower order terms we have $\mathscr{E}_k(u) \geq E_k(u)$. To gain control on the terms involving $\|f^{-k}u(t,.)\|_j$ norms with $j = 1, 2$ and $k = N, N+1/2$, we would like to exploit the terms with $H^2(\mathbb{R}^n)$ norms having large coefficients. However, we have positive terms only with the norm $t\|f^{-N-1/2}u\|_2^2$ and, as t becomes close to 0, we cannot absorb the H_2 norm of $f^{-N-1/2}u$. This is the principal difficulty when we try to absorb the lower order terms created by $\operatorname{Im}\langle f^{-2N}b_2^\varepsilon u, a_2^\varepsilon u\rangle$ and the second order operator b_2. A way around this is to use the key inequality

$$\langle\xi\rangle^4 f^{-k} \leq t\langle\xi\rangle^4 f^{-k-1} + \langle\xi\rangle^2 f^{-k-3},$$

established in Lemma 6.1. Thus, for example, we have an estimate

$$\|f^{-N+1/2}u\|_2^2 \leq t\|f^{-N}u\|_2^2 + \|f^{-N-1}u\|_1^2.$$

Applying the above equality twice, we get also norms of the form $\|f^{-N-5/2}u\|_0$. Following this way, we have negative terms with coefficients t, but we have now generated other negative terms involving norms with *weights* $f^{-N-3/2}$ and $f^{-N-5/2}$ and these also have to be absorbed. However, the latter types of terms are not included in our **energy** expressions $E_{N+1/2}(u), E_N(u)$ given by (1.10).

To deal with them, we apply to just a fraction of the positive terms in $\psi E_{N+1/2}(u)$, having a large coefficient proportional to N, the inequality

$$\begin{aligned}\psi\|f^{-N-1/2}u''\|^2 \quad \geq \quad & \partial_t\Big(\psi\|f^{N-1}u'\|^2\Big) - \psi'\|f^{-N-1}u'\|^2 + (2N/3 - 4/3)\psi\|f^{-N-3/2}u'\|^2 \\ & + \partial_t\Big(\psi\|f^{-N-2}u\|^2\Big) - \psi'\|f^{-N-2}u\|^2 + \frac{2N+1}{3}\psi\|f^{N-5/2}u\|^2.\end{aligned} \tag{1.11}$$

A similar inequality for $\psi\|f^{-N-1}u\|_1^2$ completes our technical toolkit.

Finally, in Section 7 we show that we can absorb all non positive terms. In Section 8 we get the energy estimates with loss of $2N/3 - 2$ derivatives in x which imply, by a standard argument, the well posedeness of the Cauchy problem.

2. Hyperbolic operators with triple characteristics

In this section we use the notations of Section 1. First we will change the variables so that in the new variables the principal symbol has not a term involving τ^2. Let us write

$$p_3 = \Big(\tau + \frac{1}{3}q_1\Big)^3 + (q_2 - \frac{q_1^2}{3})\Big(\tau + \frac{1}{3}q_1\Big) + q_3 - \frac{q_1 q_2}{3} + \frac{2q_1^3}{27}.$$

Then the term without coefficient $(\tau + \frac{1}{3}q_1)$ is just $\frac{\Delta_1}{27}$. With a change of variables

$$s = t,\ y_j = f_j(t,x),\ j = 1, ..., n,$$

we may transform for small $t \geq 0$ the symbol $\tau + \frac{1}{3}q_1$ into σ. Let $\frac{1}{3}q_1 = \sum_{j=1}^n \alpha_j(t,x)\xi_j$. It is sufficient to solve the first order hyperbolic equations

$$\frac{\partial f_j}{\partial t} + \sum_{k=1}^n \alpha_k(t,x)\frac{\partial f_j}{\partial x_k} = 0,\ j = 1, ..., n$$

with initial data $f_j(0,x) = x_j$. The Jacobian $J = \det \frac{D(s,y)}{D(t,x)}$ is different from 0 for small $t \geq 0$ and $x \in \bar{U}$ and we can solve the system for $f_j, j = 1, ..., n$, for small $|t| \leq \eta$.

Now by using the same notations (t,x) for the new variables, consider the operator with principal symbol

$$\mathbf{p}_3 = \tau^3 - r_2(t,x,\xi)\tau + r_3(t,x,\xi), \tag{2.1}$$

where r_j are real-valued symbols homogeneous with respect to ξ of order $j = 2, 3$. Clearly, the symbol r_3 is just $\frac{\Delta_1}{27}$. If we have a multiple real root $\tau = \lambda(0, x_0, \xi)$ of $\mathbf{p}_3 = 0$ for $(0, x_0, \xi)$, then $\lambda(0, x_0, \xi)$ is a root of $3\tau^2 - r_2(t,x,\xi) = 0$. Thus $r_2(0, x_0, \xi) \geq 0$. Moreover, the root λ is triple if and only if $\lambda(0, x_0, \xi) = r_2(0, x_0, \xi) = 0$. If $\tau = 0$ is a triple root at $(0, x_0, \xi)$ and if the fundamental matrix of $\mathbf{p}_3$ at $(0, x_0, 0, \xi)$ has non zero real eigenvalues, then $\partial_t r_2(0, x_0, \xi) > 0$ (see Lemma 8.1 in [6]), On the other hand, the hyperbolicity of the operator implies that the discriminant Δ satisfies the inequality

$$\Delta = 4r_2^3 - 27r_3^2 \geq 0.$$

If at a point $(0, x_0, \xi_0)$ we have $r_2(0, x_0, \xi_0) < 0$, then $\Delta(0, x_0, \xi_0) < 0$ and we will have two complex conjugated roots. Thus in a neighborhood of $(0, x_0)$ we have

$$r_2(t,x,\xi) = \alpha(x,\xi) + ta_2(t,x,\xi)$$

with

$$a_2(0,x,\xi) \geq c|\xi|^2,\ c > 0,\ \alpha(x,\xi) \geq 0.$$

In the new variables, denoted by (t,x), the operator P is transformed into

$$\begin{aligned}\mathcal{P} \;=\; & D_t^3 - (ta_2(t,x,D_x) + \alpha(x,D_x))D_t + a_3(t,x,D_x) + b_2(t,x,D_x) \\ & + b_1(t,x,D_x)D_t + b_0(t,x)D_t^2 + c_1(t,x,D_x) + c_0(t,x)D_t + d_0(t,x).\end{aligned} \tag{2.2}$$

Here a_2, a_3 are homogeneous polynomials with respect to ξ respectively of order 2, 3, $b_j(t,x,\xi)$ are homogeneous polynomials with respect to ξ of order j, while b_0, c_0, d_0 are smooth functions and $c_1(t,x,\xi)$ is a first order differential operator with respect to the variable x. Without loss of generality we may assume that $a_2(t,x,D_x)$ and $\alpha(x,D_x)$ are self-adjoint positive operators. This will change the operators $b_1(t,x,D_x)D_t,\ c_0(t,x)D_t$ which is not important for our argument. According to the condition (H_2), in a neighborhood U_{x_0} of x_0 a_3 has the form

$$a_3(t,x,\xi) = t^2 b_3(t,x,D_x)$$

with a symbol $b_3(t,x,\xi)$ homogeneous of order 3 in ξ. We may also assume that $b_3(t,x,D_x)$ is a self-adjoint operator changing the lower order terms. Since we have a coefficient t^2, this is not important for the subprincipal symbol p_2' at $(0,x,\xi)$.

Next, throughout our exposition we will assume that the principal symbol $p_3(t,x,\tau,\xi)$ has a triple root $\tau=0$ for $t=0, x_0\in U, \forall \xi\in\mathbb{R}^n\setminus\{0\}$ and we examine the operator having the form (2.2) with $\alpha(x_0,\xi)=0$. We suppose in the following that $x_0=0$. Since $\alpha(x,\xi)\geq 0$, in a neighborhood of 0, we have

$$\alpha(x,\xi)=\sum_{i,j=1}^n x_i x_j f_{i,j}(x,\xi). \tag{2.3}$$

Therefore the Hamilton map F_{p_3} of p_3 for $t=0, x=x_0, \tau=0$ has non-vanishing eigenvalues if only if $a_2(0,x_0,\xi)\neq 0, \xi\in\mathbb{R}^n\setminus\{0\}$.

It is well known that the subprincipal symbol and the eigenvalues of the Hamilton map are invariant on the characteristic points $\rho\in\Sigma_1=\{\rho\in T^*(G)\setminus\{0\} : p_3(\rho)=0, dp_3(\rho)=0\}$. Thus the number Π defined by (1.7) can be expressed by the subprincipal symbol $p_2'(0,x,\xi)$ and $a_2(0,x,\xi)$.

We extend the coefficients of $a_2,\alpha,b_k, k=0,1,2,3$ and c_1,c_0,d_0 for $x\in\mathbb{R}^n$ as smooth functions. Thus in the analysis in Section 3-8 we will assume that the operator $\mathcal{P}$ is defined in $\mathbb{R}^n$. Moreover, our arguments work with small modifications if a_2,α,b_k,c_k, etc, are classical pseudodifferential operators with symbols

$$a_2(t,x,\xi)\in S^2_{1,0}(\mathbb{R}^{n+1}\times\mathbb{R}^n), b_k(t,x,\xi)\in S^k_{1,0}(\mathbb{R}^{n+1}\times\mathbb{R}^n), c_k(t,x,\xi)\in S^k_{1,0}(\mathbb{R}^{n+1}\times\mathbb{R}^n)$$

depending smoothly on the parameter t.

3. Some classes of symbols

Let

$$f(t,\xi)=\frac{t}{3}+\langle\xi\rangle^{-2/3}, \tag{3.1}$$

where $\langle\xi\rangle^2=1+|\xi|^2$. Then clearly f is a symbol in the class $S^0_{1,2/3}$, when derivatives with respect to t are considered, but it is in the class $S^0_{1,0}$ if t is just a parameter and no derivatives with respect to t are involved.

It will be convenient for us to use the Weyl calculus formalism, in the variables x, in order to establish an *a priori* estimate for the operator we deal with. From now on t will be regarded as a non-negative parameter.

Let $\varepsilon>0$ be a small positive number. We consider the metric in $T^*\mathbb{R}^n$ defined by

$$g^\varepsilon_{(x,\xi)}=\varepsilon^2|dx|^2+\langle\xi\rangle^{-2}|d\xi|^2 \tag{3.2}$$

which is almost the classical $(1,0)-$metric. In the following we will write g, when there is no ambiguity. It is well known that g is a slowly varying metric.

Let N be a positive integer. In what follows the size of N is determined in terms of the problem. We define the function

$$m^t_N(\xi)=f^{-N}(t,\xi) \tag{3.3}$$

and it is trivial to verify that m^t_N is an order function. Then we may define the classes $S(m^t_N,g^\varepsilon)$ of symbols in the standard way.

We point out explicitly that t is just a parameter at this level and that if there is no ambiguity we may omit it in our notation. We have the following

Proposition 3.1. *$f^{-N}(t,\xi) \in S(m_N^t, g^\varepsilon)$.*

Proof. Of course we must check only ξ-derivatives. We have that

$$\partial_{\xi_j} f^{-N}(t,\xi) = -N f^{-N}(t,\xi) \frac{\langle\xi\rangle^{-2/3}}{f(t,\xi)} \left(-\frac{2}{3}\right) \frac{\xi_j}{\langle\xi\rangle} \langle\xi\rangle^{-1}.$$

Hence we have the estimate $|\partial_{\xi_j} f^{-N}(t,\xi)| \leq C_N f^{-N}(t,\xi)\langle\xi\rangle^{-1}$. A simple iteration concludes the proof. □

Remark 3.1. *In particular we deduce that*

$$\partial_\xi^\alpha f^{-N}(t,\xi) = \mathcal{O}\left(N^{|\alpha|} f^{-N}(t,\xi)\langle\xi\rangle^{-|\alpha|}\right).$$

Given a symbol $a(t,x,\xi) \in S(m_N^t, g^\varepsilon)$, which we may also denote by $a^t(x,\xi)$, the Weyl pseudo-differential operator associated with it is defined by the formula

$$a^{tw}u(x) = (2\pi)^{-n} \int\int e^{i\langle x-y,\xi\rangle} a^t\left(\frac{x+y}{2}, \xi\right) u(y)dy\, d\xi.$$

We recall the composition rule for two symbols (see e.g. [5]). Define

$$g^\sigma(w) = \sup_{w'} \frac{|\sigma(w,w')|^2}{g(w')}.$$

Here σ denotes the symplectic form defined on $T(T^*\mathbb{R}^n) \times T(T^*\mathbb{R}^n)$, which in our local coordinates is given by

$$\sigma\left((x,\xi),(y,\eta)\right) = \langle y,\xi\rangle - \langle x,\eta\rangle.$$

Theorem 3.1 (Theorem 18.5.4, [5])**.** *Let g be a temperate metric with $g \leq g^\sigma$ and let m_1, m_2 be $(\sigma,\, g)$ temperate order functions. Let $a_j \in S(m_j, g)$, $j = 1, 2$. Then the composition of the associated pseudodifferential operators is associated to a symbol map $(a_1, a_2) \mapsto a = a_1 \# a_2$ from $S(m_1, g) \times S(m_2, g)$ to $S(m_1 m_2, g)$ and a is defined by*

$$a(x,\xi) = \exp\left(\frac{i}{2}\sigma(D_x, D_\xi; D_y, D_\eta)\right) a_1(x,\xi)a_2(y,\eta)\Big|_{(x,\xi)=(y,\eta)}. \tag{3.4}$$

Let

$$h(x,\xi)^2 = \sup_w \frac{g_{x,\xi}(w)}{g^\sigma_{x,\xi}(w)}, \tag{3.5}$$

then we have that for every integer M the map associating a_1, a_2 to the remainder term

$$a_1\#a_2(x,\xi) - \sum_{j<M} \frac{(i\sigma(D_x, D_\xi; D_y, D_\eta))^j}{2^j j!} a_1(x,\xi)a_2(y,\eta)$$

evaluated on the diagonal $(x,\xi) = (y,\eta)$, is continuous with values in $S(h^M m_1 m_2, g)$.

Remark 3.2. *We explicitly note that the above formula (3.4) reduces to the usual symbol composition formula (i.e. with no effect of the Weyl operator definition) if a_1 or a_2 does not depend on x; thus (3.4) reduces to*

$$a(x,\xi) = \sum_{|\alpha|\geq 0} \frac{1}{\alpha!} \partial_\xi^\alpha a_1(\xi) D_x^\alpha a_2(x,\xi),$$

or the analogous symmetric formula if a_2 *is independent of* x.

Note that a different way of writing the above formula is

$$a(x,\xi) = \exp\left(\frac{i}{2}\langle D_y, D_\xi\rangle\right) a_1(\xi)a_2(y,\eta)\big|_{(x,\xi)=(y,\eta)}.$$

Remark 3.3. *An easy and explicit calculation yields that*

$$(g^\varepsilon_{x,\xi})^\sigma = \langle\xi\rangle^2|dx|^2 + \varepsilon^{-2}|d\xi|^2, \tag{3.6}$$

and consequently the function h *is given by*

$$h(x,\xi) = \frac{\varepsilon}{\langle\xi\rangle}. \tag{3.7}$$

Evidently in this case

$$g^\varepsilon_{x,\xi} \leq (g^\varepsilon_{x,\xi})^\sigma.$$

We need to have a better control of this remainder terms in order to estimate the composition symbol with respect to the parameter N introduced above.

We recall a result due to J.–M. Bony, [1], according to which the composition $a_1\#a_2$ is written as a finite sum plus a remainder term:

$$a_1\#a_2(x,\xi) = \sum_{p=0}^{M-1}\frac{1}{p!}\left(\frac{i}{2}\sigma(D_x,D_\xi;D_y,D_\eta)\right)^p a_1(x,\xi)a_2(y,\eta)\big|_{(x,\xi)=(y,\eta)} + R_M(a_1,a_2)(x,\xi), \tag{3.8}$$

where

$$R_M(a_1,a_2)(x,\xi) = \int_0^1 \frac{(1-\theta)^{M-1}}{(M-1)!} \cdot \frac{1}{(\pi\theta)^{2n}}\int\int e^{-(2i/\theta)\sigma((x,\xi)-(t,\tau),(x,\xi)-(y,\eta))} \cdot \left(\frac{i}{2}\sigma(D_t,D_\tau;D_y,D_\eta)\right)^M a_1(t,\tau)a_2(y,\eta)dtd\tau dyd\eta d\theta \tag{3.9}$$

Proposition 3.2. *Assume that* $a_i \in S(m_i, g)$, *where* g *is a slowly varying, temperate metric such that* $g \leq g^\sigma$. *Then both* R_M *and the restriction to the diagonal of*

$$\sigma((D_x,D_\xi);(D_y,D_\eta))a_1(x,\xi)a_2(y,\eta)$$

belong to $S(m_1m_2h^M, g)$.

4. Scaling and multiplier

To obtain an a priori estimate and to deal with the lower order terms we introduce a scaling

$$t = \varepsilon^{2/3}s, \qquad x = \varepsilon y, \qquad \varepsilon > 0. \tag{4.1}$$

Multiplying by ε^2, we obtain an operator

$$\begin{aligned}\mathcal{P} = D_s^3 - sa_2(\varepsilon^{2/3}s,\varepsilon y,D_y)D_s + b_2(\varepsilon^{2/3}s,\varepsilon y,D_y) - \varepsilon^{-2/3}\alpha(\varepsilon y,D_y)D_s \\ +\varepsilon^{1/3}\Big[s^2b_3(\varepsilon^{2/3}s,\varepsilon y,D_y) + b_1(\varepsilon^{2/3}s,\varepsilon y,D_y)D_s\Big] + \varepsilon^{2/3}b_0(\varepsilon^{2/3}s,\varepsilon y)D_s^2 \\ +\varepsilon c_1(\varepsilon^{2/3}s,\varepsilon y,D_y) + \varepsilon^{4/3}c_0(\varepsilon^{2/3}s,\varepsilon y)D_s + \varepsilon^2 d_0(\varepsilon^{2/3}s,\varepsilon y).\end{aligned} \tag{4.2}$$

Here we applied (2.2) and we use for simplicity the same notation $\mathcal{P}$ for the transformed operator. Moreover, $a_2,\ \alpha, b_k, k = 0, 1, 2, 3$, etc. are the symbols of Section 2.

Since we are interested in obtaining an estimate for $0 \le t_0 \le t \le T$ with initial conditions on $t_0 = 0$, with sufficiently small $T > 0$, we may think of ε as a parameter which is going to be chosen sufficiently small; actually it will be fixed below as $\varepsilon = \mathcal{O}(\frac{1}{N})$, where $N = \frac{13}{2}\Pi + N_0$ and Π was defined in the Introduction.

We may also return to the notation (t, x) for the time and space variables respectively, without any risk of misunderstanding. Let

$$P_0 = D_t^3 - ta_2(0, \varepsilon x, D_x)D_t + b_2(0, \varepsilon x, D_x) \tag{4.3}$$

be the leading term in $\mathcal{P}$ having no factors depending on ε.

It is convenient to use the following notation:

$$a_2^\varepsilon(t, x, D_x) = a_2(\varepsilon^{2/3}t, \varepsilon x, D_x),\ \alpha^\varepsilon = \alpha(\varepsilon x, D_x), \tag{4.4}$$

and

$$b_j^\varepsilon(t, x, D_x) = b_j(\varepsilon^{2/3}t, \varepsilon x, D_x),\ c_j^\varepsilon(t, x, D_x) = c_j(\varepsilon^{2/3}t, \varepsilon x, D_x) \tag{4.5}$$

for the differential operators appearing in the definition (4.2) of $\mathcal{P}$, emphasizing the dependence on the parameter ε.

For $u,\ v \in C_0^\infty(\overline{\mathbb{R}^+} \times \mathbb{R}^n)$, we denote by

$$\langle u, v\rangle = \int_\Omega u(t, x)\bar{v}(t, x)dx,$$

the usual scalar product in $L^2(\mathbb{R}^n)$ w.r.t. the space variables x. Also we denote by $\|v(t, .)\|_k$ the norms in the spaces $H^k(\mathbb{R}^n)$.

In order to deduce an energy estimate, we need a second order multiplier operator. In what follows we use the multiplier

$$M(t, x, D_t, D_x) = \psi(t)\Big(D_t^2 - \theta ta_2^\varepsilon(t, x, D_x) - \theta\varepsilon^{-2/3}\alpha^\varepsilon(x, D_x)\Big), \tag{4.6}$$

where θ denote a positive constant to be chosen later and $\psi(t) = \frac{e^{-2\lambda t}}{t}$, $\lambda > 0$. Clearly, we have the inequalities

$$-\psi'(t) > \lambda\psi(t),\ -\psi'(t) > \frac{e^{-2\lambda t}}{t^2} > \frac{e^{-4\lambda t}}{t^2} = \psi^2(t). \tag{4.7}$$

For $u \in C_0^\infty(\overline{\mathbb{R}^+} \times \mathbb{R}^n)$ and $0 \le t_0 \le t \le T$ we compute the expression

$$-2\,\mathrm{Im}\langle f^{-2N}(t, D_x)\mathcal{P}u, Mu\rangle.$$

Here $f^{-2N}(t, D_x)$ denotes the pseudodifferential operator whose symbol is $f^{-2N}(t, \xi) \in S(m_{2N}^t, g^\varepsilon)$. We suppose in addition that $u(t_0, x) = u_t(t_0, x) = u_{tt}(t_0, x) = 0$. Thus in the case $t_0 = 0$ the terms with $\psi(t)$ have a sense for $t = 0$.

We have

$$
\begin{aligned}
-2\operatorname{Im}\langle f^{-2N}(t,D_x)\mathcal{P}u, Mu\rangle &= 2\operatorname{Re}\langle \psi f^{-2N}\left(\partial_t^3 + ta_2^\varepsilon\partial_t\right)u, \left(\partial_t^2+\theta ta_2^\varepsilon\right)u\rangle \\
&+ 2\operatorname{Re}\theta\varepsilon^{-4/3}\langle\psi f^{-2N}\alpha^\varepsilon\partial_t u, \alpha^\varepsilon u\rangle \\
&+ 2\theta\varepsilon^{-2/3}\operatorname{Re}\langle\psi f^{-2N}\left(\partial_t^3+ta_2^\varepsilon\partial_t\right)u, \alpha^\varepsilon u\rangle \\
&+ 2\operatorname{Re}\varepsilon^{-2/3}\langle\psi f^{-2N}\alpha^\varepsilon\partial_t u, \left(\partial_t^2+\theta ta_2^\varepsilon\right)u\rangle \\
&+ 2\operatorname{Im}\langle\psi f^{-2N}\varepsilon^{1/3}t^2 b_3^\varepsilon u, \left(\partial_t^2+\theta ta_2^\varepsilon+\theta\varepsilon^{-2/3}\alpha^\varepsilon\right)u\rangle \\
&+ 2\operatorname{Im}\langle\psi f^{-2N}b_2^\varepsilon u, \left(\partial_t^2+\theta ta_2^\varepsilon+\theta\varepsilon^{-2/3}\alpha^\varepsilon\right)u\rangle + \text{lower order terms} \\
&= \sum_{j=1}^{5} I_j + \sum_{j=1}^{4} J_j + \sum_{k=1}^{3} A_k + \sum_{\nu=1}^{3} B_\nu + \text{lower order terms}. \qquad (4.8)
\end{aligned}
$$

Here I_j, J_j denote the terms arising from the scalar product with the operator $D_t^3 - (ta_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)D_t$, the A_k come from the third order operator w.r.t. D_x and finally the B_ν originate from the lower order term b_2^ε. Moreover, we denoted by "lower order terms" the terms of order 1 or 2 involving the operators b_1, d_0, d_1. It will be evident after the discussion below that they do not have any influence whatsoever on the energy estimate for $\mathcal{P}$ that we are going to deduce and hence, to avoid burdening the exposition with useless details we omit a discussion of those terms.

In the next section we are going to estimate each term I_j with the purpose of putting in evidence a positive energy containing the weight f^{-N} as well as $f^{-N-1/2}$.

5. Estimate of the terms in (4.8)

5.1. **Estimate of I_1.** For the term I_1 we have

$$
\begin{aligned}
-2\operatorname{Im}\langle\psi f^{-2N}D_t^3u, D_t^2u) &= \\
2\operatorname{Re}\langle\psi f^{-2N}u''', u''\rangle &= \partial_t\left(\psi\|f^{-N}u''\|^2\right) + 2N/3\psi\|f^{-N-1/2}u''\|^2 - \psi'\|f^{-N}u''\|^2. \qquad (5.1)
\end{aligned}
$$

Here we write, as we did in the preceding section, f^{-2N} instead of $\operatorname{Op}(f^{-2N})$, for the sake of simplicity.

Note that we have used the fact that f, or rather its powers, is self adjoint as an operator w.r.t. the x variables when acting on smooth functions with compact support.

5.2. **Estimate of I_3 and J_3.** Due to Proposition 3.1, we have that, as a symbol, $f^{-2N} \in S(m_{2N}^t, g^\varepsilon)$, where ε is a positive parameter to be chosen below and the variable t can be regarded, for the time being, as a parameter. The order function m_{2N}^t has been defined in (3.3).

Taking into account that we performed a dilation by ε, we conclude that

Proposition 5.1. *The symbols a_2^ε, α^ε, b_2^ε belong to $S(\langle\xi\rangle^2, g^\varepsilon)$, as symbols in the x variables. It is then straightforward to show that actually*

$$
\varepsilon^{-\frac{2}{3}j}\partial_t^j a(t,x,\xi) \in S(\langle\xi\rangle^2, g^\varepsilon), \qquad (5.2)
$$

where a denotes $a_2^\varepsilon, \alpha^\varepsilon$ or b_2^ε.

A typical situation we encounter in the estimate of I_j is the evaluation of a norm or scalar product involving a commutator. We have

Proposition 5.2. *The commutator*

$$[a_2^\varepsilon(t,x,D_x), f^{-2N}] \tag{5.3}$$

has a symbol in $S(f^{-2N}N\varepsilon\langle\xi\rangle, g^\varepsilon)$.

Corollary 5.1. *If* $0<\varepsilon\le\varepsilon_0$, *where* ε_0 *denotes a suitably small positive number depending on* N, *then the commutator in* (5.3) *can be written as*

$$[a_2^\varepsilon(t,x,D_x), f^{-2N}] = f^{-2N}\gamma_1^\varepsilon(t,x,D_x), \tag{5.4}$$

where $\gamma_1^\varepsilon \in S(\langle\xi\rangle, g^\varepsilon)$.

Proof of Proposition 5.2. Since f^{-2N} does not depend on x, the bracket can be written as a product:

$$\mathrm{symb}\left(\left[a_2^\varepsilon(t,x,D_x), f^{-2N}(t,D_x)\right]\right) = a_2^\varepsilon f^{-2N} - f^{-2N}\# a_2^\varepsilon,$$

where symb(b) denotes the symbol of the operator b.

Using Proposition 3.2, as well as definitions (3.2) and (3.5), we obtain that the r.h.s. of the above identity belongs to $S(f^{-2N}\varepsilon\langle\xi\rangle, g^\varepsilon)$. □

Proof of Corollary 5.1. Choosing $M=1$ in (3.9), we get

$$\begin{aligned}\sigma&\left(\left[a_2^\varepsilon(t,x,D_x), f^{-2N}(t,D_x)\right]\right)(t,x,\xi)\\ &= \int_0^1 \frac{1}{(\pi\theta)^{2n}}\int\int e^{-(2i/\theta)\sigma((x,\xi)-(z,\zeta),(x,\xi)-(y,\eta))}\frac{i}{2}D_z a_2^\varepsilon(t,z,\zeta)\\ &\qquad\qquad \cdot D_\eta f^{-2N}(t,\eta)\,dz\,d\zeta\,dy\,d\eta\,d\theta.\end{aligned}$$

Since

$$\partial_z a_2^\varepsilon \partial_\eta f^{-2N} = \frac{4N}{3}\varepsilon\langle(\partial_z a_2)^\varepsilon, \frac{\eta}{\langle\eta\rangle}\rangle f^{-2N}\langle\eta\rangle^{-1}\frac{\langle\eta\rangle^{-2/3}}{t+\langle\eta\rangle^{-2/3}},$$

we see that besides the order function $f^{-2N}\varepsilon\langle\xi\rangle$ we have also a factor N, which justifies the presence of ε. Here we used the notation $(\partial_z a_2)^\varepsilon$ to denote the symbol $(\partial_z a_2)(\varepsilon^{2/3}t, \varepsilon z, \xi)$. See also definition (4.4). □

Due to the above statements we may conclude that

$$[f^{-2N}, a_2^\varepsilon] = f^{-2N}\alpha_1^\varepsilon, \tag{5.5}$$

for some first order symbol α_1^ε. Therefore

$$\begin{aligned} I_3 &= e^{-2\lambda t}\left(\langle f^{-2N}a_2^\varepsilon u', u''\rangle + \langle u'', \left(a_2^\varepsilon f^{-2N} + f^{-2N}\alpha_1^\varepsilon\right)u'\rangle\right)\\ &= e^{-2\lambda t}\left(\langle f^{-2N}a_2^\varepsilon u', u''\rangle + \langle f^{-2N}a_2^\varepsilon u'', u'\rangle + \langle f^{-2N}u'', \alpha_1^\varepsilon u'\rangle\right)\\ &= e^{-2\lambda t}\partial_t\langle f^{-2N}a_2^\varepsilon u', u'\rangle + \frac{2N}{3}e^{-2\lambda t}\langle f^{-2N-1}a_2^\varepsilon u', u'\rangle\\ &\quad - e^{-2\lambda t}\langle f^{-2N}\partial_t\left(a_2^\varepsilon\right)u', u'\rangle + e^{-2\lambda t}\langle f^{-2N}u'', \alpha_1^\varepsilon u'\rangle\\ &= \partial_t\langle e^{-2\lambda t}f^{-2N}a_2^\varepsilon u', u'\rangle + 2\lambda\langle e^{-2\lambda t}f^{-2N}a_2^\varepsilon u', u'\rangle + \frac{2N}{3}e^{-2\lambda t}\langle f^{-2N-1}a_2^\varepsilon u', u'\rangle\\ &\quad + I_{3,1} + I_{3,2}.\end{aligned} \tag{5.6}$$

Here we denoted by $u' = \partial_t u$ and $u'' = \partial_t^2 u$. Moreover $\partial_t (a_2^\varepsilon)$ denotes the operator whose symbol (or coefficients in the differential case) are the t-derivative of a_2^ε.

Repeating the same argument, we obtain

$$\begin{aligned}
\varepsilon^{4/3} I_5 &= -2\theta \operatorname{Im}\langle \psi f^{-2N} \alpha^\varepsilon D_t u, \alpha^\varepsilon u\rangle \\
&= 2\theta \operatorname{Re}\langle \psi f^{-2N} \alpha^\varepsilon u', \alpha^\varepsilon u\rangle \\
&= \theta[\langle \psi f^{-2N} \alpha^\varepsilon u', \alpha^\varepsilon u\rangle + \langle \alpha^\varepsilon u, \psi f^{-2N} \alpha^\varepsilon u'\rangle] \\
&= \theta \partial_t \langle \psi f^{-2N} \alpha^\varepsilon u, \alpha^\varepsilon u\rangle - \theta \psi' \langle f^{-2N} \alpha^\varepsilon u, \alpha^\varepsilon u\rangle + 2N/3\theta \langle \psi f^{-2N-1} \alpha^\varepsilon u, \alpha^\varepsilon u\rangle.
\end{aligned} \tag{5.7}$$

5.3. **Estimate of I_4.** Let us consider I_4. We have

$$\begin{aligned}
I_4 &= -2 \operatorname{Im} \psi \langle f^{-2N} t a_2^\varepsilon D_t u, \theta t a_2^\varepsilon u\rangle \\
&= 2 e^{-2\lambda t} \operatorname{Re}\langle f^{-2N} a_2^\varepsilon u', \theta t a_2^\varepsilon u\rangle \\
&= \theta \partial_t \Big(e^{-2\lambda t} \langle f^{-2N} a_2^\varepsilon u, t a_2^\varepsilon u\rangle \Big) + 2\theta t \lambda e^{-2\lambda t} \langle f^{-2N} a_2^\varepsilon u, a_2^\varepsilon u\rangle \\
&\quad + \theta 2N/3 t e^{-2\lambda t} \langle f^{-N-1/2} a_2^\varepsilon u, a_2^\varepsilon u\rangle - \theta e^{-2\lambda t} \langle f^{-2N} a_2^\varepsilon u, a_2^\varepsilon u\rangle \\
&\quad - \theta e^{-2\lambda t} \langle f^{-2N} (a_2^\varepsilon)_t u, t a_2^\varepsilon u) - \theta e^{-2\lambda t} \langle f^{-2N} a_2^\varepsilon u, t (a_2^\varepsilon)_t u\rangle.
\end{aligned} \tag{5.8}$$

Here we just used the fact that both f^{-2N} and a_2^ε are self adjoint in $L^2(\Omega)$, t being a parameter at this stage.

5.4. **Estimate of I_2 and J_1.** Let us consider the expression for I_2, see (4.8),

$$\begin{aligned}
I_2 &= 2 \operatorname{Re} e^{-2\lambda t} \langle f^{-2N} \partial_t^3 u, \theta a_2^\varepsilon u\rangle \\
&= \theta e^{-2\lambda t} \left(\langle f^{-2N} u''', a_2^\varepsilon u\rangle + \langle a_2^\varepsilon u, f^{-2N} u'''\rangle \right) \\
&= \theta e^{-2\lambda t} \partial_t \Big(\langle f^{-2N} u'', a_2^\varepsilon u\rangle + \langle a_2^\varepsilon u, f^{-2N} u''\rangle - \langle f^{-2N} u', a_2^\varepsilon u'\rangle \Big) \\
&\quad + \frac{2N}{3} \theta e^{-2\lambda t} \operatorname{Re} \Big(\langle f^{-2N-1} u'', a_2^\varepsilon u\rangle + \langle a_2^\varepsilon u, f^{-2N-1} u''\rangle \\
&\quad - \langle f^{-2N-1} u', a_2^\varepsilon u'\rangle \Big) + \theta e^{-2\lambda t} 2 \operatorname{Re}\langle f^{-2N} \tilde{\alpha}_1^\varepsilon u', u''\rangle \\
&\quad - \theta e^{-2\lambda t} \operatorname{Re} \Big(\langle f^{-2N} u'', (\partial_t a_2^\varepsilon) u\rangle + \langle (\partial_t a_2^\varepsilon) u, f^{-2N} u''\rangle - \langle f^{-2N} u', (\partial_t a_2^\varepsilon) u'\rangle \Big). \\
&= \theta \partial_t \Big(2 e^{-2\lambda t} \operatorname{Re}\langle f^{-2N} u'', a_2^\varepsilon u\rangle - e^{-2\lambda t} \operatorname{Re}\langle f^{-2N} u', a_2^\varepsilon u'\rangle \Big) \\
&\quad + \frac{2N}{3} \theta \Big(2 e^{-2\lambda t} \operatorname{Re}\langle f^{-2N-1} u'', a_2^\varepsilon u\rangle - e^{-2\lambda t} \operatorname{Re}\langle f^{-2N-1} u', a_2^\varepsilon u'\rangle \Big) \\
&\quad + 2\lambda \theta \Big(2 e^{-2\lambda t} \operatorname{Re}\langle f^{-2N} u'', a_2^\varepsilon u\rangle - e^{-2\lambda t} \operatorname{Re}\langle f^{-2N} u', a_2^\varepsilon u'\rangle \Big) + \sum_{k=1}^{4} I_{2,k}.
\end{aligned} \tag{5.9}$$

A few words are in order. Here $\tilde{\alpha}_1^\varepsilon$ denotes a suitable first order pseudodifferential operator originating from a commutator exactly as it occurred for the other terms above.

Moreover in deducing (5.9) the following identity has been used:

$$\begin{aligned}
&\partial_t\Big(\langle f^{-2N}u'', a_2^\varepsilon u\rangle + \langle a_2^\varepsilon u, f^{-2N}u''\rangle - \langle f^{-2N}u', a_2^\varepsilon u'\rangle\Big)\\
&\qquad = \langle f^{-2N}u''', a_2^\varepsilon u\rangle + \langle f^{-2N}u'', a_2^\varepsilon u'\rangle + \langle a_2^\varepsilon u', f^{-2N}u''\rangle + \langle a_2^\varepsilon u, f^{-2N}u'''\rangle\\
&\qquad\quad - \langle f^{-2N}u'', a_2^\varepsilon u'\rangle - \langle f^{-2N}u', a_2^\varepsilon u''\rangle + \text{terms being } \mathscr{O}(N) + \text{ terms involving } \partial_t a_2^\varepsilon.
\end{aligned}$$

Thus let us examine the first four terms in the r.h.s. above. We have

$$\begin{aligned}
&\langle f^{-2N}u''', a_2^\varepsilon u\rangle + \langle a_2^\varepsilon u', f^{-2N}u''\rangle + \langle a_2^\varepsilon u, f^{-2N}u'''\rangle - \langle f^{-2N}u', a_2^\varepsilon u''\rangle\\
&\qquad = \langle f^{-2N}u''', a_2^\varepsilon u\rangle + \langle a_2^\varepsilon u, f^{-2N}u'''\rangle + \langle [f^{-2N}, a_2^\varepsilon]u', u''\rangle\\
&\qquad = 2\,\mathrm{Re}\langle f^{-2N}u''', a_2^\varepsilon u\rangle + \langle f^{-2N}\tilde{\alpha}_1^\varepsilon u', u''\rangle.
\end{aligned}$$

To obtain the last line we used Corollary 5.1 and the fact that a_2^ε is a self-adjoint operator.

For J_1 we use the same argument and we obtain

$$\begin{aligned}
\varepsilon^{2/3}J_1 &= 2\theta\,\mathrm{Im}\langle \psi f^{-2N}D_t^3 u, \alpha^\varepsilon u\rangle\\
&= 2\theta\,\mathrm{Re}\langle \psi f^{-2N}\partial_t^3 u, \alpha^\varepsilon u\rangle\\
&= \theta\langle \psi f^{-2N}\partial_t^3 u, \alpha^\varepsilon u\rangle + \theta\langle \alpha^\varepsilon u, \psi f^{-2N}\partial_t^3 u\rangle\\
&= \theta\partial_t\Big(2\psi\,\mathrm{Re}\langle f^{-2N}u'', \alpha^\varepsilon u\rangle - \mathrm{Re}\langle \psi f^{-2N}u', \alpha^\varepsilon u'\rangle\Big)\\
&\quad - \theta\psi'\Big(2\,\mathrm{Re}\langle f^{-2N}u'', \alpha^\varepsilon u\rangle - \mathrm{Re}\langle f^{-2N}u', \alpha^\varepsilon u'\rangle\Big)\\
&\quad + \theta 2N/3\Big(2\psi\,\mathrm{Re}\langle f^{-2N-1}u'', \alpha^\varepsilon u\rangle - \mathrm{Re}\langle \psi f^{-2N-1}u', \alpha^\varepsilon u'\rangle\Big)\\
&\quad + \theta 2\psi\,\mathrm{Re}\langle f^{-2N}\beta_1 u', u''\rangle
\end{aligned}\tag{5.10}$$

with a first order operator β_1.

5.5. **Estimate of J_2 and J_4.** We have

$$\begin{aligned}
\varepsilon^{2/3}(J_2+J_4) &= -2\,\mathrm{Im}\langle \psi f^{-2N}ta_2^\varepsilon D_t u, \theta\alpha^\varepsilon u\rangle - 2\,\mathrm{Im}\langle \psi f^{-2N}\alpha^\varepsilon D_t u, \theta t a_2^\varepsilon u\rangle\\
&= 2\,\mathrm{Re}\Big[\langle \psi f^{-2N}ta_2^\varepsilon u', \theta\alpha^\varepsilon u\rangle + \langle \psi f^{-2N}\alpha^\varepsilon u', \theta t a_2^\varepsilon u\rangle\Big]\\
&= \theta 2\partial_t\Big(\mathrm{Re}\langle e^{-2\lambda t}f^{-2N}a_2^\varepsilon u, \alpha^\varepsilon u\rangle\Big) + \theta 4N/3\,\mathrm{Re}\langle e^{-2\lambda t}f^{-N-1/2}a_2^\varepsilon u, \alpha^\varepsilon u\rangle\\
&\quad + 4\theta\lambda\,\mathrm{Re}\langle e^{-2\lambda t}f^{-2N}a_2^\varepsilon u, \alpha^\varepsilon u\rangle - 2\theta\,\mathrm{Re}\langle e^{-2\lambda t}f^{-2N}(a_2^\varepsilon)_t u, \alpha^\varepsilon u\rangle.
\end{aligned}\tag{5.11}$$

5.6. **Estimate of J_3.** Consider, see (4.8),

$$\begin{aligned}
\varepsilon^{2/3}J_3 &= 2\,\mathrm{Re}\langle \psi f^{-2N}\alpha^\varepsilon u', u''\rangle\\
&= \psi\langle f^{-2N}\alpha^\varepsilon u', u''\rangle + \psi\langle u'', \alpha^\varepsilon f^{-2N}u'\rangle + \psi\,\mathrm{Re}\langle f^{-N}u'', \gamma_1 f^{-N}u'\rangle\\
&= \partial_t\langle \psi f^{-2N}\alpha^\varepsilon u', u'\rangle - \psi'\,\mathrm{Re}\langle f^{-2N}\alpha^\varepsilon u', u'\rangle + J_{3.1}\\
&\quad + 2N/3\langle \psi f^{-2N-1}\alpha^\varepsilon u', u'\rangle,
\end{aligned}$$

where

$$J_{3,1} = \mathrm{Re}\langle \psi f^{-N}u'', \gamma_1 f^{-N}u'\rangle$$

and γ_1 is a first order operator.

5.7. **Estimate of A_1.** Let us rewrite iA_1 in the following way

$$\begin{aligned} iA_1 &= 2i\,\mathrm{Im}\langle e^{-2\lambda t} f^{-2N}\varepsilon^{1/3} t b_3^{\varepsilon} u, u''\rangle \\ &= \varepsilon^{1/3} t e^{-2\lambda t}\Big(\langle f^{-2N} b_3^{\varepsilon} u, u''\rangle - \langle u'', f^{-2N} a_3^{\varepsilon} u\rangle\Big). \end{aligned}$$

We have the identity

$$\begin{aligned} \partial_t 2i\,\mathrm{Im}\langle f^{-2N} b_3^{\varepsilon} u, u'\rangle &= \langle f^{-2N} b_3^{\varepsilon} u, u''\rangle - \langle u'', f^{-2N} b_3^{\varepsilon} u\rangle + \langle f^{-2N} b_3^{\varepsilon} u', u'\rangle - \langle u', f^{-2N} b_3^{\varepsilon} u'\rangle \\ &\quad - \frac{2N}{3}\Big(\langle f^{-2N-1} b_3^{\varepsilon} u, u'\rangle - \langle u', f^{-2N-1} b_3^{\varepsilon} u\rangle\Big) \\ &\quad + \langle f^{-2N}(\partial_t b_3^{\varepsilon}) u, u'\rangle - \langle u', f^{-2N}(\partial_t b_3^{\varepsilon}) u\rangle. \end{aligned} \tag{5.12}$$

Plugging (5.12) into the above expression for iA_1, we then obtain

$$\begin{aligned} A_1 &= \varepsilon^{1/3} t e^{-2\lambda t}\partial_t\left(2\,\mathrm{Im}\langle f^{-2N} b_3^{\varepsilon} u, u'\rangle\right) + 2\varepsilon^{1/3} t e^{-2\lambda t}\frac{2N}{3}\,\mathrm{Im}\langle f^{-2N-1} b_3^{\varepsilon} u, u'\rangle \\ &\quad - \varepsilon^{1/3} t e^{-2\lambda t} 2\,\mathrm{Im}\langle f^{-2N} b_3^{\varepsilon} u', u'\rangle - 2\varepsilon^{1/3} t e^{-2\lambda t}\,\mathrm{Im}\langle f^{-2N}(\partial_t b_3^{\varepsilon}) u, u'\rangle \\ &= 2\varepsilon^{1/3}\partial_t\Big(t e^{-2\lambda t}\,\mathrm{Im}\langle f^{-2N} b_3^{\varepsilon} u, u'\rangle\Big) - 2\varepsilon^{1/3} e^{-2\lambda t}\,\mathrm{Im}\langle f^{-2N} b_3^{\varepsilon} u, u'\rangle \\ &\quad + 2\lambda\varepsilon^{1/3} t e^{-2\lambda t}\,\mathrm{Im}\langle f^{-2N} b_3^{\varepsilon} u, u'\rangle + \varepsilon^{1/3} t e^{-2\lambda t} 4N/3\,\mathrm{Im}\langle f^{-2N-1} b_3^{\varepsilon} u, u'\rangle \\ &\quad + \sum_{k=1}^{2} A_{1,k}. \end{aligned} \tag{5.13}$$

5.8. **Estimate of A_2 and A_3.** Using the calculus it is not difficult to show that there is a symbol of first order, $\hat{\alpha}_0^{\varepsilon}$, such that

$$A_2 = 2\varepsilon^{1/3}\theta t^2 e^{-2\lambda t}\,\mathrm{Im}\langle f^{-2N} b_3^{\varepsilon} u, a_2^{\varepsilon} u\rangle = \varepsilon^{1/3}\theta t^2 e^{-2\lambda t} 2\,\mathrm{Im}\langle f^{-2N}\gamma_0^{\varepsilon} a_2^{\varepsilon} u, a_2^{\varepsilon} u\rangle. \tag{5.14}$$

Here our argument is based on the fact the principal symbols of the operators $a_2^{\xi}(t,x,D_x)$ and $b_3^{\xi}(t,x,D_x)$ are real-valued. Thus we obtain

$$(a_2^{\xi}(t,x,D_x) b_3^{\xi}(t,x,D_x))^* = b_2^{\xi}(t,x,D_x) a_3^{\xi}(t,x,D_x)) + \alpha_4^{\xi}(t,x,D_x)$$

with a pseudodifferential operator α_4^{ξ} of order 4. Since $a_2^{\xi}(t,x,D_x)$ is elliptic, it is easy to find a zero order operator $\gamma_0^{\xi}(t,x,D_x)$ so that $\alpha_4^{\xi}(t,x,D_x) = (a_2^{\xi})^*\gamma_0^{\xi}(t,x,D_x) a_2^{\xi}$.

For A_3 we obtain straightforwardly that $A_3 = 2\theta\varepsilon^{-2/3}\,\mathrm{Im}\langle\psi f^{-2N}\varepsilon^{1/3} t^2 b_3^{\varepsilon} u, \alpha^{\varepsilon} u\rangle$.

6. Energies

Summarizing the expression of all terms in Section 5, we may rewrite (4.8) in the following form

$$-2\,\mathrm{Im}\langle f^{-2N}\mathcal{P}u, Mu\rangle = \partial_t\mathscr{E}_N(u) + \frac{2N}{3}\mathscr{E}_{N+1/2}(u) + 2\lambda\mathscr{E}_N(u) + \mathscr{R}, \tag{6.1}$$

where

$$\begin{aligned}\mathscr{E}_N(u) = \psi\Big[&\|f^{-N}u''\|_0^2 + (1-\theta)\operatorname{Re}\langle f^{-2N}(ta_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)u', u'\rangle \\ &+ \theta\|f^{-N}(ta_2^\varepsilon u + \varepsilon^{-2/3}\alpha^\varepsilon)u\|_0^2 + \theta 2\operatorname{Re}\langle f^{-2N}u'', (ta_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)u\rangle\Big] \\ &+ \varepsilon^{1/3}te^{-2\lambda t}2\operatorname{Im}\langle f^{-2N}b_3^\varepsilon u, u'\rangle.\end{aligned} \tag{6.2}$$

Moreover $\mathscr{R}$ includes all terms that can be considered "errors", since they do not contribute to the energy $\mathscr{E}_N$ or $\mathscr{E}_{N+1/2}$. Note that somewhat improperly we include into $\mathscr{R}$ also norms multiplied by $\frac{e^{-2\lambda t}}{t^2}$ which are positive, but require "ad hoc" treatement. Recall that we have $0 \le t_0 \le t \le T$ and we suppose that $u(t_0, x) = u_t(t_0, x) = u_{tt}(t_0, x) = 0$, so the expression with factor ψ or $\frac{e^{-2\lambda t}}{t^2}$ below make sense when $t_0 = 0$.

The quantity $\mathscr{R}$ is defined as

$$\begin{aligned}\mathscr{R} = \frac{e^{-2\lambda t}}{t^2}\Big(&\|f^{-N}u''\|_0^2 + \theta\varepsilon^{-4/3}\langle f^{-2N}\alpha^\varepsilon u, \alpha^\varepsilon u\rangle + (1-\theta)\varepsilon^{-2/3}\operatorname{Re}\langle f^{-2N}\alpha^\varepsilon u', u'\rangle \\ &+ \theta\varepsilon^{-2/3}2\operatorname{Re}\langle f^{-2N}u'', \alpha^\varepsilon u\rangle\Big) - \theta e^{-2\lambda t}\|f^{-N}a_2^\varepsilon u\|_0^2 \\ &- 2\theta\varepsilon^{-2/3}e^{-2\lambda t}\operatorname{Re}\langle f^{-2N}(a_2^\varepsilon)_t u, \alpha^\varepsilon u\rangle \\ &- 2\theta e^{-2\lambda t}\operatorname{Re}\langle f^{-2N}(a_2^\varepsilon)_t u, ta_2^\varepsilon u\rangle + e^{-2\lambda t}\left(-\langle f^{-2N}(a_2^\varepsilon)_t u', u'\rangle + \langle f^{-2N}u'', \alpha_1^\varepsilon u'\rangle\right) \\ &+ \theta e^{-2\lambda t}2\operatorname{Re}\langle f^{-2N}\tilde{\alpha}_1^\varepsilon u', u''\rangle + 2\theta\varepsilon^{-2/3}\psi\operatorname{Re}\langle\beta_1 f^{-N}u', f^{-N}u''\rangle + \varepsilon^{-2/3}\psi\operatorname{Re}\langle\gamma_1 f^{-N}u', f^{-N}u''\rangle \\ &- \theta e^{-2\lambda t}\Big(2\operatorname{Re}\langle f^{-2N}u'', (a_2^\varepsilon)_t u\rangle - \operatorname{Re}\langle f^{-2N}u', (a_2^\varepsilon)_t u'\rangle\Big) \\ &- 2\varepsilon^{1/3}e^{-2\lambda t}\operatorname{Im}\langle f^{-2N}b_3^\varepsilon u, u'\rangle - \varepsilon^{1/3}te^{-2\lambda t}2\operatorname{Im}\langle f^{-2N}b_3^\varepsilon u', u'\rangle \\ &- 2\varepsilon^{1/3}te^{-2\lambda t}\operatorname{Im}\langle f^{-2N}(\partial_t b_3^\varepsilon)u, u'\rangle \\ &+ \varepsilon^{1/3}\theta t^2 e^{-2\lambda t}2\operatorname{Im}\langle f^{-2N}\gamma_0^\varepsilon a_2^\varepsilon u, a_2^\varepsilon u\rangle \\ &+ 2\theta\varepsilon^{-1/3}e^{-2\lambda t}t\operatorname{Im}\langle f^{-2N}b_3^\varepsilon u, \alpha^\varepsilon u\rangle \\ &+ \sum_{\nu=1}^{3} B_\nu + \text{lower order terms.}\end{aligned} \tag{6.3}$$

To keep the exposition simple it is convenient to denote by $\sum_{j=1}^{10}\mathscr{R}_j$ the sum of 10 terms corresponding to the 10 lines in the expression of $\mathscr{R}$ above.

Consider the sum

$$S_k = \psi\|f^{-k}u''\|_0^2 + \theta\psi\|f^{-k}(ta_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)u\|_0^2 + 2\theta\psi\operatorname{Re}\langle f^{-k}(ta_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)u, f^{-k}u''\rangle.$$

Choose $\theta = 1/3$. Then we deduce

$$S_k = \frac{1}{3}\psi\|f^{-k}u''\|_0^2 + \frac{1}{6}\psi\|f^{-k}(ta_2^\varepsilon u + \varepsilon^{-2/3}\alpha^\varepsilon)u\|_0^2 + \frac{2}{3}\psi\|f^{-k}(u'' + \frac{1}{2}(ta_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)u))\|_0^2. \tag{6.4}$$

It is clear that

$$\|f^{-k}(u'' + \frac{1}{3}(ta_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)u)\|_0^2 \le 2\|f^{-k}(u'' + \frac{1}{2}(ta_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)u)\|_0^2 + \frac{1}{18}\|f^{-k}(ta_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)u\|_0^2$$

and we get

$$S_k \geq \frac{1}{3}\psi\|f^{-k}u''\|_0^2 + \frac{4}{27}\psi\|f^{-k}(ta_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)u)\|_0^2 + \frac{1}{3}\psi\|f^{-k}(u'' + \frac{1}{3}(ta_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)u)\|_0^2. \quad (6.5)$$

In the same way we obtain

$$\begin{aligned} \mathscr{R}_1 + \mathscr{R}_2 &= \frac{e^{-2\lambda t}}{t^2}\Big(\frac{1}{3}\|f^{-N}u''\|_0^2 + \frac{1}{6}\varepsilon^{-4/3}\|f^N\alpha^\varepsilon u\|_0^2 + \frac{2}{3}\|f^{-N}\Big(u'' + \frac{1}{2}\varepsilon^{-2/3}\alpha^\varepsilon u\Big)\|_0^2 \\ &\quad + \frac{2}{3}\varepsilon^{-2/3}\operatorname{Re}\langle f^{-2N}\alpha^\varepsilon u', \alpha^\varepsilon u'\rangle - \frac{1}{3}\|f^{-N}ta_2^\varepsilon u\|_0^2\Big) \\ &\geq \frac{e^{-2\lambda t}}{3t^2}\Big(\|f^{-N}u''\|_0^2 + \frac{4}{9}\varepsilon^{-4/3}\|f^N\alpha^\varepsilon u\|_0^2 + \|f^{-N}\Big(u'' + \varepsilon^{-2/3}\alpha^\varepsilon u\Big)\|_0^2 \\ &\quad + 2\varepsilon^{-2/3}\operatorname{Re}\langle f^{-2N}\alpha^\varepsilon u', \alpha^\varepsilon u'\rangle - \|f^{-N}ta_2^\varepsilon u\|_0^2\Big). \end{aligned}$$

To simplify the notations in the following we will write $\mathbf{a}_2^\varepsilon$ for $a_2^\varepsilon(0, x, D_x)$, while a_2^ε will denote the operator $a_2^\varepsilon(t, x, D_x)$. Therefore $a_2^\varepsilon(t, x, \xi) = \mathbf{a}_2^\varepsilon(0, x, \xi) + \varepsilon^{2/3}t\tilde{a}_2^\varepsilon(t, x, \xi)$ and we have

$$\frac{2}{3}\psi\operatorname{Re}\langle f^{-k}(ta_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)u', f^{-k}u'\rangle + \frac{1}{6}\psi\|f^{-k}(ta_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)u\|_0^2$$

$$= \frac{2}{3}\psi\operatorname{Re}\langle f^{-k}(t\mathbf{a}_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)u', f^{-k}u'\rangle + \frac{1}{6}\psi\|f^{-k}(t\mathbf{a}_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)u\|_0^2 + te^{-2\lambda t}\varepsilon^{2/3}A_k^{(2)}(u).$$

Here $t\varepsilon^{2/3}A_k^{(2)}(u)$ denote a sum of terms which have coefficient $t\varepsilon^{2/3}$. It will be easy to absorb them taking ε and t small and we will discuss this in Section 7 after the terms in $\mathscr{E}_N$ will have been conveniently prepared. To start we need the following

Proposition 6.1. *There exist positive constants C_1 and C_2, independent of the positive integer k such that for $0 < \varepsilon \leq \varepsilon_0(k)$ we have*

$$\operatorname{Re}\langle f^{-k}\mathbf{a}_2^\varepsilon v, f^{-k}v\rangle \geq C_1\|f^{-k}v\|_1^2 - C_2\|f^{-k}v\|_0^2, \quad (6.6)$$

for every $v \in C_0^\infty$. The constant C_1 depends only on the symbol $a_2^\varepsilon(0, x, \xi)$.

Proof. The proof consists in just making sure that we may commute the weight operator f^{-k} with a_2^ε and estimate the errors, which naturally depend on N. We have that

$$\langle f^{-k}a_2^\varepsilon(0, x, D_x)v, f^{-k}v\rangle = \langle\mathbf{a}_2^\varepsilon f^{-k}v, f^{-k}v\rangle + \langle[f^{-k}, \mathbf{a}_2^\varepsilon]v, f^{-k}v\rangle = X_1 + X_2.$$

Keeping in mind that a_2^ε is uniformly elliptic and using the strict Gårding inequality for it, we obtain that

$$X_1 \geq c_1\|f^{-k}v\|_1^2 - c_2\|f^{-k}v\|_0^2,$$

for two suitable positive constants c_1 and c_2 independent of k. We are thus left with X_2. By Proposition 5.2 and Corollary 5.1 we see that if ε is small enough depending on k, i.e. if $\varepsilon \leq \varepsilon_0(k)$, there is a positive constant c_3 independent of k, such that

$$|X_2| \leq c_3\|f^{-k}v\|_{1/2}^2 \leq \delta\|f^{-k}v\|_1^2 + c_3'\delta^{-1}\|f^{-k}v\|_0^2.$$

Choosing δ conveniently small, but independent of ε and k, we obtain the assertion. $\square$

To treat the negative terms, we apply the following lemma which will play a key role in the next section.

Lemma 6.1. *For $t \geq 0$ and $\xi \in \mathbb{R}^n$ we have*

$$\frac{1}{\langle\xi\rangle^2} + tf^2 \geq f^3. \tag{6.7}$$

Proof. The proof is a simple verification. In fact $f^3 = f^2 t/3 + f^2\langle\xi\rangle^{-2/3}$. The latter quantity is equal to $f^2 t/3 + \langle\xi\rangle^{-2} + \frac{2}{3}t\langle\xi\rangle^{-4/3} + \frac{t^2}{9}\langle\xi\rangle^{-2/3}$. It is clear that

$$t\Big[\frac{2}{3}\langle\xi\rangle^{-4/3} + \frac{t}{9}\langle\xi\rangle^{-2/3}\Big] \leq \frac{2}{3}tf^2$$

and this accomplishes the proof. □

To examine the term $J_{\alpha,k} = \psi\varepsilon^{-2/3}\,\mathrm{Re}\langle f^{-k}\alpha^\varepsilon u, f^{-k}u\rangle$, we use the fact that the operator α^ε is positive and write

$$J_{\alpha,k} \geq \varepsilon^{-2/3}\psi\,\mathrm{Re}\langle[f^{-k},\alpha^\varepsilon]u, f^{-k}u\rangle.$$

The symbol of the operator α^ε is $\mathscr{O}(\varepsilon^2)$ uniformly and for $0 < \varepsilon < \varepsilon_1(k)$ we obtain

$$|J_{\alpha,k}| \leq c_4\varepsilon^{1/3}\psi\|f^{-k}u\|^2_{1/2}$$

with c_4 independent on ε and k. Now an application of Lemma 6.1 yields

$$\langle\xi\rangle f^{-2k} \leq t\langle\xi\rangle f^{-2k-1} + \langle\xi\rangle^{-1}f^{-2k-3} \leq \langle\xi\rangle^{-1/3}t\langle\xi\rangle^2 f^{-2k} + \langle\xi\rangle^{-1/3}f^{-2k-2}$$

since $\langle\xi\rangle^{-2/3}f^{-1} \leq 1$. Therefore

$$|J_{\alpha,k}| \leq c_4\varepsilon^{1/3}e^{-2\lambda t}\|f^{-k}u\|^2_{1/2} + c_4\varepsilon^{1/3}\psi\|f^{-k-1}u\|^2_0$$

and for small $0 < \varepsilon \leq \varepsilon_1(k)$ taking into account (6.6), we get

$$\begin{aligned} 2\psi|\,\mathrm{Re}\langle f^{-k}(t\mathbf{a}_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)u', f^{-k}u'\rangle| \quad &\geq \quad C_3e^{-2\lambda t}\|f^{-k}u'\|^2_1 \\ -C_2e^{-2\lambda t}\|f^{-k}u'\|^2_0 - c_4\varepsilon^{1/3}\psi\|f^{-k-1}u'\|^2_0. \end{aligned} \tag{6.8}$$

We introduce now the energy by the following

Definition 6.1. *For a non negative integer k we define the k-th energy as*

$$\begin{aligned} E_k(u) = \psi\Big(&\frac{1}{3}\|f^{-k}u''\|^2_0 + \frac{2}{3}\,\mathrm{Re}\langle f^{-k}(t\mathbf{a}_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)u', f^{-k}u'\rangle \\ &+ \frac{1}{6}\|f^{-k}(t\mathbf{a}_2^\varepsilon u + \varepsilon^{-2/3}\alpha^\varepsilon)u\|^2_0 + \frac{2}{3}\|f^{-k}(u'' + \frac{1}{2}(t\mathbf{a}_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)u)\|^2_0\Big). \end{aligned}$$

For the expression of the energy $\mathscr{E}_k(u)$, $k = N$, $N + 1/2$ we have, with the notations above, the representation

$$\mathscr{E}_k(u) = E_k(u) + t\varepsilon^{2/3}e^{-2\lambda t}A_k^{(2)}(u) + 2\varepsilon^{1/3}te^{-2\lambda t}\,\mathrm{Im}\langle f^{-k}b_3^\varepsilon u, f^{-k}u'\rangle.$$

The last two terms on the right hand side can be considered as small perturbations.

Moreover in the energy $E_{N+1/2}(u)$ we have no positive terms involving

$$\|f^{-N-3/2}u\|^2_1, \quad \|f^{-N-3/2}u'\|^2_0 \quad \text{and} \quad \|f^{-N-5/2}u\|^2_0.$$

These turn out very useful in order to absorb a number of "errors" using Lemma 6.1.

For this purpose we will obtain several new positive terms exploiting a part of the energy $E_{N+1/2}(u)$. The same argument applies to the energy $E_N(u)$.

Let us now consider the following identity, where k is a positive integer and g denotes a smooth function in the same class as u:

$$\psi f^{-2k} 2\operatorname{Re} g'\bar{g} = \partial_t\Big(\psi f^{-2k}|g|^2\Big) - \psi' f^{-2k}|g|^2 + 2k/3\psi f^{-2k-1}|g|^2$$

which implies

$$\psi f^{-2k+1}|g'|^2 \geq \partial_t\Big(\psi f^{-2k}|g|^2\Big) - \psi' f^{-2k}|g|^2 + (2k/3-1)\psi f^{-2k-1}|g|^2.$$

Taking $g = \partial_t u$, $k = N+1$, we have

$$\psi f^{-2N-1}|u''|^2 \geq \partial_t\Big(\psi f^{-2N-2}|u'|^2\Big) - \psi' f^{-2N-2}|u'|^2 + (2N/3-1/3)\psi f^{-2N-3}|u'|^2, \tag{6.9}$$

while taking $g = u$, $k = N+2$, we get

$$\psi f^{-2N-3}|u'|^2 \geq \partial_t\Big(\psi f^{-2N-4}|u|^2\Big) - \psi' f^{-2N-4}|u|^2 + (2N/3+1/3)\psi f^{-2k-5}|u|^2. \tag{6.10}$$

Combining (6.9) and (6.10), we get

$$\begin{aligned}\psi\|f^{-N-1/2}u''\|^2 \geq \partial_t\Big(\psi\|f^{N-1}u'\|^2\Big) - \psi'\|f^{-N-1}u'\|_0^2 \\ + (2N/3-4/3)\psi\|f^{-N-3/2}u'\|_0^2 + \partial_t\Big(\psi\|f^{-N-2}u\|_0^2\Big) \\ - \psi'\|f^{-N-2}u\|_0^2 + \frac{2N+1}{3}\psi\|f^{N-5/2}u\|_0^2.\end{aligned} \tag{6.11}$$

Also we obtain easily the inequality

$$\begin{aligned}e^{-2\lambda t}\|f^{-N-\frac{1}{2}}\partial_t u\|_1^2 \quad\geq\quad & \partial_t\left(e^{-2\lambda t}\|f^{-N-1}u\|_1^2\right) + 2\lambda e^{-2\lambda t}\|f^{-N-1}u\|_1^2 \\ & + \frac{1}{3}(2N+1)e^{-2\lambda t}\|f^{-N-\frac{3}{2}}u\|_1^2.\end{aligned} \tag{6.12}$$

By using the calculus of pseudodifferential operators, Proposition 6.1 and (6.8), we may write

$$\begin{aligned}2\psi\operatorname{Re}\langle f^{-N-1/2}(t\mathbf{a}_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)u', f^{-N-1/2}u'\rangle \geq 2C_3 e^{-2\lambda t}\|f^{-N-1/2}u'\|_1^2 \\ - C_4 e^{-2\lambda t}\|f^{-N-1/2}u'\|_0^2 - C_5\varepsilon^{1/3}\psi\|f^{-N-3/2}u'\|_0^2.\end{aligned} \tag{6.13}$$

Next, taking into account the inequalities (6.11), (6.12) and (6.13), for small ε we get

$$\begin{aligned}&\frac{1}{3}\psi\|f^{-N-1/2}u''\|_0^2 + \frac{2}{3}\psi\operatorname{Re}\langle f^{-N-1/2}(t\mathbf{a}_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)u', f^{-N-1/2}u'\rangle \\ &\geq \frac{1}{3}\Big[\partial_t\left(\psi\|f^{-N-1}u'\|_0^2\right) - \psi'\|f^{-N-1}u'\|_0^2 \\ &+ (2N-5)/3\psi\|f^{-N-\frac{3}{2}}u'\|_0^2 + \partial_t\left(\psi\|f^{-N-2}u\|_0^2\right) - \psi'\|f^{-N-2}u\|_0^2 \\ &+ (2N+1)/3\psi\|f^{-N-\frac{5}{2}}u\|_0^2\Big] \\ &+ \frac{1}{3}C_3\Big[\partial_t\left(e^{-2\lambda t}\|f^{-N-1}u\|_1^2\right) + 2\lambda e^{-2\lambda t}\|f^{-N-1}u\|_1^2 + (2N+1)/3e^{-2\lambda t}\|f^{-N-\frac{3}{2}}u\|_1^2\Big] \\ &+ \frac{1}{3}C_3 e^{-2\lambda t}\|f^{-N-1/2}u'\|_1^2 - \frac{1}{3}C_4 e^{-2\lambda t}\|f^{-N-1/2}u'\|_0^2.\end{aligned} \tag{6.14}$$

Here the term $-C_5\varepsilon^{1/3}\psi\|f^{-N-3/2}u'\|_0^2$ has been absorbed by diminishing the coefficient in the term $\frac{2N-5}{3}\psi\|f^{-N-3/2}u'\|_0^2$ (compare with $\frac{2N-4}{3}\psi\|f^{-N-3/2}u'\|_0^2$ in the above inequality.)

Therefore, using (6.14), we have for small t and large λ the estimate

$$\begin{aligned}
&\frac{5N}{9}\psi\Big[\frac{1}{3}\|f^{-N-1/2}u'\|_0^2+\frac{2}{3}\operatorname{Re}\langle f^{-N-1/2}(t\mathbf{a}_2^\varepsilon+\varepsilon^{-2/3}\alpha^\varepsilon)u',f^{-N-1/2}u'\rangle\Big]\\
&\qquad+\frac{N}{9}\psi\left(\frac{1}{3}\|f^{-N-1/2}u''\|_0^2+\frac{2}{3}\operatorname{Re}\langle f^{-N-1/2}(t\mathbf{a}_2^\varepsilon+\varepsilon^{-2/3}\alpha^\varepsilon)u',f^{-N-1/2}u'\rangle\right)\\
&\qquad\geq\partial_t\Big(\frac{N}{27}\psi\|f^{-N-1}u'\|_0^2+\frac{N}{27}\psi\|f^{-N-2}u\|_0^2+\frac{N}{27}C_3e^{-2\lambda t}\|f^{-N-1}u\|_1^2\Big)\\
&\qquad+\frac{5N}{9}\Big[\frac{1}{3}\psi\|f^{-N-1/2}u''\|_0^2+\frac{2}{3}e^{-2\lambda t}\operatorname{Re}\langle f^{-N-1/2}(t\mathbf{a}_2^\varepsilon+\varepsilon^{-2/3}\alpha^\varepsilon)u',f^{-N-1/2}u'\rangle\Big]\\
&\qquad+\frac{N}{9}\Big[-\frac{\psi'}{3}\|f^{-N-1}u'\|_0^2+\frac{2N-5}{9}\psi\|f^{-N-3/2}u'\|_0^2-\frac{\psi'}{3}\|f^{-N-2}u\|_0^2\\
&\qquad\qquad+\frac{2N+1}{9}\psi\|f^{-N-5/2}u\|_0^2+\frac{2C_3}{3}\lambda e^{-2\lambda t}\|f^{-N-1}u\|_1^2\\
&\qquad\qquad+\frac{(2N+1)C_3}{9}e^{-2\lambda t}\|f^{-N-3/2}u\|_1^2+\frac{C_3}{3}e^{-2\lambda t}\|f^{-N-1/2}u'\|_1^2\Big].
\end{aligned}\tag{6.15}$$

Going back to the operator $\mathcal{P}$, we have

$$\begin{aligned}
-2\operatorname{Im}\langle f^{-2N}\mathcal{P}u,Mu\rangle&=2\psi\operatorname{Im}\langle f^{-2N}\mathcal{P}u,(u''+\frac{1}{3}\varepsilon^{-2/3}\alpha^\varepsilon u)\rangle+\frac{2}{3}e^{-2\lambda t}\operatorname{Im}\langle f^{-2N}\mathcal{P}u,a_2^\varepsilon u\rangle\\
&\leq 7e^{-2\lambda t}\|f^{-N}\mathcal{P}u\|_0^2+\frac{e^{-2\lambda t}}{6t^2}\Big(\|f^{-N}(u''+\frac{1}{3}\varepsilon^{-2/3}\alpha^\varepsilon u)\|_0^2+\frac{2}{3}\|f^{-N}ta_2^\varepsilon u\|_0^2\Big).
\end{aligned}$$

Therefore, exploiting (6.6), we obtain

$$\begin{aligned}
7e^{-2\lambda t}\|f^{-N}\mathcal{P}u\|^2&\geq\partial_t\mathscr{E}_N(u)+\frac{2N}{3}\mathscr{E}_{N+1/2}(u)+2\lambda\mathscr{E}_N(u)+\mathscr{R}\\
&\qquad-\frac{e^{-2\lambda t}}{6t^2}\Big[\frac{1}{3}\|f^{-N}(u''+\frac{1}{3}\varepsilon^{-2/3}\alpha^\varepsilon u)\|_0^2+\frac{2}{3}e^{-2\lambda t}\|f^{-N}a_2^\varepsilon u\|_0^2\Big]\\
&=\partial_t\mathscr{E}_N(u)+\frac{2N}{3}\mathscr{E}_{N+1/2}(u)+2\lambda\mathscr{E}_N(u)+\mathscr{Q}_1+\sum_{j=3}^{10}\mathscr{R}_j+\sum_{j=\nu}^{3}B_\nu+\text{lower order terms}.
\end{aligned}\tag{6.16}$$

Here

$$\begin{aligned}
\mathscr{Q}_1=\frac{e^{-2\lambda t}}{t^2}\Big(&\frac{1}{3}\|f^{-N}u''\|_0^2+\varepsilon^{-4/3}\frac{4}{27}\|f^{-N}\alpha^\varepsilon u\|_0^2+\frac{5}{18}\|f^{-N}(u''+\frac{1}{3}\varepsilon^{-2/3}\alpha^\varepsilon u)\|_0^2\\
&+\frac{2}{3}\varepsilon^{-2/3}\langle f^{-N}\alpha^\varepsilon u',f^{-N}u'\rangle-\frac{4}{9}\|f^{-N}ta_2^\varepsilon u\|_0^2\Big).
\end{aligned}$$

Finally, taking into account (6.4), (6.15), (6.16), we obtain the energy estimate

$$
\begin{aligned}
7e^{-2\lambda t}\|f^{-N}\mathcal{P}u\|_0^2 \geq \partial_t\Big(&\mathscr{E}_N(u) + \frac{N}{27}\psi\|f^{-N-1}u'\|_0^2 + \frac{N}{27}\psi\|f^{-N-2}u\|_0^2 \\
&+ \frac{N}{27}C_3e^{-2\lambda t}\|f^{-N-1}u\|_1^2\Big) \\
&+ 2\lambda\mathscr{E}_N(u) + \varepsilon^{1/3}te^{-2\lambda t}\frac{4N}{3}\operatorname{Im}\langle f^{2N-1}b_3u, u'\rangle + \varepsilon^{2/3}tA^{(2)}_{N+1/2}(u) \\
&+ N\psi\Big[\frac{5}{27}\|f^{-N-1/2}u''\|_0^2 + \frac{10}{27}\operatorname{Re}\langle f^{-N-1/2}(t\mathbf{a}_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)u', f^{-N-1/2}u'\rangle \\
&+ \frac{4}{9}\|f^{-N-1/2}(u'' + \frac{1}{2}(t\mathbf{a}_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)u)\|_0^2 + \frac{1}{9}\|f^{-N-1/2}(t\mathbf{a}_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)u\|_0^2\Big] \\
&+ \frac{N}{9}\Big[-\frac{\psi'}{3}\|f^{-N-1}u'\|_0^2 + \frac{2N-5}{9}\psi\|f^{-N-3/2}u'\|_0^2 - \frac{\psi'}{3}\|f^{-N-2}u\|_0^2 \\
&+ \frac{2N+1}{9}\psi\|f^{-N-5/2}u\|_0^2 + \frac{2C_3}{3}\lambda e^{-2\lambda t}\|f^{-N-1}u\|_1^2 \\
&+ \frac{(2N+1)C_3}{9}e^{-2\lambda t}\|f^{-N-3/2}u\|_1^2 + \frac{C_3}{3}e^{-2\lambda t}\|f^{-N-1/2}u'\|_1^2\Big] \\
&+ \mathscr{Q}_1 + \sum_{j=3}^{10}\mathscr{R}_j + \sum_{\nu=1}^{3}B_\nu + \text{lower order terms}. \qquad (6.17)
\end{aligned}
$$

7. Estimate of the error terms in the energy inequality

The last line of (6.17) contains a number of terms grouping the "errors" that must be dominated with the other positive terms. We point out that the B_ν are just those "errors" associated with the lower order terms containing pure second order x-derivatives and did not play any role up to now. It is convenient to write $a_2^\varepsilon(t,x,\xi) = a_2^\varepsilon(0,x,\xi) + t\varepsilon^{2/3}\tilde{a}_2^\varepsilon(t,x,\xi)$ and to replace the operator $a_2^\varepsilon(t,x,D_x)$ by the operator $\mathbf{a}_2^\varepsilon$ with symbol $a_2^\varepsilon(0,x,\xi)$. This will add a few terms similar to $t\varepsilon^{2/3}A_N^{(2)}(u)$. Consequently, we have to deal with lower order terms which can be treated choosing ε small,.

For the analysis of the terms B_ν, $\nu = 1,2,3$ we apply a similar procedure. First we write $b_2^\varepsilon(t,x,\xi) = b_2^\varepsilon(0,x,\xi) + t\varepsilon^{2/3}\tilde{b}_2^\varepsilon(t,x,\xi)$. The terms with the factor $t\varepsilon^{2/3}$ are similar to $t\varepsilon^{2/3}A_N^{(2)}(u)$ and can be treated choosing ε small. We will analyze these terms in subsection 7.6. To keep the notation simple, we denote by $\mathbf{b}_2^\varepsilon$ the operator with symbol $b_2^\varepsilon(0,x,\xi)$. The modified terms B_ν will be denoted by $\tilde{B}_\nu$, $\nu = 1,2,3$.

7.1. Estimate of $\|\mathbf{b}_2^\varepsilon f^{-N+1/2}u\|_0$.

The subprincipal symbol of the operator $\mathcal{P}$ for $t = 0$ and $\tau = 0$ has the form

$$p_2'(0,x_0,\xi_0) = -\frac{i}{2}a_2^\varepsilon(0,x_0,\xi_0) + b_2^\varepsilon(0,x_0,\xi_0).$$

If we have a triple point $\rho = (0,x_0,\xi)$ for the symbol $p_3(t,x,\tau,\xi)$, then $t = \tau = 0$. Thus

$$b_2^\varepsilon(0,x,\xi) = \Big[\frac{1}{2}i + \frac{p_2'(0,x,\xi)}{a_2^\varepsilon(0,x,\xi)}\Big]a_2^\varepsilon(0,x,\xi).$$

Let us introduce the number

$$\Pi = \frac{2}{3} + \max_{x \in \bar{U}_{x_0},\, \alpha(x,\xi)=0,\, |\xi|=1} \Big| \frac{p_2'(0,x,\xi)}{a_2^\varepsilon(0,x,\xi)} \Big|.$$

which correspond to (1.7). Here $U_{x_0} \subset \mathbb{R}^n$ is the open set defined in the hypothesis (H_2). Notice that we could have only one point $y \in U_{x_0}$ such that $\alpha(y,\xi) = 0$. It is clear that for $V_{x_0} \Subset U_{x_0}$ sufficiently small we have

$$\sup_{x \in \bar{V}_{x_0},\, |\xi|=1} \Big| \frac{1}{2} i + \frac{p_2'}{a_2^\varepsilon}(0,x,\xi) \Big| \leq \Pi.$$

In the following we can assume that $u(t,x)$ has a support with respect to x included in V_{x_0}. Let $\chi \in C^\infty(\mathbb{R}^n)$ be a function such that $0 \leq \chi(x) \leq 1$, $\chi(x) = 0$ for $|x| \leq 1$ and $\chi(x) = 1$ for $|x| \geq 2$. We write

$$a_2^\varepsilon(0,x,\xi) = \chi(\xi\delta) a_2^\varepsilon(0,x,\xi) + (1 - \chi(\xi\delta)) a_2^\varepsilon(0,x,\xi)$$

with $\delta > 0$. The operator with symbol $(1 - \chi(\xi\delta)) a_2^\varepsilon(0,x,\xi)$ is smoothing and the analysis of the corresponding term is covered by using the argument for lower order terms. On the other hand, the norm in $\mathcal{L}(L^2(V_{x_0}))$ of the zero order operator

$$\Big[\frac{1}{2} i + \frac{p_2'}{a_2^\varepsilon}(0,x,D_x)\Big] \chi(D_x \delta) \tag{7.1}$$

is not greater than Π if δ is chosen small enough depending on the symbols a_2 and p_2' (see Theorem 18.1.15 in Hörmander, [5].)

Thus we have

$$\begin{aligned} \|\mathbf{b}_2^\varepsilon f^{-N+1/2} u\|_0 &\leq \Big\| \Big[\frac{1}{2} i + \frac{p_2'}{a_2^\varepsilon}(0,x,D_x)\Big] a_2^\varepsilon(0,x,D_x) f^{-N+1/2} u \Big\|_0 + \|R_0\|_0 \\ &\leq \Pi \|\mathbf{a}_2^\varepsilon f^{-N+1/2} u\|_0 + \|R_0\|_0, \end{aligned}$$

where R_0 is a lower order term including $\alpha_1 f^{-N+1/2}$ with some first order pseudodifferential operator α_1.

We are going to study the term

$$- \|\mathbf{a}_2^\varepsilon f^{-N+1/2} u\|_0^2 = -\Big[\mathrm{Re}\langle (a_2^\varepsilon)^2(0,x,D_x) f^{-N+1/2} u, f^{-N+1/2} u\rangle + |F_1|\Big], \tag{7.2}$$

where $|F_1| \leq C_6 \|f^{-N+1/2} u\|_{3/2}^2$ and $(a_2^\varepsilon)^2(0,x,D_x)$ means $\mathrm{Op}\,\big((a_2^\varepsilon)^2\big)$. We have the inequality

$$(a_2^\varepsilon)^2 \leq t (a_2^\varepsilon)^2 f^{-1} + a_2^\varepsilon \alpha_0 f^{-3} \leq t^2 f^{-1} (a_2^\varepsilon)^2 f^{-1} + 2t f^{-2} a_2^\varepsilon \alpha_0 f^{-2} + f^{-3} \alpha_0^2 f^{-3}, \tag{7.3}$$

where $\alpha_0(x,\xi) = \frac{a_2^\varepsilon(0,x,\xi)}{\langle \xi \rangle^2}$. We can apply the sharp Gårding inequality and we estimate

$$\begin{aligned} &- \mathrm{Re}\langle (a_2^\varepsilon)^2 f^{-N+1/2} u, f^{-N+1/2} u\rangle - |F_1| \geq -t^2 \,\mathrm{Re}\langle (a_2^\varepsilon)^2 f^{-N-1/2} u, f^{-N-1/2} u\rangle \\ &\quad -2t\, \mathrm{Re}\langle \alpha_0 \mathbf{a}_2^\varepsilon f^{-N-3/2} u, f^{-N-3/2} u\rangle - A_3^2 \|f^{-N-5/2} u\|_0^2 - Y_1 \|f^{-N+1/2} u\|_{3/2}^2 \\ &\geq -t^2 \|\mathbf{a}_2^\varepsilon f^{-N-1/2} u\|_0^2 - 2t\, \mathrm{Re}\langle \alpha_0 \mathbf{a}_2^\varepsilon f^{-N-3/2} u, f^{-N-3/2} u\rangle - A_3^2 \|f^{-N-5/2} u\|_0^2 \\ &\quad - Y_2 \|f^{-N+1/2} u\|_{3/2}^2 = \sum_{j=1}^4 \Gamma_j. \end{aligned}$$

Here we have used the fact that

$$t^2 \operatorname{Re}\langle (a_2^\varepsilon)^2 f^{-N-1/2}u, f^{-N-1/2}u\rangle = t^2\|\mathbf{a}_2^\varepsilon f^{-N-1/2}u\|_0^2 + F_2,$$

where $|F_2| \leq A_4 t^2 \|f^{-N-1/2}u\|_{3/2}^2$. Since $t^2 f^{-2} \leq 9$, we included the term F_2 in the above sum taking $Y_2 \geq Y_1$. Notice that Y_1, Y_2 depend only on the symbol $a_2^\varepsilon(0, x, \xi)$.

It is convenient to transform the term Γ_4. For this purpose we use the inequality

$$\begin{aligned}\langle\xi\rangle^3 f^{-2N+1} &\leq t\langle\xi\rangle^3 f^{-2N} + \langle\xi\rangle f^{-2N-2} \leq t^2\langle\xi\rangle^3 f^{-2N-1} + t\langle\xi\rangle f^{-2N-3} + \langle\xi\rangle^2 f^{-2N-2}\\ &\leq \delta_1 t^2 \langle\xi\rangle^4 f^{-2N-1} + D_{\delta_1} t^2 f^{-2N-1} + 2t\langle\xi\rangle^2 f^{-2N-3} + f^{-2N-5}.\end{aligned}$$

Thus

$$\begin{aligned}\|f^{-N+1/2}u\|_{3/2}^2 &\leq \delta_1 t^2 \|f^{-N-1/2}u\|_2^2 + 2t\|f^{-N-3/2}u\|_1^2 + \|f^{-N-5/2}u\|_0^2 + D_{\delta_1} t^2 \|f^{-N-1/2}u\|_0^2\\ &\leq \delta_1 C^2 t^2 \|\mathbf{a}_2^\varepsilon f^{-N-1/2}u\|_0^2 + 2t\|f^{-N-3/2}u\|_1^2 + \|f^{-N-5/2}u\|_0^2 + D'_{\delta_1} t^2 \|f^{-N-1/2}u\|_0^2.\end{aligned}$$

We take $\delta_1 > 0$ small enough so that $\delta_1 C^2 Y_2 \leq \delta$ and we couple the term with the factor t^2 with that also involving t^2 in Γ_1. Next we fix δ_1 and for small t we have $D'_{\delta_1} Y_2 t^2 \|f^{-N-1/2}u\|_0^2 \leq \|f^{-N-5/2}u\|_0^2$. Notice that we can choose $\delta > 0$ as small as we wish. We sum this term with Γ_3. Consequently, we get

$$\begin{aligned}\sum_{j=1}^4 \Gamma_4 \quad \geq \quad & -(1+\delta)t^2\|\mathbf{a}_2^\varepsilon f^{-N-1/2}u\|_0^2 - 2t(C_{a_2} + Y_2)\|f^{-N-3/2}u\|_1^2\\ & -(A_3^2 + 2Y_2)\|f^{-N-5/2}u\|_0^2.\end{aligned} \tag{7.4}$$

Here we have used that

$$\operatorname{Re}\langle \mathbf{a}_2^\varepsilon f^{-N-3/2}u, f^{-N-3/2}u\rangle \leq C_{a_2}\|f^{-N-3/2}u\|_1^2$$

and the constant C_{a_2} depends on $a_2^\varepsilon(0, x, \xi)$, while $A_3 = \|\alpha_0(x, D_x)\|_{L^2(U)\to L^2(U)}$.

Summarizing we obtain the following

Lemma 7.1. *Let $0 \leq t_0 \leq t \leq T$ and let $D_t^k u(t_0, x) = 0$, $x \in V_{x_0}$, $k = 0, 1, 2$. For every fixed small number $\delta > 0$ there exist constants C_{a_2}, Y_2, A_3 such that modulo lower order term ψR_0 we have*

$$\begin{aligned}\psi\|\mathbf{a}_2^\varepsilon f^{-N+1/2}u\|_0^2 \leq (1+\delta)te^{-2\lambda t}\|\mathbf{a}_2^\varepsilon f^{-N-1/2}u\|_0^2 + 2e^{-2\lambda t}(C_{a_2} + Y_2)\|f^{-N-3/2}u\|_1^2\\ + (A_3^2 + 2Y_2)\psi\|f^{-N-5/2}u\|_0^2.\end{aligned} \tag{7.5}$$

We turn to the analysis of the term

$$\begin{aligned}K_{N+1/2} &= \varepsilon^{-2/3}e^{-2\lambda t} 2\operatorname{Re}\langle f^{-2N-1}a_2^\varepsilon u, \alpha^\varepsilon u\rangle\\ &= \varepsilon^{-2/3}e^{-2\lambda t}2\Big[\operatorname{Re}\langle a_2^\varepsilon(f^{-N-1/2}u), \alpha^\varepsilon(f^{-N-1/2}u)\rangle + \operatorname{Re}\langle \alpha_1 f^{-N-1/2}u, f^{-N-1/2}\alpha^\varepsilon u\rangle\\ &\quad + \operatorname{Re}\langle a_2^\varepsilon f^{-N-1/2}u, \beta_1 f^{-N-1/2}u\rangle\Big]\end{aligned}$$

for some first order operators α_1, β_1. First, notice that for the principal symbol of α^ε we have $\alpha^\varepsilon = \varepsilon^2\alpha_0^\varepsilon$ with a second order non-negative symbol α_0^ε and that the operator $\alpha_0^\varepsilon(x, D_x)$ is self-adjoint. We replace α^ε by $\varepsilon^2\alpha_0^\varepsilon$ in the above equality and we obtain a small factor $\varepsilon^{1/3}$. In fact for the two terms on the right hand side we have factor $\varepsilon^{4/3}$, while for the last one we have $\varepsilon^{1/3}$ related to the commutator $[\alpha^\varepsilon, f^{-N-1/2}]$. For the term $\operatorname{Re}\langle \alpha_0^\varepsilon a_2^\varepsilon f^{-N-1/2}u, f^{-N-1/2}u\rangle$ we can apply the sharp Gårding inequality since the principal symbol of $\alpha_0^\varepsilon a_2^\varepsilon$ is non-negative. The other terms in $K_{N+1/2}$ involve third order operators and we get

$$K_{N+1/2} \geq -C\varepsilon^{1/3} e^{-2\lambda t} \|f^{-N-1/2} u\|^2_{3/2}.$$

An application of Lemma 6.1 yields

$$\langle\xi\rangle^3 f^{-2N-1} \leq t\langle\xi\rangle^3 f^{-2N-2} + \langle\xi\rangle f^{-2N-4} \leq c_1 \langle\xi\rangle^{-1/3} \Big[t\langle\xi\rangle^4 f^{-2N-1} + \langle\xi\rangle^2 f^{-2N-3}\Big]$$

since $\langle\xi\rangle^{-2/3} f^{-1} \leq c$. Thus

$$K_{N+1/2} \geq -C_1\varepsilon^{1/3} e^{-2\lambda t}\Big(t\|f^{-N-1/2}u\|_2^2 + \|f^{-N-3/2}u\|_1^2\Big).$$

On the other hand,

$$e^{-2\lambda t}\|f^{-N-1/2}u\|_2^2 \leq C_5 e^{-2\lambda t}\|\mathbf{a}_2^\varepsilon f^{-N-1/2}u\|_0^2 + C_6 e^{-2\lambda t}\|f^{-N-1/2}u\|_1^2$$

and we deduce

$$K_{N+1/2} \geq -C_7\varepsilon^{1/3} e^{-2\lambda t}\Big[t\|\mathbf{a}_2^\varepsilon f^{-N-1/2}u\|_0^2 + t\|f^{-N-1/2}u\|_1^2 + \|f^{-N-3/2}u\|_1^2\Big].$$

Combining this with (7.5), we deduce for small ε the estimate

$$\begin{aligned} \psi\|\mathbf{a}_2^\varepsilon f^{-N+1/2}u\|_0^2 \quad &\leq \quad (1+\delta)\psi\|(t\mathbf{a}_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon) f^{-N-1/2}u\|_0^2 \\ &\quad +2e^{-2\lambda t}(C_{a_2} + Y_2 + C_8\varepsilon^{1/3})\|f^{-N-3/2}u\|_1^2 \\ &\quad +(A_3^2 + 2Y_2)\psi\|f^{-N-5/2}u\|_0^2. \end{aligned} \tag{7.6}$$

7.2. **Estimate of the sum $\sum_{\nu=1}^3 \tilde{B}_\nu$.** We have

$$\begin{aligned} \sum_{\nu=1}^3 \tilde{B}_\nu &= 2\,\mathrm{Im}\,\psi\langle f^{-N}\mathbf{b}_2^\varepsilon u, f^{-N}\Big(u'' + \frac{1}{3}(t\mathbf{a}_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)u\Big)\rangle \\ &= 2\,\mathrm{Im}\,\psi\langle f^{-N}\mathbf{b}_2^\varepsilon u, f^{-N}\Big(u'' + \frac{1}{2}(t\mathbf{a}_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)u\Big)\rangle \\ &\quad - 2\,\mathrm{Im}\,\psi\langle f^{-N}\mathbf{b}_2^\varepsilon u, \frac{1}{6}(t\mathbf{a}_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)u\rangle \\ &= Z_1 + Z_2. \end{aligned}$$

Taking into account (7.6), we obtain

$$\begin{aligned} |Z_1| &\leq \eta\Pi\psi\|\mathbf{a}_2^\varepsilon f^{-N+1/2}u\|_0^2 + \frac{1}{\eta}\Pi\psi\|f^{-N-1/2}(u'' + \frac{1}{2}(t\mathbf{a}_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)u)\|_0^2 + |R_1| \\ &\leq (1+\delta)\eta\Pi\psi\|(t\mathbf{a}_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)f^{-N-1/2}u\|_0^2 + 2\Pi\eta e^{-2\lambda t}(C_{a_2} + Y_2 + C_8\varepsilon^{4/3})\|f^{-N-3/2}u\|_1^2 \\ &\quad + \eta(A_3^2 + 2Y_2)\Pi\psi\|f^{-N-5/2}u\|_0^2 + \frac{1}{\eta}\Pi\psi\|f^{-N-1/2}(u'' + \frac{1}{2}(t\mathbf{a}_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)u)\|_0^2 + |R_1| \end{aligned}$$

with $\eta > 0$ which will be chosen below. Here and below we note by $R_j, j = 0, 1, 2, ...$ lower order terms which include first order operators. The analysis of these terms will be considered in the next subsection.

On the other hand, for Z_2, we get

$$\begin{aligned}|Z_2| &\le \frac{1}{3}\psi|\operatorname{Im}\langle f^{-N}\mathbf{b}_2^\varepsilon u, f^{-N}(t\mathbf{a}_2^\varepsilon+\varepsilon^{-2/3}\alpha^\varepsilon)u\rangle|\\ &\le \frac{1}{3}\psi\|\mathbf{b}_2^\varepsilon f^{-N+1/2}u\|_0\|(t\mathbf{a}_2^\varepsilon+\varepsilon^{-2/3}\alpha^\varepsilon)f^{-N-1/2}u\|_0+|R_2|\\ &\le \frac{1}{3}\Pi\psi\|\mathbf{a}_2^\varepsilon f^{-N+1/2}u\|_0\|(t\mathbf{a}_2^\varepsilon+\varepsilon^{-2/3}\alpha^\varepsilon)f^{-N-1/2}u\|_0+|R_2|.\end{aligned}$$

According to (7.6), we have

$$\begin{aligned}&\frac{1}{3}\Pi\psi\|\mathbf{a}_2^\varepsilon f^{-N+1/2}u\|_0\|(t\mathbf{a}_2^\varepsilon+\varepsilon^{-2/3}\alpha^\varepsilon)f^{-N-1/2}u\|_0\\ &\le \frac{1}{3}\Pi\psi\Big[\sqrt{1+\delta}\|(t\mathbf{a}_2^\varepsilon+\varepsilon^{-2/3}\alpha^\varepsilon)f^{-N-1/2}u\|_0+\sqrt{2(C_{a_2}+Y_3)}\sqrt{t}\|f^{-N-3/2}u\|_1\\ &\qquad +\sqrt{A_3^2+2Y_2}\|f^{-N-5/2}u\|_0\Big]\|(t\mathbf{a}_2^\varepsilon+\varepsilon^{-2/3}\alpha^\varepsilon)f^{-N-1/2}u\|_0\\ &\le \frac{1}{3}\Pi\psi\Big(\sqrt{1+\delta}+2\delta_1\Big)\|(t\mathbf{a}_2^\varepsilon+\varepsilon^{-2/3}\alpha^\varepsilon)f^{-N-1/2}u\|_0^2+\frac{2}{3}\Pi\delta_1^{-1}(C_{a_2}+Y_3)e^{-2\lambda t}\|f^{-N-3/2}u\|_1^2\\ &\qquad +\delta_1^{-1}2(A_3^2+2Y_2)/3\Pi\psi\|f^{-N-5/2}u\|_0^2\end{aligned}$$

with some constant Y_3 and $\delta_1>0$ such that $\sqrt{1+\delta}+2\delta_1=1+\delta$. To absorb the leading terms in the sum $\sum_{\nu=1}^3\tilde{B}_\nu$, the inequalities

$$\frac{1}{4\eta}\Pi\le\frac{N}{9},\qquad (\frac{1}{3}+\eta)(1+\delta)\Pi\le\frac{N}{9}.$$

must be satisfied to compensate for the terms

$$\frac{1}{\eta}\Pi\psi\|f^{-N-1/2}(u''+\frac{1}{2}(t\mathbf{a}_2^\varepsilon+\varepsilon^{-2/3}\alpha^\varepsilon)u\|_0^2\quad\text{and}\quad(\frac{1}{3}+\eta)(1+\delta)\Pi\psi\|(t\mathbf{a}_2^\varepsilon+\varepsilon^{-2/3}\alpha^\varepsilon)f^{-N-1/2}u\|_0^2.$$

In order to optimize the choice of η, we take $\eta=\frac{\sqrt{10}-1}{6}$. Then for $N\ge\frac{13}{2}\Pi$ and small δ we satisfy the above inequalities. For example, for the first one we have

$$\frac{3}{\sqrt{10}-1}\le\frac{13}{9}\Leftrightarrow 27<13\sqrt{10}-13\Leftrightarrow 40<13\sqrt{10},$$

while the second one we get

$$\frac{\sqrt{10}+1}{6}(1+\delta)\le\frac{13}{18}\Leftrightarrow 3(\sqrt{10}+1)(1+\delta)\le 13\Leftrightarrow 3\sqrt{10}+\mathcal{O}(\delta)\le 10.$$

Now we fix δ and so the constants A_3, Y_2, Y_3 are fixed. To absorb $-\frac{2}{3}\delta_1^{-1}(A_3^2+2Y_2)\Pi\psi\|f^{-N-5/2}u\|_0^2$, we exploit the corresponding term in (6.17) and arrange things so that

$$\frac{2}{3}\delta_1^{-1}(A_3^2+2Y_2)\Pi\le\frac{4}{3}\delta_1^{-1}(A_3^2+2Y_2)\frac{N}{13}\le\frac{N(2N+1)}{81},$$

that is $\frac{27.4}{13}\delta_1^{-1}(A_3^2+2Y_2)\le 2N+1$. Since $N=\frac{13}{2}\Pi+N_0$, we can do this choosing N_0 large. In the same way, we arrange the inequality

$$\frac{4}{11}\delta_1^{-1}(C_{a_2}+Y_2)\le\frac{(2N+1)C_3}{27}$$

and we absorb the term $-\frac{4}{3}\delta_1^{-1}(C_{a_2}+Y_3)\Pi e^{-2\lambda t}\|f^{-N-3/2}u\|_1^2$ by the corresponding term in (6.17).

7.3. **Analysis of $R_j, j = 0, 1, 2$.** We will prove that we can absorb the terms R_j choosing λ large enough.

The term R_1 has the form $2\psi \operatorname{Im}\langle \alpha_1 f^{-N} u, f^{-N}(u'' + \frac{1}{3}(t\mathbf{a}_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)u)\rangle$ with a first order pseudodifferential operator α_1. We have

$$R_1 \geq -\mu\frac{e^{-2\lambda t}}{t^2}\|f^{-N}(u'' + \frac{1}{3}(t\mathbf{a}_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)u)\|_0^2 - \frac{D_1}{\mu}e^{-2\lambda t}\|f^{-N}u\|_1^2$$

$$\geq -2\mu\frac{e^{-2\lambda t}}{t^2}\Big(\|f^{-N}(u'' + \frac{1}{3}\varepsilon^{-2/3}\alpha^\varepsilon u)\|_0^2 + \frac{1}{9}\|f^{-N}t\mathbf{a}_2^\varepsilon u\|_0^2\Big) - \frac{D_1}{\mu}e^{-2\lambda t}\|f^{-N}u\|_1^2$$

The term involving $\|f^{-N}(u'' + \frac{1}{3}\varepsilon^{-2/3}\alpha^\varepsilon u)\|_0^2$ is absorbed by the corresponding term in $\mathcal{Q}_1$ choosing $\mu > 0$ small, the term $-\frac{2}{9}\mu\|f^{-N}t\mathbf{a}_2^\varepsilon u\|_0^2$ is added to the term $\mathcal{Q}_1$ which will be estimated below. Finally, the last term is absorbed by $\frac{2C_3}{27}\lambda e^{-2\lambda t}\|f^{-N-1}u\|_1^2$ in (6.17) taking λ large. For R_2 we use the inequality

$$R_2 \geq -\delta_3\frac{e^{-2\lambda t}}{t^2}\|f^{-N}(t\mathbf{a}_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)u\|_0^2 - {\delta_3}^{-1}D_2 e^{-2\lambda t}\|f^{-N}u\|_1^2$$

$$\geq -2\delta_3\frac{e^{-2\lambda t}}{t^2}\Big[t^2\|f^{-N}\mathbf{a}_2^\varepsilon u\|_0^2 + \varepsilon^{-4/3}\|f^{-N}\alpha^\varepsilon u\|_0^2\Big] - \delta_3^{-1}D_2 e^{-2\lambda t}\|f^{-N}u\|_1^2.$$

Taking δ_3 small, we add the first term on the right hand side to the term $\mathcal{Q}_1$ which we will estimate below. Also the term with $\varepsilon^{-4/3}\|f^{-N}\alpha^\varepsilon u\|_0^2$ can be absorbed by $\mathcal{Q}_1$ choosing δ_3 small. The third term on the right is handled as above. The term R_0 is easy to be treated since we have $f^{-N+1/2}u$ instead of $f^{-N}u$ and we may repeat the argument applied for R_2. Notice that we choose λ large depending on the norms of first order operators but we keep the dependence of N on Π.

7.4. **Analysis of $t\varepsilon^{2/3}A^{(2)}_{N+1/2}(u)$.** The term $2Nt\varepsilon^{2/3}A^{(2)}_{N+1/2}(u)$ is a sum of terms. They can be estimated following our previous arguments but we can take advantage of the factor $t\varepsilon^{2/3}$. Consider a typical term:

$$L_4 = 2N\varepsilon^{2/3}te^{-2\lambda t}\operatorname{Re}\langle f^{-2N-1}\tilde{a}_2^\varepsilon u, u''\rangle.$$

We have modulo lower order terms

$$|L_4| \leq N\varepsilon^{2/3}e^{-2\lambda t}\Big(\|f^{-N-1/2}u''\|_0^2 + t^2 C_{a_2}\|f^{-N-1/2}u\|_2^2\Big).$$

We absorb this term involving u'' by (6.17) taking ε small and using the term $N\psi\frac{5}{27}\|f^{-N-1/2}u''\|_0^2$ with small t. For the other term we take ε small to arrange $\varepsilon C_{a_2} < 1$ and we apply Proposition 6.1 to reduce the analysis to an estimate of $t^2C\|\mathbf{a}_2^\varepsilon f^{-N-1/2}u\|_0^2$ where we have a factor t^2 and the term can be handled by term in (6.17) involving $t\|\mathbf{a}_2^\varepsilon f^{-N-1/2}u\|_0^2$.

Next consider the term

$$L_3 = 2N\varepsilon^{2/3}t\operatorname{Re}\langle f^{-2N-1}\tilde{a}_2^\varepsilon u', u'\rangle.$$

Modulo lower order terms we have

$$|L_3| \leq Nt\varepsilon^{2/3}C_{a_2}\|f^{-N-1/2}u'\|_1^2$$

and this can be absorbed by the corresponding terms in (6.17) taking ε and t small.

7.5. **Analysis of $\mathcal{Q}_1, \mathscr{R}_3$.** In $\mathcal{Q}_1$ we have only one negative term. We use the inequality

$$\begin{aligned} e^{-2\lambda t}\|f^{-N}\mathbf{a}_2^\varepsilon u\|_0^2 &= e^{-2\lambda t}\langle f^{-N+1/2}\mathbf{a}_2^\varepsilon u, f^{-N-1/2}\mathbf{a}_2^\varepsilon u\rangle \\ &\leq \psi\|\mathbf{a}_2^\varepsilon f^{-N+1/2}u\|_0^2 + e^{-2\lambda t}t\|\mathbf{a}_2^\varepsilon f^{-N-1/2}u\|_0^2 + R_3. \end{aligned}$$

For the first term on the right hand side we apply Lemma 7.1 and we obtain a leading term $D_3 t\|\mathbf{a}_2^\varepsilon f^{-N-1/2}u\|_0^2$ which can be absorbed taking N_0 large. The other terms can be treated as above exploiting the corresponding terms in (6.17). Passing to the term $\mathscr{R}_3$, we get

$$\begin{aligned} |\mathscr{R}_3| &\leq \frac{2}{3}\varepsilon^{-2/3}e^{-2\lambda t}\|f^{-N}(a_2^\varepsilon)_t u\|_0\|f^{-N}\alpha^\varepsilon u\|_0 \\ &\leq \frac{1}{3}\varepsilon^{2/3}\Big(e^{-2\lambda t}t\|f^{-N}\partial_t(a_2)u\|_0^2 + \varepsilon^{-4/3}\frac{e^{-2\lambda t}}{t^2}\|f^{-N}\alpha^\varepsilon u\|_0^2\Big). \end{aligned}$$

Here the factor $\varepsilon^{2/3}$ comes from the derivative with respect to t of $a_2(\varepsilon^{2/3}t, \varepsilon x, \xi)$. For small ε we may absorb the term involving $\alpha^\varepsilon u$ using $\mathcal{Q}_1$, while for the other term we write $\partial_t(a_2) = \gamma_0 a_2$ with a zero order operator γ_0 and absorb this term with the corresponding term in $\lambda\mathscr{E}_N(u)$ taking λ large.

7.6. **Analysis of $-2\varepsilon^{1/3}\frac{4N}{3}te^{-2\lambda t}\langle f^{-2N-1}b_3^\varepsilon u, u'\rangle$ and $\mathscr{R}_j, j = 7, 8$.** It is clear that we have

$$\begin{aligned} &\varepsilon^{1/3}\frac{4N}{3}te^{-2\lambda t}|\langle f^{-2N-1}b_3^\varepsilon u, u'\rangle| \\ &\leq \varepsilon^{1/3}\frac{4N}{3}te^{-2\lambda t}\||D_x|^{-1}b_3^\varepsilon f^{-N-1/2}u\|_0\||D_x|f^{-N-1/2}u'\|_0 + |R_4| \\ &\leq \varepsilon^{1/3}Ne^{-2\lambda t}\Big[t^2 D_3\Big(\|\mathbf{a}_2^\varepsilon f^{-N-1/2}u\|_0^2 + \|f^{-N-1/2}u\|_0^2\Big) + \frac{4}{3}\|f^{-N-1/2}u'\|_1^2\Big] + |R_4|. \end{aligned}$$

We may absorb the term $e^{-2\lambda t}t^2D_3\|\mathbf{a}_2^\varepsilon f^{-N-1/2}u\|_0^2$ on the right hand side by the corresponding terms in (6.17) choosing ε and t small so that $\varepsilon^{1/3}ND_3t < 1$. The term with $t^2\|f^{-N-1/2}u\|_0^2$ is also easily absorbed. Finally, the term $\|f^{-N-1/2}u'\|_1^2$ is absorbed by $\frac{10N}{27}e^{-2\lambda t}\langle\mathbf{a}_2^\varepsilon f^{-N-1/2}u', f^{-N-1/2}u'\rangle$ in (6.17) applying Proposition 6.1 and taking ε small. The rest R_4 involves a second order operator γ_2 in the place of b_3. We apply the same argument and we are going to absorb a term $e^{-2\lambda t}t^2D_3N\|f^{-N-1/2}u\|_1^2$ choosing t small or by using the term $\mathcal{O}(N^2)e^{-2\lambda t}\|f^{-N-3/2}u\|_1^2$ in (6.17).

The term $\mathscr{R}_8 = -2\varepsilon^{1/3}te^{-2\lambda t}\operatorname{Im}\langle f^{-2N}(\partial_t b_3^\varepsilon)u, u'\rangle$ can be treated as above. Here we do not have a coefficient N to deal with and, moreover, we have the operator f^{-2N} instead of $f^{-N-1/2}$. We get

$$|\mathscr{R}_8| \leq \varepsilon^{1/3}e^{-2\lambda t}\Big[t^2D_3\Big(\|\mathbf{a}_2^\varepsilon f^{-N}u\|_0^2 + \|f^{-N}u\|_0^2\Big) + \|f^{-N}u'\|_1^2\Big] + |R_5|.$$

We may absorb all terms taking λ sufficiently large, ε small by using the positive terms in $\lambda E_N(u)$. Passing to the term $\mathscr{R}_7$, consider first

$$\varepsilon^{1/3}e^{-2\lambda t}|\operatorname{Im}\langle f^{-2N}b_3^\varepsilon u, u'\rangle| \leq \varepsilon^{1/3}D_4e^{-2\lambda t}\Big[\|\mathbf{a}_2^\varepsilon f^{-N+1/2}u\|_0^2 + \|f^{-N-1/2}u'\|_1^2\Big] + |R_6|,$$

where R_6 includes second order operator coming from the commutator with b_3. For the term $e^{-2\lambda t}\|\mathbf{a}_2^\varepsilon f^{-N-1/2}u\|_0^2$ we apply Lemma 7.1 and we reduce the analysis to an estimate of $\varepsilon^{1/3}D_4t^2(1+\delta)e^{-2\lambda t}\|f^{-N-1/2}u\|_0^2$ plus lower order terms. Next we absorb the leading term taking $\varepsilon^{1/3}D_4$ and t small and using (6.17), where we have positive terms multiplied by N_0. The analysis of the other terms follows the same argument as above. To deal with second summand in $\mathscr{R}_7$, we use the fact that $b_3^\varepsilon(t,x,D_x)$ is self-adjoint. Then

$$2\,\mathrm{Im}\langle f^{-2N}b_3^\varepsilon u', u'\rangle = -i\Big[\langle b_3^\varepsilon f^{-N}u', f^{-N}u'\rangle - \langle f^{-N}u', b_3^\varepsilon f^{-N}u'\rangle$$
$$+\langle \gamma_2 f^{-N}u', f^{-N}u'\rangle + \langle f^{-N}u', \gamma_3 f^{-N}u'\rangle\Big] = -i\Big[\langle \gamma_2 f^{-N}u', f^{-N}u'\rangle + \langle f^{-N}u', \gamma_3 f^{-N}u'\rangle\Big]$$

with some second order operators γ_2, γ_3. The analysis of the terms of the right hand side is easy taking the parameter λ large in $\lambda E_N(u)$.

7.7. **Analysis of $\mathscr{R}_j, j = 9, 10$.** For the term $\mathscr{R}_9$ we have

$$|\mathscr{R}_9| = \frac{2}{3}\varepsilon^{1/3}t^2 e^{-2\lambda t}|\,\mathrm{Im}\langle f^{-N}\gamma_0^\varepsilon a_2^\varepsilon u, f^{-N}a_2^\varepsilon u\rangle| \le C\varepsilon^{1/3}e^{-2\lambda t}t^2\|f^{-N}a_2^\varepsilon u\|_0^2$$

and the right hand term is easy to be absorbed by the positive terms in $\lambda E_N(u)$ taking λ large. Next we obtain

$$\mathscr{R}_{10} = \frac{2}{3}t\varepsilon^{-1/3}e^{-2\lambda t}\,\mathrm{Im}\langle f^{-2N}b_3^\varepsilon u, \alpha^\varepsilon u\rangle = \frac{2}{3}t\varepsilon^{-1/3}e^{-2\lambda t}\Big[\mathrm{Im}\langle b_3^\varepsilon f^{-N}u, \alpha^\varepsilon f^{-N}u\rangle$$
$$+\,\mathrm{Im}\langle \gamma_2 f^{-N}u, \alpha^\varepsilon f^{-N}u\rangle + \mathrm{Im}\langle f^{-N}b_3^\varepsilon u, \gamma_1 f^{-N}u\rangle\Big]$$

with operators γ_k of order $k = 1, 2$ coming from the commutators of b_3 and α^ε with f^{-N}. Since the symbol α^ε has coefficient ε^2, for the last two terms on the right we obtain a factor $\varepsilon^{2/3}$ and terms $C\varepsilon^{2/3}t\|f^{-N}u\|_2^2$ which can be estimated by $C\varepsilon^{2/3}t\|\mathbf{a}_2^\varepsilon f^{-N}u\|_0^2$ and lower order terms. The leading contribution is absorbed by $\lambda E_N(u)$ with large λ. To treat the term with b_3^ε and α^ε, we exploit the fact that these operators are self-adjoint and we reduce the analysis to an estimation of

$$te^{-1/3}e^{-2\lambda t}\langle [b_3^\varepsilon, \alpha^\varepsilon]f^{-N}u, f^{-N}u\rangle.$$

From the commutator $[b_3^\varepsilon, \alpha^\varepsilon]$ we obtain a factor ε and a forth order operator. Thus we are going to estimate $Ct\varepsilon^{2/3}e^{-2\lambda t}\|f^{-N}u\|_2^2$ and we proceed as above.

7.8. **Analysis of $\mathscr{R}_j, j = 4, 5, 6$.** We have

$$\mathscr{R}_6 = -\frac{1}{3}e^{-2\lambda t}\Big(2\,\mathrm{Re}\langle f^{-2N}u'', (a_2)_t u\rangle - \mathrm{Re}\langle f^{-2N}u', (a_2)_t u'\rangle\Big).$$

The term involving u' is easy to be treated by using $\lambda E_N(u)$. For the term with u'' we write

$$2e^{-2\lambda t}\,\mathrm{Re}\langle f^{-N}u'', f^{-N}(a_2)_t u\rangle = 2e^{-2\lambda t}\,\mathrm{Re}\langle f^{-N}u'', (a_2)_t f^{-N}u\rangle + R_7.$$

Next

$$2e^{-2\lambda t}|\,\mathrm{Re}\langle f^{-N}u'', (a_2)_t f^{-N}u\rangle| \le \psi\|f^{-N}u''\|_0^2 + Cte^{-2\lambda t}\|f^{-N}u\|_2^2.$$

We may absorb both terms on the right hand side taking λ large in $\lambda E_N(u)$ and using Proposition 6.1 to estimate $t\|f^{-N}u\|_2^2$ by $C_1 t\|a_2^\varepsilon f^{-N}u\|_0^2$ plus lower order terms. The lower order term R_7 is easy to be treated by a similar argument.

Passing to the analysis of $\mathscr{R}_5$, notice that a typical term is $\varepsilon^{-2/3}\psi\,\mathrm{Re}\langle \beta_1 f^{-N}u', f^{-N}u''\rangle$. Here the first order operator β_1 comes from the operator $\varepsilon^{-2/3}\psi\,\mathrm{Re}\langle \beta_1 f^{-N}u', f^{-N}u''\rangle$. and we get a power of ε from the commutator $f^{2N}[f^{-2N}, \alpha^\varepsilon]$ (see (5.10)). Thus we must estimate

$$\varepsilon^{1/3}\Big(C\delta^{-1}e^{-2\lambda t}\|f^{-N}u'\|_1^2 + \delta\frac{e^{-2\lambda t}}{t^2}\|f^{-N}u''\|_0^2\Big),\ \delta > 0.$$

For the term involving $\|f^{-N}u''\|_0^2$ we take δ small and use the corresponding positive term in $\mathscr{R}_1+\mathscr{R}_2$, while for the term including $\|f^{-N}u'\|_1^2$ we exploit $\lambda E_N(u)$ with large λ.
For $\varepsilon^{-2/3}\psi\,\mathrm{Re}\langle\gamma_1 f^{-N}u', f^{-N}u''\rangle$ we repeat the same argument since first order operator γ_1 comes from $[f^{-2N},\alpha^\varepsilon]f^{2N}$. The analysis of $\frac{4}{3}e^{-2\lambda t}\,\mathrm{Re}\langle f^{-2N}\hat{\alpha}_1^\varepsilon u', u''\rangle$ is easer since we do not have a factor ψ. Finally, the analysis of $\mathscr{R}_4$ follows the same argument as above and all terms in $\mathscr{R}_4$ can be absorbed by $\lambda E_N(u)$.

Thus we finished the analysis of all terms in $\sum_{j=1}^{10}\mathscr{R}_j+\sum_{\nu=1}^{3}B_\nu$.

7.9. **Analysis of lower order terms.** The analysis of lower order term in (6.17) is easy since they are generated by lower order terms including only derivatives D_t^2, D_{t,x_j}^2, D_t, D_{x_j}, etc. For this purpose we may use a part of $\lambda E_N(u)$ for example $\frac{\lambda}{3}E_N(u)$ and take into account the estimate (6.5), where we have a term

$$\psi\|f^{-k}(u''+\frac{1}{3}(ta_2^\varepsilon+\varepsilon^{-2/3}\alpha^\varepsilon)u)\|_0^2$$

which will appear with a big coefficient $\frac{\lambda}{3}$. For example, we have

$$|2\,\mathrm{Im}\,\psi\langle f^{-2N}u_{t,x_j}, u''+\frac{1}{3}(ta_2^\varepsilon+\varepsilon^{-2/3}\alpha^\varepsilon)u\rangle|$$

$$\leq\psi\|f^{-N}u'\|_1^2+\psi\|f^{-N}(u''+\frac{1}{3}(ta_2^\varepsilon+\varepsilon^{-2/3}\alpha^\varepsilon)u)\|_0^2$$

and we have a control with big parameter $\mathcal{O}(\lambda)$ of both terms in the right hand side of $\frac{\lambda}{3}E_N(u)$. The analysis of other terms is completely similar and we leave the details to the reader. The terms with second derivatives D_{x_i,x_j}^2 cannot be treated by this argument and for this purpose we have examined $\sum_{\nu=1}^{3}B_\nu$ by a more sophisticated technical tools. This completes the estimate of the errors terms.

As a consequence,(6.17) can be rewritten as a true energy estimate as

$$\begin{aligned}7e^{-2\lambda t}\|f^{-N}\mathcal{P}u\|^2\geq\partial_t\Big(&\mathscr{E}_N(u)+\frac{N}{27}\psi\|f^{-N-1}u'\|_0^2+\frac{N}{27}\psi\|f^{-N-2}u\|_0^2\\&+\frac{N}{27}C_3e^{-2\lambda t}\|f^{-N-1}u\|_1^2\Big)+\lambda K_1E_N(u)\\&+K_2\psi\Big[\|f^{-N-1/2}u''\|_0^2+\|f^{-N-1/2}u'\|_1^2\;+t\|f^{-N-1/2}u\|_2^2\Big]\\&+NK_3\Big[\psi\|f^{-N-1}u'\|_0^2+N\psi\|f^{-N-3/2}u'\|_0^2+\psi\|f^{-N-2}u\|_0^2\\&+\psi N\|f^{-N-5/2}u\|_0^2+te^{-2\lambda t}\|f^{-N-3/2}u\|_1^2\Big]\\&+\lambda NK_4\psi\Big\{\|f^{-N-1}u'\|_0^2+\|f^{-N-2}u\|_0^2+t\|f^{-N-1}u\|_1^2\Big\},\end{aligned}\tag{7.7}$$

where $K_j,\ j=1,...,4$, are suitable positive constants independent of λ and N.

7.10. **Estimates of the terms on the boundary** $s = T$**.** Assume that $0 \le s < T \le 1$ and $u = D_t u = D_t^2 u = 0$ when $t = s$. We take T sufficiently small and we integrate in (7.7) from t to T with respect to s. As a result we obtain integrals $\int_t^T (...)ds$ and for $s = T$ the following boundary terms

$$\mathscr{E}_N((u(T,.)) + \frac{N}{27}\Big(\psi(T)\|f^{-N-1}u'(T,.)\|_0^2 + \psi(T)\|f^{-N-2}u(T,.)\|_0^2 \\ + C_3 e^{-2\lambda T}\|f^{-N-1}u(T,.)\|_1^2\Big). \tag{7.8}$$

Recall that the argument of the previous section yields

$$\mathscr{E}_N(u(T,.) \ge E_N(u) + T\varepsilon^{2/3}A_N^{(2)}(u(T,.)) + 2e^{-2\lambda T}\varepsilon^{1/3}T\,\mathrm{Im}\langle f^{-N}b_3^{\varepsilon}u(T,.), f^{-N}u'(T,.)\rangle. \tag{7.9}$$

It is clear that $(T/3 + \langle\xi\rangle^{-2/3})^{-N} \le c_2(T/3 + \langle\xi\rangle^{-2/3})^{-N-1}$ with $c_2 > 0$ independent on ξ and T. Hence we have a control on the norms $\|(f^{-N}u)(T,.)\|_1$, $\|(f^{-N}u')(T,.)\|_0$, etc. On the other hand, in $E_N(u(T,.))$ we have a positive term $T\|\mathbf{a}_2^{\varepsilon}(f^{-N}u)(T,.)\|_0^2$. Thus we can repeat the argument of subsection 7.6 to absorb the term involving b_3. Since we do not have a big coefficient N in (7.9), it suffices to take ε and T small. For the term $T\varepsilon^{2/3}A_N^{(2)}(u(T,.))$ we are going to repeat the analysis of subsection 7.4. Taking ε and T small and exploiting the term $T\|\mathbf{a}_2^{\varepsilon}(f^{-N}u(T,.)\|_0^2$ in $E_N(u(T,.)$, we absorb this term.

Finally, the contribution of the boundary terms is bounded from below by a positive constant and we may neglect them.

8. A priori estimate

For the function f^{-1} we apply the inequalities

$$f^{-1} = \frac{(1+|\xi|^2)^{1/3}}{\frac{t}{3}(1+|\xi|^2)^{1/3}+1} \le \frac{(1+|\xi|^2)^{1/3}}{1+\frac{t}{3}},\ t \ge 0,$$

$$f^{-1} \ge \frac{1}{1+t},\ 0 \le t \le T \le 1.$$

Therefore from (7.7) and the analysis in the Section 7 we deduce for $\lambda \ge \lambda_0$ the estimate

$$\lambda\int_t^T e^{-2\lambda s - 2N\log(1+s)}\Big(\sum_{k=0}^{2} s^{1-k}\|\partial_t^k u(s,.)\|_{(2-k)}^2 + \sum_{k=0}^{1}\|\partial_t^k u(s,.)\|_{(1-k)}^2\Big)ds \\ \le C_0\int_t^T e^{-2\lambda s - 2N\log(1+\frac{s}{3})}\|\mathcal{P}u(s,.)\|_{(2N/3)}^2 ds, \tag{8.1}$$

where $\|.\|_{(m)}$ is the $H_{(m)}$ norm in $\mathbb{R}^n$ for fixed m.

Of course, we have a negative power of s only in front of the norm $\|u''(s,.)\|_0^2$ and we may estimate from below this term without a power of s. On the other hand, the norm $\|u(s,.)\|_{(2)}^2$ appears with a coefficient s.

Remark 8.1. *It is not useful to use the estimate $f^{-1} \le \frac{3}{t}$ to bound the term $\|f^{-N+1/2}\mathcal{P}u\|^2$ in (7.7). If we did then, in (8.1), we would have the integral*

$$\int_t^T e^{-2\lambda s}\frac{1}{s^{2N}}\|\mathcal{P}u(s,.)\|_{(0)}^2 ds$$

and as $t \to 0$ this would produce no uniform estimates with respect to $t \ge 0$.

To estimate the high order derivatives with respect to x, consider the operator $(1+|D_x|^2)^{\frac{1}{2}p} = \Lambda_p,\ p>0$ and write

$$\Lambda_p Pu = D_t^3(\Lambda_p u) - t\Big(a_2 + [\Lambda_p, a_2]\Lambda_p^{-1}\Big)D_t(\Lambda_p u) + \Big(b_2 + [\Lambda_p, b_2]\Lambda_p^{-1}\Big)(\Lambda_p u)$$

$$+t\Big(a_1 + [\Lambda_p, a_1]\Lambda_p^{-1})D_t^2(\Lambda_p u) + ...$$

Moreover, we observe that the "perturbations" $[\Lambda_p, a_2]\Lambda_p^{-1}$, $[\Lambda_p, b_2]\Lambda_p^{-1}$, $[\Lambda_p, a_1]\Lambda_p^{-1}$ have order lower than the terms a_2, b_2 and a_1, respectively. Then $v = \Lambda_p u$ satisfies an equation of the type studied above and, moreover, $(D_t^2 v)(t,x) = (D_t v)(t,x) = v(t,x) = 0$. Going back to the differential operator P, we get the following

Theorem 8.1. *Assume that $(D_t^2 u)(t,x) = (D_t u)(t,x) = u(t,x) = 0$ and let $0 \leq t \leq s \leq T$ with a small $T>0$. Then for every $p \in \mathbb{R}$ there exist Δ_p and a constant C_p so that for $\lambda \geq \Delta_p$, $N = \frac{13}{2}\Pi + N_0$ and $u \in C_0^\infty(\mathbb{R}^{n+1})$ we have the estimate*

$$\lambda \int_t^T e^{-2\lambda s - 2N\log(1+s)}\Big(\sum_{k=0}^{2} s^{1-k}\|\partial_t^k u(s,.)\|^2_{(2-k+p)} + \sum_{k=0}^{1}\|\partial_t^k u(s,.)\|^2_{(1-k+p)}\Big)ds$$
$$\leq C_p \int_t^T e^{-2\lambda s - 2N\log(1+\frac{s}{3})}\|Pu(s,.)\|^2_{(2N/3+p)}ds. \tag{8.2}$$

Next we discuss the estimates for functions $u(t,x)$ satisfying the boundary conditions $D_t^2 u(T,x) = D_t u(T,x) = u(T,x) = 0$ for $T>0$. To do this, we proceed along the same lines as above. We use the function $\varphi(t) = te^{2\lambda t}$ and we take the scalar product of $f^{2N}(t,D_x)\mathcal{P}u$ with the operator $Lu = \varphi(t)(D_t^2 - \frac{1}{3}(ta_2^\varepsilon u + \varepsilon^{-2/3}\alpha^\varepsilon u))$. Thus we consider

$$2\,\mathrm{Im}\langle\varphi(t)f^{2N}(t,D_x)\mathcal{P}u, D_t^2 - \frac{1}{3}(ta_2^\varepsilon u + \varepsilon^{-2/3}u)\rangle.$$

Notice that we have changed $-\lambda$ to λ, f^{-2N} to f^{2N} and we have a $+$ sign in front of the scalar product. We then handle the terms in the same way as we did in Sections 5-8. For example,

$$2\,\mathrm{Im}\,\frac{1}{i}\langle\varphi(t)f^{2N}\partial_t^3 u, \partial_t^2 u\rangle = -2\,\mathrm{Re}\langle\varphi(t)f^{2N}\partial_t^3 u, \partial_t^2 u\rangle$$

$$= -\partial_t\Big(\varphi(t)\|f^{2N}\partial_t^2 u\|^2\Big) + 2N\varphi(t)\|f^N\partial_t^2 u\|^2 + \varphi'(t)\|f^N\partial_t^2 u\|^2.$$

Thus we obtain an analog of (6.17) in the form

$$
\begin{aligned}
7e^{2\lambda t}\|f^N\mathcal{P}u\|^2 \geq -\partial_t\Big(&\mathscr{E}_N^*(u) + \frac{N}{27}e^{2\lambda t}\|f^{N-1}u'\|_0^2 + \frac{N}{27}e^{2\lambda t}\|f^{N-2}u\|_0^2 \\
&+\frac{N}{27}C_3te^{2\lambda t}\|f^{N-1}u\|_1^2\Big) \\
&+2\lambda\mathscr{E}_N^*(u) + \varepsilon^{1/3}te^{2\lambda t}\frac{4N}{3}\operatorname{Im}\langle f^{2N-1}b_3u, u'\rangle \\
+N\varphi\Big[&\frac{5}{27}\|f^{N-1/2}u''\|_0^2 + \frac{10}{27}\operatorname{Re}\langle f^{N-1/2}(t\mathbf{a}_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)u', f^{N-1/2}u'\rangle \\
+\frac{4}{9}\|f^{N-1/2}&(u'' + \frac{1}{2}(t\mathbf{a}_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)u)\|_0^2 + \frac{1}{9}\|f^{N-1/2}(t\mathbf{a}_2^\varepsilon u + \varepsilon^{-2/3}\alpha^\varepsilon u)\|_0^2\Big] \\
&+\frac{N}{9}\Big[\frac{\varphi'}{3}\|f^{N-1}u'\|_0^2 + \frac{2N-5}{9}\varphi\|f^{N-3/2}u'\|_0^2 + \frac{\varphi'}{3}\|f^{N-2}u\|_0^2 \\
&+\frac{2N+1}{9}\varphi\|f^{N-5/2}u\|_0^2 + \frac{2C_3}{3}\lambda e^{2\lambda t}\|f^{-N-1}u\|_1^2 \\
&+\frac{(2N+1)C_3}{9}e^{2\lambda t}\|f^{N-3/2}u\|_1^2 + \frac{C_3}{3}e^{2\lambda t}\|f^{N-1/2}u'\|_1^2\Big] \\
&+\mathcal{Q}_1^* + \sum_{j=3}^{10}\mathscr{R}_j^* + \sum_{\nu=1}^{3}B_\nu^* + \text{lower order terms},
\end{aligned}
\tag{8.3}
$$

where $\mathcal{Q}_1^*$, $\mathscr{R}_j^*$, B_ν^* are obtained from the corresponding terms $\mathcal{Q}_1$, $\mathscr{R}_j$, B_ν changing N by $-N$ and

$$
\begin{aligned}
\mathscr{E}_N^*(u) = \varphi\Big[&\|f^Nu''\|^2 + (1-\theta)\operatorname{Re}\langle f^{2N}(ta_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)u', u'\rangle \\
+\theta\|f^N(ta_2^\varepsilon u &+ \varepsilon^{-2/3}\alpha^\varepsilon)u\|_0^2 + \theta 2\operatorname{Re}\langle f^{2N}u'', (ta_2^\varepsilon + \varepsilon^{-2/3}\alpha^\varepsilon)u\rangle\Big] \\
&+\varepsilon^{1/3}t^3e^{2\lambda t}2\operatorname{Im}\langle f^{2N}b_3^\varepsilon u, u'\rangle.
\end{aligned}
\tag{8.4}
$$

We repeat the argument of the previous sections and we integrate with respect to s from t to T, assuming $0 \leq t < T \leq 1$. Thus we obtain an a priori estimate involving the "weights" f^{2N-k}, $-1 \leq k \leq 5/2$.

On the other hand,

$$
\begin{aligned}
&f^{2N} \leq (t+1)^{2N},\ 0 \leq t < T \leq 1, \\
&f^{2N} \geq (1+\frac{t}{3})^{2N}(1+|\xi|^2)^{-2N/3}.
\end{aligned}
$$

Consequently, for $\lambda \geq \lambda_0 > 0$, we deduce

$$
\begin{aligned}
\lambda\int_t^T e^{2\lambda s+2N\log(1+\frac{s}{3})}\Big(\sum_{k=0}^{2}s^{3-k}\|\partial_t^ku(s,.)\|_{(2-k-2N/3)}^2 + \sum_{k=0}^{1}\|\partial_t^ku(s,.)\|_{(1-k-2N/3)}^2\Big)ds \\
\leq C_0\int_t^T e^{2\lambda s+2N\log(1+s)}s^2\|\mathcal{P}u(s,.)\|_{(0)}^2ds.
\end{aligned}
\tag{8.5}
$$

Finally, we may make a shift in the Sobolev indices for this estimate and consider $\|.\|_{(p)}$ norms. Thus we eventually obtain the following

Theorem 8.2. *Assume that $(D_t^2u)(T,x) = (D_tu)(T,x) = u(T,x) = 0$ and let $0 \leq t \leq s \leq T$ with a small $T > 0$. Then for every $p \in \mathbb{R}$ there exist Δ_p and a constant C_p so that for $\lambda \geq \Delta_p$, $N = \frac{13}{2}\Pi + N_0$ and $u \in C_0^\infty(\mathbb{R}^{n+1})$ we have the estimate*

$$\lambda \int_t^T e^{2\lambda s+2N\log(1+\frac{s}{3})}\Big(\sum_{k=0}^{2} s^{3-k}\|\partial_t^k u(s,.)\|^2_{(2-k+p)} + \sum_{k=0}^{1}\|\partial_t^k u(s,.)\|^2_{(1-k+p)}\Big)ds$$

$$\leq C_p \int_t^T e^{2\lambda s+2N\log(1+s)} s^2 \|Pu(s,.)\|^2_{(2N/3+p)}ds. \tag{8.6}$$

For the local uniqueness result it is more convenient to have estimates for the operator $\mathcal{P}^*$, the adjoint to $\mathcal{P}$. We have

$$\mathcal{P}^* = D_t^3 u - (ta_2 + \alpha)D_t + t^2 b_3 + \bar{b}_2 + ia_2 + t\alpha_2 + \text{lower order terms},$$

where α_2 is a second order operator with respect to x. The subprincipal symbol of $\mathcal{P}^*$ for $\rho = (0,x,\xi)$ has the form

$$\frac{i}{2}a_2(0,x,\xi) + \bar{b}_2(0,x,\xi) = \overline{p_2'}(0,x,\xi).$$

Thus the number Π^* corresponding to $\mathcal{P}^*$ coincides with Π and Theorems 8.1 and 8.2 hold for the operator $\mathcal{P}^*$ changing, if it is necessary, Λ_p and C_p.

Applying Theorems 8.1 for P and Theorem 8.2 for P^*, we can establish an existence and uniqueness results for the Cauchy problem in $G = \{(t,x): \ 0 \leq t \leq T, x \in U_{x_0}\}$ with sufficiently small T. To fix the notations, we say that $f \in H^{loc}_{(q,s)}(G)$ if $\varphi f \in H_{(q,s)}(G)$ for all $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$ and $g \in H_{(q,s)}(\mathbb{R}^{n+1})$ if

$$\|g\|^2_{(q,s)} = (2\pi)^{-(n+1)}\int (1+|\tau|^2)^q(1+\xi|^2)^s|\hat{g}(\tau,\xi)|^2 d\tau d\xi < \infty.$$

Since P is strictly hyperbolic for $0 < t \leq T$, we may repeat with minor modifications the proof of Theorem 23.4.5 in [5] to obtain the following

Theorem 8.3. *Let P be a differential operators with C^∞ coefficients in $G = [0,T] \times U_{x_0}$ satisfying the hypothesis $(H_0) - (H_2)$ and let $V_{x_0} \subset U_{x_0}$. For T sufficiently small and for $f \in H^{loc}_{(0,s)}(G)$ having support in $\bar{G}$ one can find an unique $u \in H^{loc}_{(2,s+2-2N/3)}(G)$ with support in $\bar{G}$ so that $Pu = f$ in $(0,T) \times V_{x_o}$.*

We leave the details to the reader.

In conclusion the conjecture for strongly hyperbolic operators with triple characteristics is true for operators satisfying $(H_0) - (H_2)$.

9. Appendix

In this appendix we discuss the existence of a factorization

$$((\tau - \beta(t,x,\xi))^2 - D(t,x,\xi))(\tau - \gamma(t,x,\xi)) \tag{9.1}$$

of the principal symbol $p_3(t,x,\tau,\xi)$ having the form (2.1). We suppose that in (9.1) the symbols γ and β are smooth functions homogeneous of order 1 in ξ, while D is smooth and homogeneous of order 2 in ξ. It is clear that the root γ must be real-valued. We suppose that (9.1) holds in a conic neighborhood of a point $(0,x_0,\xi_0)$ or for fixed $\xi_0 \neq 0$ and (t,x) in a neighborhood of

$(0, x_0)$. For the counterexamples we will discuss the non existence of a factorization for fixed ξ_0. The problem is to see if there exists a smooth real root $\gamma(t, x, \xi)$ of $p_3 = 0$ in a neighborhood of $(0, x_0, \xi_0)$.

Consider the symbol

$$p_3 = \tau^3 - (ta_2(t, x, \xi) + \alpha(x, \xi))\tau + t^2 b_3(t, x, \xi) \tag{9.2}$$

with $a_2(t, x, \xi) \geq c|\xi|^2,\ \alpha(x, \xi) \geq 0$ and $\alpha(0, \xi_0) = 0,\ 4(ta_2 + \alpha)^3 \geq 27t^4 b_3^2$ for $t \geq 0$.

Proposition 9.1. *Assume that* $b_3(0, 0, \xi_0) \neq 0$. *Then if* $p_3(t, x, \tau, \xi_0)$ *is factorizable for* (t, x) *in a neighborhood of* $(0, 0)$, *there exists a neighborhood* $U \subset \mathbb{R}^n$ *of* 0 *such that* $\alpha(x, \xi_0) = 0,\ \forall x \in U$.

Proof. Assume that there exists a real-valued function $\gamma(t, x, \xi_0)$ which is a solution of $p_3(t, x, \gamma, \xi_0) = 0$. Assume that there exists a sequence $x_m \to 0$ such that $a(x_m, \xi_0) \neq 0$. For $t = 0$ we obtain the equation $\gamma^3 - \alpha\gamma = 0$ and there are two possibilities.

(i) $\gamma(0, x, \xi_0) = 0$. Then $\gamma = t\rho(t, x, \xi_0)$ with a continuous $\rho(t, x, \xi_0)$ and we get for $t > 0$ the equality

$$t\rho^3 - \rho - \frac{\alpha}{t}\rho + b_3 = 0.$$

If $\lim_{(t,x)\to(0,0)} \rho(t, x, \xi_0) \neq 0$, choosing $t = |\alpha(x_m, \xi_0)|^2$ and letting $m \to 0$, we obtain a contradiction. If $\lim_{(t,x)\to(0,0)} \rho(t, x, \xi_0) = 0$, we take $t = |\alpha(x_m, \xi_0)|$ and passing to the limit $m \to \infty$, we obtain a contradiction with the fact that $b_3(0, 0, \xi_0) \neq 0$.

(ii) Let $\gamma(0, x, \xi_0) = \sqrt{\alpha(x, \xi_0)}$, provided $\sqrt{\alpha}$ smooth. Then we have $\gamma = \sqrt{\alpha(x, \xi_0)} + t\rho(t, x, \xi_0)$. In this case we obtain for $t > 0$ the equality

$$2\frac{\alpha}{t}\rho + 3\sqrt{\alpha}\rho^2 + t\rho^3 - \rho - \frac{\sqrt{\alpha}}{t} + b_3 = 0.$$

We choose $t = |\alpha(x_m, \xi_0)|$ and passing to the limit $m \to \infty$ we obtain a contradiction.

It is interesting that we have an inverse result.

Proposition 9.2. *The symbol* (9.2) *with* $\alpha(x, \xi) \equiv 0$ *is factorizable in a neighborhood of* $(0, x_0, \xi_0)$.

Proof. Clearly, the discriminant $\Delta = 4t^3 a_2^3 - 27t^4 b_3^2$ is positive for small $t > 0$. The three roots of the equation $p_3(t, x, \tau, \xi) = 0$ with respect to τ have the form (see for instance, [15])

$$x_k = -\frac{1}{3}\Big(u_k C + \frac{3ta_2}{u_k C}\Big),\ k = 1, 2, 3. \tag{9.3}$$

where u_k are the three roots of the equation $u^3 = 1$ and C has the form

$$C = \Big(\frac{27t^2 b_3 + \sqrt{-27\Delta}}{2}\Big)^{1/3}.$$

Our goal is to show that we have a real C^∞ smooth root $\gamma(t, x, \xi)$ of $p_3 = 0$ defined for $|t| \leq \varepsilon$ and (x, ξ) in a conic neighborhood of (x_0, ξ_0).

In our case C becomes

$$C = 2^{-1/3}\Big(27t^2 b_3 + 27\sqrt{-\frac{4}{27}t^3 a_2^3 + t^4 b_3^3}\Big)^{1/3}$$

$$= 2^{-1/3}3\Big(t^2b_3 + \sqrt{-t^3\alpha^3 + t^4b_3}\Big)^{1/3} = 2^{-1/3}3\Big(i(\alpha t)^{3/2}(1 - \frac{b_3^2}{\alpha^3}t)^{1/2} + t^2b_3\Big)^{1/3},$$

where $\alpha = \frac{4^{1/3}a_2}{3}$. Thus

$$C = 2^{-1/3}3(\alpha t)^{1/2}\Big(i\Big(1 - \frac{b_3^2}{\alpha^3}t\Big)^{1/2} + \frac{b_3}{\alpha^{3/2}}t^{1/2}\Big)^{1/3}.$$

To obtain a real root $\gamma(t,x,\xi)$, we take $u_k = 1$ in (9.3) and one deduces

$$C + \frac{3ta_2}{C} = \frac{C^2 + 3ta_2}{C}.$$

Now since $i^{2/3} = (-1)^{1/3} = -1$, we get

$$C^2 + 3ta_2 = \Big[-9\alpha 4^{-1/3}\Big((1 - \frac{b_3^2}{\alpha^3}t)^{1/2} - \frac{ib_3}{\alpha^{3/2}}t^{1/2}\Big)^{2/3} + 3a_2\Big]t$$

$$= t\Big[-9\alpha 4^{-1/3} + 3a_2 + \frac{4^{-1/3}6ib_3}{\alpha^{1/2}}t^{1/2} + \mathcal{O}(t)\Big] = t^{3/2}\Big[\frac{4^{-1/3}6ib_3}{\alpha^{1/2}} + \mathcal{O}(t^{1/2})\Big].$$

Dividing by C, we get

$$\gamma = -\frac{1}{3}\frac{C^2 + 3ta_2}{C} = t\Big[\frac{4^{-1/3}2^{1/3}2b_3}{3\alpha} + \mathcal{O}(t^{1/2})\Big] = t\Big[\frac{b_3}{a_2} + \mathcal{O}(t^{1/2})\Big].$$

Consequently, the real root $\gamma(t,x,\xi)$ is derivable at $t = 0$ and $\partial_t\gamma|_{t=0} = \frac{b_3}{a_2}$. Let $\gamma = t\rho$. Therefore $t\rho^3 - a_2\rho + b_3 = 0$.

Next, consider the function $F(\rho,t,a_2,b_3) = t\rho^3 - a_2\rho + b_3$. Since

$$\frac{\partial F}{\partial \rho}\Big|_{t=0} = -a_2 \neq 0,$$

by the implicit function theorem we conclude that for small t the function $\rho(t,a_2,b_3)$ is smooth. This implies that the function $\gamma(t,x,\xi)$ is smooth and we have a factorization

$$p_3 = ((\tau - a(t,x,\xi))^2 - b(t,x,\xi))(\tau - \gamma(t,x,\xi))$$

with $a = -\frac{\gamma}{2}$ and $b = ta_2 - 3a^2$.

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